An improved lower bound for the complementation of
Rabin automata

The MIT Faculty has made this article openly available. Please share how this access benefits you. Your story matters.

<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>As Published</td>
<td><a href="http://dx.doi.org/10.1109/LICS.2009.13">http://dx.doi.org/10.1109/LICS.2009.13</a></td>
</tr>
<tr>
<td>Publisher</td>
<td>Institute of Electrical and Electronics Engineers</td>
</tr>
<tr>
<td>Version</td>
<td>Final published version</td>
</tr>
<tr>
<td>Citable Link</td>
<td><a href="http://hdl.handle.net/1721.1/58827">http://hdl.handle.net/1721.1/58827</a></td>
</tr>
<tr>
<td>Terms of Use</td>
<td>Article is made available in accordance with the publisher's policy and may be subject to US copyright law. Please refer to the publisher's site for terms of use.</td>
</tr>
<tr>
<td>Detailed Terms</td>
<td></td>
</tr>
</tbody>
</table>
An Improved Lower Bound for the Complementation of Rabin Automata

Yang Cai
MIT CSAIL
The Stata Center, 32-G580
Cambridge, MA 02139 USA
ycai@csail.mit.edu

Ting Zhang
Iowa State University
226 Atanasoff Hall
Ames, IA 50011 USA
tingz@cs.iastate.edu

Haifeng Luo
Princeton University
201 Sherrerd Hall
Princeton, NJ 08544 USA
haifengl@princeton.edu

Abstract

Automata on infinite words (ω-automata) have wide applications in formal language theory as well as in modeling and verifying reactive systems. Complementation of ω-automata is a crucial instrument in many of these applications, and hence there have been great interests in determining the state complexity of the complementation problem. However, obtaining nontrivial lower bounds has been difficult. For the complementation of Rabin automata, a significant gap exists between the state-of-the-art lower bound \( 2^{\Omega(N \log N)} \) and upper bound \( 2^{O(N \log N)} \), where \( k \), the number of Rabin pairs, can be as large as \( 2^N \). In this paper we introduce multidimensional rankings to the full automata technique. Using the improved technique we establish an almost tight lower bound for the complementation of Rabin automata. We also show that the same lower bound holds for the determinization of Rabin automata.

1 Introduction

Automata on infinite words (ω-automata) have wide applications in formal language theory as well as in modeling and verifying reactive systems. In many of these applications a crucial instrument is complementation, which is to construct an automaton \( CA \) from a given automaton \( A \), such that \( CA \) accepts an infinite word if and only if \( A \) does not accept. For instance, in automata-theoretic model checking, to find out whether a system represented by an automaton \( B \) satisfies a specification represented by another automaton \( A \), one checks if \( L(B) \subseteq \overline{L(A)} \), which reduces to \( L(B) \cap \overline{L(CA)} = \emptyset \) [Kur94, VW94]. Thus, determining the state complexity of the complementation problem of ω-automata has a significant value in practice and it has attracted great interests in the last four decades [Var07].

Büchi first invented a kind of ω-automata (now called Büchi automata) as a tool to study decision problems of second-order arithmetic [Büc62]. Over the years many variants of ω-automata have been proposed, including Rabin automata and Streett automata. These common variants only differ at the definition of acceptance conditions and they all recognize ω-regular languages. Although equivalent in expressiveness, ω-automata with rich acceptance conditions, such as Streett automata and Rabin automata can express properties more easily and succinctly than Büchi automata. For example, the strong fairness condition [FK84, Fra86] that every infinitely enabled transition in a run is also taken infinitely often, can be expressed straightforwardly by a Streett acceptance condition. On the other hand, Rabin condition, defined as the dual of Streett condition, can directly express unfairness. The fair termination problem [Fra86, KK91], that is, whether all fair computations terminate, can be easily encoded into a Rabin condition that requires all infinite computations be either unfair or eventually forever idling. For these reasons, tightening the bounds for the complementation problem of other types of ω-automata also has great practical value.

The complementation problem of Büchi automata have been investigated for over 40 years. The current best algorithm has \( O(N^2((0.76 + c_0)N)^N) \) state blow-up (for a fixed \( c_0 \in (0, 1) \)) [Sch09], which tightly matches the best lower bound \( \Omega(((0.76 + c_0)N)^N) \) (for the same \( c_0 \in (0, 1) \)) [Yan06]. However, for the complementation of Rabin automata, a huge gap still exists between the state-of-the-art lower bound \( 2^{\Omega(N \log N)} \) [Yan06] and upper bound \( 2^{O(N \log N)} \) [KV05a], where \( k \), the number of Rabin pairs, can be as large as \( 2^N \). A similar huge gap exists for the complementation of Streett automata [KV05a, Yan06].

In this paper we generalize the full automata technique [Yan06] to incorporate multiple dimensional ranking functions. Using the generalized method we show that for any \( \epsilon > 0 \), the lower bound for the com-
plementation of Rabin automata with $N$ states, $k$ Rabin pairs, and an alphabet of size $N^2$ is $2^{O(N \lg N)}$ if $k \leq 2^{N(1-\varepsilon)}$, and is $2^{O(2^{N(1-\varepsilon)} \lg N)}$ if $k > 2^{N(1-\varepsilon)}$. For a Rabin automaton with $N$ states, the number of effective Rabin pairs can be at most $2^N$ because we can merge two Rabin pairs $(G_0, B)$ and $(G_1, B)$ into one pair $(G_0 \cup G_1, B)$ without changing the recognized language. In this sense, our lower bound is almost tight in terms of big-$O$ notation.

The condition of using alphabets of unbounded cardinality can be removed via an encoding trick. We show that for any $\varepsilon > 0$, a constant $d$ exists such that the above lower bound holds for any alphabet of size $d$. We can further reduce the alphabet size to a fixed small constant (independent of $\varepsilon$) with the lower bound adjusted to $2^{O(N \lg N)}$ for $k \leq 2^{\frac{N}{2}(1-\varepsilon)}$ and to $2^{O(2^{\frac{N}{2}(1-\varepsilon)} \lg N)}$ for $k > 2^{\frac{N}{2}(1-\varepsilon)}$. We also show that these lower bounds apply to the determinization of Rabin automata. Note that all lower bounds in this paper apply to any complementation or determinization algorithm which outputs $\omega$-automata of common types (see Acceptance Conditions in Section 2).

Related Work and Comparison. The first complementation construction for Büchi automata was given by Büchi and that construction requires $2^{2^{(N)}}$ states [Büc62]. The construction was improved to $2^{O(N)}$ states by Sistla, Vardi and Wolper [SVV87]. Safra gave a determinization construction for Büchi automata, from which a complementation construction with $2^{O(N \lg N)}$ states was obtained [Sa88]. This upper bound matches well with the lower bound $N! \approx (0.36N)^N = 2^{O(N \lg N)}$ proved by Michel [Mic88, Löd99]. Klarlund later gave a construction with $2^{O(N \lg N)}$ states without using determinization [Kla91]. Klarlund’s construction relies on quasi co-Büchi measure, which is a ranking function on states in a run graph, measuring the progress of a run toward being accepted. Kupferman and Vardi proposed a complementation construction that uses co-Büchi ranking (similar to quasi co-Büchi measure). The construction is essentially the same as Klarlund’s, but provides a better lower bound $O((6N)^N) = 2^{O(N \lg N)}$ [KV01]. With refined constructions, the upper bound was further improved to $O((0.9624N)^N)$ by Friedgut, Kupferman and Vardi [FKV06], and then most recently to $O(N^2((0.76 + c_0)N)^N))$ (for a fixed $c_0 \in (0, 1)$) by Schewe [Sch09]. In 2006, Yan introduced rankings into full automata technique and obtained a sequence of sharper lower bounds for complementation and determinization of $\omega$-automata [Yan06]. In particular, Yan sharpened the lower bound for the complementation of Büchi automata to $\Omega\left(\left(0.76 + c_0\right)N\right)^N$ (for the same $c_0 \in (0, 1)$), which now is only quadratically smaller than the best upper bound obtained by Schewe. In [Yan06], Yan also showed that the lower bound holds for any complementation construction whose output automata are of common types. This immediately gives a $2^{O(N \lg N)}$ lower bound for the complementation of Rabin automata because a Büchi automaton can be viewed as a Rabin automaton by simply interpreting the Büchi condition as a Rabin condition. In the contrast, the state-of-the-art complementation construction for Rabin automata, introduced by Kupferman and Vardi in [KV05a], requires $2^{O(N \lg N)}$ states.

Sakoda and Sipser introduced the full automata technique and used it to obtain several completeness and lower bound results on transformations involving 2-way finite automata [SS78]. Full automata operate on unconventional and large alphabets; in a full automaton of $N$ states, every possible unit transition graph (bipartite graph with $2N$ vertices) is identified with a letter, and words are nothing but potential runs of the automaton. Using this technique, in [SS78] Sakoda and Sipser also proved the classic result that the lower bound of complementing finite automata (on finite words) is $2^N$. In the proof, the difficult word serving as the witness of this lower bound is just a difficult run that cannot be accepted by any automaton with less than $2^N$ states. The power of full automata technique, however, does not rely on using alphabets of unbounded cardinality; by an encoding trick, a large alphabet can be mapped to a small alphabet containing only a few letters, with little comprise to the lower bound results [Sip79, Yan06].

To the best of our knowledge, although the full automata technique offers a systematic way for constructing difficult witnesses, it has never been applied to obtain lower bounds for transformations of $\omega$-automata until Yan extended the technique with rankings [Yan06]. Rankings used in [Yan06] bear certain similarities to those used in the work of Klarlund, Friedgut, Kupferman and Vardi [Kla91, KV01, FKV06, KV05a]. However, these two kinds of rankings are designed for different purposes. In [Kla91, KV01, FKV06, KV05a], a word is in the complementary language of a Büchi automaton if and only if there exists an odd co-Büchi ranking (or quasi co-Büchi measure) on the run graph of the word. A complementary automaton is so constructed to recognize run graphs with odd co-Büchi rankings (or quasi co-Büchi measures). In [Yan06], the rankings are designed to show that for a family of full automata $\mathcal{F}_n$, a family of difficult

---

1The term was first coined in [Yan06] where the definition is slightly different from that used in [SS78].
words (difficult runs) \(|a_n|\) exists such that for each \(n\), 
\(a_n\) is not recognized by \(\mathcal{F}_n\) or by any “small” 
complementary automaton of \(\mathcal{F}_n\). Our lower bound 
proof relies on rankings in the same vein as Yan’s. 
To obtain tighter lower bounds, however, we use a 
type of multi-dimensional rankings, which result in 
a construction considerably more sophisticated than 
that used in [Yan06]. Our rankings are also different 
from the old Streett rankings that were used in [KV05a] 
for the complementation of Rabin automata.

Our generalization relies on two key notions: (1) 
\(Q_k\)-rankings and (2) \(Y\)-graphs. A \(Q_k\)-ranking is 
k-dimensional function mapping states to \(k\)-tuples of 
integers. In fact a \(Q_k\)-ranking can be viewed as \(k\) 
independent bijective functions on states. A transition 
graph is called \(Q_k\)-ranked if every level of the graph is 
associated with a \(Q_k\) ranking. A \(Y\)-graph is a special 
\(Q_k\)-ranked transition graph that satisfies four proper-
ties designed for constructing difficult words. These 
properties are parameterized with a pair of states and 
an index value in between 1 and \(k\). It is not hard to 
construct a \(Y\)-graph that satisfies the four properties for 
a specific instantiation of the parameters. The technical 
difficulty, however, lies in how to accommodate the 
four properties for each pair of states and for each in-
dex simultaneously. Our solution is to use “bypasses” 
called \(\text{Refuge} \) and \(\text{Tunnel} \) to concatenate a sequence 
of \(Y\)-graphs each of which satisfies the four properties 
for a specific pair of states and for a specific index. The 
bypasses make the concatenation behaves like a par-
allel composition so that properties satisfied by each 
fragment are all preserved in the final concatenation.

Paper Organization. Section 2 presents notations 
used in this paper and basic terminology in automata 
theory. Section 3 generalizes the full automata tech-
nique and proves the lower bound. Section 4 presents 
the detailed construction with examples. Section 5 
concludes with a discussion of future work. Due to 
space limitation, figures and the proofs for technical 
lemmas and theorems are omitted. They are available 
in the full version of this paper, which is available on 
authors’ webpages.

2 Preliminaries

Basic Notations. \(\mathbb{N}\) denotes natural numbers. We 
write \([i..j]\) for \([k \in \mathbb{N} \mid i \leq k \leq j]\) and \([n]\) to mean 
\([0..n-1]\). If \(u\) be a sequence, we use \(|u|\) to denote 
the length of \(u\), \(u(i)\) \((i \in [u])\) to denote the object at 
the \(i\)-th position, and \(u[i..j]\) \((i, j \in [|u|])\) to denote the 
subsequence of \(u\) from position \(i\) to position \(j\). We 
use \(u \circ v\) to denote the concatenation of \(u\) and \(v\) and 
when little confusion is present, we simply use the 
 juxtaposition \(uv\).

\(\omega\)-automata. A finite automaton on infinite words 
is a tuple \(\mathcal{A} = (\Sigma, I, \Delta, \mathcal{F})\) where \(\Sigma\) is an al-
phabet, \(S\) is a finite set of states, \(I \subseteq S\) is a set of initial 
states, \(\Delta \subseteq S \times \Sigma \times S\) is a set of transition relations, 
and \(\mathcal{F}\) is an acceptance condition. We say that \(\mathcal{A}\) is 
deterministic when \(|\ell| = 1\) and for all \(p \in S, q \in \Sigma, 
|q| \in S \mid (p, q) \in \Delta\) \(|\leq 1\). Unless stated otherwise, all 
automata we consider in this paper are nondetermi-

Run of Automata. An infinite word \(\omega\)-words\) over 
\(\Sigma\) is an infinite sequence of letters in \(\Sigma\). A run of \(\mathcal{A}\) 
on an \(\omega\)-word \(w\) is an infinite sequence of states in 
\(S\) such that \(\rho(0) \in I\) and, \((\rho(i), w(i), \rho(i+1)) \in \Delta\) 
for \(i \in \mathbb{N}\). We use \(\rho[v_1,v_2]\) to denote the subsequence 
\(\rho(v_1)\rho(v_1+1)\cdots\rho(v_2)\). Let \(Occ(\rho)\) be the set of states 
 occurring in \(\rho\) and \(Inf(\rho)\) the set of states that occur 
ininitely many times in \(\rho\). A finite run of \(\mathcal{A}\) from state 
\(p\) to state \(q\) over a finite word \(w\) is a finite sequence of 
states \(\rho = (\rho(0)\rho(1)\cdots\rho(|w|))\) such that \(\rho(0) = p, \rho(|w|) = q\) 
and \((\rho(i), w(i), \rho(i+1)) \in \Delta\) for all \(i \in [|w|]\).

Acceptance Conditions. Automata on infinite words 
are classified according to acceptance conditions. We 
say that a type of \(\omega\)-automata is common if the accep-
tance condition of this type is defined solely with 
respect to \(Inf(\rho)\). Below we list some common types of 
\(\omega\)-automata relevant to this paper.

- Büchi automata, where \(\mathcal{F} \subseteq S\), and \(\rho\) is accepting 
iff \(Inf(\rho) \cap \mathcal{F} \neq \emptyset\).
- Rabin automata, where \(\mathcal{F} = \{(G_1,B_1),\ldots,(G_k,B_k)\}\), 
and \(\rho\) is accepting iff for some \(i \in [1..k]\), we have 
that \(Inf(\rho) \cap G_i \neq \emptyset\) and \(Inf(\rho) \cap B_i = \emptyset\). States in \(G_i\) 
and \(B_i\) are called, respectively, reconfirming states 
and invalidating states.
- Streett automata, where \(\mathcal{F} = \{(G_1,B_1),\ldots,(G_k,B_k)\}\), 
and \(\rho\) is accepting iff for all \(i \in [1..k]\), if \(Inf(\rho) \cap G_i \neq \emptyset\), 
then \(Inf(\rho) \cap B_i \neq \emptyset\).

We say that an \(\omega\)-word \(w\) is accepted by \(\mathcal{A}\) if there 
exists an accepting run of \(\mathcal{A}\) over \(w\). By \(\mathcal{L}(\mathcal{A})\) we 
denote the set of \(\omega\)-words accepted by \(\mathcal{A}\).

Δ-Graph. Let \(\mathcal{A} = (\Sigma, S, I, \Delta, \mathcal{F})\) be an automaton. A 
\(\Delta\)-graph of an \(\omega\)-word \(w\) under \(\mathcal{A}\) is a directed graph 
\(\mathcal{G}_w = (V,E)\) where \(V = S \times \mathbb{N}\) and \(E = \{(p,i),(q,i+1)\} \in 
V \times V \mid p,q \in S, i \in \mathbb{N}, (p,w(i),q) \in \Delta\) By the \(i\)-th 
level, we mean the vertex set \(S \times \{i\}\). The \(\Delta\)-graph of
a finite word is defined similarly. Let \( w \) be a finite word. By \( |w| \) we denote the length of \( w \), which is the same as \(|w|\). We call a path in \( \gamma_{\mathcal{A}} \) a full path if the path goes from level 0 to level \(|\gamma_{\mathcal{A}}|\). By \( \gamma_{\mathcal{A}} \circ \gamma'_{\mathcal{A}} \), we mean the concatenation of \( \gamma_{\mathcal{A}} \) and \( \gamma'_{\mathcal{A}} \), which is the graph obtained by merging the last level of \( \gamma_{\mathcal{A}} \) with the first level of \( \gamma'_{\mathcal{A}} \). Note that \( \gamma_{\mathcal{A}} \circ \gamma'_{\mathcal{A}} = \gamma_{\mathcal{A} \cup \mathcal{A}'} \).

\( \gamma \) is a visualization of the complete behavior of \( \mathcal{A} \) over \( w \). It is easily seen that \( \Delta \) can be identified with a function \( \Delta' : \Sigma \to 2^{\times \times S} \) such that \((p, q, g) \in \Delta'(\sigma) \) iff \((p, q, g) \in \Delta \) for any \( \sigma \in \Sigma \). With indices dropped, \( \gamma_{\mathcal{A}} \) is the graph of a letter \( \sigma \), is just the graph of the relation \( \Delta'(\sigma) \). By abusing notation, we identify \( \Delta'(\sigma) \) with \( \gamma_{\mathcal{A}} \) and \( \gamma'_{\mathcal{A}} \) (where \( w = \sigma_0 \sigma_1 \ldots \)) with \( \Delta'(w) = \Delta'(\sigma_0) \circ \Delta'(\sigma_1) \circ \cdots \).

Let \( w \) be a finite word. For \( v_0, v_1 \in \mathbb{N}, p_0, p_1 \in S \) we write \( (p_0, v_0) \twoheadrightarrow (p_1, v_1) \) to mean that there exists a run \( \rho \) of \( \mathcal{A} \) such that \( p_0 = \rho(p_0, v_0) = (p_1, v_1) \) to mean, in addition, that \( p_0 \rho(p_0, v_0) = (p_1, v_1) \) contains an \( S_0 \)-state but no \( S_1 \)-state. We write \( (p_0, v_0) \twoheadrightarrow (p_1, v_1) \) to mean that \( p_0 \rho(p_0, v_0) = (p_1, v_1) \) does not contain an \( S_1 \)-state. In case the indices of a run are of no importance, we simply drop them and write \( p_0 \twoheadrightarrow p_1, p_0 \twoheadrightarrow p_1, p_0 \twoheadrightarrow p_1 \).

### Full Automata

A full automaton \( \mathcal{A} = (\Sigma, S, I, \Delta, \mathcal{F}) \) is an automaton such that \( \Sigma = 2^{\times \times S} \), \( \Delta \subseteq S \times 2^{\times \times S} \times S \), and for all \( p, q \in S, \sigma \in \Sigma, (p, q, \sigma) \in \Delta \) if and only if \( (p, q, \sigma) \in \Delta \) (Lemma 2). By this transitivity property, we can construct \( \alpha_n \) by repeatedly enumerating \( Q_n \)-rankings. Property (3.1) guarantees that \( \alpha_n \notin \mathcal{L}(\mathcal{FR}_n) \), and Properties (3.2) and (3.3) imply that for any \( \omega \)-automata \( \mathcal{A} \) that accepts \( \alpha_n \), if \( \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{FR}_n) = \emptyset \), then the number of states of \( \mathcal{A} \) must exceed the number of distinct \( Q_n \)-rankings.

**Definition 1** (Full Rabin Automata). We define a family of full Rabin automata \( \mathcal{FR}_n = (\Sigma, S, I, \Delta, \mathcal{F}) \) such that

1. \( S = I \cup R \cup T \cup G \cup B \) where \( I, R, T, G \) and \( B \) are pairwise disjoint and assume the following forms:
   
   \[ I = \{s_0, \ldots, s_{n-1}\}, \quad T = \{t_0, \ldots, t_{n-1}\}, \quad R = \{r_0, \ldots, r_{n-1}\}, \quad B = \{b_1, \ldots, b_{2^\gamma}\}, \quad G = \{g\}; \]

2. \( \mathcal{F} = \{(G, B_1), \ldots, (G, B_k)\} \) where \( B_i \subseteq B, |B_i| = \gamma \) and \( B_i \neq B_j \) if \( i \neq j \).

Note that \( \mathcal{FR}_n \) is defined with three parameters: \( n, \gamma \) and \( k \), all required to be positive integers. Thus all notions derived from \( \mathcal{FR}_n \) should also be parameterized with the three. But for notation simplicity, we selectively omit all or some of them unless there is a chance of confusion. Now let \( N \) denote the number of states of \( \mathcal{FR}_n \).

\( \mathcal{FR}_n \) is designed to accommodate sophisticated properties of \( \mathcal{G} \)-graphs (Definition 3) to be introduced below. To reduce the construction complexity, we divide the state set \( S \) to five pairwise disjoint subsets \( I, R, T, G \) and \( B \), each designated for a different task. \( I \) is intended to be the domain of \( Q_n \)-rankings. \( R \) and \( T \), called Refuge and Tunnel, respectively, are solely for...
building by-passes. \( G \) and \( B \) are pools from which re-confirming states and invalidating states are chosen to form \( G \)'s and \( B \)'s, respectively. The exact lower bound actually is the number of total \( Q \)-rankings which will be shown to be \( \Omega(n^k) \) (Lemma 1). To get the desired lower bound \( 2\Omega(n^k \log n) \), we want \( n \) to be as close to \( n \) as possible and \( k \) to be as close to \( 2^n \) as possible. It turns out that it suffices to let \( G \) be a singleton and then let all \( G_i \)'s be the same as \( G \). To make \( k \) close to \( 2^n \), we require that \( B_i \)'s be pairwise unequal subsets of \( B \), all with the cardinality \( |B|/2 \). In this way, \( k \) can be as large as \( \binom{2^n}{2} \).

Now we introduce \( Q_k \)-ranking, a function that associates a \( k \)-tuple of integers with each state in \( I \).

**Definition 2** (\( Q_k \)-Ranking). A \( Q_k \)-ranking for \( FR_n \) is a function \( f : I \rightarrow [n]^k \) such that for each \( i \in [1..k] \), \( f^{(i)} : I \rightarrow [n] \), the projection of \( f \) on the \( i \)-th coordinate (i.e., \( f^{(i)}(p) = f(p)(i) \) for \( p \in I \)), is bijective.

A \( Q_k \)-ranking can be viewed \( k \) independent bijective functions from \( I \) into \([n] \). Let \( N \) denote the number of \( Q_k \)-rankings (again, for simplicity we omit the parameters \( n \) and \( k \)). We have

**Lemma 1.** \( N = \Omega(\Omega(n^k \log n)) \).

**Proof.** By \( N \) we denote the number of \( Q_1 \)-rankings. By definition, a \( Q_1 \)-ranking is a bijective function from \( I \) to \([n] \), and therefore \( N = n! = 2^{\Omega(n \log n)} \). Note that a \( Q_0 \)-ranking consists of \( k \) independent \( Q_1 \)-rankings. Therefore \( N = (N)^k = 2^{\Omega(n \log n)} \). \( \square \)

A \( \Delta \)-graph is called \( Q_k \)-ranked if its every level is associated with a \( Q_k \)-ranking. We write \( \text{rank} \) for the \( Q_k \)-ranking at level \( j \) and \( \text{rank}_j \) to denote the \( j \)-th projection of \( \text{rank} \) at level \( j \). Let \( X \) be a subset of \( S \). We call a vertex \((p, v) \) in a \( \Delta \)-graph of \( FR_n \) an \( X \)-vertex if \( p \in X \). When there is no confusion, we just write \( p \) for \((p, v) \). We are interested in a special kind of \( Q_k \)-ranked \( \Delta \)-graphs.

**Definition 3** (\( \gamma \)-Graph). We say that a \( Q_k \)-ranked \( \Delta \)-graph \( \mathcal{G} \) is an \( \gamma \)-graph if the following conditions hold.

**3.1** For any \( p, q \in I \), if in \( \mathcal{G} \) there exists a path \( l \) from \((p, v_0) \) to \((q, v_1) \) (for some \( v_0, v_1 \)) such that \( l \) contains no other I-vertex, then for each \( i \in [1..k] \),

3.1a if \( l \) contains a \( G \)-vertex, then \( \text{rank}_v(p) > \text{rank}_v(q) \);

3.1b if \( \text{rank}_v(p) < \text{rank}_v(q) \), then \( l \) contains a \( B_i \)-vertex.

**3.2** For any \( p, q \in I \), if \( \text{rank}_v(p) > \text{rank}_v(q) \), then \( p \xrightarrow{(\mathcal{G}, i)} q \).

**3.3** For any \( p, q \in I \), if \( \text{rank}_v(p) = \text{rank}_v(q) \), then \( p \xrightarrow{(\mathcal{G}, -i)} q \).

**3.4** In \( \mathcal{G} \) only I-vertices have outgoing edges at the first level and incoming edges at the last level, and for every \( \gamma \)-vertex at the first level there exists an outgoing path from that vertex to an I-vertex at the last level. In particular, for any \( p, q \in S \), if \( p \xrightarrow{(\mathcal{G})} q \), then \( p, q \in I \).

Property (3.1) is of local and universal nature; it requires that all paths in between I-vertices satisfy certain conditions, which induces a Streett condition on any infinite path in \( \alpha_n \) so that \( \alpha_{n} \notin \mathcal{L}(FR_n) \) (Lemma 3). In the contrast, Property (3.2) is of global and existential nature; it ensures that in \( \alpha_n \), from a vertex with higher rank to a vertex with lower rank (with respect to some index), there exists a “bad” finite path which, if repeated forever, generates an infinite path that satisfies the dual Rabin condition. This property is intended to show that it is hard to separate \( \alpha_n \) from \( \mathcal{L}(FR_n) \), in the sense that we can construct a word \( \alpha_n' \) from \( \alpha_n \) such that \( \alpha_n' \) is accepted by \( FR_n \) and no “small” automaton can distinguish \( \alpha_n' \) from \( \alpha_n \). Property (3.3) is also of global and existential nature; it ensures that in \( \alpha_n \), in between two vertices of the same rank (with respect to certain index \( i \)), there is a path that does not visit \( B_i \). This makes sure that Property (3.2) can be maintained during concatenation. Property (3.4) is mainly technical; it guarantees graph connectivity under concatenation.

**Definition 4** (\( R \)-Compatibility). We say that a word \( \mathcal{G} \in (\Sigma^*)^\ast \) is \( R \)-compatible with an ordered pair of \( Q_k \)-rankings \( (f, g) \) for \( FR_n \) if there exists a \( \gamma \)-graph \( \mathcal{G} \) in which the first level and the last level are ranked by \( f \) and \( g \), respectively.

**Theorem 1.** For any pair of \( Q_k \)-rankings \( (f, g) \), there exists a \( \gamma \)-graph \( \mathcal{G} \) that is \( R \)-compatible with \( (f, g) \).

This is the key theorem of this paper. We leave its proof to Section 4.

**Lemma 2** (Transitivity). Let \( f, g, h \) be \( Q_k \)-rankings for \( FR_n \), and \( \mathcal{G}_{(f, g)} \) and \( \mathcal{G}_{(g, h)} \) be \( \gamma \)-graphs that are \( R \)-compatible with \( (f, g) \) and \( (g, h) \), respectively. Then \( \mathcal{G}_{(f, g)} \circ \mathcal{G}_{(g, h)} \) is \( R \)-compatible with \( (f, h) \).

Theorem 1 and Lemma 2 allow us to construct an \( \omega \)-word which does not belong to \( \mathcal{L}(FR_n) \). Let \( f_0, f_1, \ldots \) be a repeated enumeration of \( Q_k \)-rankings such that (1) for any \( i, j \in \mathbb{N}, f_i \neq f_j \) and for any \( i, j \in \mathbb{N}, f_i = f_{j+i} \). Now define \( \alpha_{n} \) to be the \( \omega \)-word \( \mathcal{G}_{0} \mathcal{G}_{1} \mathcal{G}_{2} \cdots \).
such that for \( i \geq 0 \), \( \mathcal{G}_i = \mathcal{G}_{f_i} \), the word compatible with \( \langle f_i, f_{i+1} \rangle \).

**Lemma 3.** We have \( \alpha_n \notin \mathcal{L}(\mathcal{F}_R) \).

**Proof.** Suppose that \( \alpha_n \in \mathcal{L}(\mathcal{F}_R) \). Let \( \rho \) be a successful run of \( \mathcal{F}_R \) over \( \alpha_n \). By Property (3.4), infinitely many I-vertices occur in \( \rho \). So we can assume \( \rho \) to be of the form \( \rho(k_0) \cdots \rho(k_1) \cdots \rho(k_2) \cdots \) where \( k_0 = 0 \), and for any \( i \in \mathbb{N} \), \( \rho(k_i) \in I \) and no intermediate states from \( \rho(k_i) \) to \( \rho(k_{i+1}) \) belong to I. Since \( \rho \) is accepted by \( \mathcal{F}_R \), there is an \( i' \in [1,k] \) and a sufficiently large \( j \), such that for any \( j' \geq j \), \( \rho(j') \not\in B_\mu \). According to Property (3.1b), we have \( \text{rank}_{k_i}(\rho(k_j)) \geq \text{rank}_{k_{i+1}}(\rho(k_{j+1})) \geq \text{rank}_{k_{i+2}}(\rho(k_{j+2})) \geq \cdots \), in which strict inequality can only occur finitely many times. As a consequence, by Property (3.1a), the G-vertex can only appear finitely often in \( \rho \), a contradiction. \( \square \)

Although \( \alpha_n \) is not recognized by \( \mathcal{F}_R \), it closely “resembles” a word in \( \mathcal{L}(\mathcal{F}_R) \) in the sense that any \( \omega \)-regular language that separates \( \alpha_n \) from \( \mathcal{L}(\mathcal{F}_R) \) cannot be recognized by an automaton with less than \( \mathcal{F} \) states. This is established by the following lemma which is tailored for Rabin automata from Lemma 5 in [Yan06]. The proof relies on a counting argument similar to that of Pumping Lemma.

**Lemma 4.** Let \( \mathcal{A} \) be an \( \omega \)-automaton of any common type defined on the same alphabet as \( \mathcal{F}_R \) such that \( \alpha_n \in \mathcal{L}(\mathcal{A}) \) and \( \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{F}_R) = \emptyset \). Then \( \mathcal{A} \) must have more than \( \mathcal{F} \) states.

**Proof.** Suppose the contrary. Let \( \mathcal{A} = (\Sigma, S', \text{I}', \Delta', F^\prime) \) be an \( \omega \)-automaton such that \( |S'| < \mathcal{F} \), and \( \rho = \rho(0) \rho(1) \cdots \in (S')^\omega \) be an accepting run of \( \mathcal{A} \) over \( \alpha_n \). We shall show that \( \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{F}_R) = \emptyset \).

By the construction of \( \alpha_n \), we know there exists an infinite sequence of indices \( k_0, k_1, \ldots \) such that \( k_0 = 0 \), and for any \( i \geq 0 \), \( k_{i+1} - k_i = |\mathcal{G}| \) and \( \rho(k_i) \not\in S_{\mu} \). For each \( i < \mathcal{F} \), let

\[ S_i = \{ s^* \in S' \mid \rho(k_{i+1}) = s^* \text{ for infinitely many } j \in \mathbb{N} \} . \]

Obviously, for each \( i < \mathcal{F} \), \( S_i \neq \emptyset \). Since \( \mathcal{A} \) has less than \( \mathcal{F} \) states, there exist two different indices \( \mu, \nu < \mathcal{F} \) such that \( S_\mu \cap S_\nu \neq \emptyset \).

Since \( f_\mu \neq f_\nu \), without loss of generality, we assume that for some state \( q \in I \) and \( t \in [1,k] \), we have \( f_\mu(t)(q) > f_\nu(t)(q) \). Let \( s^* \in S_\mu \cap S_\nu \). Because there are infinitely many \( I \) satisfying \( \rho(k_{i+1}) = s^* \), there exists a sufficiently large \( \eta \in \mathbb{N} \), such that (1) \( \rho(k_{i+1}+\eta) = s^* \), (2) for every \( I > \mathcal{F} \), \( \nu(l) \in \text{Inf}(\rho) \). For the same reason we can find a sufficiently large \( \eta \) such that (1)

\[ k_{\mathcal{F}+\eta} \geq k_{\mathcal{F}+\mu} \]

and (3) \( \text{Inf}(\rho) = \text{Occ}(p[k_{\mathcal{F}+\mu}k_{\mathcal{F}+\nu}]) \). Let \( \mathcal{G}_\nu = \mathcal{G}_{f_\nu} \mathcal{G}_{f_{\nu+1}} \cdots \mathcal{G}_{f_{\nu+\eta}} \). It is easily seen that \( \text{Inf}(\rho) = Inf(p[k_{\mathcal{F}+\mu}k_{\mathcal{F}+\nu}]) \). Therefore, \( \rho' = \rho[0,k_{\mathcal{F}+\nu}] \circ (p[k_{\mathcal{F}+\mu}k_{\mathcal{F}+\nu}])^\omega \) is a run of \( \mathcal{A} \) over \( \alpha'_n \) such that \( \text{Inf}(\rho') = \text{Inf}(\rho) \). So \( \alpha'_n \in \mathcal{L}(\mathcal{A}) \).

Next we show that \( \alpha'_n \in \mathcal{L}(\mathcal{F}_R) \). Recall that there exists \( t \in [1,k] \) such that \( f_\mu(t)(q) > f_\nu(t)(q) \). By Lemma 2, \( \mathcal{G}_\nu \) is R-compatible with \( \langle f_0, f_{\mathcal{F}+\mu} \rangle = \langle f_0, f_{\nu+1} \rangle \), and \( \mathcal{G}_\nu \) is R-compatible with \( \langle f_{\mathcal{F}+\mu}, f_{\mathcal{F}+\nu} \rangle = \langle f_{\nu+1}, f_{\nu+2} \rangle \). Since \( f_{\nu+1} \) is bijective, there is a state \( q_0 \in I \) such that \( f_{\nu+1}(q_0) = f_{\nu+1}(q) \). By Property (3.3), we have \( q_0 \not\in q \). Also, because \( f_{\nu+1}(q) > f_{\nu+1}(q) \), Property (3.2) gives us \( q \not\in S_{\nu} \). Now \( q_0 \not\in q \). By Lemma 3, an accepting run of \( \mathcal{F}_R \) over \( \alpha_n \). Thus \( \alpha'_n \in \mathcal{L}(\mathcal{F}_R) \). \( \square \)

Note that in Lemma 4, \( \mathcal{A} \) only needs to be an automaton of a common type.

Now we are ready for the lower bound proof.

**Theorem 2.** For any \( \epsilon > 0 \), the lower bound for the complementation problem of Rabin automata with \( N \) states and \( k \) Rabin pairs is \( 2^{2\Omega(\log N)} \) if \( k \leq 2^{N-\epsilon} \), and is \( 2^{2\Omega(2^{N-\epsilon}\log N)} \) if \( k > 2^{N-\epsilon} \).

**Proof.** By Lemmas 1 and 4, the lower bound of complementing \( \mathcal{F}_R \) is \( 2^{2\Omega(\log N)} \) if \( k \leq 2^{N-\epsilon} \). Let \( c \) be a constant and \( \gamma = cn \). So \( N = 3n + 2cn + 1 = O(n) \) and we are left to show that if \( k \leq 2^{N-\epsilon} \), then \( k \leq 2^{N-\epsilon} \). It follows from Stirling’s inequality ([Rob55]) that for any \( m > 0 \), \( 2^{m-\epsilon} > \frac{2^{m-\epsilon}}{2m} < \frac{(N-1)\epsilon}{2N+3} < \left( \frac{N-1)\epsilon}{2N+3} \right) \). As stated in the introduction, no generality is lost by using alphabets of unbounded cardinalities. We can map large alphabets to an alphabet of constant size, with little compromise to our lower bounds.

**Theorem 3.** For any \( \epsilon > 0 \), there exists a constant \( d > 0 \) such that the lower bound for the complementation problem of Rabin automata over an alphabet of size \( d \) and with \( N \) states and \( k \) Rabin pairs is \( 2^{2\Omega(\log N)} \) if \( k \leq 2^{N-\epsilon} \), and is \( 2^{2\Omega(2^{N-\epsilon}\log N)} \) if \( k > 2^{N-\epsilon} \). For an alphabet of small constant size, the lower bound is \( 2^{2\Omega(\log N)} \) if \( k \leq 2^{N-\epsilon} \), and is \( 2^{2\Omega(2^{N-\epsilon}\log N)} \) if \( k > 2^{N-\epsilon} \).

As Lemma 4, Theorems 2 and 3 apply to any complementation algorithm that outputs automata of common types. The same lower bounds hold for the determination of Rabin automata.
Corollary 1. The lower bounds in Theorems 2 and 3 apply to the determinization of Rabin automata.

Proof. Suppose the contrary. Let \( \mathcal{A} \) be a Rabin automaton and \( \mathcal{B} \) a deterministic \( \omega \)-automaton of a common type \( T \) such that \( \mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{B}) \) and the state size of \( \mathcal{B} \) is below our lower bounds. Let \( T' \) denote the dual type of \( T \) such that the acceptance condition of \( T' \) is just the negation of the acceptance condition of \( T \). Obviously, \( T' \) is also a common type. Since \( \mathcal{B} \) is deterministic, we can obtain an automaton \( \mathcal{C} \) of type \( T' \) that complements \( \mathcal{B} \) by simply negating the acceptance condition of \( \mathcal{B} \). Now \( \mathcal{C} \) complements \( \mathcal{A} \) with state size below our lower bounds, a contradiction. \( \square \)

4 Construction of Difficult Words

In this section we prove Theorem 1. We shall show that for any pair of \( Q_c \)-rankings \( (f, g) \), there exists a \( \Upsilon \)-graph \( \mathcal{G}(f,g) \) that is \( R \)-compatible with \( (f, g) \).

Proof Plan. We need a construction to simultaneously satisfy all properties in Definition 3. The key challenge lies in making Property (3.2) in harmony with other properties. Our solution is to divide \( \mathcal{G}(f,g) \) into two sequential subgraphs \( \mathcal{G}_f \) and \( \mathcal{G}_{f,g} \). \( \mathcal{G}_f \) is a \( \Upsilon \)-graph with respect to \( (f, f) \) while \( \mathcal{G}_{f,g} \) is almost \( \Upsilon \)-graph with respect to \( (f, g) \) except that it does not satisfy Property (3.2). However, it turns out that Property (3.2) will hold in \( \mathcal{G}_f \circ \mathcal{G}_{f,g} \) as follows.

For any \( p, q \in I, i \in \{1..k\}, \) suppose that rank\(_0\)(\( p \)) > rank\(_{\mathcal{G}_f}(q) \). Since \( \mathcal{G}_f \) is a \( \Upsilon \)-graph for \( (f, f) \), we have rank\(_0\) = rank\(_{\mathcal{G}_f}(\mathcal{G}_f) \). So we can find a vertex \( r \in I \) in the boundary of \( \mathcal{G}_f \) and \( \mathcal{G}_{f,g} \) such that rank\(_0\)(\( p \)) > rank\(_{\mathcal{G}_{f,g}}(r) \). By Property (3.2), there exists a path \( I_f \) from \( (p, 0) \) to \( (r, f_{\mathcal{G}_f}) \) such that \( I_f \) visits \( G \) but no \( B_i \). By Property (3.3), there exists a path \( I_{f,g} \) from \( (r, f_{\mathcal{G}_f}) \) to \( (q, f_{\mathcal{G}_{f,g}}) \) such that \( I_{f,g} \) visits no \( B_i \)-vertices either. Therefore, \( I_f I_{f,g} \) is the desired path from \( (p, 0) \) to \( (q, f_{\mathcal{G}_{f,g}}) \).

\( \mathcal{G}_f \) and \( \mathcal{G}_{f,g} \) are constructed in a similar manner. First, we construct a sequence of graph fragments, each of which has a portion for building bypasses, and satisfies the corresponding properties with respect to a specific combination of \( i \in \{1..k\}, p, q \in I \). Second, we concatenate these fragments in such a way that bypasses are in place to guarantee properties of these fragments are all preserved under concatenation. Let us take a look at an example before going into the details. (The reader should refer to the full version of this paper for the figures used in this example and the examples to follow.)

Example 1 (\( \Upsilon \)-Graph \( \mathcal{G}(f,g) \)). Consider \( \mathcal{F}^\mathcal{R}_1 \) where \( \gamma = 1, k = 2 \), and

\[
\begin{align*}
I &= \{s_0, s_1, s_2, s_3\}, & R &= \{r_0, r_1, r_2, r_3\}, & B_1 &= \{b_1\}, \\
T &= \{t_0, t_1, t_2, t_3\}, & G &= \{\tilde{g}\}, & B_2 &= \{b_2\}.
\end{align*}
\]

Consider the following pair of \( Q_c \)-rankings \( f \) and \( g \).

\[
\begin{align*}
f(s_0) &= (3, 3) & f(s_1) &= (1, 2) & f(s_2) &= (2, 0) & f(s_3) &= (0, 1) \\
g(s_0) &= (0, 3) & g(s_1) &= (3, 2) & g(s_2) &= (1, 1) & g(s_3) &= (2, 0)
\end{align*}
\]

The first subgraph \( \mathcal{G}_f \) is shown in Figure 1 where we omit state sets \( R \) and \( T \) as they have no use. The second subgraph \( \mathcal{G}_{f,g} \) has two parts, which are shown, respectively, as \( \mathcal{G}_1 \) in Figure 2 and \( \mathcal{G}_2 \) in Figure 3. The complete graph \( \mathcal{G}(f,g) \) has 45 levels (0..44) in total. Since \( f((s_0, 0)) = (3, 3) \) and \( g((s_2, 44)) = (1, 1) \), we have \( (s_0, 0) \to (G, B_1) \to (s_2, 44) \) and \( (s_0, 0) \to (G, B_2) \to (s_2, 44) \). We mark a red (double dotted) path that satisfies the former and a green (double lined) path that satisfies the latter.

The existence of the first subgraph and the second subgraph is established, respectively, by Lemma 5 and Lemma 6. Due to space limitation, for each lemma, we only present a construction for \( \mathcal{F}^\mathcal{R}_n \) with \( \gamma = 1 \) (denoted by \( \mathcal{F}^\mathcal{R}_n(1) \)) and leave the generalization and detailed proof to the appendix.

Lemma 5. For any \( Q_c \)-rankings \( f \), there exists a \( \Upsilon \)-graph \( \mathcal{G}_f \) that is \( R \)-compatible with \( (f, f) \).

Proof Sketch. We show the construction of \( \mathcal{G}_f \) for \( \mathcal{F}^\mathcal{R}_n(1) \). Since \( \gamma = 1 \), we have \( k = 2 \), but we keep \( k \) as a parameter for the sake of later generalization. Also in this setting \( B_i \)'s \( i \in \{1..k\} \) are singletons, and hence with no loss of generality we assume that \( B_i = \{b_i\} \).

As mentioned before, \( \mathcal{G}_f \) is obtained by concatenating a sequence of graph fragments, each of which satisfies Properties (3.1)-(3.4) with respect to a specific combination of \( i \in \{1..k\}, p, q \in I \). It turns out that \( I \) suffices for building bypasses and so there is no need of \( R \) or \( T \) in \( \mathcal{G}_f \).

The construction uses the following letters:

\[
\begin{align*}
Id(I) &= \{ (p, p) \mid p \in I \} & ToG(p) &= Id(I) \cup \{(p, \tilde{g})\} \\
GtoB_i &= Id(I) \cup \{(\tilde{g}, b_i)\} & B_i toB_j &= Id(I) \cup \{(b_i, b_j)\} \\
FrG(p) &= Id(I) \cup \{(\tilde{g}, p)\} & FrB_i(p) &= Id(I) \cup \{(b_i, p)\}
\end{align*}
\]

where \( p \in I \) and \( i, j \in \{1..k\} \). For \( i \in \{1..k\}, p, q \in I \) we define \( d^{(i)}(p, q) \) to be

\[
ToG(p) \circ GtoB_i \circ \cdots \circ B_{i+1} toB_{i-1} \circ \cdots \circ B_2 toB_1 \circ FrB_1(q)
\]

173
Note that $d^{(0)}(p, q) = T \circ G(\rho) \circ B_{k-1} \circ \cdots \circ B_2 \circ B_1 \circ F_{\rho}(\theta)$ and similar adjustment for $d^{(0)}(p, q)$. It is not hard to verify that for fixed $i, p, q$, if $f_i^{(0)}(p) > f_i^{(0)}(q)$, then $d^{(0)}(p, q)$ satisfies Properties (3.1)-(3.4). Indeed, since $G$ is visited in $d^{(0)}(p, q), Property (3.2)$ is satisfied. Property (3.3) holds trivially because $B_i$ is not visited in $d^{(0)}(p, q)$. Property (3.1) also holds trivially because of the assumption $f_i^{(0)}(p) > f_i^{(0)}(q)$. Property (3.4) obviously holds because all letters contain $Id(i)$.

Formally, $G_i$ is a concatenation of subgraphs $G_1, \ldots, G_k$ where each $G_i$ in turn is a concatenation of graphs of the form $d^{(0)}(p, q)$ where $p, q$ satisfy $f_i^{(0)}(p) > f_i^{(0)}(q)$. Every level of $G_i$ is ranked by $f_i$. By the definition of $Q_i$-rankings, for each $i \in [1..k], j \in [n]$, there is one and only one state $p_j^i \in I$, such that $f_i^{(0)}(p_j^i) = j$. So the set $\{p_j^0, p_j^1, \ldots, p_j^{n_i-1} \mid j \in [n], j_1 > j_2 \}$ contains all the combinations of $(p_j^i, q)$ that we need. It is not hard to see that the set $\{p_j^{n_i-1}, p_j^{n_i} \mid j \in [n-1]\}$ just serves our purpose as long as we let a path go down through ranks step by step. Put all together, we have $G = G_1 \cdots \circ G_k$ where for $i \in [1..k], G_i$ is

$$d^{(0)}(p_j^0, p_j^1) \circ d^{(0)}(p_j^1, p_j^2) \circ \cdots \circ d^{(0)}(p_j^{n_i-1}, p_j^{n_i}).$$

We show how to select a path $l_i$ from $(p_0, 0)$ to $(q_i, [G_i])$ that satisfies Property (3.2) for a fixed index $i$. Beginning from $(p_0, 0), l_i$ takes horizontal edges passing through $G_0, \ldots, G_i$ until it enters $G_i$ and reaches $d^{(0)}(p, p')$ for some $p' \in I$ such that $f_i^{(0)}(p) = f_i^{(0)}(p') + 1$. Then $l_i$ takes the only path that leads to $p'$ in another horizontal track, decreasing the $i$-th rank by 1. Repeating this process, each step going to a vertex whose $i$-th rank is one less, eventually $l_i$ reaches $q$ at some level and from there it only takes horizontal edges till reaching $(q_i, [G_i])$.

**Example 2 (Y-Graph $G_i$).** Let us revisit Example 1 and take a close look at $G_i$ which is shown in Figure 1. Every level of $G_i$ is ranked by $f_i$ and $G_i$ has two parts: $G_i$ and $G_0$. In $G_i$, after a path leaves an $I$-vertices, it visits $G$, then $B_2$ and comes back to an I-vertex whose $i$-th rank is 1 less than that of the last visited I-vertex. Similarly, in $G_0$, after a path leaves an I-vertices, it visits $G$, then $B_1$ and comes back to an I-vertex whose $2$-nd rank is 1 less than that of the last visited I-vertex. Property (3.1) therefore holds for both indices 1 and 2. The red (double dotted) path is a witness for Property (3.2) with respect to $(s_0, 0), (s_1, 18)$ and index 1, and the green (double lined) path is a witness for Property (3.2) with respect to $(s_0, 0), (s_3, 18)$ and index 2. Property (3.3) is satisfied by any horizontal path from level 0 to level 18. Property (3.4) is obvious.

**Lemma 6.** For any pair of $Q_i$-rankings $(f, g)$, there exists a $Q_i$-ranked $\Delta$-graph $G_{f,g}$ such that the first level and the last level of $G$ is ranked by $f$ and $g$ respectively, and $G_{f,g}$ satisfies Properties (3.1), (3.2), and (3.4).

**Proof Sketch.** We show the construction of $G_{f,g}$ for $T \circ R_n$. As before we keep $k$ as a parameter for the later generalization and we assume that $B_i = (b_i)$ for $i \in [1..k]$.

The idea underlying the construction is the same as before. The desired graph $G_{f,g}$ for $T \circ R_n$ is a concatenation of subgraphs $G_1, \ldots, G_k$ where each $G_i$ fulfills the requirements with respect to the $i$-th index. In each $G_i$, two kinds of bypasses help preserve properties of all subgraphs under concatenation. $T (Tunnel)$ is intended to build paths that satisfy Property (3.3) for this index $i$, while $R (Refuge)$ is to let pass through $G_i$ those paths that are obliged to satisfy Property (3.3) for indices other than $i$.

To simplify the construction of each $G_i$, we introduce a sequence of $k - 1$ transitional $Q_i$-rankings that gradually bridges the difference between $f$ and $g$, in such a way that any two adjacent rankings differ at exactly one coordinate. In each $G_i$, the $Q_i$-rankings of all levels but the last one are the same and they differ from the $Q_i$-ranking of the last level only at the $i$-th coordinate. Formally, we define, for each $i \in [k]$,

$$(f, g)_i = (f^{(i)}, \ldots, f^{(0)}, f^{(i+1)}, \ldots, f^{(k)}).$$

Note that $(f, g)_0 = f$, $(f, g)_k = g$, and $(f, g)_{i-1} = (f, g)_i$ for $j \neq i$. We will assign $(f, g)_i$ to the last level of $G_i$ and $(f, g)_{i-1}$ to all other levels of $G_i$.

We also need to satisfy other properties. Property (3.4) is easily seen satisfied once the whole construction is given. Property (3.1a) is also easy to accommodate because no other properties require that the G-vertex be visited. So we simply do not use the G-vertex and Property (3.1a) trivially holds. Property (3.1b) indeed needs more care (see below).

Now we begin the formal construction. For $i, j \in [1..k]$, we define the following letters.

$$S' = I \cup T \cup R \cup \{s_j | j \in [n]\} \cup \{r_j | j \in [n]\}$$

As mentioned before, each $G_i$ is to fulfill Property (3.3) for index $i$. That means that for every pair of states $(p, q)$ $(p, q \in I)$ such that $(f, g)^{(i)}_i(p) = (f, g)^{(i)}_i(q)$, $G_i$ should contain a full path from $(p, 0)$ to $(q, [G_i])$ such that no $B_i$-vertex appears on the path. This is done by using $Tunnel$. For each $j \in [n]$, there exists one and exactly one $j' \in [n]$ such that $(f, g)^{(i)}_i(s_j) = (f, g)^{(i)}_i(s_{j'})$.
So let \( h : [n] \rightarrow [n] \) be such a bijective function and then let \( \text{toTunnel}_i \) be
\[
S_i \circ \text{toB}_i \circ \text{R} = B_i \circ \text{toB}_i \circ \cdots \circ B_{i-1} \circ \text{toB}_i \circ B_i \circ \text{toT}_h(i),
\]
where \( i_1, \ldots, i_t \) is a decreasing enumeration of
\[
I_i = \{ l \in [n] \mid \langle f, g \rangle_{i-1}(s_l) < \langle f, g \rangle_i(s_{h(l)}) \}.
\]
Note that if \( I_i = \emptyset \), we have \( \text{toTunnel}_i = S_i \circ \text{toB}_i \circ \text{R} \). In \( \text{toTunnel}_i \), no \( B_i \)-vertex is visited, and before a path jumps to the \( h(i) \)-th track in \( \text{toTunnel}_i \), the path first visits a \( B_j \)-vertex for every \( l \in I_i \), which is to respect Property (3.1b) for indices other than \( i \). These visits surely do not violate Property (3.3) for this \( i \) because \( i \notin I_i \). Now let
\[
\text{ToTunnel}_i = \text{toTunnel}_0 \circ \text{toTunnel}_1 \circ \cdots \circ \text{toTunnel}_{n-1},
\]
which contains a tunnel for each \( j \in [n] \).

We are done if we only consider Properties (3.1) and (3.3) for \( \mathcal{G}_i \). However, to guarantee these properties hold for \( \mathcal{G}_f \), we need to create a bypass in each \( \mathcal{G}_i \) such that a full path in \( \mathcal{G}_f \) that is obliged to satisfy these properties for coordinate \( i \) will take the corresponding bypass in every \( \mathcal{G}_i \) with \( j \neq i \). This is achieved by using \( \text{Refuge} \). We define \( \text{toRefuge}_i \) to be
\[
\begin{align*}
S_i \circ \text{toB}_i \circ \text{R} & \quad \text{if } \langle f, g \rangle_{i-1}(s_l) < \langle f, g \rangle_i(s_{h(l)}) \\
S_i \circ \text{toRefuge}_j & \quad \text{otherwise}
\end{align*}
\]
which makes sure that there is a bypass from \( s_j \) to \( s_i \) that visits \( B_i \) if \( \langle f, g \rangle_{i-1}(s_l) < \langle f, g \rangle_i(s_{h(l)}) \). Therefore, Property (3.1b) for this \( i \) is respected. Obviously Property (3.1b) for other indices are not violated because \( \langle f, g \rangle_{i-1} = \langle f, g \rangle_i \) for \( i \neq j \). Now let \( \text{toRefuge}_i \) be
\[
\text{toRefuge}_0 \circ \text{toRefuge}_1 \circ \cdots \circ \text{toRefuge}_{n-1},
\]
which contains a refuge for each \( j \in [n] \).

Put all together we define
\[
\mathcal{G}_i = \text{Id}(I) \circ \text{toRefuge}_0 \circ \text{toTunnel}_0 \circ \text{Converge},
\]
where \( \text{Id}(I) \) is to force paths to enter \( \mathcal{G}_i \) only through \( I \)-vertices, and \( \text{Converge} \) is to synchronize both \( T \)-vertices and \( R \)-vertices with the corresponding \( I \)-vertices before they leave \( \mathcal{G}_i \). Note that in each \( \mathcal{G}_i \), \( \text{Converge} \) forces that any full path from \( s_i \) to \( s_j \) either goes through \( \text{Tunnel} \) or goes through \( \text{Refuge} \). In the former case, we have \( \langle f, g \rangle_{i-1}(s_l) = \langle f, g \rangle_i(s_{h(l)}) \), and in the latter case, we have \( j = j' \).

We show how to select a path \( l_{f,g} \) from \( (s_r,0) \) to \( (s_f,|\mathcal{G}_{f,g}|) \) that satisfies Property (3.3) for a fixed index \( i \). Beginning from \( (p,0), l_{f,g} \) uses \( \text{Refuge} \) to pass through \( \mathcal{G}_0, \ldots, \mathcal{G}_{i-1} \) until it enters \( \mathcal{G}_i \). Inside \( \mathcal{G}_i \), \( l_{f,g} \) takes the horizontal track to reach \( \text{toTunnel}_i \) and from there it takes a path to enter \( \text{Tunnel} \). It continues to move through \( \text{Tunnel} \) until it reaches \( s_f \) at the last level of \( \mathcal{G}_i \). Then again \( l_{f,g} \) uses \( \text{Refuge} \) to pass through the remaining \( \mathcal{G}_{i+1}, \ldots, \mathcal{G}_n \) until reaching \( (s_f,|\mathcal{G}_{f,g}|) \).

Example 3 (Y-Graph \( \mathcal{G}_{f,g} \)). Let us revisit Example 1 to have a close look at \( \mathcal{G}_{f,g} \), which has two parts, \( \mathcal{G}_1 \) shown in Figure 2, and \( \mathcal{G}_2 \) shown in Figure 3. All levels but the last one of \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are ranked, respectively, by \( f \) and \( (f,g)_1 \). The last levels of \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are ranked, respectively, by \( (f,g)_1 \) and \( g \). In \( \mathcal{G}_1 \), a path starting from \( (s_1,18) \) \( i \in [1..k] \) has to go through either \( \text{Refuge} \) or \( \text{Tunnel} \). In the former case, the path ends at \( (s_1,32) \) and in the latter case, the path ends at \( (s_2,32) \) for some \( j \in [1..k] \) such that \( f_1(s_i) = (f,g)_1(s_i) \). Similarly for paths in \( \mathcal{G}_2 \).

A short path from \( (s_3,23) \) to \( (s_3,25) \) via \( (b_1,24) \) (in \( \text{toRefuge}_1 \)) is intended for any path passing through \( \text{Refuge} \) to respect Property (3.1b) because \( f_1((s_3,20)) < (f,g)_1(s_3,32) \). Similarly for the path from \( (s_1,20) \) to \( (r_1,22) \) via \( (b_1,21) \) (in \( \text{toRefuge}_1 \)) and the path from \( (s_2,35) \) to \( (r_2,37) \) via \( (b_2,36) \) (in \( \text{toRefuge}_2 \)).

The red (double dotted) path from \( (s_1,18) \) to \( (s_2,44) \) satisfies Property (3.3) for index 1 and the green (double lined) path from \( (s_3,18) \) to \( (s_2,44) \) satisfies Property (3.3) for index 2. The former path uses the \( \text{Tunnel} \) in \( \mathcal{G}_1 \) and \( \text{Refuge} \) in \( \mathcal{G}_2 \) while the latter uses \( \text{Refuge} \) in \( \mathcal{G}_1 \) and \( \text{Tunnel} \) in \( \mathcal{G}_2 \). Property (3.1b) holds for the reason stated above. Property (3.1a) holds vacuously because no \( G \)-vertex is visited. Property (3.4) is obvious as usual.

We are ready to prove the key theorem of this paper.

**Theorem 1.** For any pair of \( \mathcal{G}_r \)-rankings \( (f,g) \), there exists a Y-graph \( \mathcal{G}_{(f,g)} \) that is \( R \)-compatible with \( (f,g) \).

**Proof.** Let \( \mathcal{G} \) be an Y-graph that satisfies Lemma 5, and let \( \mathcal{G}_{f,g} \) be a \( \mathcal{G}_r \)-ranked A-graph that satisfies Lemma 6. Let \( \mathcal{G}_{(f,g)} = \mathcal{G}_f \circ \mathcal{G}_{f,g} \). We show that \( \mathcal{G}_{(f,g)} \) satisfies Properties (3.1)-(3.4). Let \( p, q \in I, i \in [1..k] \).

Suppose that \( f_0(p) = g^0(q) \). By Lemma 5 (Property (3.3)), \( p \xrightarrow{\mathcal{G}_f} p \) and by Lemma 6 (Property (3.3)), \( p \xrightarrow{\mathcal{G}_{(f,g)}} q \). Therefore, \( p \xrightarrow{\mathcal{G}_{(f,g)}} q \), which gives us Property (3.3) for \( \mathcal{G}_{(f,g)} \).

Suppose that \( f_0(p) > g^0(q) \). Because \( f_0 \) is bijective, there exists \( r \in I \) such that \( f_0(p) > f_0(r) = g^0(q) \). By Lemma 5 (Property (3.2)), \( p \xrightarrow{\mathcal{G}_f} r \) and by Lemma 6...
(Property (3.3)), \( r_{Gf,q} \) holds for the determinization of Rabin automata. Note that our lower bounds can be further improved. In the proof, a \( Q_k \)-ranking is defined to be a sequence of \( k \) independent bijective functions from \([n]\) to \([n]\), and hence the number of \( Q_k \)-rankings is \( (n!)^k \approx (0.36n)^{kn} \).

Our proof can be adapted to use a new type of \( Q_k \)-rankings, each of which is a sequence of \( k \) independent tight level rankings as defined in [FKV06, Yan06]. In this way, the number of new \( Q_k \)-rankings is at least \( (0.76n)^{kn} \). But this change does not affect the lower bounds expressed in the big-O notation, and hence we chose the current definition for simplicity of the proof.

We plan to use the improved technique to tighten bounds for other types of automata transformations. In particular, we are interested in investigating the complementation problem of Streett automata. The current best lower bound for this problem is \( \Omega(2^{\frac{1}{2}}n^{2kN}) \) ([Yan06]) while the best upper bound is \( 2^{\frac{1}{2}n^{2kN}} \) ([KV05a]), where \( k \), the number of Streett pairs, can be as large as \( 2^N \).

**Acknowledgements.** We are grateful to Professor Tian Liu of Peking University for reviewing the early draft of this paper and providing many valuable suggestions to improve it. We thank Qiqi Yan for comments concerning the upper bound results in [FKV06, Sch09] and for pointing out an error. We also thank anonymous reviewers for their valuable comments. Part of the work was done when the second author was at Microsoft Research Asia, and the first and the third author were at Peking University.

**References**


