Gapped Two-Body Hamiltonian Whose Unique Ground State Is Universal for One-Way Quantum Computation

The MIT Faculty has made this article openly available. Please share how this access benefits you. Your story matters.
Many-body entanglement is fundamental to the understanding of complex condensed matter systems, as well as a primary resource for quantum computation. A surprising result in quantum computation is that certain entangled states can be employed to perform arbitrary quantum information processing tasks, merely by measuring single sites in different bases, in a method known as “one-way” quantum computation [1]. If such universal resource states are available, this approach potentially simplifies experimental requirements by employing only measurements and not multiqubit gates normally needed. Exotic physical properties may arise in these states due to their large amount of entanglement; many methods have been developed in condensed matter theory to study such systems, including the matrix product state formalism [2] or more generally the projected entangled pair states (PEPS) representation [3].

The special entangled states which make arbitrary one-way quantum computation possible unfortunately seem to be difficult to realize naturally. Ideally, such universal resource states could be obtained as the unique ground state of a physically reasonable Hamiltonian made of two-body, nearest-neighbor interactions. We introduce such a state, composed of six-state particles on a hexagonal lattice, and describe a general method for analyzing its properties based on its projected entangled pair state representation.

Many-body entangled quantum states studied in condensed matter physics can be primary resources for quantum information, allowing any quantum computation to be realized using measurements alone, on the state. Such a universal state would be remarkably valuable, if only it were thermodynamically stable and experimentally accessible, by virtue of being the unique ground state of a physically reasonable Hamiltonian of two-body, nearest-neighbor interactions. We introduce such a state, composed of six-state particles on a hexagonal lattice, and describe a general method for analyzing its properties based on its projected entangled pair state representation.
on a square lattice a cluster state defined on a square lattice \( |\psi_{\text{Cluster}}^{\text{Sqr}}\rangle \) is also universal [14]. On a hexagonal lattice of singlet pairs [Fig. 1(b)], the projector defining this cluster state is \( P_{\text{Hex}}^{\text{Cluster}} = |1\rangle\langle 1| + |\bar{1}\rangle\langle \bar{1}| \), giving \( |\psi_{\text{Cluster}}^{\text{Hex}}\rangle \approx P_{\text{Cluster}}^{\text{Hex}} |\psi_{\text{PEPS}}^{\text{Hex}}\rangle \), where the labels denote left-right-up and left-right-down virtual qubits on sites in sublattices A and B, respectively. (2) PEPS states defined with projectors \( P' = |0\rangle\langle 0| + |1\rangle\langle 1| \) or \( P'' = |0\rangle\langle 1| + |1\rangle\langle 0| \) only differ from \( |\psi_{\text{Cluster}}^{\text{Hex}}\rangle \) by local Pauli \( Z \) operations, as \( \tilde{Z}P = P' (x \otimes z \otimes I) = P'' (z \otimes x \otimes I) \) and \( |\psi_{\text{PEPS}}^{\text{Hex}}\rangle \) is invariant with \( (x \otimes z \otimes I) \) or \( (z \otimes x \otimes I) \) applied to every site. Hence, a modified local measurement scheme still exists, allowing these states to also be universal.

We now introduce a new state, the tricluster state \( |\psi_{\text{Tric}}\rangle \), which is motivated by these two insights, and has properties we desire. This is defined in the PEPS representation on a two-dimensional hexagonal lattice [Fig. 1(b)], with projectors

\[
P_{\text{Tric}} = |0\rangle\langle 0| + |\bar{1}\rangle\langle \bar{1}| + |\bar{2}\rangle\langle 2| + |3\rangle\langle 3| + |4\rangle\langle 4|
\]

\[
\times |1\rangle\langle 0| + |\bar{5}\rangle\langle 10|,
\]

using the same labeling scheme as above, such that \( |\psi_{\text{Tric}}\rangle \approx P_{\text{Tric}} |\psi_{\text{PEPS}}^{\text{Hex}}\rangle \). Hence, at each lattice site there lives a 6-dimensional particle.

Intuitively, \( |\psi_{\text{Tric}}\rangle \) is universal because it is closely related to the standard cluster state. Specifically, \( |\psi_{\text{Tric}}\rangle \) projected into the subspace spanned by \( \{0, \bar{1}\} \) is the same as \( |\psi_{\text{Cluster}}\rangle \), as are also the states given by \( |\psi_{\text{Tric}}\rangle \) projected into \( \{\bar{2}, \bar{3}\} \) and \( \{4, \bar{5}\} \), up to local Pauli errors. Thus, \( |\psi_{\text{Tric}}\rangle \) is like a “superposition” of three cluster states. Computational qubits are encoded in the virtual qubits and operated upon by measuring the physical particles. Although the three subspaces of \( |\psi_{\text{Tric}}\rangle \) cannot be decoupled physically, they may be employed independently in processing encoded qubits with a suitable choice of measurement basis, as detailed later.

The most interesting nontrivial feature of \( |\psi_{\text{Tric}}\rangle \) is that it is the unique ground state of a gapped two-body Hamiltonian, and we begin with that.

Uniqueness and gap.—The fact that \( |\psi_{\text{Tric}}\rangle \) occurs as the unique ground state of a gapped two-body Hamiltonian is very surprising, as on the one hand the ground states of two-body Hamiltonians are rarely exactly known and on the other hand simply constructed states are not always ground states of simple Hamiltonians. Even the one-dimensional cluster state requires three-body interactions in its parent Hamiltonian. Below, we give a two-body nearest-neighbor Hamiltonian \( H_{\text{Tric}} \) which has \( |\psi_{\text{Tric}}\rangle \) as its ground state. Furthermore, we prove that \( |\psi_{\text{Tric}}\rangle \) is the
only ground state of $H_{\text{Tric}}$ and the Hamiltonian has a constant gap independent of system size.

The central step in constructing $H_{\text{Tric}}$ and studying its properties is to find the support space $S_{ab}$ of the reduced density matrix of any two nearest-neighbor particles $a$ and $b$ in the state $(a, b$ are in two sublattices, $A, B$, respectively). This is accomplished by first finding the corresponding support space $S^\text{PEPS}_{ab}$ of the six virtual qubits on sites $a$ and $b$, in the PEPS picture, and then computing $S_{ab} \propto P_{\text{Tric}} S^\text{PEPS}_{ab}$. For example, when $a$ is to the left of $b$ (Fig. 2), virtual qubits 1 to 6 on those sites are only connected to virtual qubits $a$ to $d$ elsewhere. By tracing out $a$ to $d$ from the 5 singlet pairs, we find $S^\text{PEPS}_{ab}$ for virtual qubits 1 to 6 to be spanned by $|\pm\rangle_1 |\pm\rangle_3 |\varphi\rangle_2 |\pm\rangle_5 |\pm\rangle_6$, where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ and $|\varphi\rangle$ is the singlet state. This 16-dimensional space is then projected to give $S_{ab}$ for the depicted lattice site. $S_{ab}$ is different for the three bond directions in a hexagonal lattice, i.e., $a$ to the left of, to the right of, and below $b$.

Providing a two-body Hamiltonian with $|\Psi_{\text{Tric}}\rangle$ being a ground state is straightforward. The Hilbert space of two neighboring sites $a$, $b$ is 36-dimensional, larger than the dimension of $S_{ab}$. Therefore we may choose any non-hermitian Hamiltonian operator $h_{ab}$ on the two sites that has $S_{ab}$ as its null space, such that $h_{ab} |\Psi_{\text{Tric}}\rangle = 0$ for every $h_{ab}$. Thus, $|\Psi_{\text{Tric}}\rangle$ is a ground state of the two-body Hamiltonian $H_{\text{Tric}} = \sum_{ab} h_{ab}$, where the summation is over all nearest-neighbor pairs. However, the key is to construct $H_{\text{Tric}}$ such that $|\Psi_{\text{Tric}}\rangle$ is the unique ground state, and it turns out the above procedure does work.

Specifically, let $h_{ab}$ be the projection operator $h_{ab}^p$ which projects onto the 36 - 16 = 20 dimensional subspace orthogonal to $S_{ab}$, giving the total Hamiltonian

$$H_{\text{Tric}} = \sum_{aA} (h_{ab}^p + h_{ba}^p + h_{a}^p).$$

The summation is over sites $a$ in sublattice $A$ and the three terms $h_{ab}^p$, $h_{ba}^p$, $h_{a}^p$ correspond, respectively, to three bond directions. The Hamiltonian is hence invariant under translation along sublattice $A$. An explicit expression for $H_{\text{Tric}}$ in terms of spin operators is given in [15].

The specific $H_{\text{Tric}}$ we have presented has $|\Psi_{\text{Tric}}\rangle$ as its unique ground state. This is shown by verifying the condition [13] that for any region $R$ of spins in $|\Psi_{\text{Tric}}\rangle$, the support space $S_R$ of the reduced density matrix on $R$ satisfies $S_R = \bigcap_{ab} S_{ab} \otimes I_{R_{ab}}$ where the intersection is taken over all neighboring pairs $ab$ and $I_{R_{ab}}$ is the full Hilbert space of all spins in region $R$ except $a$ and $b$. For every possible configuration containing three or four connected sites in $|\Psi_{\text{Tric}}\rangle$ the condition is confirmed by direct calculation. To check the condition for larger regions, it is useful to notice that any region in $|\Psi_{\text{Tric}}\rangle$ containing more than one site is injective [13]. By Lemma 2 of [13], (1) if regions $R_1$ and $R_3$ are not connected and $R_2$ and $R_3$ are injective, then $S_{R_1 \cup R_2 \cup R_3} = (S_{R_1 \cup R_2} \otimes I_{R_3}) \cap (S_{R_2 \cup R_3} \otimes I_{R_1})$, and (2) if regions $R_1$, $R_2$, $R_3$ are all injective, then $S_{R_1 \cup R_2 \cup R_3} = (S_{R_1 \cup R_2} \otimes I_{R_3}) \cap (S_{R_2 \cup R_3} \otimes I_{R_1}) \cap (S_{R_1 \cup R_3} \otimes I_{R_2})$. Hence for a region $R$ containing more than 4 sites in $|\Psi_{\text{Tric}}\rangle$, $S_R$ is the intersection of all four-body support spaces in $R$. By induction, it follows that the required condition is satisfied on $|\Psi_{\text{Tric}}\rangle$ for any $R$. Therefore, $|\Psi_{\text{Tric}}\rangle$ is the unique ground state of $H_{\text{Tric}}$.

$H_{\text{Tric}}$ is also gapped; an energy gap $\eta$ above the ground state exists, which is constant as the system size goes to infinity. The existence of this gap guarantees protection of $|\Psi_{\text{Tric}}\rangle$ against thermal noise. $\eta$ can be bounded. First, we show that $\eta$ is greater than $\lambda$, the gap of another Hamiltonian $K$ which also has $|\Psi_{\text{Tric}}\rangle$ as its unique ground state, but has four-body terms instead of only two-body terms. We then bound $\lambda$ above a positive constant value.

Consider the Hamiltonian $K$ for a relabeled version of $|\Psi_{\text{Tric}}\rangle$, in which particles are regrouped into disjoint blocks each containing two nearest neighbors (Fig. 3). Let $K = \sum_{mn} k_{mn}$, where $m, n$ denote two connected blocks, each containing two particles $m^l$, $m^r$ and $n^l$, $n^r$, respectively, and $k_{mn}$ is the projection onto the orthogonal space of the four-body reduced density matrix on $m^l$, $m^r$, $n^l$, $n^r$ (assuming $m^l$ and $n^l$ are connected). Then

$$H_{\text{Tric}} = \sum_{ab} h_{ab}^p \geq \frac{1}{4} \sum_{mn} (h_{ml}^p h_{nl}^p + h_{ml}^p h_{nl}^p + h_{ml}^p h_{nl}^l + h_{ml}^l h_{nl}^l + k_{mn}) \geq \frac{1}{4} \sum_{mn} \mu k_{mn} = \frac{1}{4} \mu K.$$

$h_{ml}^p$ and $k_{mn}$ are both non-operators with the same null space, so the last inequality holds for some positive number $\mu$. Assume that the gaps of the projectors $h_{ab}^p$ and $k_{mn}$ are both 1. Direct calculation gives $\mu = \frac{1}{2}$. As

FIG. 2. Two representative sites $a, b$, and neighboring boundary, in the hexagonal lattice of $|\Psi_{\text{PEPS}}\rangle$. Filled circles connected by solid lines represent virtual singlets $|\varphi\rangle$, and dashed circles indicate sites projected to obtain the physical state.

FIG. 3. Regrouping of lattice sites in tricluster state into disjoint blocks, each containing two sites.
discussed in the uniqueness proof, $K$ also has $|\Psi_{\text{Tric}}\rangle$ as its unique ground state. Using this, we find $\eta \equiv \frac{1}{2}\mu \lambda = \frac{1}{2} \lambda$.

The gap $\lambda$ can be bounded by showing that $K^2 \geq cK$ for some positive constant $c$. Specifically, $K^2 = (\sum_{mn} k_{mn})^2 = K + \sum_{mn,m'n'} (k_{mn}k_{m'n'} + k_{m'n'}k_{mn}) \geq K + \sum_{n,mn} k_{mn}k_{m'n'} + k_{m'n'}k_{mn}$, and $n_i$ and $n_j$ are blocks connected to $m$. The last inequality holds because when region $mn$ and region $m'n'$ do not intersect $k_{mn}k_{m'n'} + k_{m'n'}k_{mn} \geq 0$. Direct calculation shows that (Fig. 3) $k_{mn}k_{m'n'} + k_{m'n'}k_{mn} \geq 0$ for $(i, j) = (1, 2), (1, 3), (2, 4), (3, 4)$, and $k_{mn}k_{m'n'} + k_{m'n'}k_{mn} \geq -\frac{1}{2} k_{mn} - \frac{1}{2} k_{mn}$ for $(i, j) = (1, 4)$ or $(2, 3)$. Summing over all consecutive $n_i$, $m$, and $n_j$ gives $\sum_{n,mn} k_{mn}k_{m'n'} + k_{m'n'}k_{mn} \geq -\frac{1}{2} \sum_{mn} k_{mn}$. Therefore $K^2 \geq \frac{1}{2} K$, giving $\lambda \geq \frac{1}{2}$. Finally, we find a lower bound on the gap $\eta$ of $H_{\text{Tric}}$ of $\eta \geq \frac{1}{2} \lambda \geq \frac{1}{2}$.

Universality.—$|\Psi_{\text{Tric}}\rangle$ is a universal resource state. Similar to a cluster state, computational qubits are encoded in the virtual qubits, and the active computational state flows along the lattice as measurements on the physical states are performed. In contrast, however, with $|\Psi_{\text{Tric}}\rangle$ extra Pauli errors occur, thus necessitating additional analysis. Below, we describe the steps necessary, focusing on initialization and readout, one-qubit gates, and a two-qubit gate sufficient for universality.

1. Initialization and readout: Just as with the cluster state, with $|\Psi_{\text{Tric}}\rangle$, measurement in the six-state basis $\{|0\cdots 5\rangle\}$ accomplishes several tasks. First, such measurement detaches unnecessary sites from their neighbors (up to a known Pauli error). Next for state initialization, it gives a postmeasurement state with an encoded qubit projected into $|+\rangle$ (when the outcome is $0, 3, 4$) or $|-\rangle$ (for outcomes $1, 2, 5$). At the end of computation, the encoded qubit can also be read out in this way, giving $0$ (for $0, 2, 5$) and $1$ (for $1, 2, 5$).

2. One-qubit gates: Similar to gate implementations with the cluster state, once a line in the lattice has been detached from the rest, appropriately measuring a site in the line performs a single qubit rotation, up to a known Pauli error. Specifically, measuring in the basis $\{|0\rangle \pm e^{i\theta}|1\rangle, |2\rangle \pm e^{i\theta}|3\rangle, |4\rangle \pm e^{-i\theta}|5\rangle\}$ implements operation $\{HZ(\theta), XHZ(\theta), ZHZ(\theta), YHZ(\theta), ZHZ(\theta), YHZ(\theta)\}$, respectively, on the encoded qubit [using standard notation for qubit gates, with $Z(\theta)$ denoting a rotation about $\hat{z}$ by angle $\theta$, up to preexisting Pauli frame errors from detaching the line.

3. Two-qubit controlled-$Z$ gate: Measurement of two vertically connected particles $a$ and $b$ implements the final ingredient needed for universality, a controlled-$Z$ gate $CZ_{ab}$. Specifically, measuring in basis $\{|0\cdots 5\rangle = |0\rangle \pm |1\rangle, |2\rangle \pm |3\rangle, |4\rangle \pm |5\rangle\}$ implements the two-qubit operation $(X_a^u Z_{a,b}^u H_a)(X_b^u Z_{b,a}^u H_b) \otimes (X_a^w X_b^w CZ_{ab} X_a^u X_b^w)$ on the two adjacent encoded qubits. For $x \in \{a, b\}, u_x = 1$ for $x$ measurement outcomes $1, 3, 5$; $v_x = 1$ for outcomes $2, 3, 4, 5$; $w_x = 1$ for $4, 5$; and $u_x, v_x, w_x$ are 0 otherwise. More complicated configurations for use in a larger circuit is given in [15].

Conclusion.—$|\Psi_{\text{Tric}}\rangle$ is a remarkable entangled many-body state which is universal for both one-way quantum computation and the unique ground state of a gapped Hamiltonian $H_{\text{Tric}}$, made of local two-body terms. While imperfect, due to the use of six-state spins, the triclider model steps closer to physical realizability than previous models. Moreover, the methods introduced here, based on the PEPS representation, are very general. These analysis methods lead directly to a number of additional universal states, and deepen connections between the study of complex condensed matter systems and quantum information science.