Chiral anomaly in soft collinear effective theory

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Anomalies have infrared and ultraviolet ingredients, and are often realized in effective theories in a nontrivial way. We study the chiral anomaly in soft collinear effective theory (SCET), where the anomaly equation has terms contributing at different orders in the power expansion. The chiral anomaly equations in SCET are computed up to next-to-next-to-leading order in the power counting with external collinear and/or ultrasoft gluons. We do this by expanding the QCD anomaly equation, using the tree level (leading order in $\alpha_s$) relations between QCD and SCET fields. The validity of this correspondence between the anomaly equations is confirmed by direct computation of the one-loop diagrams in SCET.

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\section*{I. INTRODUCTION}

In calculations with a hierarchy of scales it is often useful to work with an effective field theory, in which ultraviolet degrees of freedom have been integrated out. Examples of this are heavy quark effective theory and soft collinear effective theory (SCET). Or as a more well known example, integrating out a heavy quark.

Anomalies [1,2] come from ultraviolet divergences, but they are also an infrared effect in the sense that only massless particles contribute [3,4]. If masses are generated through a Higgs mechanism as in the standard model, fields are massless above the symmetry breaking scale and can contribute to anomalies too.

Because of the infrared nature of anomalies one would expect effective field theories to reproduce the anomalies of the full theory in some way, but this is not always a trivial matter. As an illustration we consider the case of integrating out a heavy quark.

Another example is the axial anomaly in chiral perturbation theory. By itself chiral perturbation theory would not include reactions such as $\pi^0 \rightarrow \gamma \gamma$ or $K \bar{K} \rightarrow \pi^+ \pi^- \pi^0$. They do not show up because of the symmetry $M \rightarrow -M$, where $M$ is the matrix of Goldstone bosons, which is not a symmetry of QCD. Again one needs to add Wess-Zumino terms and additional terms that couple to the gauge fields [7,8].

The goal of this paper is to study how the chiral anomaly is realized in SCET. SCET has been introduced to describe the dynamics of energetic light hadrons and provides a systematic way to separate the hard, collinear, and soft scales [9–12]. It captures the long-distance physics in terms of collinear, soft, and ultrasoft (usoft) fields. The short-distance physics is absorbed into Wilson coefficients. So in SCET a quark field gets replaced by these different fields, which can each run around in loops. Currents in SCET often generate $\frac{1}{\Lambda^2}$ divergences at one loop that are removed by renormalization. We will see that such divergences show up in individual SCET anomaly diagrams, but cancel when they are summed. Also in SCET there are vertices with two quark fields and more than one gluon, which lead to nontriangle anomaly diagrams. These graphs turn out to be important.

Which SCET fields one needs to consider depends on the physical process one has in mind. We will be looking at SCET$_1$ with one collinear direction $n^\mu$. The fields in this theory are collinear quarks $\xi_n$ and gluons $A_n^\mu$ and usoft quarks $q_{us}$ and gluons $A_{us}^\mu$. The momenta of the collinear fields scale like $p_n^\mu = (n \cdot p, n \cdot p, p_\perp) \sim Q(\lambda^2, 1, \lambda)$ and for usoft fields as $p_{us}^\mu \sim Q \lambda^2$. Here $Q$ is the hard scale, $\lambda$ the SCET expansion parameter, and $n^\mu$ and $\bar{n}^\mu$ are lightcone basis vectors satisfying $n^2 = \bar{n}^2 = 0$ and $n \cdot \bar{n} = 2$. One can, for example, take $n^\mu = (1, 0, 0, 1)$ and $\bar{n}^\mu = (1, 0, 0, -1)$. For a typical process $\lambda^2 = \Lambda_{QCD}/Q$. We will also need the power counting of the fields, which are listed in Table I.

To get a systematic expansion in the power counting, collinear fields have both a label momentum $p$ containing the collinear momentum and a (usoft) coordinate $x$, $\xi_{n,p}(x)$. The label momenta are picked out by the label operator $\mathcal{P}$, e.g. $\mathcal{P}_n^\mu \xi_{n,p} = p_n^\mu \xi_{n,p}$. The collinear covariant derivatives are then defined as

\[ i\bar{n} \cdot D_n = \bar{n} \cdot \mathcal{P} + g\bar{n} \cdot A_{n,q}, \quad iD_n^{\perp \mu} = \mathcal{P}_\perp + gA_{n,q}^{\perp \mu}, \]

\[ in \cdot D = in \cdot \partial + gn \cdot A_{n,q} + gn \cdot A_{us}, \quad \text{(1)} \]

and the usoft covariant derivative as

\[ iD_{us}^\mu = i\partial^\mu + gA_{us}^\mu. \quad \text{(2)} \]

Note that these covariant derivatives are each homogenous in the power counting. We are now ready to write down the
leading order Lagrangian for a collinear quark $\xi_n$ \cite{10},
\begin{equation}
L^{(0)}_{\xi_n,\xi} = \bar{\xi}_n(x) \left( i n \cdot D + i \psi_\perp \frac{1}{i n \cdot D_c} i \psi \right) \frac{\hat{\mu}}{2} \xi_n(x),
\end{equation}
where it is understood that the (suppressed) label momenta are summed over and that the label momenta of each term in $\mathcal{L}$ are conserved.

This paper is organized as follows: We start in Sec. II A by matching the chiral anomaly equation in QCD onto SCET, using the tree level relations between fields. For simplicity we first restrict ourselves to leading order (LO) in the power counting, postponing the matching up to next-to-next-to-leading order (NNLO) to Sec. III A. Whenever we speak of e.g. LO and next-to-leading order (NLO) in this paper, we will always be referring to the order in the power expansion $\lambda$ and not to radiative corrections. No contributions are expected beyond one loop due to the Adler and Bardeen theorem \cite{13}. We then verify these SCET anomaly equations by computing the one-loop anomaly diagrams, we will (as usual) restrict ourselves to two outgoing gluons. Gauge invariance then determines anomaly matrix elements for more gluons. Let $p$, $q$ be the momenta of these two gluons. For simplicity we assume that they have no component in the $\perp$ direction, $p_\perp = q_\perp = 0$. We will also assume that the gluons are $\perp$ polarized. The matrix element of $\mathcal{J}$ is then given by
\begin{equation}
\langle gg|\mathcal{J}|0\rangle = \frac{i}{2} n \cdot (p + q) \langle gg|n \cdot J_5|0\rangle
+ \frac{i}{2} n \cdot (p + q) \langle gg|n \cdot J_3|0\rangle.
\end{equation}
The tree level relation between the QCD and SCET (collinear) field is \cite{10}
\begin{equation}
\Psi = \frac{\xi_n}{\lambda} + \frac{1}{i n \cdot D_c} \frac{1}{2} \frac{\hat{\mu}}{\xi_n} + \ldots,
\end{equation}
with the power counting as indicated. This leads to
\begin{equation}
\langle gg|\mathcal{J}^{(4)}|0\rangle = \frac{i}{2} n \cdot (p + q)\langle gg|n \cdot J_5^{(2)}|0\rangle
+ \frac{i}{2} n \cdot (p + q)\langle gg|n \cdot J_3|0\rangle.
\end{equation}
We introduce the following notation for the left- and right-hand side of Eq. (4),
\begin{equation}
\mathcal{J} = \partial_\mu J_5^\mu,
\mathcal{F} = -\frac{g^2}{16\pi^2} \epsilon_{\alpha\beta\mu\nu} \text{tr}[F^{\alpha\beta} F^{\mu\nu}].
\end{equation}
and write their expansions as
\begin{equation}
\mathcal{J} = \mathcal{J}^{(4)} + \mathcal{J}^{(5)} + \mathcal{J}^{(6)} + \ldots,
\end{equation}
\begin{equation}
\mathcal{F} = \mathcal{F}^{(4)} + \mathcal{F}^{(5)} + \mathcal{F}^{(6)} + \ldots,
\end{equation}
where the order in $\lambda$ is indicated in brackets. When calculating the one-loop anomaly diagrams, we will (as usual) restrict ourselves to two outgoing gluons. Gauge invariance then determines anomaly matrix elements for more gluons. Let $p$, $q$ be the momenta of these two gluons. For simplicity we assume that they have no component in the $\perp$ direction, $p_\perp = q_\perp = 0$. We will also assume that the gluons are $\perp$ polarized. The matrix element of $\mathcal{J}$ is then given by
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\end{equation}
with the power counting as indicated. This leads to
\begin{equation}
\langle gg|\mathcal{J}^{(4)}|0\rangle = \frac{i}{2} n \cdot (p + q)\langle gg|n \cdot J_5^{(2)}|0\rangle
+ \frac{i}{2} n \cdot (p + q)\langle gg|n \cdot J_3|0\rangle.
\end{equation}

II. THE SCET ANOMALY AT LO

A. Matching the QCD anomaly onto SCET

In this section we will calculate the chiral anomaly in SCET at LO. We start by expanding the left- and right-hand side of the QCD anomaly equation
\begin{equation}
\partial_\mu J_5^\mu = -\frac{g^2}{16\pi^2} \epsilon_{\alpha\beta\mu\nu} \text{tr}[F^{\alpha\beta} F^{\mu\nu}]
\end{equation}
in the power counting, where $J_5^\mu = \bar{\Psi} \gamma^\mu \gamma_5 \Psi$. We use the tree level relations between fields, effectively matching QCD onto SCET at tree level. In the next subsections we will explicitly check that the SCET anomaly equation we find holds at one loop.

We introduce the following notation for the left- and right-hand side of Eq. (4),

<table>
<thead>
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<th>Operator</th>
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<th>Covariant derivatives</th>
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<td>Power counting</td>
<td>$\xi_n$</td>
<td>$\bar{n} \cdot A_n$</td>
<td>$A_n^{\mu}$</td>
<td>$n \cdot A_n$</td>
<td>$\xi_n$</td>
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<tr>
<td>$\lambda$</td>
<td>$\lambda^0$</td>
<td>$\lambda$</td>
<td>$\lambda^2$</td>
<td>$\lambda^3$</td>
<td>$\lambda^2$</td>
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TABLE I. Power counting of the various fields and operators in SCET.

\( J \equiv \partial_\mu J_5^\mu, \quad F = -\frac{g^2}{16\pi^2} \epsilon_{\alpha\beta\mu\nu} \text{tr}[F^{\alpha\beta} F^{\mu\nu}] \),
where \( J = J^{(4)} + J^{(5)} + J^{(6)} + \ldots, \)
\( F = F^{(4)} + F^{(5)} + F^{(6)} + \ldots, \)

However, it is important to note that \( n \cdot J_5^{(4)} \) contributes at the same order as the $\bar{n} \cdot J_5$ term in the anomaly equation because of the power counting for collinear momenta,
\begin{equation}
\langle gg|\mathcal{J}^{(4)}|0\rangle = \frac{i}{2} n \cdot (p + q)\langle gg|n \cdot J_5^{(2)}|0\rangle
+ \frac{i}{2} n \cdot (p + q)\langle gg|n \cdot J_3|0\rangle.
\end{equation}

We will first study these terms separately by imposing $\bar{n} \cdot (p + q) = 0$ or $n \cdot (p + q) = 0$ in the following subsections. With these additional assumptions we have to
keep gluons off shell, \( p^2, q^2 \neq 0 \), because everything would otherwise trivially vanish. At the end we will drop these additional assumptions, effectively combining the two cases.

Now we consider the right-hand side of the anomaly equation. With our assumptions the important terms are 

\[
\begin{align*}
\epsilon_{\alpha\beta\mu} F^{\alpha\beta} F_{\mu} &= -\frac{4}{g^2} \epsilon_{\alpha\beta\mu} \epsilon^{\alpha\beta \mu} \\
\end{align*}
\]

with \( \epsilon^{\alpha\beta \mu} \) the only contribution we need to consider at \( O(\lambda^3) \) is

\[
\mathcal{F}^{(4)} = -\frac{g^2}{16\pi^2} \times -\frac{4}{g^2} \epsilon_{\alpha\beta\mu} \bar{\alpha}^\alpha \bar{\beta}^\beta \bar{\mu}^\mu \\
\times \bar{\gamma}_{\lambda} \gamma_{\rho} A^{\alpha}_{\rho q} + i n \cdot \partial g A^{\alpha}_{\rho q} + \bar{\gamma}_{\lambda} \gamma_{\rho} A^{\alpha}_{\rho q} + i n \cdot \partial g A^{\alpha}_{\rho q}. \\
\]

\[
\mathcal{F}^{(4)} = \mathcal{F}^{(4)}, \\
\]

we will verify this result by calculating the one-loop diagrams.

**B. LO with \( \bar{n} \cdot (p + q) = 0 \)**

We start by considering the simplest case, namely \( \bar{n} \cdot (p + q) = 0 \) for the momenta of the external gluons. The axial current is then given by

\[
\langle gg | \mathcal{F}^{(4)} | 0 \rangle = \frac{i}{2} \bar{n} \cdot (p + q) \langle gg | \bar{n} \cdot J_5^{(2)} | 0 \rangle \\
= \frac{i}{2} \bar{n} \cdot (p + q) \langle gg | \bar{\gamma}_5 \bar{\gamma}_5 | 0 \rangle, \\
\]

and there are two diagrams that contribute at this order, shown in Fig. 1. As mentioned before, taking the gluons on shell would make everything vanish because of \( \bar{n} \cdot (p + q) = 0 \). We therefore keep the gluons off shell, which generally takes care of any IR divergences as well. Because we assume \( \bar{n} \cdot (p + q) = 0 \), we still need to be careful when combining denominators as \( (l + p)^2 \) and \( l^2 \). We will elaborate on this for the bubble diagram.

In dimensional regularization with \( d = 4 - 2\epsilon \), the triangle diagram is given by

\[
A_T = \frac{1}{2} \bar{n} \cdot (p + q) (-1) \int d^d l \text{tr} \left[ \gamma_{\lambda} \gamma_{\rho} \bar{n} \cdot (l - q) \right] \\
\times \text{tr} \left[ T^{B} A^B \right] \\
\times \text{tr} \left[ T^{A} A^A \right], \\
\]

where \( \gamma_5 \) in dimensional regularization correctly requires some care, for example, using the 't Hooft-Veltman scheme. We avoid such complications by never (anti)commuting with \( \gamma_5 \) in our calculations. Since the external momenta have no \( \perp \) component, we can immediately replace \( \bar{\gamma}_{\lambda} \gamma_{\rho} \rightarrow \bar{\gamma}_{\lambda} \gamma_{\rho} \) \( \bar{n} \cdot (l + p) \), \( \bar{n} \cdot (l - q) \) and \( \bar{n} \cdot (l - q) \).

\[
\epsilon^{\perp}_{\mu \nu} = \frac{i}{2} \epsilon_{\alpha\beta\mu} \bar{\alpha}^\alpha \bar{n}^\beta \bar{n}^\mu = \frac{i}{2} \text{tr} \left[ \gamma_5 \bar{n} \gamma_5 \right], \quad \epsilon_{\perp} = \frac{1}{2} \bar{\gamma}_{\lambda} \gamma_{\rho} \bar{n}, \quad (\perp, q, \bar{q}, \bar{\gamma}_5), \\
\]

\[
\mathcal{F}^{(4)} = \mathcal{F}^{(4)}, \\
\]

we will verify this result by calculating the one-loop diagrams.
This leads to
\[ A_T = -ig^2 n \cdot (p + q) \delta^{AB} \epsilon_{\rho \nu}^\perp (I_1 + I_2 + I_3) - (1 + 2\epsilon)I_4 + (p, \rho, A) \leftrightarrow (q, \nu, B) \]
\[ = \frac{g^2}{8\pi^2} n \cdot (p + q) \bar{n} \cdot p \delta^{AB} \epsilon_{\rho \nu}^\perp. \] (22)

where
\[ I_1 = \int dl \frac{\bar{n} \cdot (l - q) \bar{n} \cdot (l + p)}{(l - q)^2(l + p)^2}, \]
\[ I_2 = \int dl \frac{\bar{n} \cdot (l - q) \bar{n} \cdot (l + p)}{(l - q)^2(l + p)^2}, \]
\[ I_3 = \int dl \frac{\bar{n} \cdot (l + p)}{(l - q)^2(l + p)^2}, \]
\[ I_4 = \int dl \frac{\bar{n} \cdot l \bar{n} \cdot (l - q) \bar{n} \cdot (l + p)}{(l - q)^2(l + p)^2}. \] (23)

We computed these integrals using an appropriate Feynman parametrization. All the necessary integrals can be found in the appendix.

The contribution from the bubble diagram (right panel of Fig. 1) is
\[ A_B = \frac{1}{2} \bar{n} \cdot (p + q)(-1) \int dt \left[ \bar{n} \cdot \gamma_5 \left( \frac{1}{2} \frac{l}{l - q} \right) \bar{n} \cdot \gamma_5 \left( \frac{1}{2} \frac{l}{l - q} \right) \right] \]
\times \left[ \bar{n} \cdot (l + p) \bar{n} \cdot (l - q) \right]
\[ = ig^2 n \cdot (p + q) \delta^{AB} \epsilon^\perp_{\rho \nu}, \]
\times \int dl \left( \frac{1}{\bar{n} \cdot l} - \frac{1}{\bar{n} \cdot (l + p - q)} \right) \bar{n} \cdot (l - q) \bar{n} \cdot (l + p)
\] (24)

By sending \( l \rightarrow -l - p + q \) we see that the two terms are the same (giving a factor of 2, rather than canceling each other). We do not have a second contribution from \( (p, \rho, A) \leftrightarrow (q, \nu, B) \) for this diagram. As mentioned before, we need to be careful in dealing with IR divergences here. Writing \( \delta = \bar{n} \cdot (p + q) \rightarrow 0 \), we encounter \( \delta^{-\epsilon} \) when doing this integral. Obviously the order of taking
\[ \delta \to 0 \text{ and } \epsilon \to 0 \] matters, and we should first take \( \epsilon \to 0 \) with \( \delta \neq 0 \) to regulate any IR divergences. Proceeding along this way, we still find \( A_B = 0 \).

We now add \( A_T \) and \( A_B \) and compare with \( \langle gg|J^{(4)}|0 \rangle \) as given in (16). They agree. This was expected since at this order only collinear fields are involved, which is just a boosted version of QCD.

**C. LO with \( n \cdot (p + q) = 0 \)**

We move on to the next case: \( n \cdot (p + q) = 0 \). This time the relevant term of the axial current comes from \( n \cdot J_5^{(4)} \):
\[ \langle gg|J^{(4)}|0 \rangle = \frac{1}{2} \bar{n} \cdot (p + q) \left( \frac{g}{2} \gamma_5 + \frac{g}{2} \gamma_5 \right) \frac{1}{\bar{n} \cdot D_c} \bar{n} \cdot \xi_5 |0 \rangle. \] (25)
It can be part of a diagram in various different ways; for example, a gluon can now come out of the current. These different ways are pictured in Fig. 2, and the corresponding expressions are
\[ (1) - \bar{n}^{\perp} \gamma_5 \frac{f_{\perp}}{\bar{n} \cdot (l - q)} \bar{n} \cdot (l - q) \] (26)
\[ (2) - \bar{n}^{\perp} \gamma_5 \frac{f_{\perp}}{\bar{n} \cdot (l - p)} \bar{n} \cdot (l - p) \]
\[ (3) - \bar{n}^{\perp} \gamma_5 \frac{gT^B}{\bar{n} \cdot (l - q)} \bar{n} \cdot (l - q) \]
\[ (4) - \bar{n}^{\perp} \gamma_5 \frac{gT^B}{\bar{n} \cdot (l - p)} \bar{n} \cdot (l - p) \]

First of all we get a triangle [Fig. 3(a)] and bubble diagram [Fig. 3(b)] coming from (1)
\[ \tilde{A}_A + \tilde{A}_B = ig^2 \bar{n} \cdot (p + q) \delta^{AB} \epsilon_{\rho \nu}^\perp ((1 + 2\epsilon)I_1 - \epsilon I_2) - \epsilon I_3 - I_4 + (p, \rho, A) \leftrightarrow (q, \nu, B), \] (27)

where

**FIG. 2.** \( n \cdot J_5^{(4)} \) can be inserted into diagrams in various ways.
We also have diagrams [Fig. 3(c1) with (2) and Fig. 3(c2) with (3)] where one gluon comes out of the current. This turns out to be zero.

Finally there is also a diagram [Fig. 3(d)] where both gluons come out of the current, which involves (4). This turns out to be zero, $\tilde{A}_P = 0$.

We simplify the remaining integrals using

$$I_2 = \int \frac{d^3l}{\hat{n} \cdot (l - q)^2 l^2(l + p)^2},$$

$$I_7 = \int \frac{d^3l}{\hat{n} \cdot (l - q)^2 l^2(l + p)^2}.$$  \hspace{1cm} (30)

Finally there is also a diagram [Fig. 3(d)] where both gluons come out of the current, which involves (4). This turns out to be zero. $\tilde{A}_P = 0$.

We simplify the remaining integrals using

$$I_2 = \int \frac{d^3l}{\hat{n} \cdot (l - q)^2 l^2(l + p)^2 - n \cdot (l + p) n \cdot (l + q)},$$

$$I_7 = \int \frac{d^3l}{\hat{n} \cdot (l - q)^2 l^2(l + p)^2}.$$  \hspace{1cm} (31)

where $\vec{l}_1 = -\vec{l}_1$ if $g^{\mu\nu} = \text{diag}(+ - - -)$. For example,

$$\tilde{I}_2 = \int \frac{d^3l}{\hat{n} \cdot (l + p)(l - q)^2 l^2 - n \cdot (l + p) n \cdot (l + q)}.$$

$$= I_{2A} + I_{2B}.$$  \hspace{1cm} (32)

Doing the math, we find

\begin{align*}
\langle gg|\mathcal{J}^{(4)}|0\rangle &= \tilde{A}_A + \tilde{A}_B + \tilde{A}_{C1} + \tilde{A}_{C2} + \tilde{A}_D \\
&= -\frac{g^2}{8\pi^2} \hat{n} \cdot (p + q) n \cdot p \delta^{AB} \epsilon_{\mu\nu}^\perp. \hspace{1cm} (33)
\end{align*}

Again this agrees with $\langle gg|\mathcal{J}^{(4)}|0\rangle$, as expected because the calculation so far only involved collinear fields.

D. LO general case

Finally we now drop the additional assumptions [e.g. $n \cdot (p + q) = 0$] and study the general case. These assumptions allowed us to look at one term of $\langle gg|\mathcal{J}^{(4)}|0\rangle$ in (11) at the time, so we basically have to add the anomalies of Secs. II B and II C. The only other place these assumptions were used is in simplifying the Feynman integrals. We can now take the gluons on shell, and we will assume

$$n \cdot p = 0, \quad \hat{n} \cdot q = 0.$$  \hspace{1cm} (34)

After repeating the above analysis and performing the Feynman integrals we find

$$\langle gg|\mathcal{J}^{(4)}|0\rangle = \frac{g^2}{8\pi^2} \hat{n} \cdot p n \cdot q \delta^{AB} \epsilon_{\mu\nu}^\perp = \langle gg|\mathcal{J}^{(4)}|0\rangle.$$  \hspace{1cm} (35)

Thus the SCET anomaly equation at LO $\mathcal{J}^{(4)} = \mathcal{J}^{(4)}$, which we derived by expanding, is correct at one loop. In the previous sections the bubble diagrams in Figs. 3(c1) and 3(c2) contributed, but the bubble diagram with the two gluon vertex in Figs. 1 and 3(b) turned out to be zero. However, in this general case with on-shell momenta we do get a genuine contribution from them.

III. THE SCET ANOMALY AT NLO

A. Matching the QCD anomaly onto SCET at higher orders

We will now move on to calculating the chiral anomaly in SCET beyond LO in the power counting, by taking into account the higher order terms in Eq. (6). Again we start by matching the QCD anomaly equation onto SCET at tree level. The derivation is similar to that in Sec. II A, but this time we need the tree level relation between the QCD and
We find the anomaly from higher order currents such as \( \tilde{g} g \). Expanding the other side of the QCD anomaly equation found this way is correct, \( \gamma_5 \). We already checked at LO that the SCET anomaly equation at NLO and NNLO reads

\[
\mathcal{T}\{\mathcal{J}^4_i L^{(1)}) + \mathcal{J}^5 = \mathcal{T}\{\mathcal{J}^4_i L^{(1)}) + \mathcal{J}^5, \quad (45)
\]

\[
\mathcal{T}\{\mathcal{J}^4_i L^{(1)}) + \mathcal{J}^5 = \mathcal{T}\{\mathcal{J}^4_i L^{(1)}) + \mathcal{J}^5 + \mathcal{J}^6
\]

\[
= \mathcal{T}\{\mathcal{J}^4_i L^{(1)}) + \mathcal{T}\{\mathcal{J}^4_i L^{(1)}) + \mathcal{T}\{\mathcal{J}^5_i L^{(1)}) + \mathcal{T}^6
\]

\[
+ \mathcal{J}^6), \quad (46)
\]

where \( \mathcal{T}\{\mathcal{J}^4_i L^{(1)}) = \int d^4 x \mathcal{T}\{\mathcal{J}^4_i L^{(1)}(x), \text{etc.} L^{(1)} \) and \( L^{(2)} \) refer to all terms in the quark and gluon action of the respective order.

Let us consider the right-hand side of these equations. Calculating the anomaly at one loop corresponds to tree level diagrams for \( \mathcal{J}^{(4)} \). The only nontrivial diagrams come from insertions on external glue lines. If one would include insertions on external gluon lines, one would get contributions on both sides of these equations, which are equal because these insertions are not part of the loop. We will not consider such diagrams, and so the right-hand side of (45) and (46) reduces to \( \mathcal{J}^{(5)} \) and \( \mathcal{J}^{(6)} \).

We list the subleading Lagrangians we need below [14–20]:

\[
L^{(1)}_{\xi_n} = (\varepsilon_n W) i \bar{\psi}_a \hspace{1em} (W^i i \psi^c_\perp \frac{\not{\xi}_n}{2} + (\varepsilon_n i \psi^c_\perp W)
\]

\[
\times \frac{1}{i \not{n} \cdot D} i \bar{\psi}_a \hspace{1em} (W^i i \psi^c_\perp \frac{\not{\xi}_n}{2} + (\varepsilon_n i \psi^c_\perp W)
\]

\[
L^{(2)}_{\xi_n} = (\varepsilon_n W) i \bar{\psi}_a \hspace{1em} (W^i i \psi^c_\perp \frac{\not{\xi}_n}{2} + (\varepsilon_n i \psi^c_\perp W)
\]

\[
\times 1 (i \not{n} \cdot D) i \bar{\psi}_a \hspace{1em} (W^i i \psi^c_\perp \frac{\not{\xi}_n}{2} + (\varepsilon_n i \psi^c_\perp W)
\]

\[
L^{(3)}_{\xi_q} = \frac{1}{i \not{n} \cdot D} i g \bar{\psi}_c W q_a + H.c., \quad (49)
\]

where \( ig \bar{\psi}_c = [i \not{n} \cdot D, i \psi^c_\perp] \).

**B. NLO with \( n \cdot (p + q) = 0 \)**

At NLO things become more interesting, because both collinear and ultrasoft fields are involved. To get the right power counting one of the outgoing gluons must be collinear and the other ultrasoft. The only additional assumption that makes sense here is

\[
n \cdot (p + q) = 0,
\]

because \( n \cdot p \sim 1 \gg \lambda^2 \sim n \cdot q \).

The axial current gives rise to...
The various diagrams lead to contributions with \( n \cdot J_5^{(0)} \), one \( L^{(1)} \) vertex and \( L^{(0)} \) vertices, and diagrams with \( n \cdot J_5^{(5)} \) and only \( L^{(0)} \) vertices. There is no term in \( L^{(0)} \) for a \( \perp \) -polarized usoft gluon, so the usoft gluon has to come out of the \( L^{(1)} \) vertex or the \( n \cdot J_5^{(5)} \) current. This leads to the diagrams in Figs. 4 and 5. Obviously we do not have \((p, p, A) \leftrightarrow (q, \nu, B)\) terms in this case, since one gluon is collinear and one usoft. As it turns out, we already computed all the necessary integrals in the LO \( n \cdot (p + q) = 0 \) calculation. Here we have the additional simplification that we can generally drop \( \bar{n} \cdot q \) for the usoft momentum \( \bar{q} \), because \( \bar{n} \cdot q \ll \bar{n} \cdot p, \bar{n} \cdot l \).

The various diagrams lead to contributions

\[
A^\text{NLO}_{A_1} + A^\text{NLO}_{A_2} = 2g^2 \bar{n} \cdot p \delta^{AB} \epsilon_{\rho\nu}^A (1 + \epsilon) (\bar{I}_1 - \bar{I}_2),
\]

\[
A^\text{NLO}_{B_1} + A^\text{NLO}_{B_2} = 2g^2 \bar{n} \cdot p \delta^{AB} \epsilon_{\rho\nu}^B (1 + \epsilon) \bar{I}_{2A},
\]

\[
A^\text{NLO}_{C_1} + A^\text{NLO}_{C_2} = 2g^2 \bar{n} \cdot p \delta^{AB} \epsilon_{\rho\nu}^C (\bar{l}_6 + \epsilon \bar{l}_7),
\]

\[
A^\text{NLO}_{D_1} + A^\text{NLO}_{D_2} = 0,
\]

where \( \bar{I}_{2A} \) was defined in (32). Adding the pieces we get

\[
\langle gg | T\{J^{(4)} i L^{(1)} + J^{(5)}\} | 0 \rangle = \frac{1}{2} i \bar{n} \cdot (p + q) \langle gg | T\{n \cdot J_5^{(4)} i L^{(1)} + n \cdot J_5^{(5)}\} | 0 \rangle = \frac{1}{2} i \bar{n} \cdot (p + q) \langle gg | \left( -\frac{\bar{\xi}_n i \bar{\psi}_{\perp} \gamma_5 \frac{1}{\bar{n} \cdot D_c} \tilde{\xi}_n}{i \bar{n} \cdot D_c} i \bar{\psi}_{\perp} \xi_n \right) i L^{(1)} \rangle 
\]

\[
+ \left( -2i \bar{\xi}_n i \bar{\psi}_{\perp} \gamma_5 \left( W_{q\nu} + W_1 \frac{1}{\bar{n} \cdot D_c} i \bar{\psi}_{\perp} \frac{\bar{\gamma}_5}{2} W^\dagger \xi_n \right) + \text{H.c.} \right) | 0 \rangle.
\]

So the SCET anomaly equation is correct at NLO too.

**IV. THE SCET ANOMALY AT NNLO**

We will pursue our calculation to one higher order in \( \lambda \). At this order the loop momentum can have either collinear or ultra-soft scaling.

Let us start by observing that there is a freedom in what you call label and residual momentum. Consequently SCET should be invariant under (for example) the following reparametrization:

\[
\bar{n} \cdot p_c \rightarrow \bar{n} \cdot p_c + \bar{n} \cdot k, \quad \bar{n} \cdot p_r \rightarrow \bar{n} \cdot p_r - \bar{n} \cdot k,
\]

where \( p_c \) is the (collinear) label momentum, \( p_r \) the (usoft) residual momentum, and \( k \) some constant usoft momentum. This ties together \( J^{(4)} \) and \( J^{(6)} \), basically predicting the latter. It is easy to check that (42) + (44) satisfies this. We will also calculate the loop diagrams, to verify the SCET anomaly equation at this order. Since both gluons are collinear this time, either \( \bar{n} \cdot (p + q) = 0 \) or \( n \cdot (p + q) = 0 \) can be imposed. We will restrict ourselves to the

**FIG. 5.** Diagrams coming from \( n \cdot J_5^{(5)} \) contributing to the NLO anomaly.
former. As can be seen from the expression for $\mathcal{J}^{(6)}$ in (44), we need to keep the residual momentum components of the external momenta

$$p = p_e + p_r = (0, \bar{n} \cdot p_c, p_c^\perp = 0)$$

$$+ (n \cdot p_r, \bar{n} \cdot p_r, p_r^\perp = 0)$$  \hspace{1cm} (58)$$

where $\mathcal{L}^{(1)}$ involves the residual loop momentum $\bar{n}$ only involved the residual loop momenta e.g. get diagrams that also depend on other components of $\bar{n}$. Therefore we cannot make a diagram using linear loops for which we combined label and residual momenta $\Sigma_{l_e} \int dl_e \rightarrow \int dl$. This was possible because these diagrams only involved $\bar{n} \cdot l$, and $n \cdot l$. Now we can get diagrams that also depend on other components of the residual loop momenta e.g. $l^\perp$. This requires us to do $\int dl^\perp(\ldots)$ separately, which yields zero for a collinear loop since all factors of $l^\perp$ are in the numerator. This is not the case when the loop momentum is usoft, because then $l^\perp$ appears in the denominator as well. Diagrams with collinear loops may still have nonzero contributions, for example, $\Sigma_{l_e} \int dl_e \bar{n} \cdot (l_e + p_r)(\ldots) = \bar{n} \cdot p_r \Sigma_{l_e} \int dl_e(\ldots)$. Because we take the external $p_r^\perp = q^\perp = 0$, many diagrams vanish immediately.

We cannot make a diagram using $\bar{n} \cdot J^{(4)}_5$ and only $\mathcal{L}^{(0)}$ vertices, since the $\bar{n} \cdot J^{(4)}_5$ current has a collinear and a usoft quark coming out of it. Therefore the only contributions come from $\bar{n} \cdot J^{(2)}_5$.

There is a bit of a technical issue with loop integrals that we need to discuss here. At LO and NLO we had collinear loops for which we combined label and residual momenta $\Sigma_{l_e} \int dl_e \rightarrow \int dl$. This was possible because these diagrams only involved $\bar{n} \cdot l$, and $n \cdot l$. Now we can get diagrams that also depend on other components of the residual loop momenta e.g. $l^\perp$. This requires us to do $\int dl^\perp(\ldots)$ separately, which yields zero for a collinear loop since all factors of $l^\perp$ are in the numerator. This is not the case when the loop momentum is usoft, because then $l^\perp$ appears in the denominator as well. Diagrams with collinear loops may still have nonzero contributions, for example, $\Sigma_{l_e} \int dl_e \bar{n} \cdot (l_e + p_r)(\ldots) = \bar{n} \cdot p_r \Sigma_{l_e} \int dl_e(\ldots)$. Because we take the external $p_r^\perp = q^\perp = 0$, many diagrams vanish immediately.

$\langle gg|T\{\mathcal{J}^{(4)}_4, \mathcal{L}^{(1)} + \mathcal{J}^{(4)}_4, \mathcal{L}^{(1)} + \mathcal{J}^{(5)}_4, \mathcal{L}^{(1)} + \mathcal{J}^{(6)}_4\}|0\rangle$

$$= \frac{1}{2} i n \cdot (p + q) \langle gg|T\{\bar{n} \cdot J^{(4)}_5, l^\perp, l^\perp\}|0\rangle + \langle gg|T\{\bar{n} \cdot J^{(5)}_5, l^\perp, l^\perp\}|0\rangle + \langle gg|T\{\bar{n} \cdot J^{(6)}_5, l^\perp, l^\perp\}|0\rangle$$

$$= \frac{1}{2} i n \cdot (p + q) \langle gg|T\{\bar{\xi}_n \gamma_5 \xi_n, l^\perp, l^\perp\}|0\rangle + \langle gg|T\{\bar{\xi}_n \gamma_5 \xi_n, l^\perp, l^\perp\}|0\rangle + \langle gg|T\{\bar{\xi}_n \gamma_5 \xi_n, l^\perp, l^\perp\}|0\rangle$$

First of all we get the same diagrams as at LO, but with an extra $\mathcal{L}^{(2)}$ insertion [Figs. 6(a1), 6(a2), 6(a3), 6(b1), and 6(b2)]:

$$A_{A1}^{NNLO} + A_{A2}^{NNLO} + A_{A3}^{NNLO} + A_{B1}^{NNLO} + A_{B2}^{NNLO}$$

$$= -2 i g^2 n \cdot (p + q) \bar{n} \cdot p_r \delta^{AB} \epsilon_r^\perp \left( \hat{I}_1 + 2 e \hat{I}_2 \right)$$

$$- (1 + 2 e) (\hat{I}_4 - \hat{I}_5) + (p, p, A) \leftrightarrow (q, q, B)$$  \hspace{1cm} (61)$$

where

$$\hat{I}_1 = \int dl \bar{n} \cdot (l - q)^2 l^\perp (l + p)^4,$$

$$\hat{I}_2 = \int dl \bar{n} \cdot (l - q)^2 l^\perp (l + p)^4,$$

$$\hat{I}_4 = \int dl \bar{n} \cdot (l - q)^2 l^\perp (l + p)^4,$$

$$\hat{I}_5 = \int dl \bar{n} \cdot (l - q)^2 l^\perp (l + p)^4.$$  \hspace{1cm} (62)$$

FIG. 6. Diagrams contributing to the anomaly at NNLO for $\bar{n} \cdot (p + q) = 0.$
We also get a triangle [Figs. 6(c1) and 6(c2)] where one of the vertices is $L^{(2)}$ and a bubble diagram [Fig. 6(d)] with an $L^{(2)}$ vertex:

$$A_{C1}^{NNLO} + A_{C2}^{NNLO} + A_{D}^{NNLO}$$

$$= 2g^2 n \cdot (p + q) \hat{n} \cdot p, \delta^{AB} \epsilon_{\mu \nu}^{\perp}(e \hat{I}_8 - (1 + 2e)\hat{I}_9$$

$$+ (p, p, A) \leftrightarrow (q, v, B) + \hat{I}_{10}),$$

with

$$\hat{I}_8 = \int dl \frac{\hat{L}_1^2}{(l - q)^2 P(l + p)^2},$$

$$\hat{I}_9 = \int dl \frac{\hat{n} \cdot l \hat{L}_1^2}{\hat{n} \cdot (l + p)(l - q)^2 P(l + p)^2},$$

$$\hat{I}_{10} = \int dl \frac{(\hat{n} \cdot (l + p))^2}{\hat{n} \cdot (l + p)(l - q)^2 P(l + p)^2}.$$

Finally there is also a mixed collinear-usoft diagram [Fig. 6(e)] with a usoft loop momentum:

$$A_{E}^{NNLO} = -ig^2 n \cdot (p + q) \delta^{AB} \epsilon_{\mu \nu}^{\perp} \int dl \frac{-\hat{n} \cdot q}{-\hat{n} \cdot q n \cdot (l - q)}$$

$$\times \frac{n \cdot l}{l^2} \frac{\hat{n} \cdot p}{\hat{n} \cdot n \cdot (l + p)} + (p, p, A) \leftrightarrow (q, v, B) = 0.$$  (66)

Rewriting the loop integrals using (31) and

$$\frac{d}{d(n \cdot p)} \frac{1}{(l + p)^2} = \frac{\hat{n}(l + p)}{(l + p)^3},$$

$$\frac{d}{d(\hat{n} \cdot p)} \frac{1}{(l + p)^2} = -\frac{n(l + p)}{(l + p)^3},$$

we get

$$\hat{I}_1 = \hat{I}_2 + \hat{I}_5 - \hat{I}_8 - n \cdot p \frac{d}{d(\hat{n} \cdot p)} \hat{I}_8,$$

$$\hat{I}_4 = \hat{I}_2 - \hat{I}_8 + \hat{I}_9 - \hat{n} \cdot p \frac{d}{d(\hat{n} \cdot p)} \hat{I}_8.$$  (69)

This reduces the calculation to

$$\langle gg|T\{J^{(4)}iL^{(1)}iL^{(1)}\} + T\{J^{(4)}iL^{(2)}\} + T\{J^{(5)}iL^{(1)}\} + J^{(6)}|0\rangle = A_{A1}^{NNLO} + A_{A2}^{NNLO} + \ldots + A_{D}^{NNLO} + A_{E}^{NNLO}$$

$$= -2ig^2 n \cdot (p + q) \hat{n} \cdot p, \delta^{AB} \epsilon_{\mu \nu}^{\perp} \left(\epsilon \hat{I}_8 - n \cdot p \frac{d}{d(n \cdot p)} \hat{I}_8 + (1 + 2e)\hat{n} \cdot p \frac{d}{d(\hat{n} \cdot p)} \hat{I}_8 - \frac{1}{2} \hat{I}_{10}\right) + (p, p, A) \leftrightarrow (q, v, B).$$  (70)

Working out $\hat{I}_{10}$, one finds zero. The remainder is fairly easy to evaluate if you take $(p, p, A) \leftrightarrow (q, v, B)$ before performing the integral over Feynman parameters. We find

$$\langle gg|T\{J^{(4)}iL^{(1)}iL^{(1)}\} + T\{J^{(4)}iL^{(2)}\} + T\{J^{(5)}iL^{(1)}\} + J^{(6)}|0\rangle$$

$$= \frac{g^2}{8\pi^2} n \cdot (p + q) \hat{n} \cdot p, \delta^{AB} \epsilon_{\mu \nu}^{\perp} = \langle gg|J^{(6)}|0\rangle.$$  (71)

Once again we see by direct computation that the SCET anomaly equation is correct.

V. CONCLUSIONS

We derived the chiral anomaly equations in SCET up to NNLO by matching the QCD anomaly equation onto SCET, using the tree level relations between fields in QCD and SCET. Gathering all the expressions, the anomaly equations up to NLO read$^2$

$$J^{(4)} = J^{(4)},$$

$$T\{J^{(4)}iL^{(1)}\} + J^{(5)} = T\{J^{(4)}iL^{(1)}\} + J^{(5)},$$

where

$$J^{(4)} = \frac{1}{2} n \cdot \partial \left[ \tilde{\xi}_n \gamma_5 \gamma_5 \xi_n \right]$$

$$+ \frac{1}{2} \hat{n} \cdot \partial \left[ -\tilde{\xi}_n \gamma_5 \gamma_5 \frac{1}{in \cdot D_c} \gamma_5 \frac{1}{in \cdot D_c} i\bar{\psi}_c \gamma_5 \xi_n \right]$$

$$+ \partial_{\mu} \left[ \tilde{\xi}_n \gamma_\mu \gamma_5 \frac{1}{in \cdot D_c} i\bar{\psi}_c \gamma_5 \frac{1}{2} \xi_n + \text{H.c.} \right].$$  (74)

$^2$We only studied some terms of the NNLO anomaly equations and would need the higher order terms in (36) to write down the full expression at NNLO.
\( \mathcal{J}^{(5)} = \frac{1}{2} \bar{n} \cdot \partial \left[ -2 \xi_n i \bar{\Phi}_c \frac{1}{i \bar{n} \cdot D_c} \right. \\
\left. \times \gamma_5 \left( W q_{us} + W - \frac{1}{n} \frac{1}{D_c i \bar{\Phi}_c \frac{1}{2} W^\dagger \xi_n \right) \right] \\
+ \frac{1}{2} \bar{\xi}_n \gamma_\mu \gamma_5 \left( W q_{us} + W - \frac{1}{n} \frac{1}{D_c i \bar{\Phi}_c \frac{1}{2} W^\dagger \xi_n \right) \\
+ \text{H.c.}, \tag{75} \)

\( \mathcal{F}^{(4)} = \frac{1}{16 \pi^2} \epsilon_{\mu \nu} \text{tr} \left[ i n \cdot D \frac{1}{i \bar{n} \cdot D_c} i D_c \frac{1}{D_c} \right] \\
\pm \text{permutations}, \tag{76} \)

\( \mathcal{F}^{(5)} = \frac{1}{8 \pi^2} \epsilon_{\mu \nu} \text{tr} \left[ i n \cdot D \frac{1}{i \bar{n} \cdot D_c} i D_c \frac{1}{D_c} \right] (W i D_{us} W^\dagger) \\
\pm \text{permutations}. \tag{77} \)

Here the sum over permutations means the sum of all different orderings of the operators multiplied by the sign of the permutation. Note that \( \mathcal{F}^{(4)} \) has an additional factor of \( \frac{1}{2} \) compared to \( \mathcal{F}^{(5)} \), which comes from the redundant interchange of the \( i D_c \). The relevant terms of \( L^{(1)} \) can be found at the end of Sec. III A.

By explicitly calculating one-loop graphs we verified these equations. Although we have not checked every possible term in the operator relations, our work suggests their correctness. At LO the anomaly only involves collinear fields, and SCET with only collinear fields is just a boosted version of QCD, so we expected agreement. It was not a priori clear how the anomaly equations would work out beyond that order.

Of course there are still higher orders to consider, which might be worth pursuing if one has in mind applying the anomaly in some specific process. \( \mathcal{F} \) contains terms up to order \( \lambda^8 \), whereas \( \mathcal{J} \) has terms of arbitrary high order. This can be seen from replacing \( 1/(in \cdot D_c) \rightarrow 1/(in \cdot D_c + \frac{1}{2}) \).

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\[ \text{APPENDIX: USEFUL INTEGRALS} \]

All the loop integrals needed for our calculations are given below:

\[ \int d^d l \frac{1}{(l^2 - m^2)^\nu} = \frac{i (-1)^\nu}{16 \pi^2} \frac{(1 + \epsilon \log 4 \pi + \ldots)}{\Gamma(\nu)} (m^2)^{2-\nu - \epsilon}, \tag{A1} \]

\[ \int d^d l \frac{l_1^2}{(l^2 - m^2)^\nu} = \frac{i (-1)^{\nu+1}}{16 \pi^2} \frac{(1 + \epsilon \log 4 \pi - 1 + \ldots)}{\Gamma(\nu)} (m^2)^{3-\nu - \epsilon}, \tag{A2} \]

\[ \int d^d l \frac{1}{\bar{n} \cdot (l + a)(l^2 - m^2)^\nu} = \frac{i (-1)^\nu}{16 \pi^2} \frac{(1 + \epsilon \log 4 \pi + \ldots)}{\Gamma(\nu)} \frac{(m^2)^{2-\nu - \epsilon}}{\bar{n} \cdot a}, \tag{A3} \]

\[ \int d^d l \frac{l_1^2}{\bar{n} \cdot (l + a)(l^2 - m^2)^\nu} = \frac{i (-1)^{\nu+1}}{16 \pi^2} \frac{(1 + \epsilon \log 4 \pi - 1 + \ldots)}{\Gamma(\nu)} \frac{(m^2)^{3-\nu - \epsilon}}{\Gamma(\nu)} \frac{\bar{n} \cdot a}{\bar{n} \cdot l}, \tag{A4} \]

where \( \int d^d l = \int d^d l (2\pi)^{-d} \) and \( d = 4 - 2\epsilon \). Note that \( l_1^2 = -l_1^2 \) when \( g^{\mu \nu} = \text{diag}(+ - - -) \).

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