delta-baryon electromagnetic form factors in lattice QCD

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A primary motivation for this work is to understand the role of deformation in baryon structure: whether any of the low-lying baryons have deformed intrinsic states and if so, why. Thus, a major achievement of this work is the development of lattice methods with sufficient precision to show, for the first time, that the electric quadrupole form factor is nonzero and hence the magnetic dipole form factor has a nonvanishing matrix element. This is particularly crucial for the latter since without the construction of an optimized source for the sequential propagator it can not be extracted with the required precision. We note that this increases the computational cost since additional inversions are needed. Our techniques are first tested in quenched QCD [12]. We then calculate form factors using two degenerate flavors of dynamical Wilson fermions, denoted by $N_F = 2$, with pion masses in the range of 700 MeV to 380 MeV [13,14]. Finally, we use a mixed action with chirally symmetric domain-wall valence quarks and dynamical staggered sea quarks. The magnetic moment of the $\Delta$ is compared with chiral effective field theory calculations and the $\Delta$ charge density distributions are discussed.

We develop techniques to calculate the four $\Delta$ electromagnetic form factors using lattice QCD, with particular emphasis on the subdominant electric quadrupole form factor that probes deformation of the $\Delta$. Results are presented for pion masses down to approximately 350 MeV for three cases: quenched QCD, two flavors of dynamical Wilson quarks, and three flavors of quarks described by a mixed action combining domain-wall valence quarks and dynamical staggered sea quarks. The magnetic moment of the $\Delta$ is compared with chiral effective field theory calculations and the $\Delta$ charge density distributions are discussed.

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quadrupole form factor \( G_{E2}(q^2) \), and the magnetic octupole form factor \( G_{M3}(q^2) \) [17], and \( \mathcal{A} \) is a known factor depending on the normalization of hadron states. These form factors can be extracted from correlation functions calculated in lattice QCD [17]. We calculate in Euclidean time the two- and three-point correlation functions in a frame where the final state \( \Delta \) is at rest:

\[
G(t, \bar{q}) = \sum_{j, \bar{j}} \sum_{j=1}^{3} e^{-i\bar{x}_j \cdot \bar{q}} \Gamma_{\alpha\beta}^{4} (J_{j\beta}(x_j) J_{j\alpha}(0))
\]

\[
G_{\sigma \tau}(\Gamma, t, \bar{q}) = \sum_{j, \bar{j}} e^{i\bar{x}_j \cdot \bar{q}} \Gamma_{\alpha\beta}^{v} (J_{\sigma\beta}(x_j) J_{\tau\alpha}(0)),
\]

where \( j^\mu \) is the electromagnetic current on the lattice, \( J \) and \( \bar{J} \) are the \( \Delta^+ \) interpolating fields constructed from smeared quarks [12], \( \Gamma^4 = \frac{1}{3} (1 + \gamma^5) \), and \( \Gamma^k = i\Gamma^4 \gamma^5 \gamma^k \). The form factors can then be extracted from ratios of three- and two-point functions in which unknown normalization constants and the leading time dependence cancel

\[
R_{\sigma \tau} = \frac{G_{\sigma \tau}(\Gamma, t, \bar{q})}{G(0, \bar{q})} = \frac{G(t_f - t, \bar{p}_f) G(0, \bar{q})}{G(t_f - t, 0) G(t_f, \bar{p}_f)},
\]

For sufficiently large \( t_f - t \) and \( t - t_i \), this ratio exhibits a plateau \( R(\Gamma, t, \bar{q}) \to \Pi(\Gamma, \bar{q}) \), from which the form factors are extracted, and we use the particular combinations

\[
\sum_{k=1}^{3} \Pi_k^a = K_1 G_{E0}(Q^2) + K_2 G_{E2}(Q^2),
\]

\[
\sum_{j,k=1}^{3} \varepsilon_{jk} \Pi_k^a = K_3 G_{M1}(Q^2),
\]

\[
\sum_{j,k=1}^{3} \varepsilon_{jk} \Pi_k^a = K_4 G_{E2}(Q^2).
\]

The connected part of each combination of three-point functions can be calculated efficiently using the method of sequential inversions [18]. These yield the isovector form factors. At present, it is not yet computationally feasible to calculate the contributions arising from disconnected diagrams. A calculation of disconnected contributions in the case of the electromagnetic form factors have shown that these are consistent with zero [19]. We note that these disconnected contributions are particularly hard to calculate not just because they require the all-to-all propagators but also because they are noise dominated [20]. We expect that, like in the case of nucleon electromagnetic form factors, the disconnected contributions to the electromagnetic \( \Delta \) form factors are small. The known kinematical coefficients \( K_1, K_2, K_3, K_4 \) are functions of the \( \Delta \) mass and energy as well as \( \mu \) and \( \bar{q} \). The combinations above are chosen such that all possible directions of \( \mu \) and \( \bar{q} \) contribute symmetrically to the form factors at a given \( Q^2 \) [21]. The over-constrained system of Eqs. (4)–(6) is solved by a least-squares analysis, and \( G_{E2}(Q^2) \) can also be isolated separately from Eq. (6).

The details of the simulations are summarized in Table I. In each case, the separation between the final and initial time is \( t_f - t_i \approx 1 \) fm and Gaussian smearing is applied to both source and sink to produce adequate plateaus by suppressing contamination from higher states having the quantum numbers of the \( \Delta(1232) \). For the mixed-action calculation, the domain-wall valence quark mass was chosen to reproduce the lightest pion mass obtained using \( N_F = 2 + 1 \) improved staggered quarks [21,23].

### III. RESULTS

The results for \( G_{E0}(Q^2) \) are shown in Fig. 1 as a function of \( Q^2 \) at the lightest pion mass for each of the three actions. For Wilson fermions, we use the conserved lattice current requiring no renormalization. The local current is used for the mixed action, and the renormalization constant \( Z_V = 1.0992(32) \) is determined by the condition that \( G_{E0}(0) \) equals the charge of the \( \Delta \) in units of \( e \). As can be seen,

<table>
<thead>
<tr>
<th>( N_{\text{conf}} )</th>
<th>( m_\sigma ) [GeV]</th>
<th>( m_\Delta ) [GeV]</th>
<th>( \sqrt{\langle r^2 \rangle} ) [fm]</th>
<th>( \mu_{\Delta^+} ) [( \mu_N )]</th>
<th>( Q_{\Delta^+}^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>0.563(4)</td>
<td>1.470(15)</td>
<td>0.6147(66)</td>
<td>1.720(42)</td>
<td>0.961(12)</td>
</tr>
<tr>
<td>200</td>
<td>0.490(4)</td>
<td>1.425(16)</td>
<td>0.6329(76)</td>
<td>1.763(51)</td>
<td>0.911(15)</td>
</tr>
<tr>
<td>200</td>
<td>0.411(4)</td>
<td>1.382(19)</td>
<td>0.6516(87)</td>
<td>1.811(69)</td>
<td>0.832(21)</td>
</tr>
<tr>
<td>( N_F = 2 ) Wilson, ( 24^3 \times 40 ) (32 for lightest pion), ( a = 0.077 ) fm</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>185</td>
<td>0.691(8)</td>
<td>1.687(15)</td>
<td>0.5279(61)</td>
<td>1.462(45)</td>
<td>0.802(21)</td>
</tr>
<tr>
<td>157</td>
<td>0.509(8)</td>
<td>1.559(19)</td>
<td>0.594(10)</td>
<td>1.642(81)</td>
<td>0.41(45)</td>
</tr>
<tr>
<td>200</td>
<td>0.384(8)</td>
<td>1.395(18)</td>
<td>0.611(17)</td>
<td>1.58(11)</td>
<td>0.46(35)</td>
</tr>
<tr>
<td>( N_F = 2 + 1 ), Mixed action, ( 28^3 \times 64 ), ( a = 0.124 ) fm [22]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>0.353(2)</td>
<td>1.533(27)</td>
<td>0.641(22)</td>
<td>1.91(16)</td>
<td>0.74(68)</td>
</tr>
</tbody>
</table>
all three calculations yield consistent results. The momentum dependence of the charge form factor is described well by a dipole form

\[ G_{E0}(Q^2) = \frac{1}{1 + (Q^2/C_0^2)} \]

To compare the slopes at \( Q^2 = 0 \), we follow convention and show in Table I the so-called “rms radius” \( \langle r^2 \rangle = -6 \frac{d}{dQ^2} G_{E0}(Q^2)|_{Q^2=0} \) [17].

The momentum dependence of \( G_{M1}(Q^2) \) is displayed in Fig. 2. In order to extract the magnetic moment an extrapolation to zero momentum transfer is necessary. Both an exponential form, \( G_{M1} e^{-Q^2/A_{m1}} \), and a dipole describe the \( Q^2 \) dependence well, and we adopt the exponential form because of its faster decay at large \( Q^2 \), in accord with perturbative arguments. The larger spatial volume for the quenched and mixed-action cases yields smaller and more densely spaced values of the lattice momenta and correspondingly more precise determination of the form factor than for the smaller volume used with dynamical Wilson fermions. In Fig. 2, we show the best exponential fit and error band for the mixed action and quenched results. As can be seen, results in the quenched theory and for \( N_f = 2 \) Wilson fermions are within the error band. The magnetic moment in natural units is given by \( \mu_\Delta = GM_1(0)e/(2m_\Delta) \), where \( m_\Delta \) is the \( \Delta \) mass measured on the lattice and \( GM_1(0) \) is from the exponential fits. In Table I, we give the values of the \( \Delta^+ \) magnetic moment in nuclear magnetons \( e/(2m_N) \), with \( m_N \) the physical nucleon mass. The magnetic moments of the \( \Delta^+ \) and \( \Delta^{++} \) are accessible to experiments [1,2], which presently suffer from large uncertainties. The magnetic moment as function of \( m_\pi^2 \) is shown in Fig. 3, together with a comparison to a chiral effective field theory (ChEFT) result [24]. The ChEFT result has one free parameter (a low-energy constant) that has been fitted to lattice data, shown by the central line. We also estimate the uncertainty of the ChEFT expansion (expansion in pion mass) by the error band in Fig. 3. The uncertainty of the ChEFT calculation vanishes in the chiral limit because in this limit one simply has the value of the low-energy constant, which lies within the broad experimental error band \( \mu_{\Delta^+} = 2.7^{+1.0}_{-1.3}(\text{stat}) \pm 1.5(\text{syst}) \pm 3.0(\text{theory})\mu_N [1] \). In this work, we do not consider the uncertainty in the fit value of the low-energy constant due to the lattice errors, as the calculations are still performed for pion masses where the \( \Delta \) is stable (on the right side of the kink). A calculation for pion mass values where the \( \Delta \)
becomes unstable will be a challenge for future calculations. The $\Delta$ moments using an approach similar to ours are calculated only in the quenched approximation [17,25,26]. Our magnetic moment results agree with recent background field calculations using dynamical improved Wilson fermions [27], which supersede previous quenched background field results [28]. The spatial length $L_s$ of our lattices satisfies $L_s m_\pi > 4$ in all cases except at the lightest pion mass with $N_F = 2$ Wilson fermions, for which $L_s m_\pi = 3.6$. For that point, the magnetic moment falls slightly below the error band, consistent with the fact that Ref. [27] shows that finite volume effects decrease the magnetic moment.

The electric quadrupole form factor is particularly interesting because it can be related to the shape of a hadron, and lattice calculations for each of the three actions are shown in Fig. 4 with exponential fits for the quenched and mixed-action cases. Just as the electric form factor for a spin-1/2 nucleon can be expressed precisely as the transverse Fourier transform of the transverse quark charge density in the infinite momentum frame [29], a proper field-theoretic interpretation of the shape of the $\Delta(1232)$ can be obtained by considering the quark transverse charge densities in this frame [30–32]. With respect to the direction of the average baryon momentum $P$, the transverse charge density in a spin-3/2 state with transverse polarization $s_\perp$ is defined as

$$
\rho_{J^3 = 0}^\Delta (b) = \int d^2\vec{q}_\perp (2\pi)^2 e^{-i\vec{q}_\perp \cdot \vec{b}} \frac{1}{2P^+} \times \left\langle p^+, \frac{\vec{q}_\perp}{2}, s_\perp \left| J^+ (0) \right| p^+, -\frac{\vec{q}_\perp}{2}, s_\perp \right\rangle ,
$$

where the photon transverse momentum $\vec{q}_\perp$ satisfies $\vec{q}_\perp = -\vec{b}$. The term proportional to $[G_{M1}(0) - 3e_\Delta]$ is an electric quadrupole moment induced in the moving frame due to the magnetic dipole moment. For a spin-3/2 particle without internal structure, $G_{M1}(0) = 3e_\Delta$, $G_{E2}(0) = -3e_\Delta$ [21,33], and the quadrupole moment of the transverse charge density vanishes. Hence, $Q^\Delta_{3/2}$, and thus the deformation of the two-dimensional transverse charge density, is only sensitive to the anomalous parts of the spin-3/2 magnetic dipole and electric quadrupole moments, and vanishes for a particle without internal structure. The analogous property holds for a spin-1 particle [32], indicating the generality of this description in terms of transverse densities.

Figure 5 shows the transverse density $\rho_{J^3 = 0}^\Delta$ for a $\Delta^+$ with transverse spin $s_\perp = +3/2$ calculated from the fit to the quenched Wilson lattice results for the $\Delta$ form factors (which has the smallest statistical errors of the three cal-

![FIG. 4 (color online). The electric quadrupole form factor. The notation is the same as that in Fig. 1. The value of $G_{E2}$, in units of $e/(2m_\pi^2)$, at $Q^2 = 0$ are $-0.810 \pm 0.291$ for the quenched calculation, $-0.87 \pm 0.67$ for the dynamical Wilson case, and $-2.06^{+1.27}_{-2.35}$ for the hybrid calculation.](014507-4)

![FIG. 5 (color online). Quark transverse charge density in a $\Delta^+$ polarized along the $x$ axis, with $s_\perp = +3/2$. The light (dark) regions correspond with the largest (smallest) values of the density.](014507-4)
It is seen that the $\Delta^+$ quark charge density is elongated along the axis of the spin (prolate). This prolate deformation is robust in the sense that the values for $Q_{3/2}^t$ obtained from Eq. (9) and given in Table I are all consistently positive. The connection between the results on deformation in the current formulation and previous ones will be discussed in a forthcoming publication [21]. In the case of the magnetic octupole form factor [21], which is related to the magnetic octupole moment $O_3$, the result from zero. The lattice calculations show that the quark density in a $\Delta^+$ of transverse spin projection $+3/2$ is elongated along the spin axis.

**IV. CONCLUSIONS**

In summary, a formalism for the accurate evaluation of the $\Delta$ electromagnetic form factors as functions of $q^2$ has been developed and used in quenched QCD and full QCD with two and $2+1$ flavors. The charge radius and magnetic dipole moment were determined as a function of $m_\Delta^2$, and the dipole moment was compared with the result of chiral effective theory. The electric quadrupole form factor was evaluated for the first time with sufficient accuracy to distinguish it from zero. The lattice calculations show that the quark density in a $\Delta^+$ of transverse spin projection $+3/2$ is elongated along the spin axis.

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