THETA BODIES FOR POLYNOMIAL IDEALS

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Abstract. Inspired by a question of Lovász, we introduce a hierarchy of nested semidefinite relaxations of the convex hull of real solutions to an arbitrary polynomial ideal, called theta bodies of the ideal. These relaxations generalize Lovász’s construction of the theta body of a graph. We establish a relationship between theta bodies and Lasserre’s relaxations for real varieties which allows, in many cases, for theta bodies to be expressed as feasible regions of semidefinite programs. Examples from combinatorial optimization are given. Lovász asked to characterize ideals for which the first theta body equals the closure of the convex hull of its real variety. We answer this question for vanishing ideals of finite point sets via several equivalent characterizations. We also give a geometric description of the first theta body for all ideals.

1. Introduction

A central concern in optimization is to understand $\text{conv}(S)$, the convex hull of the set of feasible solutions $S$, to a given problem. In many instances, the set of feasible solutions to an optimization problem is the set of real solutions to a polynomial system: $f_1(x) = f_2(x) = \cdots = f_m(x) = 0$, where $f_1, \ldots, f_m \in \mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_n]$. This set is the real variety, $V_\mathbb{R}(I)$, of the ideal $I$ in $\mathbb{R}[x]$ generated by $f_1, \ldots, f_m$, and it is often necessary to compute or represent $\text{conv}(V_\mathbb{R}(I))$ exactly or at least approximately.

Recall that $\text{cl}(\text{conv}(V_\mathbb{R}(I)))$, the closure of $\text{conv}(V_\mathbb{R}(I))$, is cut out by the inequalities $f(x) \geq 0$ as $f$ runs over all linear polynomials that are non-negative on $V_\mathbb{R}(I)$. (Call $f \in \mathbb{R}[x]$ a linear polynomial if it is affine linear of the form $f = a_0 + \sum_{i=1}^n a_i x_i$.) A classical certificate for the non-negativity of a polynomial $f$ on $V_\mathbb{R}(I)$ is the existence of a sum of squares (sos) polynomial $\sum_{j=1}^t h_j^2$ that is congruent to $f \mod I$ (i.e., $f - \sum_{j=1}^t h_j^2 \in I$), written as $f \equiv \sum_{j=1}^t h_j^2 \mod I$. If this is the case, we say that $f$ is sos mod $I$. Hence a natural relaxation of $\text{cl}(\text{conv}(V_\mathbb{R}(I)))$ is the closed convex set:

$$\{x \in \mathbb{R}^n : f(x) \geq 0 \forall f \text{ linear and sos mod } I\}.$$  

Depending on $I$, (1) may be strictly larger than $\text{cl}(\text{conv}(V_\mathbb{R}(I)))$ since there may be polynomials that are non-negative on $V_\mathbb{R}(I)$ but not sos mod $I$.
However, in many interesting cases, \((\ref{item:1})\) will equal \(\text{cl}(\text{conv}(V_R(I)))\). By bounding the degree of the \(h_j\)’s that appear in the sos representations, and gradually increasing this bound, we obtain a hierarchy of relaxations to \(\text{cl}(\text{conv}(V_R(I)))\). In \cite{LO}, Lovász asked a question that leads to the study of this hierarchy. To explain it, we first introduce some definitions.

**Definition 1.1.** Let \(f\) be a polynomial in \(\mathbb{R}[x]\), \(I\) be an ideal in \(\mathbb{R}[x]\) with real variety \(V_R(I) := \{s \in \mathbb{R}^n : f(s) = 0 \forall f \in I\}\), and let \(\mathbb{R}[x]_k\) denote the set of polynomials in \(\mathbb{R}[x]\) of degree at most \(k\).

1. The polynomial \(f\) is \(k\text{-sos}\) mod \(I\) if there exists \(h_1, \ldots, h_t \in \mathbb{R}[x]_k\) for some \(t\) such that \(f \equiv \sum_{j=1}^t h_j^2\) mod \(I\).
2. The ideal \(I\) is \(k\text{-sos}\) if every polynomial that is non-negative on \(V_R(I)\) is \(k\text{-sos}\) mod \(I\). If every polynomial of degree at most \(d\) that is non-negative on \(V_R(I)\) is \(k\text{-sos}\) mod \(I\), we say that \(I\) is \((d,k)\text{-sos}\).

**Example 1.2.** Consider the principal ideal \(I = (x_1^2x_2 - 1) \subset \mathbb{R}[x_1, x_2]\). Then \(\text{conv}(V_R(I)) = \{(s_1, s_2) \in \mathbb{R}^2 : s_2 > 0\}\), and any linear polynomial that is non-negative over \(V_R(I)\) is of the form \(\alpha x_2 + \beta\), where \(\alpha, \beta \geq 0\). Since \(\alpha x_2 + \beta \equiv (\sqrt{\alpha} x_1 x_2)^2 + (\sqrt{\beta})^2\) mod \(I\), \(I\) is \((1,2)\text{-sos}\). Check that \(x_2\) is not \(1\text{-sos}\) mod \(I\) and so, \(I\) is not \((1,1)\text{-sos}\).

In \cite{LO}, Lovász asked the following question.

**Problem 1.3.** \cite{LO} Problem 8.3 Which ideals in \(\mathbb{R}[x]\) are \((1,1)\text{-sos}\)? How about \((1,k)\text{-sos}\)?

The geometry behind the above algebraic question leads to a natural hierarchy of relaxations of \(\text{conv}(V_R(I))\) which we now introduce. The name comes from earlier work of Lovász and will be explained in Section 3.

**Definition 1.4.** (1) For a positive integer \(k\), the \(k\text{-th theta body}\) of an ideal \(I \subseteq \mathbb{R}[x]\) is \(\text{TH}_k(I) := \{x \in \mathbb{R}^n : f(x) \geq 0 \text{ for every linear } f \text{ that is } k\text{-sos} \text{ mod } I\}\).

2. An ideal \(I \subseteq \mathbb{R}[x]\) is \(\text{TH}_k\text{-exact}\) if \(\text{TH}_k(I)\) equals \(\text{cl}(\text{conv}(V_R(I)))\).

3. The \(\text{theta-rank}\) of \(I\) is the smallest \(k\) for which \(I\) is \(\text{TH}_k\text{-exact}\).

By definition, \(\text{TH}_1(I) \supseteq \text{TH}_2(I) \supseteq \cdots \supseteq \text{conv}(V_R(I))\). As seen in Example 1.2, \(\text{conv}(V_R(I))\) may not be closed while the theta bodies are. Therefore, the theta-body sequence of \(I\) can converge, if at all, only to \(\text{cl}(\text{conv}(V_R(I)))\).

A natural question at this point is whether the algebraic notion of an ideal being \((1,k)\text{-sos}\) is equivalent to the geometric notion of being \(\text{TH}_k\text{-exact}\).

**Lemma 1.5.** If an ideal \(I \subseteq \mathbb{R}[x]\) is \((1,k)\text{-sos}\) then it is \(\text{TH}_k\text{-exact}\.\)

**Proof:** Let \(I\) be \((1,k)\text{-sos}\) and \(s \in \mathbb{R}^n\) be not in \(\text{cl}(\text{conv}(V_R(I)))\). By the separation theorem \cite[Theorem III.1.3]{1} there exists a linear polynomial \(f\), non-negative over \(\text{cl}(\text{conv}(V_R(I)))\), such that \(f(s) < 0\). However, since \(I\) is \((1,k)\text{-sos}\), \(f\) is \(k\text{-sos}\) mod \(I\) and so \(s \notin \text{TH}_k(I)\). Hence \(\text{TH}_k(I) \supseteq \text{cl}(\text{conv}(V_R(I)))\) and so, \(\text{TH}_k(I)\) equals \(\text{cl}(\text{conv}(V_R(I)))\). \(\Box\)
Interestingly, the converse of Lemma 1.3 is false in general.

Example 1.6. Consider \( I = \langle x^2 \rangle \subset \mathbb{R}[x] \) with \( \mathcal{V}_\mathbb{R}(I) = \{0\} \subset \mathbb{R} \). All linear polynomials that are non-negative on \( \mathcal{V}_\mathbb{R}(I) \) are of the form \( \pm a^2 x + b^2 \) for some \( a, b \in \mathbb{R} \). If \( b \neq 0 \), then \( (\pm a^2 x + b^2) \equiv (\frac{a^2}{2b} x \pm b)^2 \mod I \). However, \( \pm x \) is not a sum of squares mod \( I \), and hence \( I \) is not \( (1,k) \)-sos for any \( k \). On the other hand, \( I \) is TH\(_1\)-exact since \( \text{conv}(\mathcal{V}_\mathbb{R}(I)) = \{0\} \) is cut out by the infinitely many linear inequalities \( \pm x + b^2 \geq 0 \) as \( b \) varies over \( b \neq 0 \).

Definition 1.7. Let \( I \) be an ideal in \( \mathbb{R}[x] \). Then \( I \) is

1. **radical** if it equals its radical ideal
   \[ \sqrt{I} := \{ f \in \mathbb{R}[x] : f^m \in I, \; m \in \mathbb{N} \setminus \{0\} \} \]
2. **real radical** if it equals its real radical ideal
   \[ \sqrt[\mathbb{R}]{I} := \{ f \in \mathbb{R}[x] : f^{2m} + g_1^2 + \cdots + g_t^2 \in I, \; m \in \mathbb{N} \setminus \{0\}, \; g_1, \ldots, g_t \in \mathbb{R}[x] \} \]
3. **zero-dimensional** if its complex variety
   \[ \mathcal{V}_\mathbb{C}(I) := \{ x \in \mathbb{C}^n : f(x) = 0 \; \forall \; f \in I \} \]

Recall that given a set \( S \subseteq \mathbb{R}^n \), its vanishing ideal in \( \mathbb{R}[x] \) is the ideal \( \mathcal{I}(S) := \{ f \in \mathbb{R}[x] : f(s) = 0 \; \forall \; s \in S \} \). Hilbert’s Nullstellensatz states that for an ideal \( I \subseteq \mathbb{R}[x] \), \( \sqrt{I} = \mathcal{I}(\mathcal{V}_\mathbb{C}(I)) \) and the Real Nullstellensatz states that \( \sqrt[\mathbb{R}]{I} = \mathcal{I}(\mathcal{V}_\mathbb{R}(I)) \). Hence, \( I \subseteq \sqrt{I} \subseteq \sqrt[\mathbb{R}]{I} \), and if \( I \) is real radical then it is also radical. See for example, [20] Appendix 2, for these notions.

We will prove in Section 2 that the converse of Lemma 1.3 holds for real radical ideals. These ideals occur frequently in applications and for them, Problem 1.3 is asking when \( I \) is TH\(_1\)-exact, or more generally, TH\(_k\)-exact.

Contents of this paper. Recall that a semidefinite program (SDP) is an optimization problem in the space of real symmetric matrices of the form:

\[
\max \left\{ c^tx : A_0 + \sum_{i=0}^{m} A_ix_i \succeq 0 \right\}
\]

where \( c \in \mathbb{R}^m \) and the \( A_i \)'s are real symmetric matrices. The notation \( A \succeq 0 \) implies that \( A \) is positive semidefinite. SDPs generalize linear programs and can be solved efficiently \[32\]. In Section 2 we prove that under a certain technical hypothesis (satisfied by real radical ideals for instance), the theta body sequence of an ideal \( I \) is a modified version of a hierarchy of relaxations for the convex hull of a basic semialgebraic set, due to Lasserre \[8\] \[9\]. In this case, each theta body is the closure of the projection of a spectrahedron (feasible region of a SDP), and an explicit representation is possible using the combinatorial moment matrices introduced by Laurent \[13\]. When \( I \) is a real radical ideal, we further prove that \( I \) is \((1,k)\)-sos if and only if \( I \) is TH\(_k\)-exact which impacts later sections.

In Section 3 we illustrate the theta body sequence for the maximum stable set and maximum cut problems in a graph which are two very well-studied
problems from combinatorial optimization. The stable set problem motivated Problem 1.3. We explain this connection in detail in Section 3.

In Section 4 we solve Problem 1.3 for vanishing ideals of finite point sets in \( \mathbb{R}^n \). This situation arises often in applications and is the typical set up in combinatorial optimization. Several corollaries follow: If \( S \subseteq \mathbb{R}^n \) is finite and its vanishing ideal \( \mathcal{I}(S) \) is \((1,1)\)-sos then \( S \) is affinely equivalent to a subset of \( \{0,1\}^n \) and its convex hull can have at most \( 2^n \) facets. If \( S \) is the vertex set of a down-closed 0/1-polytope in \( \mathbb{R}^n \), then \( \mathcal{I}(S) \) is \((1,1)\)-sos if and only if \( \text{conv}(S) \) is the stable set polytope of a perfect graph. Families of finite sets in growing dimension with \((1,1)\)-sos vanishing ideals are exhibited.

In Section 5, we give an intrinsic description of the first theta body, \( \text{TH}_1(I) \), of an arbitrary polynomial ideal \( I \) in terms of the convex quadrics in \( I \). This leads to non-trivial examples of \( \text{TH}_1 \)-exact ideals with arbitrarily high-dimensional real varieties and reveals the algebraic-geometric structure of \( \text{TH}_1(I) \). Analogous descriptions for higher theta bodies remain open.

Remark 1.8. In [10], Lasserre introduced the Schmüdgen Bounded Degree Representation (S-BDR) and the Putinar-Prestel Bounded Degree Representation (PP-BDR) properties of a compact basic semialgebraic set \( K = \{ x : g_1(x) \geq 0, \ldots, g_m(x) \geq 0 \} \) (where \( g_i \in \mathbb{R}[x] \)), defined as follows:

- \( K \) has the S-BDR property if there exists a positive integer \( k \) such that almost all linear \( f \) that are positive over \( K \) has a representation as \( f = \sum_{J \subseteq [m]} \sigma_J g_J \) where \( \sigma_J \) are sos, \( g_J := \prod_{j \in J} g_j \) and the degree of \( \sigma_J g_J \) is at most \( 2k \) for all \( J \subseteq [m] := \{1, \ldots, m\} \).
- \( K \) has the PP-BDR property if there exists a positive integer \( k \) such that almost all linear \( f \) that are positive over \( K \) has a representation as \( f = \sum_{j=0}^m \sigma_j g_j \) where \( \sigma_j \) are sos, \( g_0 := 1 \) and the degree of \( \sigma_j g_j \) is at most \( 2k \) for \( j = 0, \ldots, m \).

Call the smallest such \( k \) the S-BDR (respectively, PP-BDR) rank of \( K \). Here “almost all” means all except a set of Lebesgue measure zero. Note that the PP-BDR property implies that S-BDR property.

For an ideal \( I = \langle f_1, \ldots, f_m \rangle \subset \mathbb{R}[x] \), \( \mathcal{V}_{\mathbb{R}}(I) \) is the, possibly non-compact, basic semialgebraic set \( \{ x \in \mathbb{R}^n : \pm f_1(x) \geq 0, \ldots, \pm f_m(x) \geq 0 \} \). When \( \mathcal{V}_{\mathbb{R}}(I) \) is compact, its PP-BDR property is closely related to the \((1,k)\)-sos and \( \text{TH}_{k}\)-exact properties of \( I \). However, these notions are not exactly comparable since the PP-BDR rank of \( \mathcal{V}_{\mathbb{R}}(I) \) depends on the choice of generators of \( I \), and only the linear polynomials that are positive (as opposed to non-negative) over \( \mathcal{V}_{\mathbb{R}}(I) \). Regardless, note that if \( \mathcal{V}_{\mathbb{R}}(I) \) has PP-BDR rank \( k \), then \( I \) has theta-rank at most \( k \).

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2. Theta Bodies

In Definition 1.4, we introduced the $k$-th theta body of a polynomial ideal $I \subseteq \mathbb{R}[x]$ and observed that these bodies create a nested sequence of closed convex relaxations of $\text{conv}(\mathcal{V}_R(I))$ with $\text{TH}_k(I) \supseteq \text{TH}_{k+1}(I) \supseteq \text{conv}(\mathcal{V}_R(I))$. Lasserre [8] and Parrilo [21, 23] have independently introduced hierarchies of semidefinite relaxations for polynomial optimization over basic semialgebraic sets in $\mathbb{R}^n$ using results from real algebraic geometry and the theory of moments. We first examine the connection between the theta bodies of an ideal $I$ and Lasserre's relaxations for $\text{conv}(\mathcal{V}_R(I))$.

2.1. Lasserre's hierarchy and theta bodies.

**Definition 2.1.** Let $I$ be an ideal in $\mathbb{R}[x]$. The **quadratic module** of $I$ is

$$\mathcal{M}(I) := \{ s + I : s \text{ is sos in } \mathbb{R}[x] \}.$$ 

The **$k$-th truncation** of $\mathcal{M}(I)$ is

$$\mathcal{M}_k(I) := \{ s + I : s \text{ is } k\text{-sos} \}.$$ 

Both $\mathcal{M}(I)$ and $\mathcal{M}_k(I)$ are cones in the $\mathbb{R}$-vector space $\mathbb{R}[x]/I$. Let $(\mathbb{R}[x]/I)'$ denote the set of linear functionals on $\mathbb{R}[x]/I$ and $\pi_I$ be the projection map from $(\mathbb{R}[x]/I)'$ to $\mathbb{R}^n$ defined as $\pi_I(y) = (y(x_1 + I), \ldots, y(x_n + I))$.

Also let $\mathcal{M}_k(I)^* \subseteq (\mathbb{R}[x]/I)'$ denote the dual cone to $\mathcal{M}_k(I)$, the set of all linear functions on $\mathbb{R}[x]/I$ that are non-negative on $\mathcal{M}_k(I)$.

**Definition 2.2.** For $y \in (\mathbb{R}[x]/I)'$, let $H_y$ be the symmetric bilinear form

$$H_y : \mathbb{R}[x]/I \times \mathbb{R}[x]/I \longrightarrow \mathbb{R}$$

$$(f + I, g + I) \longmapsto y(fg + I)$$

and $H_{y,t}$ be the restriction of $H_y$ to the subspace $\mathbb{R}[x]/I$.

Recall that a symmetric bilinear form $H : V \times V \rightarrow \mathbb{R}$, where $V$ is a $\mathbb{R}$-vector space, is positive semidefinite (written as $H \succeq 0$) if $H(v, v) \geq 0$ for all non-zero elements $v \in V$. Given a basis $B$ of $V$, the matrix indexed by the elements of $B$ with $(b_i, b_j)$-entry equal to $H(b_i, b_j)$ is called the **matrix representation** of $H$ in the basis $B$. The form $H$ is positive semidefinite if and only if its matrix representation in any basis is positive semidefinite.

**Lemma 2.3.** Let $I \subseteq \mathbb{R}[x]$ be an ideal and $k$ a positive integer. Then

$$\mathcal{M}_k(I)^* = \{ y \in (\mathbb{R}[x]/I)': H_{y,k} \succeq 0 \}.$$ 

**Proof:** Note that $y \in \mathcal{M}_k(I)^*$ if and only if $y(s + I) \geq 0$ for all $k$-sos polynomials $s$. By linearity of $y$ this is equivalent to $y(h^2 + I) \geq 0$ for all $h \in \mathbb{R}[x]/I$ which is the definition of $H_{y,k}$ being positive semidefinite. 

The original Lasserre relaxations in [8] approximate $\text{conv}(S)$ for a basic semialgebraic set $S = \{ x \in \mathbb{R}^n : g_i(x) \geq 0, i = 1, \ldots, m \}$ by the sets

$$\left\{ (y(x_1), \ldots, y(x_n)) : y \in \mathbb{R}[x]', y(1) = 1, y \left( \sum_{i=0}^{m} s_i g_i \right) \geq 0 \right\}$$
where \( s_i \) are sos, \( g_0 := 1 \) and the degree of \( s_i g_i \) is bounded above by some fixed positive integer. When there are equations among the \( g_i(x) \geq 0 \), both Lasserre [9] (for 0/1 point sets) and Laurent [13] (more generally for finite varieties) propose doing computations mod the ideal generated by the polynomials defining the equations, to increase efficiency. We adopt this point of view since in our case, \( S = \mathcal{V}_R(I) \), is cut out entirely by equations, and work with the following definition of a Lasserre relaxation.

**Definition 2.4.** Let \( I \subseteq \mathbb{R}[x] \) be an ideal, \( k \) be a positive integer, and \( \mathcal{V}_1 \) be the hyperplane of all functions \( y \in (\mathbb{R}[x]/I)' \) such that \( y(1 + I) = 1 \). The \( k \)-th modified Lasserre relaxation \( Q_k(I) \) of \( \text{conv}(\mathcal{V}_R(I)) \) is

\[
Q_k(I) := \pi_I(\mathcal{M}_k(I)^* \cap \mathcal{V}_1).
\]

While \( \mathcal{M}_k(I)^* \cap \mathcal{V}_1 \) is always closed, \( Q_k(I) \) might not be (see Example 2.16). We first note that \( Q_k(I) \) is indeed a relaxation of \( \text{conv}(\mathcal{V}_R(I)) \).

**Lemma 2.5.** For an ideal \( I \) and a positive integer \( k \), \( \text{conv}(\mathcal{V}_R(I)) \subseteq Q_k(I) \).

**Proof:** For \( s \in \mathcal{V}_R(I) \), consider \( y^s \in (\mathbb{R}[x]/I)' \) defined as \( y^s(f + I) := f(s) \). Then \( y^s \in \mathcal{M}_k(I)^* \) and \( y^s(1 + I) = 1 \). Therefore, \( \pi_I(y^s) = s \in Q_k(I) \), and \( \text{conv}(\mathcal{V}_R(I)) \subseteq Q_k(I) \) since \( Q_k(I) \) is convex. \( \square \)

Since \( Q_{k+1}(I) \subseteq Q_k(I) \), these bodies create a nested sequence of relaxations of \( \text{conv}(\mathcal{V}_R(I)) \) as intended. Our main goal in this section is to establish a relationship between \( Q_k(I) \) and the \( k \)-th theta body, \( \text{TH}_k(I) \), of the ideal \( I \) (cf. Definition 1.4). We start by noting the following inclusion.

**Proposition 2.6.** For an ideal \( I \subseteq \mathbb{R}[x] \) and a positive integer \( k \), \( \text{cl}(Q_k(I)) \subseteq \text{TH}_k(I) \).

**Proof:** Since \( \text{TH}_k(I) \) is closed, it is enough to show that \( Q_k(I) \subseteq \text{TH}_k(I) \).

Pick \( p \in Q_k(I) \) and \( y \in \mathcal{M}_k(I)^* \cap \mathcal{V}_1 \) such that \( \pi_I(y) = p \). Let \( f = a_0 + \sum_{i=1}^n a_i x_i \) and \( f + I \in \mathcal{M}_k(I) \). Then, \( p \in \text{TH}_k(I) \) since

\[
f(p) = f(\pi_I(y)) = a_0 y(1 + I) + \sum_{i=1}^n a_i y(x_i + I) = y(f + I) \geq 0.
\]

\( \square \)

Theorem 2.8 will prove that if \( \mathcal{M}_k(I) \) is closed, we have the equality \( \text{cl}(Q_k(I)) = \text{TH}_k(I) \).

**Lemma 2.7.** Let \( I \subseteq \mathbb{R}[x] \) be an ideal and \( k \) be a positive integer. If \( f \in \mathbb{R}[x]_1 \) is non-negative over \( Q_k(I) \), then \( f + I \in \text{cl}(\mathcal{M}_k(I)) \).

**Proof:** Suppose \( f \in \mathbb{R}[x]_1 \) is non-negative over \( Q_k(I) \) and \( f + I \notin \text{cl}(\mathcal{M}_k(I)) \). Then by the separation theorem, there exists \( y \in \mathcal{M}_k(I)^* \) such that \( y(f + I) < 0 \). Since \( (f + r + I)^2 = (f + r)^2 + I \) lies in \( \mathcal{M}_k(I) \) for any real number \( r \), \( y \in \mathcal{M}_k(I)^* \) and \( y \) is linear, we get

\[
0 \leq y((f + r + I)^2) = y(f^2 + I) + 2ry(f + I) + r^2 y(1 + I)
\]
which implies that \( y(1 + I) > 0 \) since \( y(f + I) \neq 0 \). Scaling \( y \) such that \( y(1 + I) = 1 \), we have that \( y \in \mathcal{M}_k(I)^* \cap \mathbb{Y}_1 \). This implies that \( \pi_I(y) \in Q_k(I) \) and therefore, by hypothesis, \( f(\pi_I(y)) \geq 0 \). However, since \( f \in \mathbb{R}[x]_1 \) and \( y \) is linear, we also get \( f(\pi_I(y)) = y(f + I) < 0 \) which is a contradiction. \( \square \)

**Theorem 2.8.** Let \( I \subseteq \mathbb{R}[x] \) be an ideal. For a positive integer \( k \), if \( \mathcal{M}_k(I) \) is closed, then \( \text{cl}(Q_k(I)) = TH_k(I) \).

**Proof:** By Proposition 2.6, we need to prove that when \( \mathcal{M}_k(I) \) is closed, \( TH_k(I) \subseteq \text{cl}(Q_k(I)) \). Suppose \( p \not\in \text{cl}(Q_k(I)) \). By the separation theorem, there exists \( f \in \mathbb{R}[x]_1 \) non-negative on \( \text{cl}(Q_k(I)) \) with \( f(p) < 0 \). By Lemma 2.7, \( f + I \in \mathcal{M}_k(I) \) since \( \mathcal{M}_k(I) \) is closed by assumption and hence \( f \) is \( k \)-sos mod \( I \). Since \( f(p) < 0 \), \( p \not\in TH_k(I) \). \( \square \)

An important class of ideals for which \( \mathcal{M}_k(I) \) is closed is the set of real radical ideals which are the focus of Sections 3 and 4. We now derive various corollaries to Theorem 2.8 that apply to real radical ideals.

**Corollary 2.9.** If \( I \subseteq \mathbb{R}[x] \) is a real radical ideal then \( \text{cl}(Q_k(I)) = TH_k(I) \).

**Proof:** By [25 Prop 2.6], if \( I \) is real radical, then \( \mathcal{M}_k(I) \) is closed. \( \square \)

**Lemma 2.10.** Let \( V \) and \( W \) be finite dimensional vector spaces, \( H \subseteq W \) be a cone and \( A : V \to W \) be a linear map such that \( A(V) \cap \text{int}(H) \neq \emptyset \). Then \( (A^{-1}H)^* = A'(H^*) \) where \( A' \) is the dual operator to \( A \). In particular, \( A'(H^*) \) is closed in \( V' \).

**Proof:** This follows from Corollary 3.3.13 in [2] by setting \( K = V \). \( \square \)

**Corollary 2.11.** Let \( I \) be a real radical ideal in \( \mathbb{R}[x] \) and \( k \) be a positive integer. If there exists \( g \in \mathbb{R}[x]_1 \) such that \( g + I \) is in the interior of \( \mathcal{M}_k(I) \) (considered as a subset of \( \mathbb{R}[x]_{2k}/I \)), then \( TH_k(I) = Q_k(I) \).

**Proof:** By Corollary 2.9, it suffices to show that \( Q_k(I) \) is closed. Consider \( \mathbb{R}^{n+1} \) with coordinates indexed 0, 1, \ldots, \( n \) and the map

\[
\pi_I : (\mathbb{R}[x]/I)^* \to \mathbb{R}^{n+1} \quad \text{such that} \quad y \mapsto (y(1 + I), \pi_I(y)).
\]

Then \( \pi_I(M_k(I)^* \cap \mathbb{Y}_1) = \{(1, p) : p \in Q_k(I)\} \) and so, \( Q_k(I) \) will be closed if \( \pi_I(M_k(I)^* \cap \mathbb{Y}_1) = \pi_I(M_k(I)^*) \cap \{p \in \mathbb{R}^{n+1} : p_0 = 1\} \) is closed. Hence, it suffices to show that \( \pi_I(M_k(I)^*) \subseteq \mathbb{R}^{n+1} \) is closed.

Now consider the inclusion map \( A : \mathbb{R}[x]_1/I \to \mathbb{R}[x]_{2k}/I \) and let \( M \) denote the cone \( \mathcal{M}_k(I) \) considered as a subset of \( \mathbb{R}[x]_{2k}/I \). By assumption, \( A(\mathbb{R}[x]_1/I) \cap \text{int}(M) = (\mathbb{R}[x]_1/I) \cap \text{int}(M) \neq \emptyset \) and so by Lemma 2.10, \( A'(M^*) \) is closed. Let \( C := \{1^*, x_1^*, \ldots, x_n^*\} \) be the canonical basis of \( (\mathbb{R}[x]_1/I)^* \). Then \( A'(\tilde{y}) = (\tilde{y}(1 + I), \tilde{y}(x_1 + I), \ldots, \tilde{y}(x_n + I)) \) with respect to \( C \). Now note that if \( y \in \mathcal{M}_k(I)^* \) then its restriction \( \tilde{y} \) to \( \mathbb{R}[x]_{2k}/I \) belongs to \( M^* \) and \( A'(\tilde{y}) = \pi_I(y) \). Therefore, \( \pi_I(M_k(I)^*) \subseteq A'(M^*) \). Conversely, if \( \tilde{y} \in M^* \) and \( \tilde{y} \) is any extension to \( \mathbb{R}[x]/I \), then \( \tilde{y} \) belongs to \( \mathcal{M}_k(I)^* \). Since
$A'(\tilde{y}) = \tilde{\pi}_I(\tilde{y})$ we get $A'(M^*) \subseteq \tilde{\pi}_I(\mathcal{M}_k(I)^*)$ and so, $A'(M^*) = \tilde{\pi}_I(\mathcal{M}_k(I)^*)$ is closed.

Let $I$ be an ideal and $k$ a positive integer. In Lemma 1.5, we saw that if $I$ is $(1,k)$-sos (i.e., every linear polynomial that is non-negative on $V_R(I)$ is $k$-sos mod $I$), then $I$ is TH$_k$-exact (i.e., $\text{TH}_k(I) = \text{cl}(\text{conv}(V_R(I)))$). Example 1.6 showed that the reverse implication does not always hold.

**Corollary 2.12.** If an ideal $I \subseteq \mathbb{R}[x]$ is real radical then $I$ is $(1,k)$-sos if and only if $I$ is TH$_k$-exact.

**Proof:** If $f \in \mathbb{R}[x]_1$ is non-negative on $V_R(I)$ and $I$ is TH$_k$-exact, then $f$ is non-negative on $\text{TH}_k(I)$ and hence on $Q_k(I)$. Therefore, by Lemma 2.7, $f \in \text{cl}(\mathcal{M}_k(I))$. Suppose now that $I$ is also real radical. Then $\mathcal{M}_k(I)$ is closed and $f + I \in \mathcal{M}_k(I)$, which means that $I$ is $(1,k)$-sos.

We close with a brief discussion of ideals for which the theta body sequence is guaranteed to converge (finitely or asymptotically) to $\text{cl}(\text{conv}(V_R(I)))$.

1. If $V_R(I)$ is finite the results in [11] imply that $I$ is TH$_k$-exact for some finite $k$. If $V_C(I)$ is finite ($I$ is zero-dimensional), then $k$ can be bounded above by the maximum degree of a linear basis of $\mathbb{R}[x]/I$ (see Section 2.2). However, as in $I = \langle x^2 \rangle$, we cannot guarantee that $I$ is $(1,k)$-sos for any $k$, even when $I$ is zero-dimensional. If $I$ is zero-dimensional and radical, then in fact, $I$ is $(1,k)$-sos for finite $k$ with $k \leq |V_C(I)| - 1$ (see [22, 14 Theorem 2.4]). Better bounds are often possible as in Remark 4.3. For an ideal $I \subseteq \mathbb{R}[x]$ we summarize the above results in the following table.

<table>
<thead>
<tr>
<th>$V_C(I)$ finite</th>
<th>$V_R(I)$ finite</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I = \sqrt{I}$</td>
<td>$I = \sqrt{I}$</td>
</tr>
<tr>
<td>$I$ $(1,k)$-sos</td>
<td>$I$ TH$_k$-exact</td>
</tr>
</tbody>
</table>

2. If $V_R(I)$ is not finite but is compact, Schmüdgen’s Positivstellensatz [20, Chapter 3] implies that the theta body sequence of $I$ converges (at least asymptotically) to $\text{cl}(\text{conv}(V_R(I)))$ (i.e., $\bigcap_{k=1}^{\infty} \text{TH}_k(I) = \text{cl}(\text{conv}(V_R(I)))$).

3. If $V_R(I)$ is not compact, then the study of the theta body hierarchy becomes harder. Scheiderer [20, Chapter 2] has identified ideals $I$ with $V_R(I)$ not necessarily compact, but of dimension at most two, for which every $f \geq 0$ mod $I$ is sos mod $I$. In all these cases, the theta body sequence of $I$ converges to $\text{cl}(\text{conv}(V_R(I)))$.

The results of Schmüdgen and Scheiderer mentioned above fit within a general framework in real algebraic geometry that is concerned with when an arbitrary $f \in \mathbb{R}[x]$ that is positive or non-negative over a basic semi-algebraic set is sos modulo certain algebraic objects defined by the set. We only care about real varieties and whether linear polynomials that are non-negative over them are sos mod their ideals. Therefore, often there are ideals...
work with the truncated quadratic module $M$

**Definition 2.13.** Let $R$ be the sum of squares in Combinatorial moment matrices.

2.2. Combinatorial moment matrices. To compute theta bodies we must work with the truncated quadratic module $M_k(I)$ which requires computing sums of squares in $\mathbb{R}[x]/I$ as described in [24], or dually, using the combinatorial moment matrices introduced by Laurent in [13]. We describe the latter viewpoint here as it is more natural for theta bodies.

Consider a basis $B = \{f_0 + I, f_1 + I, \ldots\}$ for $\mathbb{R}[x]/I$, and define $\deg(f + I) := \min_{f \in B} \deg f$. For a positive integer $k$, let $B_k := \{f + I \in B : \deg(f + I) \leq k\}$, and set $f_k := (f + I : f + I \in B_k)$. We may assume that the elements of $B$ are indexed in order of increasing degree. Let $\lambda^{(g+\ell)} := (\lambda_i^{(g+\ell)})$ be the vector of coordinates of $g + I$ with respect to $B$. Note that $\lambda^{(g+\ell)}$ has only finitely many non-zero coordinates.

**Definition 2.13.** Let $y \in \mathbb{R}^B$. Then the combinatorial moment matrix $M_B(y)$ is the (possibly infinite) matrix indexed by $B$ whose $(i, j)$ entry is

$$\lambda^{(f_i f_j + I)} y_l.\]$$

The $k$-th-truncated combinatorial moment matrix $M_{B_k}(y)$ is the finite (upper left principal) submatrix of $M_B(y)$ indexed by $B_k$.

Although only a finite number of the components in $\lambda^{(f_i f_j + I)}$ are non-zero, for practical purposes we need to control exactly which indices can be non-zero. One way to do this is by choosing $B$ such that if $f + I$ has degree $k$ then $f + I \in \text{span}(B_k)$. This is true for instance if $B$ is the set of standard monomials of a term order that respects degree [3]. If $B$ has this property then $M_{B_k}(y)$ only depends on the entries of $y$ indexed by $B_{2k}$.

**Theorem 2.14.** For each positive integer $k$,

$$\text{proj}_{B_1} \{y \in \mathbb{R}^{B_{2k}} : M_{B_k}(y) \succeq 0, y_0 = 1\} = f_1(Q_k(I)),\]$$

where $y_0$ is the first entry of $y \in \mathbb{R}^{B_{2k}}$, $\text{proj}_{B_1}$ is the projection onto the coordinates indexed by $B_1$, and for $p \in \mathbb{R}^n$, $f_1(p) := (f_i(p))_{f_i \in B_1}$.

**Proof:** We may identify $y = (y_i) \in \mathbb{R}^{B_{2k}}$ with the operator $\bar{y} \in (\mathbb{R}[x]/I)'$ where $\bar{y}(f_i + I) = y_i$ if $f_i + I \in B_{2k}$ and zero otherwise. Then $M_{B_k}(y)$ is simply the matrix representation of $H_{\bar{y}, k}$ in the basis $B$, since we assumed that if $\deg(f_i + I), \deg(f_j + I) \leq k$ then $\bar{y}(f_i f_j + I)$ depends only on the value of $\bar{y}$ on $B_{2k}$. Therefore, $\text{proj}_{B_1} \{y \in \mathbb{R}^{B_{2k}} : M_{B_k}(y) \succeq 0\}$ equals

$$\{(\bar{y}(f_i + I))_{B_1} : \bar{y} \in (\mathbb{R}[x]/I)', H_{\bar{y}, k} \succeq 0\}.\]$$

Furthermore, since $f_i$ is linear whenever $f_i + I \in B_1$,

$$\bar{y}(f_i + I))_{B_1} = (f_i(\pi_I(\bar{y})))_{B_1} =: f_i(\pi_I(\bar{y})).\]$$
so by Lemma \ref{L:cl2.3} \( \text{proj}_{R^2} \{ y \in \mathbb{R}^{2^k} : M_{B_k}(y) \succeq 0, y_0 = 1 \} = f_1(Q_k(I)). \) \( \Box \)

**Corollary 2.15.** Suppose \( B_1 = \{ 1 + I, x_1 + I, \ldots, x_n + I \} \) and denote by \( y_0, y_1, \ldots, y_n \) the first \( n + 1 \) coordinates of \( y \in \mathbb{R}^{2^k} \), then

\[
Q_k(I) = \{ (y_1, \ldots, y_n) : y \in \mathbb{R}^{2^k} \text{ with } M_{B_k}(y) \succeq 0 \text{ and } y_0 = 1 \}.
\]

By Corollary \ref{C:cl2.15}, optimizing a linear function over \( Q_k(I) \), hence over \( \text{cl}(Q_k(I)) \), is an SDP and can be solved efficiently.

**Example 2.16.** Consider the ideal \( I = \langle x_1^2 x_2 - 1 \rangle \subset \mathbb{R}[x_1, x_2] \) from Example \ref{E:cl2.15} for which \( \text{conv}(V_{\mathbb{R}}(I)) = \{ (s_1, s_2) \in \mathbb{R}^2 : s_2 > 0 \} \) was not closed but \( I \) was TH\(_2\)-exact and \((1, 2)\)-sos. Note that \( B = \bigcup_{k \in \mathbb{N}} \{ x_1^k + I, x_k^2 + I, x_1 x_k^2 + I \} \) is a degree-compatible monomial basis for \( \mathbb{R}[x_1, x_2]/I \) for which

\[
B_4 = \{ 1, x_1, x_2, x_1^2, x_1 x_2, x_2^2, x_1^2 x_2, x_1, x_2 x_1, x_1 x_2, x_2, x_1^2, x_1 x_2, x_2^2 \} + I.
\]

The combinatorial moment matrix \( M_{B_2}(y) \) for \( y = (1, y_1, \ldots, y_{11}) \in \mathbb{R}^{B_4} \) is

\[
\begin{pmatrix}
1 & 1 & y_1 & y_2 & y_3 & y_4 & y_5 \\
x_1 & y_1 & y_3 & y_4 & y_6 & 1 & y_7 \\
x_2 & y_2 & y_4 & y_5 & 1 & y_7 & y_8 \\
x_1 x_2 & y_3 & y_6 & 1 & y_9 & y_1 & y_2 \\
1 & y_4 & 1 & y_7 & y_1 & y_2 & y_10 \\
x_2^2 & y_5 & y_7 & y_8 & y_2 & y_10 & y_{11}
\end{pmatrix}
\]

If \( M_{B_2}(y) \succeq 0 \), then the principal minor indexed by \( x_1 \) and \( x_1 x_2 \) implies that \( y_2 y_3 \geq 1 \) and so in particular, \( y_2 \neq 0 \) for all \( y \in Q_2(I) \). However, since \( Q_2(I) \supseteq \text{conv}(V_{\mathbb{R}}(I)) = \{ (s_1, s_2) \in \mathbb{R}^2 : s_2 > 0 \} \), it must be that \( Q_2(I) = \text{conv}(V_{\mathbb{R}}(I)) \), which shows that \( Q_2(I) \) is not closed.

**Remark 2.17.** Example \ref{E:cl2.15} can be modified to show that \( Q_k(I) \) may not be closed even if \( V_{\mathbb{R}}(I) \) is finite. To see this, choose sufficiently many pairs of points \( (\pm t, 1/t^2) \) on the curve \( x_1^2 x_2 = 1 \) to form a set \( S \) such that the ideal \( I(S) \) has a monomial basis \( B' \) in which \( B'_4 \) equals the \( B_4 \) from above. For instance, \( S = \{ (\pm t, 1/t^2) : t = 1, \ldots, 7 \} \) will work. Then \( Q_2(I(S)) \) coincides with \( Q_2(I) \) computed above and so is not a closed set.

We now show that in the particular case of vanishing ideals of 0/1 points, which are real radical ideals, the closure in Theorem \ref{P:cl2.8} \( \text{TH}_k(I) = \text{cl}(Q_k(I)) \) is not needed. Most ideals that occur in combinatorial optimization have this form and we will see important examples in Section \ref{S:comb}. Remark \ref{R:cl2.17} shows that the closure cannot be removed for arbitrary finite point sets.

**Proposition 2.18.** If \( S \) is a set of 0/1 points in \( \mathbb{R}^n \) and \( I = I(S) \) then for all positive integers \( k \), \( \text{TH}_k(I) = Q_k(I) \).

**Proof:** By Corollary \ref{C:cl2.11} it is enough to show that there is a linear polynomial \( g \in \mathbb{R}[x] \) such that \( g \equiv f_k^2 A f_k \mod I \) for a positive definite matrix.
A and some basis of $\mathbb{R}[x]/I$ with respect to which $f_k$ was determined. Let $B$ be a monomial basis for $\mathbb{R}[x]/I$ and $B_k = \{1, p_1, \ldots, p_l\} + I$. Let $c \in \mathbb{R}^l$ be the vector with all entries equal to $-2$, and $D \in \mathbb{R}^{l \times l}$ be the diagonal matrix with all diagonal entries equal to 4. Since $x_i^2 \equiv x_i \bmod I$ for $i = 1, \ldots, n$ and $B$ is a monomial basis, for any $f + I \in B$, $f \equiv f^2 \bmod I$. Therefore, the constant

$$l + 1 \equiv f_k^t \begin{bmatrix} l + 1 & c^t \\ c & D \end{bmatrix} f_k \bmod I,$$

and it is enough to prove that the square matrix on the right is positive definite. This follows from the fact that $D$ is positive definite and its Schur complement $(l + 1) - c^t D^{-1} c = 1$ is positive ([7, Theorem 7.7.6]).

3. Combinatorial Examples

An important area of application for the theta body hierarchy constructed in Section 2 is combinatorial optimization which is typically concerned with optimizing a linear function over a finite set of integer points. In this section, we compute theta bodies for two important problems in combinatorial optimization – the maximum stable set problem and the maximum cut problem in a graph. We explain the observations about the stable set problem which motivated Lovász to pose Problem 1.3. The cut problem is modeled in two different ways. The first is a non-standard approach which is described fully. For the second, more standard model of the cut problem, theta bodies provide a new hierarchy of semidefinite relaxations for the cut polytope that is studied in detail in [5]. We outline those results briefly here. A recent trend in theoretical computer science has been to study the computational complexity of approximating problems in combinatorial optimization via the standard hierarchies of convex relaxations to these problems such as those in [19] and [8, 9]. Our theta body approach provides a new mechanism to establish such complexity results.

3.1. The Maximum Stable Set Problem. Let $G = ([n], E)$ be an undirected graph with vertex set $[n] = \{1, \ldots, n\}$ and edge set $E$. A stable set in $G$ is a set $U \subseteq [n]$ such that for all $i, j \in U$, $\{i, j\} \notin E$. The maximum stable set problem seeks the stable set of largest cardinality in $G$, the size of which is the stability number of $G$, denoted as $\alpha(G)$.

The maximum stable set problem can be modeled as follows. For each stable set $U \subseteq [n]$, let $\chi^U \in \{0, 1\}^n$ be its characteristic vector defined as $(\chi^U)_i = 1$ if $i \in U$ and $(\chi^U)_i = 0$ otherwise. Let $S_G \subseteq \{0, 1\}^n$ be the set of characteristic vectors of all stable sets in $G$. Then $\text{STAB}(G) := \text{conv}(S_G)$ is called the stable set polytope of $G$ and the maximum stable set problem is, in theory, the linear program $\max \{\sum_{i=1}^n x_i : x \in \text{STAB}(G)\}$ with optimal value $\alpha(G)$. However, $\text{STAB}(G)$ is not known apriori, and so one resorts to relaxations of it over which one can optimize $\sum_{i=1}^n x_i$.

In [16], Lovász introduced, $\text{TH}(G)$, a convex relaxation of $\text{STAB}(G)$, called the theta body of $G$. The problem $\max \{\sum_{i=1}^n x_i : x \in \text{TH}(G)\}$ is
a SDP which can be solved to arbitrary precision in polynomial time in the size of $G$. The optimal value of this SDP is called the \textit{theta number} of $G$ and provides an upper bound on $\alpha(G)$. See [6, Chapter 9] and [29] for more on the stable set problem and TH($G$). The body TH($G$) was the first example of a SDP relaxation of a discrete optimization problem and snowballed the use of SDP in combinatorial optimization. See [15, 18] for surveys. Recall that a graph $G$ is \textit{perfect} if and only if $G$ has no induced odd cycles of length at least five or their complements. Lovász showed that $\text{STAB}(G) = \text{TH}(G)$ if and only if $G$ is perfect. This equality shows that the maximum stable set problem can be solved in polynomial time in the size of $G$ when $G$ is a perfect graph, and this geometric proof is the only one known for this complexity result.

The theta body TH($G$) has many definitions (see [6, Chapter 9]) but the one relevant for this paper was observed by Lovász and appears without proof in [17]. Let $I_G := \{x_j^2 - x_j \mid j \in [n], \ x_i x_j \forall \{i,j\} \in E\} \subseteq \mathbb{R}[x]$. Then check that $V_R(I_G) = S_G$ and that $I_G$ is both zero-dimensional and real radical. Lovász observed that

\begin{equation}
\text{TH}(G) = \{x \in \mathbb{R}^n : f(x) \geq 0 \ \forall \ \text{linear } f \text{ that is 1-sos mod } I_G\}.
\end{equation}

By Definition 1.4 (1), TH($G$) is exactly the first theta body, TH$_1(I_G)$, of the ideal $I_G$, and by the above discussion, $I_G$ is TH$_1$-exact (i.e., TH$_1(I_G) = \text{STAB}(G)$) if and only if $G$ is perfect. Lovász observed that, in fact, $I_G$ is (1, 1)-sos if and only if $G$ is perfect which motivated Problem 1.3 that asks for a characterization of all (1, 1)-sos ideals in $\mathbb{R}[x]$. Lovász refers to a (1, 1)-sos ideal as a \textit{perfect ideal}. A (1, 1)-sos ideal $I$ would have the property that its first and simplest theta body, TH$_1(I)$, coincides with $\text{cl}(\text{conv}(V_R(I)))$ which is a valuable property for linear optimization over $\text{conv}(V_R(I))$, especially when TH$_1(I)$ is computationally tractable.

The theta body hierarchy of the ideal $I_G$ therefore naturally extends the theta body of $G$ to a family of nested relaxations of STAB($G$). Further, the connection between TH($G$) and sums of squares polynomials motivated Definition 1.4 which extends the construction of TH($G$) to a hierarchy of relaxations of $V_R(I)$ for any ideal $I \subseteq \mathbb{R}[x]$. We now explicitly describe the $k$-th theta body of $I_G$ in terms of combinatorial moment matrices.

For $U \subseteq [n]$, let $x^U := \prod_{i \in U} x_i$. From the generators of $I_G$ it is clear that if $f \in \mathbb{R}[x]$, then $f \equiv g$ mod $I_G$ where $g$ is in the $\mathbb{R}$-span of the set of monomials $\{x^U : U$ is a stable set in $G\}$. Check that $\mathcal{B} := \{x^U + I_G : U$ stable set in $G\}$ is a basis of $\mathbb{R}[x]/I_G$ containing $1 + I_G, x_1 + I_G, \ldots, x_n + I_G$. Therefore, by Corollary 2.15 and Proposition 2.15 we have

\begin{equation}
\text{TH}_k(I_G) = \left\{ y \in \mathbb{R}^n : \begin{array}{l}
\exists M \succeq 0, M \in \mathbb{R}^{[B_k] \times [B_k]} \text{ such that} \\
M_{\emptyset} = 1, \\
M_{\emptyset \{i\}} = M_{\{i\} \emptyset} = M_{\{i\} \{i\}} = y_i \\
M_{UU'} = 0 \text{ if } U \cup U' \text{ is not stable in } G \\
M_{UU'} = M_{WW'} \text{ if } U \cup U' = W \cup W'
\end{array} \right\}.
\end{equation}
In particular, indexing the one element stable sets by the vertices of \( G \),

\[
\text{TH}_1(I_G) = \left\{ y \in \mathbb{R}^n : \begin{array}{l}
\exists M \succeq 0, M \in \mathbb{R}^{(n+1) \times (n+1)} \text{ such that } \\
M_{00} = 1, \\
M_{0i} = M_{i0} = M_{ii} = y_i \forall i \in [n] \\
M_{ij} = 0 \forall \{i,j\} \in E
\end{array} \right\}.
\]

This description of \( \text{TH}_1(I_G) \) coincides with the semidefinite description of \( \text{TH}(G) \) (see [19, Lemma 2.17] for instance) and so, \( \text{TH}(G) = \text{TH}_1(I_G) \). Corollary 2.12 confirms Lovász’s observation and adds to his other characterizations of a perfect graph as follows.

**Theorem 3.1.** [6, Chapter 9] The following are equivalent for a graph \( G \).

1. \( G \) is perfect.
2. \( \text{STAB}(G) = \text{TH}(G) \).
3. \( \text{TH}(G) \) is a polytope.
4. The complement \( \overline{G} \) of \( G \) is perfect.
5. \( I_G \) is \((1,1)\)-sos.

The usual Lasserre relaxations of the maximum stable set problem are set up from the following initial linear programming relaxation of \( \text{STAB}(G) \):

\[
\text{FRAC}(G) := \{ x \in \mathbb{R}^n : x_i \geq 0 \forall i \in [n], 1 - x_i - x_j \geq 0 \forall \{i,j\} \in E \}.
\]

Note that \( S_G = \text{FRAC}(G) \cap \{0,1\}^n \). The \( k \)-th Lasserre relaxation of \( \text{STAB}(G) \) (see [9, 12]) uses both the ideal \( \langle x_i^2 - x_i : i \in [n] \rangle \) and the inequality system describing \( \text{FRAC}(G) \), whereas in the theta body formulation, \( \text{TH}_k(I_G) \), there is only the ideal \( I_G \) and no inequalities. Despite this difference, [12, Lemma 20] proves that the usual Lasserre hierarchy is exactly our theta body hierarchy for the stable set problem. This interpretation of the Lasserre hierarchy provides new tools to understand these relaxations such as establishing the validity of inequalities over them as shown below.

Since no monomial in the basis \( B \) of \( \mathbb{R}[x]/I_G \) has degree larger than \( \alpha(G) \), for any \( G \), \( I_G \) is \((1,\alpha(G))\)-sos and \( \text{STAB}(G) = \text{TH}_{\alpha(G)}(I_G) \). However, for many non-perfect graphs the theta-rank of \( I_G \) can be a lot smaller than \( \alpha(G) \). For instance if \( G \) is a \((2k+1)\)-cycle, then \( \alpha(G) = k \) while Proposition 3.3 below shows that the theta-rank of \( I_G \) is two.

**Theorem 3.2.** [28, Corollary 65.12a] If \( G = ([n],E) \) is an odd cycle with \( n \geq 5 \), then \( \text{STAB}(G) \) is determined by the following inequalities:

\[
x_i \geq 0 \forall i \in [n], \quad 1 - \sum_{i \in K} x_i \geq 0 \forall \text{ cliques } K \text{ in } G, \quad \alpha(G) - \sum_{i \in [n]} x_i \geq 0.
\]

**Proposition 3.3.** If \( G \) is an odd cycle with at least five vertices, then \( I_G \) is \((1,2)\)-sos and therefore, \( \text{TH}_2 \)-exact.

**Proof:** Let \( n = 2k+1 \) and \( G \) be an \( n \)-cycle. Then \( I_G = \langle x_i^2 - x_i, x_i x_{i+1} \forall i \in [n] \rangle \) where \( x_{n+1} = x_1 \). Therefore, \( (1 - x_i)^2 \equiv 1 - x_i \) and \( (1 - x_i - x_{i+1})^2 \equiv \)
1 - x_i - x_{i+1} \mod I_G. This implies that, mod I_G,
\[ p_i^2 := ((1-x_1)(1-x_{2i}-x_{2i+1}))^2 \equiv p_i = 1-x_1-x_{2i}-x_{2i+1} + x_1x_{2i} + x_1x_{2i+1}. \]
Summing over \( i = 1, \ldots, k \), we get
\[ \sum_{i=1}^{k} p_i^2 = k - kx_1 - \sum_{i=2}^{2k+1} x_i + \sum_{i=3}^{2k} x_1x_i \mod I_G \]

since \( x_1x_2 \) and \( x_1x_{2k+1} \) lie in \( I_G \). Define \( g_i := x_1(1-x_{2i+1} - x_{2i+2}) \). Then \( g_i^2 - g_i \in I_G \) and mod \( I_G \) we get that
\[ \sum_{i=1}^{k-1} g_i^2 \equiv (k-1)x_1 - \sum_{i=3}^{2k} x_1x_i, \quad \text{which implies} \quad \sum_{i=1}^{k} p_i^2 + \sum_{i=1}^{k-1} g_i^2 \equiv k - \sum_{i=1}^{2k+1} x_i. \]

To prove that \( I_G \) is \((1,2)\)-sos it suffices to show that the left hand sides of the inequalities in the description of \( \text{STAB}(G) \) in Theorem \( \text{3.2} \) are \( 2 \)-sos mod \( I_G \) since by Farkas Lemma \[27\], all other linear inequalities that are non-negative over \( S_G \) are non-negative real combinations of a set of inequalities defining \( \text{STAB}(G) \). Clearly, \( x_i \equiv x_i^2 \mod I_G \) for all \( i \in [n] \) and one can check that for each clique \( K, (1 - \sum_{i \in K} x_i) \equiv (1 - \sum_{i \in K} x_i)^2 \mod I_G \). The previous paragraph shows that \( k - \sum_{i=1}^{2k+1} x_i \) is also \( 2 \)-sos mod \( I_G \).

An induced odd cycle \( C_{2k+1} \) in \( G \), yields the well-known odd cycle inequality \( \sum_{i \in C_{2k+1}} x_i \leq \alpha(C_{2k+1}) = k \) that is satisfied by \( S_G \) \[6\] Chapter 9]. Proposition \( \text{3.3} \) implies that for any graph \( G \), \( \text{TH}_2(I_G) \) satisfies all odd cycle inequalities from \( G \) since every stable set \( U \) in \( G \) restricts to a stable set in an induced odd cycle in \( G \). This general result can also be proved using results from \[19\] and \[12\]. The direct arguments used in the proof of Proposition \( \text{3.3} \) are examples of the algebraic inference rules outlined by Lovász in \[17\]. Similarly, one can also show that other well-known classes of inequalities such as the odd antihole and odd wheel inequalities \[6\] Chapter 9] are also valid for \( \text{TH}_2(I_G) \). Schoenebeck \[26\] has recently shown that there is no constant \( k \) such that \( \text{STAB}(G) = \text{TH}_k(I_G) \) for all graphs \( G \) (as expected, unless \( P=NP \)). However, no explicit family of graphs that exhibit this behaviour is known.

### 3.2. Cuts in graphs

Given an undirected connected graph \( G = ([n], E) \) and a partition of its vertex set \([n]\) into two parts \( V_1 \) and \( V_2 \), the set of edges \( \{i,j\} \in E \) such that exactly one of \( i \) or \( j \) is in \( V_1 \) and the other in \( V_2 \) is the cut in \( G \) induced by the partition \((V_1, V_2)\). The cuts in \( G \) are in bijection with the \( 2^{n-1} \) distinct partitions of \([n]\) into two sets. The maximum cut problem in \( G \) seeks the cut in \( G \) of largest cardinality. This problem is \( NP \)-hard and has received a great deal of attention in the literature. A celebrated result in this area is an approximation algorithm for the max cut problem, due to Goemans and Williamson \[3\], that guarantees a cut of size at least 0.878 of the optimal cut. It relies on a simple SDP relaxation of the problem.
We first study a non-standard model of the max cut problem. Let

\[ \text{SG} := \{ \chi^F : F \subseteq E \text{ is contained in a cut of } G \} \subseteq \{0, 1\}^E. \]

Then the \textit{weighted} max cut problem with non-negative weights \( w_e \) on the edges \( e \in E \) is \( \max \{ \sum_{e \in E} w_e x_e : x \in \text{SG} \} \), and the vanishing ideal

\[ \mathcal{I}(\text{SG}) = \langle x_e^2 - x_e, x^T : e \in E, T \text{ odd cycle in } G \rangle. \]

A basis of \( \mathbb{R}[x]/\mathcal{I}(\text{SG}) \) is

\[ \mathcal{B} = \{ x^U + I(\text{SG}) : U \subseteq E \text{ does not contain an odd cycle in } G \} \]

and \( 1 + \mathcal{I}(\text{SG}), x_e + \mathcal{I}(\text{SG}) (\forall e \in E) \) lie in \( \mathcal{B} \). Therefore,

\[
\text{TH}_k(\mathcal{I}(\text{SG})) = \left\{ y \in \mathbb{R}^E : \exists M \succeq 0, M \in \mathbb{R}^{[E_k] \times [E_k]} \text{ such that} \right. \\
M_{00} = 1, \\
M_{0(i)} = M_{(i)0} = M_{(i)(i)} = y_i, \\
M_{UU'} = 0 \text{ if } U \cup U' \text{ has an odd cycle} \\
M_{UU'} = M_{WW'} \text{ if } U \cup U' = W \cup W' \left. \right\}.
\]

In particular,

\[
\text{TH}_1(\mathcal{I}(\text{SG})) = \left\{ y \in \mathbb{R}^E : \exists M \succeq 0, M \in \mathbb{R}^{(|E|+1) \times (|E|+1)} \text{ such that} \right. \\
M_{00} = 1, \\
M_{0e} = M_{ee} = y_e \forall e \in E \left. \right\}.
\]

Note that for any graph \( G \), \( \text{TH}_1(\mathcal{I}(\text{SG})) \) is the unit cube in \( \mathbb{R}^E \) which may not be equal to \( \text{conv}(\text{SG}) \). This stands in contrast to the case of stable sets for which \( \text{TH}_1(I_G) \) is a polytope if and only if \( \text{TH}_1(I_G) = \text{STAB}(G) \).

**Proposition 3.4.** The ideal \( \mathcal{I}(\text{SG}) \) is \( \text{TH}_1 \)-exact if and only if \( G \) is a bipartite graph.

**Proof:** This follows immediately from the description of \( \text{TH}_1(\mathcal{I}(\text{SG})) \) and from the fact that \( G \) is bipartite if and only if it has no odd cycles. \( \square \)

Since the maximum degree of a monomial in \( \mathcal{B} \) is the size of the max cut in \( G \), the theta-rank of \( \mathcal{I}(\text{SG}) \) is bounded from above by the size of the max cut in \( G \).

**Proposition 3.5.** There is no constant \( k \) such that \( \mathcal{I}(\text{SG}) \) is \( \text{TH}_k \)-exact for all graphs \( G \).

**Proof:** Let \( G \) be a \((2k + 1)\)-cycle. Then \( \text{TH}_k(\mathcal{I}(\text{SG})) \neq \text{conv}(\text{SG}) \) since the linear constraint imposed by the cycle in the definition of \( \text{TH}_k(\mathcal{I}(\text{SG})) \) will not appear in theta bodies of index \( k \) or less. \( \square \)

The theta bodies of a second, more standard, formulation of the weighted max cut problem are studied in [1]. In this setup, each cut \( C \) in \( G = ([n], E) \) is recorded by its \textit{cut vector} \( \chi^C \in \{\pm 1\}^E \) with \( \chi^C_{(i,j)} = 1 \) if \( \{i, j\} \notin C \) and
The cut vectors \( \chi_{\{i,j\}} = -1 \) if \( \{i,j\} \in C \). Let \( E_n \) denote the edge set of the complete graph \( K_n \), and \( \pi_E \) be the projection from \( \mathbb{R}^{E_n} \) to \( \mathbb{R}^E \). The cut polytope of \( G \) is

\[
\text{CUT}(G) := \text{conv}\{\chi^C : C \text{ is a cut in } G\} \subseteq \mathbb{R}^E = \pi_E(\text{CUT}(K_n)),
\]

and the weighted max cut problem, for weights \( w_e \in \mathbb{R} (\forall e \in E) \) becomes

\[
\max \left\{ \frac{1}{2} \sum_{e \in E} w_e(1 - x_e) : x \in \text{CUT}(G) \right\}.
\]

In [5], the vanishing ideal \( IG \) of the cut vectors \( \{\chi^C : C \text{ is a cut in } G\} \) is described and a combinatorial basis \( I \) for \( \mathbb{R}^E/IG \) is identified. Using these, the \( k \)-th theta body, \( \text{TH}_k(IG) \), of \( IG \) can be described as:

\[
\left\{ y \in \mathbb{R}^E : M_{0,\emptyset} = 1, \ M_{F_1,F_2} = M_{F_3,F_4} \text{ if } F_1 \Delta F_2 \Delta F_3 \Delta F_4 \text{ is a cycle in } G \right\}.
\]

These theta bodies provide a new canonical set of SDP relaxations for \( \text{CUT}(G) \) that exploits the structure of \( G \) directly. It is also shown in [5] that \( IG \) is TH1-exact if and only if \( G \) has no \( K_5 \)-minor and no induced cycle of length at least five which answers Problem 8.4 posed by Lovász in [18].

**Remark 3.6.** We remark that the stable set problem and the first formula of the max cut problem discussed above are special cases of the following general setup. Let \( \Delta \) be an abstract simplicial complex (or independence system) with vertex set \([n]\) recorded as a collection of subsets of \([n]\), called the faces of \( \Delta \). The Stanley-Reisner ideal of \( \Delta \) is the ideal \( J_{\Delta} \) generated by the squarefree monomials \( x_{i_1}x_{i_2} \cdots x_{i_k} \) such that \( \{i_1,i_2,\ldots,i_k\} \subseteq [n] \) is not a face of \( \Delta \). If \( I_\Delta := J_{\Delta} + \langle x_i^2 - x_i : i \in [n] \rangle \), then \( \nu_{\mathbb{R}}(I_\Delta) = \{s \in \{0,1\}^n : \text{support}(s) \in \Delta\} \). For \( T \subseteq [n] \), recall that \( x^T := \prod_{i \in T} x_i \). Then \( \mathcal{B} := \{x^T : T \in \Delta \} + I_\Delta \) is a basis for \( \mathbb{R}[x]/I_\Delta \) containing \( 1 + I_\Delta, x_1 + I_\Delta, \ldots, x_n + I_\Delta \). Therefore, by Corollary 2.15 and Proposition 2.18, the \( k \)-th theta body of \( I_\Delta \) is

\[
\text{TH}_k(I_\Delta) = \text{proj}_{y_1,\ldots,y_n}\{y \in \mathbb{R}^{E_{2k}} : M_{B_k}(y) \succeq 0, y_0 = 1\}.
\]

Since \( \mathcal{B} \) is in bijection with the faces of \( \Delta \), and \( x_i^2 - x_i \in I_\Delta \) for all \( i \in [n] \), the theta body can be written explicitly as follows:

\[
\text{TH}_k(I_\Delta) = \left\{ y \in \mathbb{R}^n : M_{\emptyset} = 1, \ y_{i_{\{i\}}} = M_{\{i\}\emptyset} = M_{\{i\}\{i\}} = y_{i}, \ M_{U \cup U'} = 0 \text{ if } U \cup U' \notin \Delta, \ M_{U \cup U'} = M_{W \cup W'} \text{ if } U \cup U' = W \cup W' \right\}.
\]

If the dimension of \( \Delta \) is \( d - 1 \) (i.e., the largest faces in \( \Delta \) have size \( d \)), then \( I_\Delta \) is \((1,d)\)-sos and therefore, \( \text{TH}_d \)-exact since all elements of \( \mathcal{B} \) have degree at most \( d \). However, the theta-rank of \( I_\Delta \) could be much less than \( d \).
4. Vanishing ideals of finite sets of points

Recall that when \( S \subseteq \mathbb{R}^n \) is finite, its vanishing ideal \( \mathcal{I}(S) \) is zero-dimensional and real radical.

**Definition 4.1.** We say that a finite set \( S \subseteq \mathbb{R}^n \) is *exact* if its vanishing ideal \( \mathcal{I}(S) \subseteq \mathbb{R}[x] \) is \( \text{TH}_1 \)-exact.

We now answer Lovász’s question (Problem 1.3) for vanishing ideals of finite point sets in \( \mathbb{R}^n \).

**Theorem 4.2.** For a finite set \( S \subseteq \mathbb{R}^n \), the following are equivalent.

1. \( S \) is exact.
2. \( \mathcal{I}(S) \) is \( (1,1) \)-sos.
3. There is a linear inequality description of \( \text{conv}(S) \), of the form \( g_i(x) \geq 0 \) \((i = 1, \ldots, m)\), where each \( g_i \) is \( 1 \)-sos mod \( \mathcal{I}(S) \).
4. There is a linear inequality description of \( \text{conv}(S) \), of the form \( g_i(x) \geq 0 \) \((i = 1, \ldots, m)\), where each \( g_i \) is an idempotent mod \( \mathcal{I}(S) \), i.e., \( g_i^2 - g_i \in \mathcal{I}(S) \) for \( i = 1, \ldots, m \).
5. There is a linear inequality description of \( \text{conv}(S) \), of the form \( g_i(x) \geq 0 \) \((i = 1, \ldots, m)\), where each \( g_i \) takes at most two different values in \( S \), i.e., for each \( i \), \( S \) is contained in the union of the hyperplane \( g_i(x) = 0 \) and one unique parallel translate of it.

**Proof:** Since \( \mathcal{I}(S) \) is real radical, by Corollary 2.12, \((1) \iff (2)\).

The implication \((2) \Rightarrow (3)\) follows from the fact that \( \text{conv}(S) \) has a finite linear inequality description, since \( S \) is finite. The implication \((3) \Rightarrow (2)\) follows from Farkas lemma, which implies that any valid inequality on \( S \) is a non-negative real combination of the linear inequalities \( g_i(x) \geq 0 \).

Suppose \((3)\) holds and \( \text{conv}(S) \) is a full-dimensional polytope. Let \( F \) be a facet of \( \text{conv}(S) \), and \( g(x) \geq 0 \) its defining inequality in the given description of \( \text{conv}(S) \). Then \( g(x) \) is \( 1 \)-sos mod \( \mathcal{I}(S) \) if and only if there are linear polynomials \( h_1, \ldots, h_l \in \mathbb{R}[x] \) such that \( g \equiv h_1^2 + \cdots + h_l^2 \) mod \( \mathcal{I}(S) \). In particular, since \( g(x) = 0 \) on the vertices of \( F \), and all the \( h_i^2 \) are non-negative, each \( h_i \) must be zero on all the vertices of \( F \). Hence, since the \( h_i \)’s are linear, they must vanish on the affine span of \( F \) which is the hyperplane defined by \( g(x) = 0 \). Thus each \( h_i \) must be a multiple of \( g \) and \( g \equiv \alpha g^2 \) mod \( \mathcal{I}(S) \) for some \( \alpha > 0 \). We may assume that \( \alpha = 1 \) by replacing \( g(x) \) by \( g'(x) := \alpha g(x) \). If \( \text{conv}(S) \) is not full-dimensional, then since mod \( \mathcal{I}(S) \), all linear polynomials can be assumed to define hyperplanes whose normal vectors are parallel to the affine span of \( S \), the proof still holds. Therefore, \((3)\) implies \((4)\). Conversely, since if for a linear polynomial \( g \), \( g \equiv g^2 \) mod \( \mathcal{I}(S) \), then \( g \) is \( 1 \)-sos mod \( \mathcal{I}(S) \), \((4)\) implies \((3)\).

The equivalence \((4) \iff (5)\) follows since \( g \equiv g^2 \) mod \( \mathcal{I}(S) \) if and only if \( g(s)(1 - g(s)) = 0 \) for all \( s \in S \). \( \square \)

Recall from the discussion at the end of Section 2.1 that by results of Parrilo, if \( I \subseteq \mathbb{R}[x] \) is a zero-dimensional radical ideal, then the theta-rank
of $I$ is at most $|\mathcal{V}_C(I)| - 1$. Better upper bounds can be derived using the following extension of Parrilo’s theorem.

**Remark 4.3.** Suppose $S \subseteq \mathbb{R}^n$ is a finite set such that each facet $F$ of $\text{conv}(S)$ has a facet defining inequality $h_F(x) \geq 0$ where $h_F$ takes at most $t+1$ values on $S$, then $\mathcal{I}(S)$ is TH$_t$-exact: In this case, it is easy to construct a degree $t$ interpolator $g$ for the values of $\sqrt{h_F}$ on $S$, and we have $h_F \equiv g^2 \mod \mathcal{I}(S)$. The result then follows from Farkas Lemma.

**Remark 4.4.** The theta-rank of $\mathcal{I}(S)$ could be much smaller than the upper bound in Remark 4.3. Consider a $(2t+1)$-cycle $G$ and the set $S_G$ of characteristic vectors of its stable sets. Proposition 3.3 shows that $\mathcal{I}(S_G)$ is TH$_2$-exact. However, we need $t+1$ translates of the facet cut out by $\sum_{i=1}^{2t+1} x_i = t$ to cover $S_G$.

In the rest of this section we derive various consequences of Theorem 4.2.Finite point sets with property (5) in Theorem 4.2 have been studied in various contexts. In particular, Corollaries 4.5, 4.9 and 4.11 below were also observed independently by Greg Kuperberg, Raman Sanyal, Axel Werner and Günter Ziegler (personal communication). In their work, $\text{conv}(S)$ is called a **2-level polytope** when property (5) in Theorem 4.2 holds.

If $S$ is a finite subset of $\mathbb{Z}^n$ and $\mathcal{L}$ is the smallest lattice in $\mathbb{Z}^n$ containing $S$, then the lattice polytope $\text{conv}(S)$ is said to be **compressed** if every reverse lexicographic triangulation of the lattice points in $\text{conv}(S)$ is unimodular with respect to $\mathcal{L}$. Compressed polytopes were introduced by Stanley [30]. Corollary 4.5 (4) and Theorem 2.4 in [31] (see also the references after Theorem 2.4 in [31] for earlier citations of part or unpublished versions of this result), imply that a finite set $S \subset \mathbb{R}^n$ is exact if and only if $\text{conv}(S)$ is affinely equivalent to a compressed polytope.

**Corollary 4.5.** Let $S, S' \subset \mathbb{R}^n$ be exact sets. Then

1. all points of $S$ are vertices of $\text{conv}(S)$,
2. the set of vertices of any face of $\text{conv}(S)$ is again exact,
3. the product $S \times S'$ is exact, and
4. $\text{conv}(S)$ is affinely equivalent to a 0/1 polytope.

**Proof:** The first three properties follow from Theorem 4.2 (5). If the dimension of $\text{conv}(S)$ is $d \leq n$, then $\text{conv}(S)$ has at least $d$ non-parallel facets. If $a \cdot x \geq b$ cuts out a facet in this collection, then $\text{conv}(S)$ is supported by both $\{x \in \mathbb{R}^n : a \cdot x = b\}$ and a parallel translate of it. Taking these two parallel hyperplanes from each of the $d$ facets gives a parallelepiped. By Theorem 4.2 $S$ is contained in the vertices of this parallelepiped intersected with the affine hull of $S$. This proves (4).

By Corollary 4.5 (4), it essentially suffices to look at subsets of $\{0, 1\}^n$ to obtain all exact finite varieties in $\mathbb{R}^n$. In $\mathbb{R}^2$, the set of vertices of any 0/1-polytope verify this property. In $\mathbb{R}^3$ there are eight full-dimensional
0/1-polytopes up to affine equivalence. In Figures 1 and 2 the convex hulls of the exact and non-exact 0/1 configurations in $\mathbb{R}^3$ are shown.

**Example 4.6.** The vertices of the following 0/1-polytopes in $\mathbb{R}^n$ are exact for every $n$: (1) hypercubes, (2) (regular) cross polytopes, (3) hypersimplices (includes simplices), (4) joins of 2-level polytopes, and (5) stable set polytopes of perfect graphs on $n$ vertices.

**Theorem 4.7.** If $S$ is a finite exact point set then $\text{conv}(S)$ has at most $2^d$ facets and vertices, where $d = \text{dim conv}(S)$. Both bounds are sharp.

**Proof:** The bound on the number of vertices is immediate by Corollary 4.5 (4) and is achieved by $[0, 1]^d$.

For a polytope $P$ with an exact vertex set $S$, define a **face pair** to be an unordered pair $(F_1, F_2)$ of proper faces of $P$ such that $S \subseteq F_1 \cup F_2$ and $F_1$ and $F_2$ lie in parallel hyperplanes, or equivalently, there exists a linear form $h_{F_1,F_2}(x)$ such that $h_{F_1,F_2}(F_1) = 0$ and $h_{F_1,F_2}(F_2) = 1$. We will show that if $\text{dim } P = d$ then $P$ has at most $2^d - 1$ face pairs and $2^d$ facets.

If $d = 1$, then an exact $S$ consists of two distinct points and $P$ has two facets and one face pair as desired. Assume the result holds for $(d - 1)$-polytopes with exact vertex sets and consider a $d$-polytope $P$ with exact vertex set $S$. Let $F$ be a facet of $P$ which by Theorem 4.2 is in a face pair $(F, F')$ of $P$. Since exactness does not depend on the affine embedding, we may assume that $P$ is full-dimensional and that $F$ spans the hyperplane $\{x : x_d = 0\}$, while $F'$ lies in $\{x : x_d = 1\}$. By Corollary 4.5, $F$ satisfies the induction hypothesis and so has at most $(2^{d-1} - 1)$ face pairs. Any face pair of $P$ besides $(F, F')$ induces a face pair of $F$ by intersection with $F$, and every facet of $P$ is in a face pair of $P$ since $S$ is exact. The plan is to count how many face pairs of $P$ induce the same face pair of $F$ and the number of facets they contain.

Fix a face pair $(F_1, F_2)$ of $F$, with associated linear form $h_{F_1,F_2}$ depending only on $x_1, \ldots, x_{d-1}$. Suppose $(F_1, F_2)$ is induced by a face pair of $P$ with
associated linear form \( H(x) \). Since \( H \) and \( h_{F_1,F_2} \) agree on every vertex of \( F \), a facet of \( P \), \( H(x) = h_{F_1,F_2}(x_1, \ldots, x_{d-1}) + cx_d \) for some constant \( c \).

If \( h_{F_1,F_2}(x_1, \ldots, x_{d-1}) \) takes the same value \( v \) on all of \( F' \), then \( H(F') = v + c = 0 \) or \( 1 \) which implies that \( c = -v \) or \( c = 1 - v \). The two possibilities lead to the face pairs \( (\text{conv}(F_1 \cup F'), F_2) \) and \( (\text{conv}(F_2 \cup F'), F_1) \) of \( P \). Each such pair contains at most one facet of \( P \).

If \( h_{F_1,F_2}(x_1, \ldots, x_{d-1}) \) takes more than one value on the vertices of \( F' \), then these values must be \( v \) and \( v + 1 \) for some \( v \) since \( H \) takes values 0 and 1 on the vertices of \( F' \). In that case, \( c = -v \), so \( H \) is unique and we get at most one face pair of \( P \) inducing \( (F_1, F_2) \). This pair will contain at most two facets of \( P \).

Since there are at most \( 2^{d-1} - 1 \) face pairs in \( F \), they give us at most \( 2(2^{d-1} - 1) \) face pairs and facets of \( P \). Since we have not counted \( (F, F') \) as a face pair of \( P \), and \( F \) and \( F' \) as possible facets of \( P \), we get the desired result. The bound on the number of facets is attained by cross-polytopes.

\( \Box \)

**Remark 4.8.** Günter Ziegler has pointed out that our proof of Theorem 4.7 can be refined to yield that \( P \) (as used above) has \( 2^d - 1 \) face pairs if and only if it is a simplex and \( 2^d \) facets if and only if it is a regular cross-polytope.

Recall that Problem \([\text{P}3]\) was inspired by perfect graphs. Theorem 4.8 adds to the characterizations of a perfect graph (c.f. Theorem 3.1) as follows.

**Corollary 4.9.** For a graph \( G \), let \( S_G \) denote the set of characteristic vectors of stable sets in \( G \). Then the following are equivalent.

1. The graph \( G \) is perfect.
2. The stable set polytope, \( \text{STAB}(G) \), is a 2-level polytope.

A polytope \( P \) in \( \mathbb{R}^n \) is said to be **down-closed** if for all \( \mathbf{v} \in P \) and \( \mathbf{v}' \in \mathbb{R}^n_{\geq 0} \) such that \( v'_i \leq v_i \) for \( i = 1, \ldots, n \), \( \mathbf{v}' \in P \). For a graph \( G \), \( \text{STAB}(G) \) is a down-closed 0/1-polytope, and \( G \) is perfect if and only if the vertex set of \( \text{STAB}(G) \) is exact. We now prove that all down-closed 0/1-polytopes with exact vertex sets are stable set polytopes of perfect graphs.

**Theorem 4.10.** Let \( P \subseteq \mathbb{R}^n \) be a down-closed 0/1-polytope and \( S \) be its set of vertices. Then \( S \) is exact if and only if all facets of \( P \) are either defined by non-negativity constraints on the variables or by an inequality of the form \( \sum_{i \in I} x_i \leq 1 \) for some \( I \subseteq [n] \).

**Proof:** If \( P \) is not full-dimensional then since it is down-closed, it must be contained in a coordinate hyperplane \( x_i = 0 \) and the arguments below can be repeated in this lower-dimensional space. So we may assume that \( P \) is \( n \)-dimensional. Then since \( P \) is down-closed, \( S \) contains \( \{0, \mathbf{e}_1, \ldots, \mathbf{e}_n\} \).

If all facets of \( P \) are of the stated form, using that \( S \subseteq \{0, 1\}^n \), it is straight forward to check that \( S \) is exact.

Now assume that \( S \) is exact and \( g(x) \geq 0 \) is a facet inequality of \( P \) that is not a non-negativity constraint. Then \( g(x) := c - \sum_{i=1}^n a_i x_i \geq 0 \) for some
integers $c, a_1, \ldots, a_n$ with $c \neq 0$. Since $0 \in S$ and $S$ is exact, we get that $g(s)$ equals $0$ or $c$ for all $s \in S$. Therefore, for all $i$, $g(e_i) = c - a_i$ equals $0$ or $c$, so $a_i$ is either $0$ or $c$. Dividing through by $c$, we get that the facet inequality $g(x) \geq 0$ is of the form $\sum_{i \in I} x_i \leq 1$ for some $I \subseteq [n]$.

\begin{corollary}
Let $P \subseteq \mathbb{R}^n$ be a full-dimensional down-closed 0/1-polytope and $S$ be its vertex set. Then $S$ is exact if and only if $P$ is the stable set polytope of a perfect graph.
\end{corollary}

\begin{proof}
By Corollary 4.10 we only need to prove the “only-if” direction. Suppose $S$ is exact. Then by Theorem 4.11 all facet inequalities of $P$ are either of the form $x_i \geq 0$ for some $i \in [n]$ or $\sum_{i \in I} x_i \leq 1$ for some $I \subseteq [n]$. Define the graph $G = ([n], E)$ where $\{i, j\} \in E$ if and only if $\{i, j\} \subseteq I$ for some $I$ that indexes a facet inequality of $P$.

We prove that $P = \text{STAB}(G)$ and that $G$ is perfect. Let $K \subseteq [n]$ such that its characteristic vector $\chi^K \in S$. If there exists $i, j \in K$ such that $i, j \in I$ for some $I$ that indexes a facet inequality of $P$, then $1 - \sum_{i \in I} x_i$ takes three different values when evaluated at the points $0, e_i, \chi^K$ in $S$ which contradicts that $S$ is exact. Therefore, $K$ is a stable set of $G$ and $P \subseteq \text{STAB}(G)$. If $K \subseteq [n]$ is a stable set of $G$ then, by construction, for every $I$ indexing a facet inequality of $P$, $\chi^K$ lies on either $\sum_{i \in I} x_i = 0$ or $\sum_{i \in I} x_i = 0$. Therefore $\chi^K \in P$ and $\text{STAB}(G) \subseteq P$. Since all facet inequalities of $\text{STAB}(G)$ are either non-negativities or clique inequalities, $G$ is perfect by [6 Theorem 9.2.4 iii].
\end{proof}

## 5. Arbitrary $\text{TH}_1$-exact Ideals

In this last section we describe $\text{TH}_1(I)$ for an arbitrary (not necessarily real radical or zero-dimensional) ideal $I \subseteq \mathbb{R}[x]$. The main structural result is Theorem 5.4 which allows the construction of non-trivial high-dimensional $\text{TH}_1$-exact ideals as in Example 5.5.

In this study, the convex quadrics in $\mathbb{R}[x]$ play a particularly important role. These are precisely the polynomials of degree two that can be written as $F(x) = x^t A x + b^t x + c$, where $A \neq 0$ is an $n \times n$ positive semidefinite matrix, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Note that every sum of squares of linear polynomials in $\mathbb{R}[x]$ is a convex quadric.

\begin{lemma}
For $I \subseteq \mathbb{R}[x]$, $\text{TH}_1(I) \neq \mathbb{R}^n$ if and only if there exists some convex quadric $F \in I$.
\end{lemma}

\begin{proof}
If $\text{TH}_1(I) \neq \mathbb{R}^n$, there exists a degree one polynomial $f$ that is strictly positive on $\text{TH}_1(I)$, hence 1-sos modulo $I$. Then $f(x) \equiv g(x) \mod I$ for some 1-sos $g(x) \neq 0$ and $g(x) - f(x) \in I$ is a convex quadric.

Conversely, suppose $x^t A x + b^t x + c \in I$ with $A \succeq 0$. Then for any $d \in \mathbb{R}^n$, $(x + d)^t A (x + d) = x^t A x + 2d^t A x + d^t A d \equiv (2d^t A - b^t)x + d^t A d - c \mod I$. 

Therefore, since \((x + d)^t A(x + d)\) is a sum of squares of linear polynomials, the linear polynomial \((2d^t A - b^t)x + d^t Ad - c\) is 1-sos mod \(I\) and \(TH_1(I)\) must satisfy it. Since \(d\) can be chosen so that \((2d^t A - b^t) \neq 0\), \(TH_1(I)\) is not trivial.

\[\text{Lemma 5.2.} \quad \text{For an ideal } I \subseteq \mathbb{R}[x], \ TH_1(I) = \bigcap \ TH_1(\langle F \rangle), \text{ where } F \text{ varies over all convex quadrics in } I. \]

\[\text{Proof:} \quad \text{If } F \in I \text{ then } \langle F \rangle \subseteq I. \text{ Also, if } f \text{ is linear and 1-sos mod } \langle F \rangle \text{ then it is also 1-sos mod } I. \text{ Therefore, } TH_1(I) \subseteq TH_1(\langle F \rangle). \]

To prove the reverse inclusion, we need to show that if \(f\) is a linear polynomial that is nonnegative on \(TH_1(I)\), it is also nonnegative on \(\bigcap_{F \in I} TH_1(\langle F \rangle)\), where \(F\) is a convex quadric. It suffices to show that whenever \(f\) is linear and 1-sos mod \(I\), then there is a convex quadric \(F \in I\) such that \(f(x) \geq 0\) is valid for \(TH_1(F)\), or equivalently that \(f\) is 1-sos mod \(\langle F \rangle\). Since \(f\) is 1-sos mod \(I\), there is a sum of squares of linear polynomials \(g(x)\) such that \(f(x) \equiv g(x) \mod I\). But \(g\) is a convex quadric, hence so is \(g(x) - f(x)\). Thus \(f\) is 1-sos mod the ideal \(\langle g(x) - f(x) \rangle\) and we can take \(F(x) = g(x) - f(x)\).

\[\text{Lemma 5.3.} \quad \text{If } F(x) = x^t A x + b^t x + c \text{ with } A \succeq 0, \text{ then } TH_1(\langle F \rangle) = \text{conv}(\mathcal{V}_{\mathbb{R}}(F)). \]

\[\text{Proof:} \quad \text{We know that } \text{conv}(\mathcal{V}_{\mathbb{R}}(F)) \subseteq TH_1(\langle F \rangle) \text{ and, since } F \text{ is convex, } \text{conv}(\mathcal{V}_{\mathbb{R}}(F)) = \{x \in \mathbb{R}^n : F(x) \leq 0\}. \text{ Thus, if for every } x \in \mathcal{V}_{\mathbb{R}}(F) \text{ grad} F(x) \neq 0, \text{ then conv}(\mathcal{V}_{\mathbb{R}}(F)) \text{ is supported by the tangent hyperplanes to } \mathcal{V}_{\mathbb{R}}(F). \text{ In this case, to show that } TH_1(\langle F \rangle) \subseteq \text{conv}(\mathcal{V}_{\mathbb{R}}(F)), \text{ it suffices to prove that the defining (linear) polynomials of all tangent hyperplanes to } \mathcal{V}_{\mathbb{R}}(F) \text{ are 1-sos mod } \langle F \rangle. \text{ The proof of the “if” direction of Lemma 5.1 shows that it would suffice to prove that a tangent hyperplane to } \mathcal{V}_{\mathbb{R}}(F) \text{ has the form } (2d^t A - b^t)x + d^t Ad - c = 0, \text{ for some } d \in \mathbb{R}^n. \text{ The tangent at } x_0 \in \mathcal{V}_{\mathbb{R}}(F) \text{ has equation } 0 = (2A x_0 + b)^t (x - x_0) \text{ which can be rewritten as } 0 = (2x_0^t A + b^t)x - 2x_0^t A x_0 - b^t x_0 = (2x_0^t A + b^t)x - x_0^t A x_0 + c, \text{ and so setting } d = -x_0 \text{ gives the result.} \]

Suppose there is an \(x_0\) such that \(F(x_0) = 0\) and grad\(F(x_0) = 0\). By translation we may assume that \(x_0 = 0\), hence, \(c = 0\) and \(b = 0\). Therefore \(F = x^t A x = \sum h_i^2\) where the \(h_i\) are linear. Since \(\mathcal{V}_{\mathbb{R}}(\langle F \rangle) = \mathcal{V}_{\mathbb{R}}(\langle h_1, \ldots, h_m \rangle)\) it is enough to prove that all inequalities \(\pm h_i \geq 0\) are valid for \(TH_1(\langle F \rangle)\). For any \(\epsilon > 0\) we have

\[(\pm h_l + \epsilon)^2 + \sum_{i \neq l} h_i^2 = F \pm 2\epsilon h_l + \epsilon^2 \equiv 2\epsilon(\pm h_l + \epsilon/2) \mod \langle F \rangle, \]

so \(\pm h_l + \epsilon/2\) is 1-sos mod \(\langle F \rangle\) for all \(l\) and all \(\epsilon > 0\). This implies that all the inequalities \(\pm h_l + \epsilon/2 \geq 0\) are valid for \(TH_1(\langle F \rangle)\), therefore so are the inequalities \(\pm h_l \geq 0\). \(\square\)
Theorem 5.4. Let $I \subseteq \mathbb{R}[x]$ be any ideal, then
\[
\text{TH}_1(I) = \bigcap_{F \text{ convex quadric}} \text{conv}(\mathcal{V}_R(F)) = \bigcup_{F \text{ convex quadric}} \{x \in \mathbb{R}^n : F(x) \leq 0\}.
\]

Proof: Immediate from Lemma 5.2 and Lemma 5.3. \hfill \square

Example 5.5. Theorem 5.4 shows that some non-principal ideals such as $I = (x^2 - z, y^2 - z) \subseteq \mathbb{R}[x, y, z]$ are TH$_1$-exact. Since $\mathcal{V}_R(I) = \{(\pm t, \pm t, t^2) : t \in \mathbb{R}\}$, fixing the third coordinate we get the four points $(x, y, t^2)$ where $|x| = |y| = |t|$ which implies that
\[
\text{conv}(\mathcal{V}_R(I)) \supseteq \{(x, y, t^2) : |x| \leq t, |y| \leq t, t \geq 0\}.
\]
It is easy to see that the right hand side is equal to $\{(x, y, z) : x^2 \leq z, y^2 \leq z\}$ which is exactly $\text{conv}(\mathcal{V}_R(x^2 - z)) \bigcap \text{conv}(\mathcal{V}_R(y^2 - z))$ and so contains $\text{TH}_1(I)$ which contains $\text{conv}(\mathcal{V}_R(I))$. So all inclusions must be equalities and $I$ is $\text{TH}_1$-exact. This kind of reasoning allows us to construct non-trivial examples of $\text{TH}_1$-exact ideals with high-dimensional varieties.

Example 5.6. Consider the set $S = \{(0, 0), (1, 0), (0, 1), (2, 2)\}$. Then the family of all quadratic curves in $\mathcal{I}(S)$ is
\[
a(x^2 - x) + b(y^2 - y) - \left(\frac{a + b}{2}\right)xy = (x, y) \begin{pmatrix} a & -(a + b) \\ -(a + b) & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - ax - by.
\]
Since the case where both $a$ and $b$ are zero is trivial, we may normalize by setting $a + b = 1$ and get the matrix in the quadratic to be
\[
\begin{pmatrix} \lambda & -1/4 \\ -1/4 & 1 - \lambda \end{pmatrix}
\]
with $\lambda \geq 0$. This matrix is positive semidefinite if and only if $\lambda(1 - \lambda) - 1/16 \geq 0$, or equivalently, if and only if $\lambda \in [1/2 - \sqrt{3}/4, 1/2 + \sqrt{3}/4]$.

This means that $(x, y) \in \text{TH}_1(\mathcal{I}(S))$ if and only if, for all such $\lambda$,
\[
\lambda(x^2 - x) + (1 - \lambda)(y^2 - y) - \frac{1}{2}xy \leq 0.
\]
Since the right-hand-side does not depend on $\lambda$, and the left-hand-side is a convex combination of $x^2 - x$ and $y^2 - y$, the inequality holds for every $\lambda \in [1/2 - \sqrt{3}/4, 1/2 + \sqrt{3}/4]$ if and only if it holds at the end points of the interval. Equivalently, if and only if
\[
\left(\frac{1}{2} - \frac{\sqrt{3}}{4}\right) (x^2 - x) + \left(\frac{1}{2} + \frac{\sqrt{3}}{4}\right) (y^2 - y) - \frac{1}{2}xy \leq 0,
\]
and
\[
\left(\frac{1}{2} + \frac{\sqrt{3}}{4}\right) (x^2 - x) + \left(\frac{1}{2} - \frac{\sqrt{3}}{4}\right) (y^2 - y) - \frac{1}{2}xy \leq 0.
\]
But this is just the intersection of the convex hull of the two curves obtained by turning the inequalities into equalities. Figure 3 shows this intersection.
Figure 3. Example 5.6

References


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