Stable Levitation and Alignment of Compact Objects by Casimir Spring Forces

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We investigate a stable Casimir force configuration consisting of an object contained inside a spherical or spheroidal cavity filled with a dielectric medium. The spring constant for displacements from the center of the cavity and the dependence of the energy on the relative orientations of the inner object and the cavity walls are computed. We find that the stability of the force equilibrium—unlike the direction of the torque—can be predicted based on the sign of the force between two slabs of the same material.

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The Casimir force between atoms or macroscopic objects arises from quantum fluctuations of the electromagnetic field [1]. Typically, it is found that the force is attractive, as long as the space between the objects is empty, and the magnetic susceptibility of the objects is negligible compared to their electric susceptibilities. But when space is filled with a medium with electric permittivity \( \varepsilon_M \) intermediate between that of two objects, \( \varepsilon_1 < \varepsilon_M < \varepsilon_2 \), the force between the two becomes repulsive [2].

This effect has recently been verified experimentally in the large separation (retarded) regime [3]. While repulsive forces are nothing new, they become interesting for applications when they produce stable equilibria, which, for example, the Coulomb force cannot.

For an infinite cylinder enclosed in another, the Casimir force has recently been shown to have a stable equilibrium in the two directions perpendicular to the cylinder axes, when the material properties are chosen so that the force between two slabs of the same materials would be repulsive [4]. On the other hand, for a metal sphere or an electrically polarizable atom inside an otherwise empty spherical cavity there is no point of stable equilibrium [5].

In this Letter we investigate the first configuration, depicted in Fig. 1, in which a compact object levitates stably due to the Casimir force alone. We consider the following cases: a finite sphere or a small spheroid inside a spherical cavity, and a small spheroid inside a slightly deformed spherical cavity. The Casimir energy

\[ E = E_0 + \frac{1}{2} k \frac{a^2}{R^2} + \frac{1}{3!} k_3 \frac{a^3}{R^3} + \frac{1}{4!} k_4 \frac{a^4}{R^4} + \cdots \]

is characterized by the spring constant \( k \) and the coefficients \( k_n \) in a series expansion in \( a/R \), where \( a \) is the magnitude of the displacement from the center of the cavity and \( R \) the radius of the (undeformed) spherical cavity. (\( k \) has units of energy here.) Unlike the case of infinite cylinders, our case exhibits, for appropriately chosen materials, true stability in all directions and applies to realistic situations. For example, we compute the force on a metal sphere in a spherical drop of liquid surrounded by air. By determining the mean square deviation from the center, \( \langle a^2 \rangle = 3k_B T R^2/k \), the spring constant \( k \) can be measured experimentally. We can estimate that the size of the droplet \( R \) has to be smaller than \( = 3 \mu m \frac{\varepsilon^3}{k_B T} \), where \( r \) is the typical dimension of the inner object, for the thermal motion to be confined near the center of the cavity. (This length scale is obtained by balancing \( 3k_B T \) for room temperature with a rough estimate of the spring constant, \( k \sim \frac{b e^3}{R} \).) To keep the two objects nearly concentric against the gravitational force, \( R \) has to be smaller than \( \approx 1 \mu m \) for the total metal or liquid densities considered here. Our calculations show that, for example, a sphere of gold of radius \( R = \frac{3}{4} \mu m \) inside a spherical drop of ethanol of radius \( R = 0.1 \mu m \) has an rms deviation \( \sqrt{\langle a^2 \rangle} = 0.04R \) from the center due to thermal motion and a displacement from the center by \( d = 10^{-8} R \) due to gravity. A variety of technologies may benefit from our analysis, e.g., nanocarriers \( (R = 0.1 \mu m) \) for drug particles [6] or molecular cages for explosive molecules [7].

Whether the center of the cavity is a point of stable or unstable equilibrium turns out to be correlated with

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{(color online). Summary of the configurations we consider and of the results. We have assumed that the small spheroid’s zero frequency permittivity satisfies \( \varepsilon_{M,0} > \varepsilon_{O,0} \) and that it is larger in the body-fixed \( \hat{z} \) direction, so \( \alpha_{\hat{z}}^M > \alpha_{\hat{z}}^O \). Furthermore, the magnetic permeabilities are all set to one. (a) Direction of the force \( \mathbf{F} \) on such a spheroid in a spherical cavity if \( \varepsilon_{M,0} > \varepsilon_{O,0} \), and the direction of the torque \( \mathbf{\tau} \) when either \( \varepsilon_{M,0} > \varepsilon_{O,0} \) or \( \varepsilon_{M,0} \ll \varepsilon_{O,0} \). (b) A finite size sphere experiences a restoring force \( \mathbf{F} \) for the various combinations of materials listed in Table I. (c) Direction of the torque \( \mathbf{\tau} \) in the center of a slightly spheroidal cavity if either \( \varepsilon_{M,0} < \varepsilon_{O,0} \) or \( \varepsilon_{M,0} > \varepsilon_{O,0} \).}
\end{figure}
whether the Casimir force is repulsive or attractive for two parallel plates under those conditions. The direction of the torque, on the other hand, depends on the dielectric properties of the medium and the cavity walls in an unintuitive way, which cannot be predicted by the pairwise summation or proximity force approximations (PSA or PFA), see Fig. 1. In particular, this behavior is not due to dispersion effects, which explain similar phenomena reported in Ref. [4]. We calculate the torque on a small spheroid that is displaced from the center of a spherical cavity [Fig. 1(a)] or concentric with a slightly spheroidal cavity [Fig. 1(c)]. In the former case, the orientation dependence manifests itself in $k$, which captures both the torque and the orientation dependence of the total force. In the latter case, $E_0$ depends on the relative orientation of the spheroid inside the deformed cavity. (If the cavity is spherical, $E_0$ is a finite constant that can be ignored.) By choosing appropriate materials, the inside object, e.g., a nanorod, can be made to align in different ways with the cavity shape, a situation which is reminiscent of a compass needle aligning with the magnetic field of the earth.

The starting point of the analysis is the scattering theory approach. The method is explained and derived in detail in Ref. [8], where a partial overview over its precursors, e.g., [9], is provided. The Casimir energy $E = \frac{\hbar c}{2\pi} \int_0^\infty d\xi \ln \det(I - T_{\text{ee}})\mathcal{W}^{io} T_{\text{ee}}^{\text{intro}} \mathcal{V}^{io}$ is expressed in terms of the inner object’s exterior $T$ matrix, $T_{\text{ee}}^i$, and the outer object’s interior $T$ matrix, $T_{\text{io}}^o$. The exterior $T$ matrix describes the scattering of regular wave functions to outgoing waves when the source lies at infinity. The interior $T$ matrix expresses the opposite, the amplitudes of the regular wave functions, which result from scattering outgoing waves from a source inside the object. The translation matrices $\mathcal{W}^{io}$ and $\mathcal{V}^{io}$ convert regular wave functions between the origins of the outer and the inner objects; they are related by complex transpose up to multiplication by $(-1)$ of some matrix elements.

With uniform, isotropic, and frequency-dependent permittivity $\varepsilon_i(\omega c)$ and permeability $\mu_i(\omega c)$ functions ($x = I$: inner object; $x = O$: outer object; $x = M$: medium) the $T$ matrix of the sphere is diagonal. It is given by

$$F_{\text{ee}}^{\text{intro}} = F_{\text{io}}^i(\xi)$$

$$= -\frac{i(\xi)\delta(r_i z_{i\xi}) - \frac{\varepsilon_i}{\varepsilon_0} i(\xi)\delta(r_i z_{i\xi})}{k(\xi)\delta(r_i z_{i\xi}) - \frac{\varepsilon_i}{\varepsilon_0} i(\xi)\delta(r_i z_{i\xi})}$$

for $E$ (electric) polarization and by the same expression with $\frac{\mu_i}{\mu_0}$ replaced by $\frac{\varepsilon_i}{\varepsilon_0}$ for $M$ (magnetic) polarization (not to be confused with subscript $M$ indicating the medium’s response functions). In the above equation the frequency dependence of the response functions has been suppressed. The indices of refraction $n_i(\omega c) = \sqrt{\varepsilon_i(\omega c)\mu_i(\omega c)}$ of the sphere and the medium appear in the ratio $z_i(\omega c) = n_i(\omega c)/n_M(\omega c)$ and the argument $\xi = n_M(\omega c)\kappa r$. The interior $T$ matrix of the spherical cavity is obtained from the exterior $T$ matrix of the sphere by inserting the outside object’s radius and response functions in place of those of the inside object and exchanging the modified spherical Bessel functions $i_l$ and $k_l$.

However, the scattering approach is not limited to simple geometries. An array of techniques is available for calculating the scattering amplitudes of other shapes. We employ the perturbation approach to find the $T$ matrix of a deformed spherical cavity [10,11] of radius $R + \delta(1 - 3/2\sin^2 \theta)$. The deformation, indicated in Fig. 1(c), is chosen so that the volume is unchanged to first order in $\delta$. We find the $O(\delta)$ correction, $F^{(1)}$, to the $T$ matrix in a perturbation series expansion, $F = F^{(0)} + F^{(1)} + \ldots$, by matching the regular and outgoing fields according to the Maxwell boundary conditions along the deformed object’s surface [12]. On the other hand, for a small object (compared to the wavelength of the radiation), we can approximate the $T$ matrix to lowest order in $\kappa$ using the static polarizability tensor $F^{(0)}_{\text{M},\text{E}} = 2/3(n_{M,0}\kappa^3)\alpha_{\text{M,EE}} + O(\kappa^2)$, where the subscript $0$ indicates the static ($\alpha_{\text{M}} = 0$) limit and $P$ is the polarization label. The $T$-matrix elements involving higher angular momenta $l > 1$ are higher order in $\kappa$. For a small ellipsoid, in particular, the electric polarizability tensor $\alpha^E$ is diagonal in a coordinate system aligned with the ellipsoid’s body axes, $\alpha_{\text{M}}^E = \frac{Y}{4\pi} \frac{\epsilon_{\text{L},0} - \epsilon_{\text{M},0}}{\epsilon_{\text{L},0} + \epsilon_{\text{M},0}}$, where $i \in \{x, y, z\}$ [13]. The larger the semi-axis in direction $i$, the smaller the depolarization factor $n_i$ (not to be confused with the index of refraction). The magnetic polarizability tensor $\alpha^M$ is obtained by exchanging $\mu_{\text{M},0}$ for $\epsilon_{\text{M},0}$ in the expression for $\alpha^E$. In the small size limit the polarizability tensor of a perfect metal ellipsoid is obtained by taking both $\epsilon_{\text{M},0} \to \infty$ and $\mu_{\text{M},0} \to 0$.

For simplicity we specialize to a spheroid, which has two equal semi-axes. We choose the semi-axes along $\hat{x}$ and $\hat{y}$ to be equal; therefore, $\alpha_{\text{M}}^E = \alpha_{\text{M}}^E + \alpha_{\text{M}}^E$. We fix the $z$ axis of the lab frame to be along the direction of displacement of the spheroid from the center of the cavity. $\theta$ denotes the angle between the spheroid’s and the lab’s $z$ axes. For such a small spheroid inside a spherical cavity of radius $R$ [Fig. 1(a)], the spring constant is obtained by expanding the log determinant in the expression for $E$ to first order,

$$k_{\text{R},\to} = \frac{\hbar c}{R^4 n_{M,0} \left[ \text{Tr} \alpha_{\text{M}} E \right.}$$

$$+ (\alpha_{zz}^E - \alpha_{\text{M}}^E) \frac{3\cos^2 \theta - 1}{2} f_{2}^E + E \to M \right]$$

where the material dependent functions

$$f_{1}^E = \int_0^\infty \frac{\xi^2 d\xi}{9\pi} \left[ F_{\text{M},\text{E}}^{ii} - 2F_{\text{M},\text{E}}^{iO} - F_{\text{M},\text{E}}^{ii} \right]$$

$$f_{2}^E = \int_0^\infty \frac{\xi^2 d\xi}{9\pi} \left[ 4F_{\text{M},\text{E}}^{iO} - 4F_{\text{M},\text{E}}^{ii} - F_{\text{M},\text{E}}^{ii} \right]$$

express the rotation invariant and the orientation dependent parts of the energy, respectively. $f_{1}^M$ and $f_{2}^M$ are obtained from $f_{1}^E$ and $f_{2}^E$ by exchanging $E$ and $M$ everywhere. This
result is valid for asymptotically large \( R \); it involves only the zero frequency \((i\epsilon \kappa = 0)\) response functions since \( R \) is greater than any material dependent length scales as well as the size of the inner object. Notice that only the \( I = 1, 2 \) scattering amplitudes of the cavity walls appear in Eqs. (2) and (3).

The behavior of the functions \( f_E^I \), depicted in Fig. 2 for \( \mu_{M,0} = \mu_{O,0} \), is as expected: \( f_E^1 \) is monotonic, positive when \( \epsilon_{M,0} > \epsilon_{O,0} \); negative when \( \epsilon_{M,0} < \epsilon_{O,0} \), and \( f_E^1 \) always has the opposite sign of \( f_E^1 \). When \( f_E^1 \) is positive, a small object with \( \epsilon_{1,0} > \epsilon_{M,0} \) is levitated stably, when \( f_E^1 \) is negative, \( \epsilon_{M,0} > \epsilon_{1,0} \) has to hold. Thus, stability occurs under the same conditions as repulsion for two half spaces. The opposite sign of \( f_E^1 \) is expected from equivalent expressions for the two-infinite-slab geometry [14]. When the dielectric contrast between the medium and the outer sphere is taken to small or large limits, stability or instability is maximized.

To verify whether stability is observable for realistic materials and object sizes, we evaluate the energy \( E \) numerically for a sphere of radius \( r \) inside a spherical cavity of radius \( R \) filled with various liquids [Fig. 1(b)] [15]. The coefficients \( k, k_3, \) and \( k_6 \) in the series expansion in Eq. (1) are listed in Table I. For comparison, the asymptotic result \( k_{R \to \infty} \) is also included. If both inside and outside object are spherically symmetric, the series in Eq. (1) does not contain terms \( \sim \frac{r^0}{R^0} \) with \( n \) odd.

The three materials were chosen so that the sequence of permittivities \( \epsilon_r, \epsilon_M, \epsilon_O \) either increases or decreases for the imaginary frequencies that contribute most to the energy. Contrary to the prediction of the PSA and PFA the force is not symmetric with respect to exchange of the inner and outer permittivities. In the same medium, a high dielectric sphere is held more stably in the center of a cavity with low dielectric walls than a low dielectric sphere inside a cavity with high dielectric walls.

The asymptotic result \( k_{R \to \infty} \) yields a good approximation of \( k \) for \( \frac{r}{R} = \frac{1}{4} \). But from Eq. (2) one would expect \( k_{R \to \infty} \) to grow linearly with the volume of the inner sphere, since polarizability is proportional to the volume. In fact, \( k \)

<table>
<thead>
<tr>
<th>Inside-medium-outside</th>
<th>( R )</th>
<th>( r/R )</th>
<th>( k )</th>
<th>( k_{R \to \infty} )</th>
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<tr>
<td>Gold-bromobenzene-vacuum</td>
<td>0.1</td>
<td>1/4</td>
<td>0.040</td>
<td>0.054</td>
</tr>
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<td></td>
<td>3/4</td>
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<td>4200</td>
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<td>1/4</td>
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<td>0.054</td>
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<td></td>
<td>3/4</td>
<td>1.4</td>
<td>18000</td>
<td>1.0 \times 10^7</td>
</tr>
<tr>
<td>Gold-ethanol-vacuum</td>
<td>0.1</td>
<td>1/4</td>
<td>0.050</td>
<td>0.037</td>
</tr>
<tr>
<td></td>
<td>3/4</td>
<td>0.99</td>
<td>5200</td>
<td>2.2 \times 10^6</td>
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<tr>
<td></td>
<td>1.0</td>
<td>1/4</td>
<td>0.045</td>
<td>0.037</td>
</tr>
<tr>
<td></td>
<td>3/4</td>
<td>0.99</td>
<td>18000</td>
<td>1.2 \times 10^7</td>
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<tr>
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<td>0.015</td>
<td>0.012</td>
</tr>
<tr>
<td></td>
<td>3/4</td>
<td>0.33</td>
<td>1900</td>
<td>8.4 \times 10^5</td>
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<td></td>
<td>1.0</td>
<td>1/4</td>
<td>0.019</td>
<td>0.012</td>
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<tr>
<td></td>
<td>3/4</td>
<td>0.33</td>
<td>12000</td>
<td>7.6 \times 10^6</td>
</tr>
</tbody>
</table>

FIG. 3 (color online). \( f_E^2 \) and \( f_M^2 \) describe the part of the spring constant \( k_{R \to \infty} \), which changes with the orientation of the inside spheroid. In this plot, \( \mu_{M,0} = \mu_{O,0} \).
direction of the total force and the stability of the equilibrium can be determined based on the relative magnitudes of the permittivities, the torque cannot be. The PSA and PFA predict the orientation with the lowest energy in the vicinity of $\frac{\varepsilon_{oO}}{\varepsilon_{M0}} = 1$ correctly but not when $\frac{\varepsilon_{oO}}{\varepsilon_{M0}} \gg 1$.

Furthermore, the second sign change of $f_2^E$ and $f_2^M$ raises the question whether calculations of the Casimir torque for infinite conductivity metals are “universal” in the sense that they produce the correct qualitative results for real materials.

The orientation dependent part of the energy for the configurations discussed thus far vanishes, of course, when the small spheroid is in the center of the spherical cavity. If the cavity is slightly deformed, however, the energy, $E_0$, depends on the relative orientation of the spheroid and the cavity [Fig. 1(c)]. We deform the spherical cavity as described earlier and obtain to first order in $\delta/R$,

$$E_0 = \frac{\hbar \cos^2 \theta}{R^4} \frac{\delta}{n_{M0}} \left[ (\alpha_z^E - \alpha_{\perp}^E) g^E + E \rightarrow M \right],$$

where the orientation-independent part of the energy has been dropped. For $\mu_{oO} = \mu_{M0}$, $g^E$ and $g^M$ are given by

$$g^E = \int_0^\infty \frac{\xi^3 d\xi}{10\pi} \left( \frac{i_1 k_0 + i_0 k_1}{(k_0 i_0 + k_1 i_0 z_0)} \right) \left( z_0 \xi \right) (1 - z_0^2) \right]$$

$$g^M = \int_0^\infty \frac{\xi^4 d\xi}{10\pi} \left( \frac{i_1 k_0 + i_0 k_1}{(k_0 i_0 + k_1 i_0 z_0)} \right) \left[ z_0^2 - 1 \right]$$

where the arguments of the modified spherical Bessel functions and of $z_0(0)$ are suppressed. $i_1$ and $k_1$ are functions of $\xi$, and $k_0$ stands for $k_0(z_0(0))$, where $z_0(0) = n_0(ikc)/n_M(ikc)$ is the ratio of the permittivities of the cavity walls and the medium. The material dependent functions $g^E$ and $g^M$ are plotted in Fig. 4.

Again, $g^E$ has the same sign for $\frac{\varepsilon_{oO}}{\varepsilon_{M0}} \rightarrow 0$ (left in Fig. 4) and $\frac{\varepsilon_{oO}}{\varepsilon_{M0}} \rightarrow \infty$ (right). In addition to the root at $\frac{\varepsilon_{oO}}{\varepsilon_{M0}} = 1$, $g^E$ also vanishes at $\frac{\varepsilon_{oO}}{\varepsilon_{M0}} = 0.46$.

The rich orientation dependence of the energy is expected to collapse as the size of the inside object grows to fill the cavity and the PFA becomes applicable. Based on the stability analysis for finite inside spheres, though, we expect the asymptotic results, $f_2^E$ and $g^E$, to predict the orientation dependence for reasonably small inside spheroids. For comparison with real experiments, of course, various corrections to the idealized shapes considered here have to be taken into account.

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[15] For the permittivity function of gold we use $\epsilon(\omega c) = 1 + \frac{\omega_p^2}{\omega^2 c^2}$, where $\omega_p = 1.14 \times 10^{16}$ Hz and $\gamma = 9.27 \times 10^{-14}$ Hz [16] as used in Ref. [3]. For the other materials we use an oscillator model, $\epsilon(\omega c) = 1 + \sum_{n=1}^{N} \frac{1}{(\omega c/c_n - i\gamma)^2}$, with $[\omega_n]_{n=1}^{N} = [6.6, 114] \times 10^{14}$ Hz and $[C_n]_{n=1}^{N} = [23.84, 0.852]$ for ethanol [17] as used in Ref. [4], with $[\omega_n]_{n=1}^{N} = [5.47, 128.6] \times 10^{14}$ Hz and $[C_n]_{n=1}^{N} = [2.967, 1.335]$ for bromobenzene [17] as used in Ref. [3], and with $[\omega_n]_{n=1}^{N} = [0.867, 1.508, 2.026, 203.4] \times 10^{14}$ Hz and $[C_n]_{n=1}^{N} = [0.829, 0.995, 0.798, 1.098]$ for silica (SiO$_2$) [18].