**Effective transport properties of random composites: Continuum calculations versus mapping to a network**

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Effective transport properties of random composites: Continuum calculations versus mapping to a network

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The effective transport properties and percolation of continuum composites have commonly been studied using discrete models, i.e., by mapping the continuum to a lattice or network. In this study we instead directly solve the continuum transport equations for composite microstructures both analytically and numerically, and we extract the continuum percolation threshold and scaling exponents for the two-dimensional square tile system. We especially focus on the role of corner contacts on flux flow and further show that mapping such “random checkerboard” systems to a network leads to a spurious secondary percolation threshold and causes shifts in the critical scaling exponents of the effective transport properties.

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The macroscopic effective transport properties of continuum composites—such as electrical and thermal conductivities, mass diffusivity, or permeability—are largely determined by phase connectivity and percolation. Percolation effects, however, have not been successfully incorporated in traditional homogenization schemes, e.g., effective-medium theories, for continuum composites, mainly due to the lack of understanding of continuum percolation for transport processes. Existing studies on continuum percolation for the most part have not rigorously considered transport through the continuum, per se, but instead mapped the phase connectivity to a discrete network or lattice. The mapping approach entirely relies on a connectivity criterion decided beforehand, e.g., whether phase domain contacts at corners (as well as edge contacts in three dimensions) are deemed valid transport paths and what local transport coefficients to assign to these “secondary” contacts. One recent study used finite element analysis to solve for the effective conductivities of composite microstructures, but only after applying a corner contact rule similar to those described above on discrete lattices. To the authors’ knowledge, to date there has been any work systematically studying continuum percolation of transport processes by strictly solving continuum transport equations on the original continuum microstructure, nor has there been any detailed study of the viability of secondary contacts as continuum transport paths. It is the purpose of this Rapid Communication to take some steps in approaching this open problem by analytical derivation and large-scale numerical calculations. We find that mapping continuum composites onto discrete networks will, in general, not yield correct effective transport properties and percolation parameters, and we attribute the discrepancy to strong variations in transport fields around geometric irregularities, i.e., secondary contacts in the present context, that mapped regular networks cannot capture.

We begin by elucidating how transport occurs at corner contacts for a simple two-dimensional example. Consider the quartered geometry shown in Fig. 1(a) with corner contacts between regions (ii) and (iii) of phase 1 and between regions (i) and (iv) of phase 2, with perfect bonding everywhere. Working in the context of mass diffusion, we assign phase 1 a low diffusivity and phase 2 a higher diffusivity, and we relate the diffusion flux to the concentration field by Fick’s first law. Solving the Laplace equation for steady state and further requiring and the normal component of to be continuous across phase boundaries leads to an analytical form for the concentration field, where the summation runs over all values permitted by the definition of as given below in Eq. (2), are constants to be determined by external boundary conditions (which are irrelevant for the present discussion), and expressed in polar coordinates, is

![Schematic of a corner-corner contact](https://example.com/figure1)

**FIG. 1.** (Color online) (a) Schematic of a corner-corner contact in a square domain with phases 1 and 2 assigned diffusivities and assigned fluxes and radial flux, respectively, calculated from Eq. (1). (b)–(g) show the concentration distribution from continuum FEM calculations with gradually refined mesh sizes [total elements used increases from 36 in (e) to 66 in (f) and 250 in (g)].
\[ C_\mu(r, \theta) = \begin{cases} 
- r^\mu \sin\left(\frac{\pi}{4} \mu \cos\left(\theta - \frac{3\pi}{4} \mu\right)\right), & \theta \in \left[\frac{\pi}{2}, \pi\right] \text{ in (i)} \\
- r^\mu \cos\left(\frac{\pi}{4} \mu \sin\left(\theta - \frac{3\pi}{4} \mu\right)\right), & \theta \in \left[0, \frac{\pi}{2}\right] \text{ in (ii)} \\
+ r^\mu \cos\left(\frac{\pi}{4} \mu \sin\left(\theta + \frac{3\pi}{4} \mu\right)\right), & \theta \in \left[-\pi, -\frac{\pi}{2}\right] \text{ in (iii)} \\
+ r^\mu \sin\left(\frac{\pi}{4} \mu \cos\left(\theta + \frac{3\pi}{4} \mu\right)\right), & \theta \in \left[-\frac{\pi}{2}, 0\right] \text{ in (iv)}, 
\end{cases} \]

with
\[ \mu = \frac{2}{\pi} \arcsin\left(\frac{2 \sqrt{D_1 D_2}}{D_1 + D_2}\right) + 4n = \mu_0 + 4n, \quad n = 0, 1, 2, \ldots, \]

where \(\mu_0\) lies between 0 and 1. The angular and the radial components of flux are \(J \mu = -D \frac{\partial C_\mu}{\partial \theta} \propto r^\mu\) and \(J_r = -D \frac{\partial C_\mu}{\partial r} \propto r^{\mu-1}\). As \(r \rightarrow 0\), \(J_0\mu_0\) and \(J_0\mu_0\) become the dominant flux terms (\(J_{\mu_0}\) in fact diverges) as all other \(J_{\mu_0}\) and \(J_{\mu_0}\) vanish in the limit. The distributions of \(C=1+C_{\mu_0}, J_{\mu_0}\), and \(J_{\mu_0}\) very near the corner are shown in Figs. 1(b)–1(d), respectively. Flux flows around the corner in phase 1, as shown in Fig. 1(c), and also across the corner-contact phase 2, as illustrated by arrows in Fig. 1(d).

We determine the local effective diffusivity \(D_{\text{eff}}\) of a circular region with radius \(R\) surrounding the corner contact using the distributions of \(C, J_{\mu_0}\), and \(J_{\mu_0}\):
\[ D_{\text{eff}} = -\frac{\langle J \rangle}{\langle \nabla C \rangle} = \frac{\mu_0 R + \mu_0 + 1}{\mu_0 + \mu_0 R} \frac{D_1 D_2}{R - 1} = \frac{\mu_0}{\mu_0 + \mu_0} \frac{D_1 D_2}{R - 1}. \]

This non-zero local effective diffusivity verifies that diffusion can occur across corner contacts from a continuum perspective. A well-known corroboration of the above result can be found in the regular checkerboard microstructure, which is a periodic distribution of the square domain in Fig. 1(a). Many theoretical studies founded on symmetry arguments have shown that the effective transport coefficient of a regular checkerboard is the geometric mean of those of the two phases, i.e., \(D_{\text{eff}} = \sqrt{D_1 D_2}\) [12–18]. Since only corner contacts provide phase contiguity in the checkerboard, this value of \(D_{\text{eff}}\) suggests that corner contacts must allow for diffusive flow; otherwise, \(D_{\text{eff}}\) for discontinuous composites would instead be close to the Hashin-Shtrikman lower bound [19,20].

Consider a related microstructure: the so-called random checkerboard generated by assigning square tiles randomly as either phase 1 or 2. Sheng and Kohn [21] evaluated the effective dielectric constant \(\varepsilon_\text{eff}\) of a random checkerboard by assuming that a local effective dielectric constant of \(\sqrt{\varepsilon_1 \varepsilon_2}\) would apply at each corner contact, and then using the effective-medium approach [22]. The resulting \(\varepsilon_\text{eff}\) exhibits two percolation thresholds at \(p_c \approx 0.449\) and 0.551, with the former being attributed to the emergence of a percolation cluster relying on both corner and edge contacts and the latter being attributed to the formation of a percolation cluster fully connected through edges. Later, Nettelblad et al. [23] as well as Martensson and Gafvert [24,25] mapped the three-dimensional random checkerboard onto a network, with edge contacts between two \(\sigma_1\) cubes replaced with bonds with conductivity \(\sigma_1\sigma_2\) and corner contacts neglected. By solving the Kirchhoff equations for the network, they also observed two percolation thresholds for the effective conductivity. However, we note that the continuum transport in the vicinity of an individual corner or edge contact in random checkerboards depends on more details of the nearby phase distributions and should be different from that in a regular checkerboard. We therefore do not expect that the geometric mean can be strictly applied at corners in random checkerboards and suggest that prior observations of two percolation thresholds and the associated percolation properties reported by such studies may not be truly relevant for the problem of continuum transport. To explore this hypothesis further, we now directly calculate continuum effective diffusivities using the finite element method (FEM) and compare with results obtained from the network mapping procedure.

We first run test calculations on the domain containing four cells shown in Fig. 1(a). The concentration on external surfaces is required to be the same as that in Fig. 1(b), and the concentration in the domain interior is solved in the commercial finite element solver ABAQUS using quadratic elements. The model is remeshed adaptively several times according to previous solution distributions to progressively reduce the element size near the corner; the resulting mesh is highly nonuniform with extremely small elements at corner contacts. Figures 1(e)–1(g) show that, with finer meshes, the concentration distribution converges to the analytical result in Fig. 1(b), confirming that such FEM calculations can capture transport processes in complex microstructures with corner contacts.

We next apply the same FEM technique (with adaptive remeshing) to 100×100 random checkerboards. We apply a high diffusivity contrast ratio, \(D_2 / D_1 = 10^5\), in order to clearly reveal the percolation transition. The diffusant concentration is set to be a constant \(C_0\) on the left surface and zero on the right, while a zero-flux condition is imposed at the top and bottom. The calculated effective diffusivities \(D_{\text{eff}}\) are plotted against \(p\), the fraction of the phase with diffusivity \(D_2\), in Fig. 2(a) as red circles. The abrupt increase in \(D_{\text{eff}}\) signifies a percolation transition, which occurs only once across the entire range of \(p\). We obtain the percolation probability \(\Pi(p)\) by considering the system “percolating” when \(D_{\text{eff}} > \sqrt{D_1 D_2}\) and otherwise “not percolating” and averaging over many microstructure realizations at the same \(p\). As \(d\Pi / dp\) is a Gaussian distribution with a standard deviation \(\Delta\) [26], integrating \(d\Pi / dp\) from 0 to \(p\) gives
\[ \Pi(p) = \frac{1}{2} \operatorname{erf}\left(\frac{p - p_c}{\sqrt{2} \Delta}\right) + \frac{1}{2} \operatorname{erf}\left(\frac{p - p_c}{\sqrt{2} \Delta}\right). \]
Effective transport properties of random Kirchhoff network equations to obtain the same size percolating cluster; this represents an apparently minor im-
dure. Further reflection offers an explanation for the false sec-
ond threshold introduced by mapping the continuum to a
etwork. Specifically, mapping introduces an artificial third
property scale, the geometric mean \( \sqrt{D_1 D_2} \), for corner
contacts; this in essence renders what should be a binary con-
tinuum problem a ternary network problem. Transport on
three-component networks with \( D_1 < D_2 < D_3 \) has been
shown to reveal two sets of percolation thresholds [29]: the
first at \( f_2 + f_3 = f_c \) and the second at \( f_2 = f_c \), with \( f_c \) the bond
percolation threshold for binary networks and the other \( f_i \)
denoting the phase fractions of diffusivity \( D_i \). These two sets
of percolation thresholds reduce to two percolation thresh-
holds when described in terms of the continuum phase frac-
tions \( p \). A further significant implication of this result is that
many excess fictitious critical points will be introduced when
geometrically more complex composites are represented by
multicomponent networks.

In the mapping approach, the presence of more than one
percolation threshold also makes it difficult to extract correct
scaling exponents \( s \) and \( t \), which characterize the divergence of
\( D_{\text{eff}} \) as \( p_i \) is approached from below and above, respec-
tively, i.e., \( D_{\text{eff}} \propto (p_i - p)^{-s} \) for \( p_p < p_i \) and \( D_{\text{eff}} \propto (p_i - p_p)^t \) for
\( p > p_p \). For the network data presented in Fig. 2(a), the ex-
ponents fitted on the ranges \( 0.03 \leq |p - p_{c1}| \leq 0.2 \) and
\( 0.03 \leq |p - p_{c2}| \leq 0.2 \) are \( (s_1, t_1) = (1.17, 1.57) \) and
(0.69, 2). These two sets of exponents result from using a
second mapping criterion, by which a checkerboard normalized by
the diffusivity \( D \) function of the fraction \( f \) fitted to Eq. (4). (b) Scaling
analysis of the same data, with exponents \( s \) and \( t \) fitted within the range
\( 0.03 \leq |p - p_i| \leq 0.2 \); the values in parentheses are the exponents
fitted on the whole curve.

FIG. 2. (Color online) (a) Effective diffusivity \( D_{\text{eff}} \) of random checkerboards normalized by the diffusivity \( D_2 \) of phase 2 as a function of the fraction \( p \) of phase 2. The data points plotted as red hollow circles are FEM results for the continuum microstructure, while the solid points plotted in blue (dark gray) and green (light gray) are from two network mapping approaches illustrated in the bottom insets. In these insets, gray tiles are phase 1; thick lines represent bonds with diffusivity \( D_2 \), thin lines \( D_3 \), and dashed lines \( \sqrt{D_1 D_2} \). The top insets are the percolation probability \( P \) extracted from the numerical FEM calculations and fitted to Eq. (4). (b) Scaling analysis of the same data, with exponents \( s \) and \( t \) fitted within the range
\( 0.03 \leq |p - p_i| \leq 0.2 \); the values in parentheses are the exponents
fitted on the whole curve.

FIG. 3. (Color online) (a) \( D_{\text{eff}} \) of random checkerboards normalized by the diffusivity \( D_2 \) of phase 2 as a function of the fraction \( p \) of phase 2. The data points plotted as red hollow circles are FEM results for the continuum microstructure, while the solid points plotted in blue (dark gray) and green (light gray) are from two approaches of binary network mapping illustrated in the bottom insets. (b) Scaling analysis of the same data, with exponents \( s \) and \( t \) fitted over the range
\( 0.03 \leq |p - p_i| \leq 0.2 \); the values in parentheses are the exponents
fitted on the whole curve.
for the second type of mapping (in blue) and are \((s_1, t_1) = (1.04, 1.74)\) and \((s_2, t_2) = (1.97, 1.20)\) for the second type of mapping (in green). It is not clear which among these exponents should be compared to the exponents established for lattice percolation in order to evaluate the universality. The continuum exponents extracted from the same range of phase fractions are \((s, t) = (1.25, 1.42)\), which are notably different from those obtained from the network results. As shown in Fig. 2(b), while \(s_1\) and \(s\) might be considered equivalent exponents, neither \(t_1\) nor \(t_2\) is a good match to the continuum exponent \(t\). These differences between the exponents in the network and continuum approaches further discourage the use of network mapping to study continuum percolation of effective properties. Furthermore, the continuum values of \(s\) and \(t\) are slightly asymmetric and are also slightly different from their lattice counterparts, which are symmetric with the value \(s_t = 1.3^{\pm 0.1}\) [30]. Our data thus open the possibility that continuum transport percolation is not universal, and that the mapping approaches used in the literature that transform binary continua containing geometric irregularities into multi-component network problems may generally find different critical scaling behaviors.

If, as we suggest above, the principal problem with conventional “ternary” mappings is the introduction of spurious secondary percolation thresholds, then an obvious remedy to this problem would be to find a binary mapping that does not introduce an artificial third component. We explore this possibility in Fig. 3(a) by using a mapping in which the four bonds surrounding a corner contact are assigned diffusivity \(D_2\) (in blue). Although this mapping gives a percolation threshold at \(p_c \approx 0.41\) similar to that from the FEM analysis on continuum microstructures, the resulting scaling exponent \(t = 1.08\) is very different from the continuum result \(t = 1.42\), as shown in Fig. 3(b). The other type of binary mapping whereby the four bonds surrounding a corner contact are instead assigned diffusivity \(D_1\) does not lead to a matching percolation threshold or scaling exponent \(t\), as shown in Fig. 3 (in green). The scaling exponent \(s\) for either network is similar to the continuum result; the specific treatment of connectivity across corner contacts mainly affects the scaling exponent \(t\) that characterizes \(D_{\text{eff}}\) above \(p_c\).

We close by noting that, while we have focused much of our discussion here on corner contacts in continuum microstructures, we believe that similar considerations should apply to other microstructural features that lead to strong variations in the local flux and transport coefficients, such as overlapping spheres vs peripherally touching spheres, a distribution of neck widths in Swiss-cheese models [8,9,31,32], or conduction via tunneling processes [33,34]. This class of continuum percolation problems might be akin to systems where anomalous distributions of local transport coefficients induce nonuniversality [35,36]. We suggest that more studies are needed in which transport equations are explicitly solved on the continuum, in order to elucidate the universality of continuum percolation of transport properties.

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