U(1) X U(1) XI Z(2) Chern-Simons theory and Z(4) parafermion fractional quantum Hall states

The MIT Faculty has made this article openly available. Please share how this access benefits you. Your story matters.

| As Published | http://dx.doi.org/10.1103/PhysRevB.81.045323 |
| Publisher | American Physical Society |
| Version | Final published version |
| Accessed | Wed Apr 10 03:39:01 EDT 2019 |
| Citable Link | http://hdl.handle.net/1721.1/58842 |
| Terms of Use | Article is made available in accordance with the publisher's policy and may be subject to US copyright law. Please refer to the publisher's site for terms of use. |
| Detailed Terms | |
The fractional quantum Hall (FQH) states are the fractional quantum Hall theory with integral coupling constants $(k,l)$ and its relation to certain non-Abelian fractional quantum Hall (FQH) states. For the $U(1) \times U(1) \times \mathbb{Z}_2$ Chern-Simons theory, we show how to compute the dimension of its Hilbert space on genus $g$ surfaces and how this yields the quantum dimensions of topologically distinct excitations. We find that $Z_2$ vortices in the $U(1) \times U(1) \times \mathbb{Z}_2$ Chern-Simons theory carry non-Abelian statistics and we show how to compute the dimension of the Hilbert space in the presence of $n$ pairs of $Z_2$ vortices on a sphere. These results allow us to show that $l=3$ $U(1) \times U(1) \times \mathbb{Z}_2$ Chern-Simons theory is the low-energy effective theory for the $Z_4$ parafermion (Read-Rezayi) fractional quantum Hall states, which occur at filling fraction $\nu = \frac{2}{7+3}$. The $U(1) \times U(1) \times \mathbb{Z}_2$ Chern-Simons theory is more useful than an alternative $SU(2)_4 \times U(1)/U(1)$ Chern-Simons theory because the fields are more closely related to physical degrees of freedom of the electron fluid and to an Abelian bilayer phase on the other side of a two-component single-component quantum phase transition. We discuss the possibility of using this theory to understand further phase transitions in FQH systems, especially the $\nu = 2/3$ phase diagram.

DOI: 10.1103/PhysRevB.81.045323

PACS number(s): 73.43.-f, 11.25.Hf, 11.15. −q

I. INTRODUCTION

One of the most exciting breakthroughs in condensed-matter physics has been the discovery that there exist quantum phases of matter at zero temperature that cannot be described by their pattern of symmetry breaking. The prototypical and perhaps most well-studied examples of these phases are the fractional quantum Hall (FQH) states, which exhibit a different kind of order, called topological order. Topologically ordered phases are currently the subject of intense interest because of the possibility of detecting excitations that exhibit non-Abelian statistics and subsequently manipulating these non-Abelian excitations for robust quantum information storage and processing.

One way to improve our understanding of topological order in the fractional quantum Hall states is to study phase transitions between states with different topological orders. While much is known about phase transitions between phases with different patterns of symmetry breaking, much less is known about phase transitions between phases with different topological orders. Aside from its intrinsic interest, such information may be useful in identifying the topological order of a certain FQH state, which is currently a significant challenge. The experimental observation of a continuous phase transition in a FQH system may help us identify the topological order of one of the phases if we know theoretically which topologically ordered phases can be connected to each other through a continuous phase transition and which cannot. Ultimately, we would like to have an understanding of all of the possible topological orders in FQH states and how they can be related to each other through continuous phase transitions.

We may hope to understand a phase transition between two phases if we have a field theory that describes each phase and we know how the field theories of the two phases are related to each other. In the case of the fractional quantum Hall states, it is well known that the long-distance low-energy behavior is described by certain topological field theories in 2+1 dimensions, called Chern-Simons theories. For the Laughlin states and other Abelian FQH states, such as the Halperin states, the hierarchy states, and Jain states, the long-wavelength behavior is described by Chern-Simons theories with a number of $U(1)$ gauge fields.

For the non-Abelian FQH states, the corresponding Chern-Simons (CS) theory has a non-Abelian gauge group. The most well-studied examples of non-Abelian FQH states are the Moore-Read Pfaffian state and some of its generalizations, the Read-Rezayi (or $Z_k$ parafermion) states. The bosonic $\nu = 1$ Pfaffian is described by $SU(2)_2$ Chern-Simons theory, or alternatively, by $SO(5)_1$ Chern-Simons theory, while the effective theories for the other states are less well understood. It has been proposed that the Read-Rezayi $Z_k$ parafermion states are described by $SU(2)_k \times U(1)/U(1)$ Chern-Simons theory.

In this paper, we show that Chern-Simons theory with gauge group $U(1) \times U(1) \times \mathbb{Z}_2$ describes the long-wavelength properties of the $Z_4$ parafermion Read-Rezayi FQH state.

The significance of this result is that there is a bilayer state, the $(k,k,k-3)$ Halperin state at $\nu = \frac{2}{7+3}$, which may undergo a bilayer to single-layer quantum phase transition to the $Z_4$ parafermion state as the interlayer tunneling is increased.

The bilayer phase is described by a $U(1) \times U(1)$ Chern-Simons theory. This new formulation of the Chern-Simons theory for the $Z_4$ parafermion state may therefore be useful in understanding the phase transition because the gauge groups $U(1) \times U(1) \times \mathbb{Z}_2$ and $U(1) \times U(1)$ are closely related, and because the fields in the $U(1) \times U(1) \times \mathbb{Z}_2$ theory are more closely related to physical degrees of freedom of the electron fluid than they are in the proposed alternative $SU(2)_4 \times U(1)/U(1)$ theory.

In addition to aiding us in understanding this phase transition, this study shows how to compute concretely various topological properties of a Chern-Simons theory with a disconnected gauge group. For Chern-Simons theories at level $k$, where the gauge group is a simple Lie group $G$, there is a straightforward prescription to compute topological proper-
ties. The different quasiparticles are labeled by the integrable highest weight representations of the affine Lie algebra $\hat{g}_k$, where $g$ is the Lie algebra of $G$, while the quasiparticle fusion rules are given by the Clebsch-Gordon coefficients of the integrable representations of $\hat{g}_k$. In contrast, when the gauge group is disconnected and is of the form $G \times H$, where $H$ is a discrete automorphism group of $G$, it is much less straightforward to compute the topological properties of the Chern-Simons theory directly. One reason for this is that discrete gauge theories are most easily studied (and defined) on a lattice, while it is difficult to formulate lattice versions of Chern-Simons theories. This complicates the study of Chern-Simons theories with disconnected gauge groups.

In the case where the gauge group is $U(1) \times U(1) \times Z_2$, we show how to compute the ground-state degeneracy on genus $g$ surfaces and how this yields the quantum dimensions of the quasiparticles. We find that the $Z_2$ vortices carry non-Abelian statistics and we show how to compute the degeneracy of states in the presence of $n$ pairs of $Z_2$ vortices. The results, for a certain choice of coupling constants, agree exactly with results obtained in other ways for the $Z_4$ parafermion FQH state.

II. MOTIVATION AND BACKGROUND

One interesting way of obtaining the Pfaffian quantum Hall states is by starting with a bilayer ($k, k, k-2$) quantum Hall state and taking the interlayer tunneling to infinity. The bilayer state is at a filling fraction $\nu = \frac{1}{k-1}$ and is described by the wave function $\Psi = \Phi(\{z_i\}, \{w_i\})\exp[-i2\sum_i(z_i^2 + |w_i|^2)]$, with

$$\Phi = \prod_{i<j}^{N} (z_i - z_j)^{k-1} \prod_{i<j}^{N} (w_i - w_j)^{k-1} \prod_{i<j}^{N} (z_i - w_j)^{k-2}.$$  \hspace{1cm} (1)

Here, $z_i = x_i + iy_i$ is the complex coordinate of the $i$th electron in one layer and $w_i$ is the complex coordinate for the $i$th electron in the other layer.

As the tunneling is taken to infinity, we effectively end up with a single-layer state. The particles in the two layers become indistinguishable and so we might expect that the resulting wave function is the $(k, k, k-2)$ bilayer wave function but (anti)symmetrized between the $\{z_i\}$ and $\{w_i\}$ coordinates. The resulting wave function happens to be the Pfaffian state,

$$\Psi_P(\{z_i\}) = P\left(1_{z_i - z_j}^{2N} \prod_{i<j} (z_i - z_j)^{k-1} = S(\Psi(\{z_i\}, \{w_i\})) \right)$$ \hspace{1cm} (2)

where $S(\cdots)$ refers to symmetrization or antisymmetrization over $z_i$ and $w_i$ depending on whether the particles are bosons or fermions. Here we have set $z_{N+i} = w_i$. Indeed, the $(k, k, k-2)$ bilayer states undergo a continuous quantum phase transition to the single-layer state $\nu = \frac{1}{k-1}$ Pfaffian states as the interlayer tunneling is increased.\[14,16\]

In a similar fashion, the $(k, k, k-3)$ bilayer wave functions, when (anti)symmetrized over the coordinates of the particles in the two layers, yield the $Z_4$ parafermion states at filling fraction $\nu = \frac{1}{2k-3}$.\[14,16\] One way to verify this statement is through an operator algebra approach that also naturally suggests $U(1) \times U(1) \times Z_2$ as the appropriate gauge group for the corresponding Chern-Simons theory (see Appendix A). This observation suggests that as the interlayer tunneling is increased, there may be a region of the phase diagram where there is a phase transition from the bilayer $(k, k, k-3)$ state to the single-layer non-Abelian $Z_4$ parafermion state. For $k=3$, this is a phase transition at $\nu = \frac{2}{3}$, the phase diagram of which has attracted both theoretical and experimental attention.

Given this perspective, we might expect that we can understand the low-energy effective field theory of the Pfaffian and $Z_4$ parafermion states by gauging a discrete $Z_4$ symmetry associated with the $Z_2$ symmetry of interchanging the two layers. The effective field theories for the bilayer states are the $U(1) \times U(1)$ Chern-Simons theories with the field strength of one $U(1)$ gauge field describing the electron density for one layer and the field strength of the other gauge field for the other layer. This perspective suggests that the topological properties of these non-Abelian states can be described by a $U(1) \times U(1) \times Z_2$ Chern-Simons theory. This is a $U(1) \times U(1)$ Chern-Simons theory with an additional local $Z_2$ gauge symmetry. The semidirect product $\times$ here indicates that the $Z_2$ acts on the group $U(1) \times U(1)$; the $Z_2$ group element does not commute with elements of $U(1) \times U(1)$. In other words, elements of the group are $(a, p)$, where $a \in U(1) \times U(1)$ and $p \in Z_2$, and multiplication is defined by $(a_1, p_1) \times (a_2, p_2) = (a_1a_2, p_1p_2)$. This expectation for $U(1) \times U(1) \times Z_2$ Chern-Simons theory turns out to be correct for the $Z_4$ parafermion states but not quite correct for the Pfaffian states, as we will discuss.

We already have a field theory that correctly describes the topological properties of the bosonic $\nu = 1$ Pfaffian quantum Hall state. This is the $SU(2)_2$ Chern-Simons theory described in Ref. 13 or the $SO(5)_1$ Chern-Simons theory described in Ref. 12. [The Pfaffian quantum Hall state at other filling fractions are described by $SU(2)_2 \times U(1)/U(1)$ or $SO(5)_1 \times U(1)/U(1)$ Chern-Simons theory.\[12\]] Similarly, the $SU(2)_2 \times U(1)/U(1)$ Chern-Simons theories described in Ref. 15 encapsulate in some sense the topological properties of the $Z_4$ parafermion states. A possible shortcoming of those theories, however, is that it can be unclear how to connect the degrees of freedom of the field theory to the physical degrees of freedom of the electron liquid. In contrast, the $U(1) \times U(1) \times Z_2$ makes clearer the connection between the gauge fields and various physical degrees of freedom. It also makes clearer the relation to the bilayer state on the other side of the phase transition. Given this closer contact to the physical degrees of freedom of the electron fluid and to the bilayer Abelian phase, it is possible that this point of view may aid us in understanding physical properties of these quantum Hall states, such as the quantum phase transition between two topologically ordered phases: the bilayer Abelian phases and the non-Abelian single-layer phases.

The fact that such a Chern-Simons theory might describe the Pfaffian and/or $Z_4$ parafermion FQH states might also be expected from another point of view. It is known that the $Z_4$ parafermion conformal field theory (CFT), which is used in constructing the $Z_4$ parafermion FQH states, is dual to the rational $Z_2$ orbifold at a certain radius.\[20\] The rational $Z_2$ orbifold at radius $R$ is the theory of a scalar boson $\varphi$ compact-
tified on a circle of radius $R$, i.e., $\varphi \sim \varphi + 2\pi R$, and that is
gauged by a $Z_2$ action $\varphi \sim -\varphi$. Furthermore, the $Z_2$ orbifold
at a different radius is dual to two copies of the Ising CFT, which is used to construct the Pfaffian states. The Chern-
Simons theory corresponding to the $Z_2$ orbifold CFT has
gauge group $O(2)$, which we can think of as $U(1) \times Z_2$. This line of thinking is what led the authors of Ref. 22 to first
mention that $U(1) \times O(2)$ Chern-Simons theories are related to
the Pfaffian and $Z_2$ paraxeron states. In the $Z_2$ paraxeron case, the relation to $U(1) \times O(2)$ is suggestive but in-
complete because the $U(1)$ and the $O(2)$ need to be “glued”
together in an appropriate way; we elaborate more on this
point in Appendix B. The proper formulation is the $U(1)
\times U(1) \times Z_2$ theory that we present here and for which we
calculate many topological properties.

Let us first discuss the $U(1) \times U(1)$ Chern-Simons theo-
ries that describe the $(k,k,k-l)$ bilayer states. These are de-


defined by the Lagrangian

$$
L = \frac{k}{4\pi} \int_M \left( a \bar{\partial} a + \alpha \bar{\partial} \alpha \right) + \frac{k-l}{4\pi} \int_M \left( a \bar{\partial} \alpha + \alpha \bar{\partial} a \right),
$$

where $M$ is a two-dimensional manifold and $a(x,y,t)$ and \( \bar{a}(x,y,t) \) are the two $U(1)$ gauge fields defined on $M \times \mathbb{R}$. $M$
describes space and $\mathbb{R}$ describes time. The electron current
density in the top and bottom layers, $j_\mu$ and $\bar{j}_\mu$, respectively, are given by

$$
j_\mu = \frac{1}{2\pi} e^{ik} \partial_\mu \alpha,
$$

$$
\bar{j}_\mu = \frac{1}{2\pi} e^{ik} \partial_\mu \bar{\alpha}.
$$

In the $U(1) \times U(1)$ Chern-Simons theory, we package the two gauge fields in the following way:

$$
A_\mu = \begin{pmatrix} a_\mu & 0 \\ 0 & \bar{a}_\mu \end{pmatrix}.
$$

The gauge group $G = U(1) \times U(1)$ consists of the $U(1)$
\times $U(1)$ part, which we can write as

$$
U = \begin{pmatrix} e^{i\gamma} & 0 \\ 0 & e^{i\gamma} \end{pmatrix},
$$

and the $Z_2$ part, which contains the identity and the nontrivial element $\sigma_1$,

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

Thus, in addition to the usual $U(1) \times U(1)$ gauge symmetry
associated with the two gauge fields, there is a local $Z_2$ gauge
symmetry, which can be thought of in the following way.

The space of physical configurations at a certain space-time
point $(x,y,t)$ is to be described by the unordered pair $[a_\mu(x,y,t), \bar{a}_\mu(x,y,t)]$. The action of the $Z_2$ is to interchange
$a_\mu(x,y,t)$ and $\bar{a}_\mu(x,y,t)$ at the point $(x,y,t)$. Physically, we
may envision this as an electron from one layer and an electron from the other layer being interchanged. In order
to define a sensible action, we need to be dealing with differ-

tentiable gauge fields. So, we require the gauge fields to be
smooth functions on $M$, thus automatically gauge fixing the local $Z_2$ and leaving behind a residual global $Z_2$ symmetry
associated with interchanging $a$ and $\bar{a}$ at every point in


time-space. In this sense, we can use the action given by Eq.

(3) to describe our $U(1) \times U(1) \times Z_2$ Chern-Simons theory.

Although the $U(1) \times U(1)$ Chern-Simons theory and $U(1) \times U(1) \times Z_2$ Chern-Simons theory formally share the

same Lagrangian, their gauge structure is different. This is
why the same Lagrangian actually describes two different

theories. This example demonstrates that the Lagrangian is

not a good symbol for a one-to-one labeling of different topo-

logical field theories.

### III. GROUND-STATE DEGENERACY FOR $U(1) \times U(1) \times Z_2$

Chern-Simons Theory

The first check that a field theory correctly describes a

given topologically ordered phase is whether it correctly re-

produces the ground-state degeneracy of the system on sur-

faces of higher genus. Accordingly, we begin our study of

$U(1) \times U(1) \times Z_2$ by calculating the ground-state degeneracy

on a torus. We then calculate the degeneracy on surfaces of

arbitrary genus, from which we deduce the quantum dimen-

sions of the quasiparticles. Finally, we study the quasiparti-

cles.

Gauge theory with gauge group $G$ on a manifold $M$

is most generally defined by starting with a principal $G$ bundle

on $M$ and defining the gauge field, a Lie algebra-valued one-

form, as a connection on the bundle. Often, one is concerned

with situations in which $M = \mathbb{R}^n$, in which case there is a

global coordinate system and the gauge field can be written

in coordinates everywhere as $a_\mu dx^\mu$, where $a_\mu$ is a Lie

algebra-valued function on $\mathbb{R}^n$. In these cases, we do not

need to be concerned with the more general fiber bundle

definition in order to compute quantities of interest. The situa-

tion is more complicated in general, when $M$ does not have a

global coordinate system, in which case we can only lo-

cally define $a_\mu dx^\mu$ in any given coordinate chart. In these

situations, it is often convenient, when possible, to view the

gauge field as a function defined on $\mathbb{R}^n$, where $n$ is the di-

mension of $M$, and to impose suitable periodicity conditions.

This allows us to work in a global coordinate system and may

simplify certain computations. For example, for $U(1)$

gauge theory on a torus, we can choose to work with a gauge

field $a_\mu(x,y)$ defined over $\mathbb{R}^2$, but with periodic boundary

conditions,

$$
a_\mu(x,y) = a_\mu(x + L_x, y) = a_\mu(x, y + L_y).
$$

In the case where $G = U(1) \times U(1) \times Z_2$, the $Z_2$ gauge

symmetry allows for the possibility of twisted sectors: configu-

rations in which the gauge field is periodic up to conjugacy

by an element of $Z_2$. On a torus, there are four sectors and

the ground-state degeneracy is controlled by the degeneracy

within each sector. In more mathematical terms, there are

four distinct classes of $U(1) \times U(1) \times Z_2$ bundles on a torus,

distinguished by the four possible elements in the group

$[\text{Hom}: \pi_1(T^2) \to Z_2]$, which is the group of homomor-

isms.
phisms from the fundamental group of $T^2$ to $Z_2$, mod $Z_2$. Thus, we can think of $A_\mu(x,y,t)$ as defined on $\mathbb{R}^3$, with the following periodicity conditions:

$$A_\mu(x + L_\mu,y) = \sigma_1^\mu A_\mu(x,y) \sigma_1^\mu,$$

$$A_\mu(x,y + L_\nu) = \sigma_2^\nu A_\mu(x,y) \sigma_2^\nu,$$  \hspace{1cm} (9)

where $\epsilon_1$ and $\epsilon_2$ can each be 0 (untwisted) or 1 (twisted). Furthermore, in each of these sectors, the allowed gauge transformations $U(x,y)$ take the form (time index is suppressed)

$$U(x,y) = \begin{pmatrix} e^{i\phi(x,y)} & 0 \\ 0 & e^{i\theta(x,y)} \end{pmatrix}$$ \hspace{1cm} (10)

and must preserve the boundary conditions on $A_\mu$,

$$U(x + L_\mu,y) = \sigma_1^\mu U(x,y) \sigma_1^\mu,$$

$$U(x,y + L_\nu) = \sigma_2^\nu U(x,y) \sigma_2^\nu.$$ \hspace{1cm} (11)

These transform $A_\mu$ in the usual way,

$$A_\mu \rightarrow UA_\mu U^{-1} + iU\partial_\mu U^{-1}.$$ \hspace{1cm} (12)

The formulation of the theory on higher genus surfaces is similar. On a genus $g$ surface, there are $2^{2g}$ different sectors, characterized by whether there is a $Z_2$ twist along various noncontractible loops. Across these twists, the two gauge fields $a$ and $\tilde{a}$ transform into each other. The gauge transformations also obey these same twisted boundary conditions; this implies that the boundary conditions on the gauge fields are preserved under gauge transformations. The connection between this formulation and the definition of a principal $G$ bundle on a compact Riemann surface can be made more precise by considering local coordinate charts, transition functions, etc., but here we do not pursue any further mathematical precision.

A. Ground-state degeneracy on a torus

As mentioned above, there are four sectors on a torus, one untwisted sector and three twisted sectors. We now proceed to compute the ground-state degeneracy in each sector. We follow the approach in Ref. 23, which was applied to continuous and connected gauge groups.

1. Untwisted sector

In the untwisted sector, the ground states are the $Z_2$ invariant states of a $U(1) \times U(1)$ Chern-Simons theory with the Lagrangian of Eq. (3). We partially fix the gauge by setting $a_0 = \tilde{a}_0 = 0$. The equations of motion for $a_0$ and $\tilde{a}_0$ act as constraints that require zero field strength: $f = \partial_\mu a_\nu - \partial_\nu a_\mu = 0$ and $\tilde{f} = \partial_\mu \tilde{a}_\nu - \partial_\nu \tilde{a}_\mu = 0$. This implies that gauge-inequivalent configurations are completely specified by the holonomies of the gauge fields along noncontractible loops of the torus, $\phi_a \cdot dl$ and $\phi_{\tilde{a}} \cdot dl$. This is a special case of the more general statement that flat $G$ bundles are characterized by $[\text{Hom}: \pi_1(M) \rightarrow G] / G$. We can parametrize this configuration space in the following way:

$$a_1(x,y,t) = \frac{2\pi}{L} X(t), \quad \tilde{a}_1(x,y,t) = \frac{2\pi}{L} \tilde{X}(t),$$

$$a_2(x,y,t) = \frac{2\pi}{L} Y(t), \quad \tilde{a}_2(x,y,t) = \frac{2\pi}{L} \tilde{Y}(t).$$ \hspace{1cm} (13)

The large gauge transformations $a \rightarrow a + iU^{-1}dU$ with $U(x,y) = e^{i2\pi s_\mu/t + 2\pi\theta/t}$ take $(X,Y) \rightarrow (X + m, Y + n)$. Thus $(X,Y)$ and $(\tilde{X}, \tilde{Y})$ take values on a torus. Substitution into the action yields, up to total time derivatives,

$$L = 2\pi k(X\tilde{Y} + \tilde{X}Y) + 2\pi(k - l)(X\tilde{Y} + \tilde{X}Y).$$ \hspace{1cm} (14)

The Hamiltonian vanishes. The momenta conjugate to $Y$ and $\tilde{Y}$ are

$$p_Y = \frac{\delta L}{\delta \dot{Y}} = 2\pi kX + 2\pi(k - l)\tilde{X},$$

$$p_{\tilde{Y}} = \frac{\delta L}{\delta \dot{\tilde{Y}}} = 2\pi k\tilde{X} + 2\pi(k - l)X.$$ \hspace{1cm} (15)

The wave functions for this system can be written as a sum of plane waves,

$$\psi(Y,\tilde{Y}) = \sum_{n,m} c_{n,m} e^{i2\pi m Y + 2\pi n \tilde{Y}}.$$ \hspace{1cm} (16)

In momentum space, the wave function becomes

$$\phi(p_Y, p_{\tilde{Y}}) = \sum_{n,m} c_{n,m} \delta(p_Y - 2\pi m) \delta(p_{\tilde{Y}} - 2\pi n).$$ \hspace{1cm} (17)

or, equivalently,

$$\varphi(X, \tilde{X}) = \sum_{n,m} c_{n,m} \delta(kX + (k - l)\tilde{X} - n) \delta(k\tilde{X} + (k - l)X - m).$$ \hspace{1cm} (18)

Using the fact that $X \sim X + 1$ and $\tilde{X} \sim \tilde{X} + 1$, we find that

$$c_{n,m} = \delta_{n,-k,m-k+1} = \delta_{n+k,m-k}.$$ \hspace{1cm} (19)

There are $|k - l|$ independent coefficients $c_{n,m}$ which explains why the $(k,k-l)$ quantum Hall state has a degeneracy of $|k - l|$ on a torus.

We can label the quantum states by $(n,m)$. The ground states in our $U(1) \times U(1) \times Z_2$ theory will be the $Z_2$ invariant subspace of this Hilbert space; it will contain the diagonal states $(n,n)$ and ones of the form $(n,m) \pm (m,n)$. A simple count of the $Z_2$ invariant states, using identifications (19), yields a total of

$$|k - l| + 1\right]\right)/2$$ \hspace{1cm} (20)

states in this untwisted sector.

2. Twisted sectors

There are three $Z_2$ twisted sectors, corresponding to twisting in either the $x$ direction, the $y$ direction, or both. Since modular transformations, i.e., diffeomorphisms that are not
continuously connected to the identity, are symmetries that can take one twisted sector to another, we expect that all twisted sectors should have the same degeneracy. This can be verified explicitly by computing the degeneracy in each case. Here we will only consider the case where the gauge fields are twisted in the y direction. More precisely this means that the gauge fields obey the following boundary conditions:

\[ a_i(x, y + L) = a_i(x, y), \quad \tilde{a}_i(x, y + L) = a_i(x, y), \]

\[ a_i(x + L, y) = a_i(x, y), \quad \tilde{a}_i(x + L, y) = a_i(x, y). \]  

(21)

Given these twisted boundary conditions, we can consider a new field \( c_\mu(x, y) \) defined on a space that is doubled in length in the y direction,

\[ c_\mu(x, y) = \begin{cases} a_\mu(x, y), & 0 \leq y \leq L \\ \tilde{a}_\mu(x, y - L), & L \leq y \leq 2L. \end{cases} \]  

(22)

Observe that \( c \) has the periodicity

\[ c_\mu(x, y) = c_\mu(x + L, y) = c_\mu(x, y + 2L). \]  

(23)

The allowed gauge transformations that act on \( c \) are of the form \( W(x, y) = e^{i\tilde{b}c(x, y)} \), where \( W(x, y) \) need only be periodic on the doubled torus,

\[ W(x + L, y) = W(x, y + 2L) = W(x, y), \]  

(24)

c transforms as a typical \( U(1) \) gauge field,

\[ c \to c - \partial \tilde{b}. \]  

(25)

In particular, there are large gauge transformations \( W(x, y) = e^{i(2\pi n/L)x + i(2\pi m/2L)y} \) that change the zero mode of \( c \),

\[ c_i \to c_i + \frac{2\pi m}{L} + \frac{2\pi n}{2L}. \]  

(26)

In terms of \( c \), the Lagrangian becomes

\[ L = \int_0^L dx \int_0^{2L} dy \left[ \frac{k}{4\pi} c \partial \tilde{c} + \frac{k - l}{4\pi} c(x, y) \partial \tilde{c}(x, y - L) \right]. \]  

(27)

Note that this Lagrangian is actually nonlocal in the field \( c \), but this does not pose any additional difficulty. We can set temporal gauge \( c_0 = 0 \), i.e., \( a_0 = \tilde{a}_0 = 0 \), and view the equation of motion for \( c \) as a constraint that forces the field strength for \( c \) to be zero. Thus, the gauge-inequivalent configurations can be parametrized as

\[ c_i(x, y, t) = \frac{2\pi}{L_i} X_i(t), \]  

(28)

where \( L_1 = L \) and \( L_2 = 2L \). Inserting this expansion into the Lagrangian gives, up to total time derivatives,

\[ L = 2\pi(2k - l) X_1 X_2. \]  

(29)

Due to the existence of the large gauge transformations, we find that the zero modes \( X \) take values on a torus,

\[ (X_1, X_2) \sim (X_1 + 1, X_2) \sim (X_1, X_2 + 1). \]  

(30)

Thus, using the same techniques used in the previous section, we conclude that the ground-state degeneracy in this sector is \( 2k - l \). There are three different twisted sectors, so we find in total

\[ 3|2k - l| \]  

(31)

states in the twisted sectors of the \( U(1) \times U(1) \rtimes Z_2 \) theory.

### 3. Total ground-state degeneracy on torus

Adding the degeneracies from the twisted and the untwisted sectors, we find that the total ground-state degeneracy on a torus in the \( U(1) \times U(1) \rtimes Z_2 \) theory is

\[ \text{ground-state Deg. on torus} = (|l| + 7)|2k - l|/2. \]  

(32)

For \( l = 2 \) and \( k > 1 \), the filling fraction is \( \nu = \frac{1}{2k-1} \) and the above formula gives \( 9(k-1) \) states on a torus. Compare this to the torus degeneracy of the \( \nu = \frac{1}{2k-1} \) Pfaffian state, which is \( 3(k-1) \). We see that the \( U(1) \times U(1) \rtimes Z_2 \) Chern-Simons theory for \( l = 2 \) has a torus ground-state degeneracy that is three times that of the Pfaffian state. So the \( U(1) \times U(1) \rtimes Z_2 \) Chern-Simons theory for \( l = 2 \) cannot directly describe the Pfaffian state. In Appendix B, we argue that, for \( l = 2 \), \( U(1) \times U(1) \rtimes Z_2 \) Chern-Simons theory describes the Pfaffian state plus an extra copy of the Ising model.

For \( l = 3 \) and \( k > 2 \), the filling fraction is \( \nu = \frac{3}{2k-3} \) and Eq. (32) gives \( 5(2k-3) \) ground states on a torus. The \( \nu = \frac{3}{2k-3} \) \( Z_4 \) parafermion state also gives rise to same torus degeneracy of \( 5(2k-3) \). Thus, we would like to propose that the \( U(1) \times U(1) \rtimes Z_2 \) Chern-Simons theory for \( l = 3 \) and \( k > 2 \) describes the \( Z_4 \) parafermion quantum Hall states. As a more nontrivial check on these results, we now turn to the calculation of the ground state degeneracy on surfaces of arbitrary genus.

#### B. Ground-state degeneracy for genus \( g \)

The ground-state degeneracy on a genus \( g \) surface of the \( Z_4 \) parafermion quantum Hall state at filling fraction \( \nu = \frac{3}{2k-3} \), where \( k > 2 \), is given by

\[ (k - 3/2)2^g - [1(3^g + 1) + (2^g - 1)(3^{g-1} + 1)]. \]  

(33)

Note that the second factor, \( 2^g - [1(3^g + 1) + (2^g - 1)(3^{g-1} + 1)] \), is the dimension of the space of conformal blocks on a genus \( g \) surface in the \( Z_4 \) parafermion CFT [see Eq. (C2)]. The degeneracy for the corresponding quantum Hall state is \( (k - 3/2)2^g \nu^g \) times this factor.

Let us consider the ground-state degeneracy on a genus \( g \) surface for the \( U(1) \times U(1) \rtimes Z_2 \) Chern-Simons theory. Let \( \{a_i\} \) and \( \{b_i\} \), with \( i = 1, \ldots, g \), be a basis for the homology cycles (see Fig. 1). The \( a_i \) and \( b_j \) do not intersect each other, while \( a_i \) and \( b_j \) intersect if \( i = j \). That is, the \( a_i \) and \( b_i \) form a canonical homology basis. There can be a \( Z_2 \) twist along any combination of these noncontractible loops. Thus there are \( 2^g \) different sectors; one of them is untwisted while the other \( 2^g - 1 \) sectors are twisted. Let us first analyze the untwisted sector.
It is known that the \((k,k,k-l)\) bilayer FQH states, which are described by the \(U(1) \times U(1)\) Chern-Simons theory of Eq. (3) have a degeneracy of \(\det K|^{1/2}\), where the \(K\) matrix is

\[
K = \begin{pmatrix} k & k-l \\ k-l & k \end{pmatrix}.
\] (34)

Thus the degeneracy for these bilayer states is \(|P(2k-l)|^2\).

These states may be written as

\[
\otimes |n_i,m_i\rangle,
\]

where the \(n_i\) and \(m_i\) are integers, \(i=1,\ldots,g\), and with the identifications [see Eq. (19)]

\[
(n_i,m_i) \sim (n_i+k-l,m_i+k) \sim (n_i,k,m_i+k-l)
\]

for each \(i\). The action of the \(Z_2\) on these states is to take

\[
\otimes |n_i,m_i\rangle \rightarrow \otimes |m_i,n_i\rangle.
\]

(37)

We must project onto the \(Z_2\) invariant states. There are \(2k-l\) diagonal states of the form \(\otimes |n_i,n_i\rangle\). These are invariant under the \(Z_2\). There are \(|P(2k-l)|^2 - 2k-l\) off-diagonal states, and exactly half of them are \(Z_2\) invariant. This gives a total of

\[
(2k-l)\left(\frac{2k-l}{2}\right) = \frac{(2k-l)^2}{2} - \frac{l(l+1)}{2}
\]

(38)

different states, which for \(l=3\) and \(k \geq 1\) corresponds to the first term of Eq. (33).

Now consider the twisted sectors. To begin, suppose that there is a \(Z_2\) twist along the \(a_g\) cycle, and no twists along any of the other cycles. Let \(\Sigma_g\) refer to the genus \(g\) surface. Let us consider the double cover \(\Sigma_{2g-1}\) of \(\Sigma_g\), which is a genus \(2g-1\) surface. It can be constructed as follows. Take two copies of \(\Sigma_g\), referred to as \(\Sigma_g^1\) and \(\Sigma_g^2\), and cut both of them along their \(a_g\) cycle. Glueing them together in such a way that each end of the cut on one copy lands on the opposite end of the cut on the other copy leaves the \(2g-1\) surface \(\Sigma_{2g-1}\) (see Fig. 2). The sheet exchange \(R\) is a map from \(\Sigma_{2g-1}\) to itself that satisfies \(R^{-1}R=1\) and which takes \(\Sigma_g^1 \rightarrow \Sigma_g^2\) and vice versa.25

We can now define a new, continuous gauge field \(c\) on \(\Sigma_{2g-1}\) as follows:

\[
c(p) = \begin{cases} a(p), & p \in \Sigma_g^1 \\ a[R(p)], & p \in \Sigma_g^2. \end{cases}
\]

(39)

Notice that because the gauge transformations get twisted also, \(c\) now behaves exactly as a typical \(U(1)\) gauge field on a genus \(2g-1\) surface. In particular, there are large gauge transformations which change the value of \(\oint_{a_g} c \cdot dl\) or \(\oint_{b_g} c \cdot dl\) by \(2\pi\).

In terms of \(c\), action (3) becomes

\[
L = \frac{k}{4\pi} c(p) \partial c(p) + \frac{k-l}{4\pi} c(p) \partial \det [R(p)].
\]

(40)

In terms of \(c\), the Lagrangian is nonlocal; however this poses no difficulty. Fixing the gauge \(c_0=0\), the equation of motion for \(c_0\) is a constraint that enforces \(c\) to have zero field strength; that is, \(c\) is a flat connection.

Let \(\{a_i\}\) and \(\{b_i\}\) be a basis of canonical homology cycles on \(\Sigma_{2g-1}\), with \(i=1,\ldots,2g-1\). We can choose \(a_i\) and \(b_i\) in such a way that the sheet exchange \(R\) acts on these cycles as follows:

\[
R a_i = a_{i+g-1}, \quad R b_i = b_{i+g-1},
\]

(41)

where \(i=1,\ldots,g-1\). The dual basis is the set of one forms \(\omega_i\) and \(\eta_j\), which satisfy

\[
\int_{a_i} \omega_j = \delta_{ij}, \quad \int_{b_i} \omega_j = 0,
\]

\[
\int_{a_i} \eta_j = 0, \quad \int_{b_i} \eta_j = \delta_{ij}
\]

(42)

Since \(c\) must be a flat connection, we can parametrize it as

\[
c = c_1 dx^1 + c_2 dx^2 = 2\pi(x^{\prime} \omega_i + y^{\prime} \eta_j).
\]

(43)

Two connections \(c\) and \(c^{\prime}\) are gauge equivalent if

\[
x^{\prime} - x^i = \text{integer}, \quad y^{\prime} - y^j = \text{integer}.
\]

(44)

Furthermore, from the definition of \(c\) [Eq. (39)], we see that the \(Z_2\) action is the same as the action of the sheet exchange \(R\),

\[
(x^i, y^j) \rightarrow (x^{R(i)}, y^{R(i)}),
\]

(45)

where

\[
R(i) = \begin{cases} i + g - 1 & \text{for } i = 1, \ldots, g-1 \\ i - g + 1 & \text{for } i = g, \ldots, 2g-2 \\ 2g-1 & \text{for } i = 2g-1. \end{cases}
\]

(46)

Substituting into action (40) and using the fact that \(\int_{\tilde{\Sigma}_{2g-1}} \omega_l \wedge \eta_k = \delta_{lk}\) and \(\int_{\tilde{\Sigma}_{2g-1}} \omega_l \wedge \eta_k = \int_{\tilde{\Sigma}_{2g-1}} \eta_l \wedge \eta_k = 0\), we obtain
Apart from the variables with \( i = 2g - 1 \), this action looks like the action for a bilayer \((k, k, k - l)\) state on a genus \( g - 1 \) surface. Therefore, we can easily deduce that quantizing this states in this particular twisted sector. We can write the states as

\[
|n_{2g-1}\rangle \otimes |n_{r(R(i)}\rangle, \tag{48}
\]

for \( i = 1, \ldots, g - 1 \) and with the identifications

\[
n_{2g-1} \sim n_{2g-1} + 2k - l, \tag{49}
\]

\[
(n_{r(R(i)} \sim (n_{r} + k_{r}(n_{R(i)} + k - l) \sim (n_{r} + k - l, n_{R(i)} + k). \tag{50}
\]

Note the \( n_{r} \) are all integer. Now we must project onto the \( Z_{2} \) invariant sector. The action of the \( Z_{2} \) is to take

\[
|n_{2g-1}\rangle \otimes |n_{r(R(i)}\rangle \rightarrow |n_{2g-1}\rangle \otimes |n_{R(i)}(n_{r}). \tag{51}
\]

Suppose \( n_{r} = n_{R}(i) \) for each \( i \). Such states are already \( Z_{2} \) invariant; there are \([2k - l] \times [2k - l]^{2}\) of them. The remaining states for which \( n_{r} \neq n_{R}(i) \) for at least one \( i \) always change under the \( Z_{2} \) action. The \( Z_{2} \) invariant combination is

\[
|n_{2g-1}\rangle \otimes (|n_{R}(n_{r(i)}) + |n_{R}(i), n_{r})\rangle. \tag{52}
\]

There are \([2k - l] \times [2k - l]^{2}\) of these. In total therefore there are

\[
[2k - l]^{2} + 1 \rightarrow k - l/2(|l|^{2} + 1) + 1 = k. \tag{53}
\]

states in this particular twisted sector.

Now it turns out that each of the \( 2^{2g-1} \) twisted sectors (which generically has many \( Z_{2} \) twists along many different noncontractible loops) yield the same number of twisted states as the sector in which there is a single twist along just the \( a_{g} \) cycle. One can understand this by considering the modular group, or mapping class group, of \( \Sigma_{g} \). This is the group of diffeomorphisms on \( \Sigma_{g} \) modulo those that are continuously connected to the identity. They are generated by “Dehn twists,” which correspond to cutting the surface along some noncontractible loop, rotating one side by \( 2\pi \), and gluing the two sides back together. The mapping class group of \( \Sigma_{g} \) can be generated by Dehn twists along the loops \( a_{g}, b_{g}, \) and \( c_{g} \), shown in Fig. 3. Elements of the mapping class group are symmetries of the topological field theory, which means that they are represented by unitary operators on the quantum Hilbert space. In particular, the dimension of the space of states for a given twisted sector is equivalent to that of a different twisted sector if they can be related by the action of an element of the mapping class group. In the following we sketch how, using Dehn twists, one can go from any arbitrary twisted sector to the sector in which there is a single \( Z_{2} \) twist along only the \( a_{g} \) cycle.

First note that a \( Z_{2} \) twist along some cycle \( \gamma \) is equivalent to having a \( Z_{2} \) twist along \(-\gamma \), and that a \( Z_{2} \) twist along \( \gamma \) + \( \gamma \) is equivalent to having no \( Z_{2} \) twist at all. Since we are here concerned only with the properties of the \( Z_{2} \) twists, we use these properties in the algebra below. In other words, the algebra below will be defined over \( Z_{2} \) because we are only concerned with \( Z_{2} \) twists along various cycles.

Let us call \( A_{i}, B_{i}, \) and \( C_{i} \) the Dehn twists that act along the \( a_{i}, b_{i}, \) and \( c_{i} \) cycles. Notice that a \( Z_{2} \) twist along \( a_{i} \) and \( a_{i+1} \) is equivalent to a \( Z_{2} \) twist along \( c_{i} \). Let us consider the action of \( A_{i}, B_{i}, \) and \( C_{i} \) on \( Z_{2} \) twists along the \( a_{i} \) and \( b_{i} \) cycles,

\[
A_{i}; a_{i} \rightarrow a_{i},
\]

\[
b_{i} \rightarrow b_{i} + a_{i},
\]

\[
B_{i}; a_{i} \rightarrow a_{i} + b_{i},
\]

\[
b_{i} \rightarrow b_{i},
\]

\[
C_{i}; a_{i} \rightarrow a_{i},
\]

\[
b_{i} \rightarrow b_{i} + c_{i} = b_{i} + a_{i} + a_{i+1},
\]

\[
a_{i+1} \rightarrow a_{i+1},
\]

\[
b_{i+1} \rightarrow b_{i+1} + c_{i} = b_{i+1} + a_{i} + a_{i+1}. \tag{54}
\]

\( Z_{2} \) twists along all other cycles are left unchanged. Notice in particular that \( A_{i}B_{i}; a_{i} \rightarrow b_{i} \), so that a \( Z_{2} \) twist along \( a_{i} \) is equivalent to one along \( a_{i} + b_{i} \), which is also equivalent to one along \( b_{i} \). As a result, we can see that the configuration of \( Z_{2} \) twists can be labeled only by considering which of the \( g \) handles have any twists at all. Furthermore, since we can rearrange the holes without changing the topology, the configuration of \( Z_{2} \) twists is actually labeled by considering how many of the \( g \) handles have twists.

Suppose that two of the \( g \) handles have \( Z_{2} \) twists. Since we have freedom to rearrange the holes, we can consider the situation in which two neighboring handles each have a \( Z_{2} \) twist. Since twists along \( a_{i} \) and \( b_{i} \) are all equivalent, let us suppose that one handle has a twist along its \( b \) cycle, while the other handle has a twist along its \( a \) cycle. That is, we are considering the situation in which there is a twist along \( b_{i} + a_{i+1} \). Now, performing the Dehn twist \( C_{i} \), we have

\[
C_{i}; b_{i} + a_{i+1} \rightarrow b_{i} + a_{i} + a_{i+1} + a_{i+1} = b_{i} + a_{i}. \tag{55}
\]

Thus we see that the case with \( Z_{2} \) twists for two handles is equivalent to that for a \( Z_{2} \) twist along a single handle. From this, it follows that the case with \( n \) handles having \( Z_{2} \) twists is equivalent to the case where only a single handle has a \( Z_{2} \) twist.

Therefore, under actions of the Dehn twists, any arbitrary twisted sector goes into the sector in which there is a single
twist along the $a_2$ cycle. This means that the dimension of the Hilbert space is the same for each of the $(2^{2^k} - 1)$ twisted sectors and in particular is equal to that for the sector in which there is a single twist along $a_2$. We computed that situation explicitly [see Eq. (53)], so we can conclude that the number of ground states on a genus $g$ surface for the $U(1) \times U(1) \rtimes \mathbb{Z}_2$ Chern-Simons theory is

$$S_g(k,l) = |k - l|/2|2^{2^k}1|(|l|/2 + 1) + (2^{2^k} - 1)(|l|/2^k + 1).$$

(56)

For $l=3$, this corresponds to the degeneracy of the $Z_4$ parafermion quantum Hall state that we expect from a CFT calculation [see Eq. (33)]. When $l=2$, we get

$$S_g(k,2) = |k - 1|/2|2^{2^k}1|(|l|/2^k + 1)^2,$$

(57)

which corresponds to the degeneracy of the $\nu = 1/2$ Pfaffian quantum Hall state times an extra factor of $2|2^{2^k}1|$, which is the dimension of the space of conformal blocks of the Ising CFT on a genus $g$ surface. This again confirms the notion that for $l=2$, this theory corresponds to the Pfaffian state with an extra copy of the Ising model.

**IV. QUANTUM DIMENSIONS OF QUASIPARTICLES FROM GROUND-STATE DEGENERACY**

In the last section we found the ground-state degeneracy, $S_g$, of the $U(1) \times U(1) \rtimes \mathbb{Z}_2$ Chern-Simons theory on a surface of genus $g$. From $S_g$ we can deduce some topological properties of the quasiparticles. It is well known, for example, that $S_1$, the ground-state degeneracy on a torus, is equal to the number of topologically distinct quasiparticles. Here we show that from $S_g$ we can also obtain the quantum dimensions of each of the quasiparticles.

The quantum dimension $d_\gamma$ of a quasiparticle denoted by $\gamma$ has the following meaning. For $\nu$ quasiparticles of type $\gamma$ at fixed positions, the dimension of the Hilbert space grows as $d_\gamma^\nu$. For Abelian quasiparticles at fixed positions, there is no degeneracy of states, so the quantum dimension of an Abelian quasiparticle is 1. The quantum dimension $d_\gamma$ can be obtained from the fusion rules of the quasiparticles, $N_{\gamma'\gamma}^\gamma$: $d_\gamma$ is the largest eigenvalue of the fusion matrix $N_{\gamma'\gamma}$, where $(N_{\gamma'\gamma})_\gamma = N_{\gamma'\gamma}$. From the quantum dimensions $d_\gamma$, we can obtain $S_g$ through the formula,

$$S_g = D^{2|2^k|} \sum_{\gamma=0}^{N-1} d_\gamma^{2|2^k|},$$

(58)

where $N$ is the number of quasiparticles, $d_\gamma$ is the quantum dimension of quasiparticle $\gamma$, and $D = \sqrt{\sum d_\gamma}$ is the “total quantum dimension.” Remarkably, this formula also implies that if we know $S_g$, then we can uniquely determine all of the quantum dimensions $d_\gamma$. To see how, let us first order the quasiparticles so that $d_{\gamma_1} \geq d_{\gamma_2}$. Notice that the identity has unit quantum dimension, $d_0 = 1$, and suppose that $d_i \geq 1$ for $i = 0, \ldots, i_0$ ($i_0 \geq 0$), $d_{i_0+1} > 1$. Now consider

$$
\lim_{g \to \infty} \frac{S_g}{S_{g + 1}} = D \lim_{g \to \infty} \left( \begin{array}{c} N-1 \\
\gamma_{n+1} \end{array} \right) d_{\gamma_{n+1}}^{2^n} = D^2.
$$

(59)

We see that the total quantum dimension $D$ can be found by computing $\lim_{g \to \infty} S_g/S_{g+1}$. Now define

$$
\overline{S}_g^{(1)} = \frac{S_g}{D^{2(g-1)} - 1} = \sum_{\gamma_{n+1}} d_{\gamma_{n+1}}^{2(g-1)}
$$

and suppose that $\gamma_1, \ldots, \gamma_l$ all have the same quantum dimension. Now consider the following limit:

$$
\lim_{g \to \infty} \frac{\overline{S}_{g+1}^{(1)}}{\overline{S}_g^{(1)}} = \lim_{g \to \infty} \frac{d_{\gamma_{n+1}}^{2^n}}{d_{\gamma_{n+1}}^{2^{n+1}}} = \frac{d_1}{d_{1+1}}.
$$

(60)

We see that $d_1$ can be determined by computing $\lim_{g \to \infty} \overline{S}_g^{(1)}$. This allows one to define

$$
\overline{S}_g^{(2)} = \overline{S}_g^{(1)} - d_1^{2(g-1)} = \sum_{\gamma_{n+2}} d_{\gamma_{n+2}}^{2(g-1)},
$$

(62)

and in turn we find $d_1^2 = \lim_{g \to \infty} \overline{S}_g^{(2)}/\overline{S}_g^{(1)}$. Proceeding in this way, one can obtain $d_l$, then define

$$
\overline{S}_g^{(l+1)} = \overline{S}_g^{(l)} - d_l^{2(g-1)} = \sum_{\gamma_{n+1}} d_{\gamma_{n+1}}^{2(g-1)},
$$

(63)

and then compute $d_{l+1}$ from $\overline{S}_g^{(l+1)}$.

$$
d_{l+1} = \lim_{g \to \infty} \frac{\overline{S}_{g+1}^{(l+1)}}{\overline{S}_g^{(l+1)}}.
$$

(64)

Thus we can see that in this way all of the quantum dimensions of the quasiparticles can be obtained from the formula for the ground-state degeneracy on a genus $g$ surface.

Carrying out this procedure for the $U(1) \times U(1) \rtimes \mathbb{Z}_2$ Chern-Simons theory, we find that when $|l| < 4$, the quantum dimensions of the quasiparticles take one of three different values. $2|2k-|l|$ of them have quantum dimension 1, $2|2k-|l|$ of them have quantum dimension 2, and $N - 2|2k-|l|/2$ of them have quantum dimension 2. The total quantum dimension, for all $(k, l)$, is

$$D = 4|2k-2|l|.
$$

(65)

For $l=3$ this coincides exactly with the quantum dimensions of the quasiparticles in the $\nu = 1/2$ $Z_4$ parafermion FQH states.
V. QUASIPARTICLES

When we refer to quasiparticles in a Chern-Simons theory, we are referring to topological defects in the configuration of the gauge fields. For instance, for a Chern-Simons theory at level \( k \) with a simple Lie group \( G \), a quasiparticle is represented by a unit of flux in an integrable representation of the affine Lie algebra \( \hat{g} \), where \( g \) is the Lie algebra of \( G \). The partition function of the Chern-Simons theory in the presence of external sources of quasiparticles is

\[
Z(\{C_i, R_i\}) = \int DA \prod_i W_R(C_i) e^{\mathcal{S}_C[A]},
\]

where the Wilson loop operator \( W_R(C) \) is defined as

\[
W_R(C) = \text{Tr}_R \mathcal{P} \exp \left[ i \oint_C A \cdot dl \right].
\]

\( \text{Tr}_R \) is a trace in the representation \( R \), \( \mathcal{P} \) refers to path ordering, and \( C \) is a loop describing the world line of the quasiparticle. Furthermore, the action of the quantum operator \( W_R(C) \) is to take one ground state to another when \( C \) is a noncontractible loop in space.

In the \( U(1) \times U(1) \times Z_2 \) Chern-Simons theory, there are several types of quasiparticles to consider. Some of the quasiparticles are related to the Wilson loop operators for the \( U(1) \) gauge fields; some are neutral under the \( Z_2 \) gauge field while others carry \( Z_2 \) charge. There are also \( Z_2 \) vortices, which we explicitly analyze in the following section.

A. \( Z_2 \) vortices

One basic excitation in a theory with a \( Z_2 \) gauge symmetry is a \( Z_2 \) vortex. In the context of \( U(1) \times U(1) \times Z_2 \) Chern-Simons theory, a \( Z_2 \) vortex is, roughly speaking, a point around which the \( U(1) \) gauge fields transform into each other. Here we compute the degeneracy of states in the presence of \( n \) pairs of \( Z_2 \) vortices at fixed positions; we find that this degeneracy grows like \( n^n \), and therefore the \( Z_2 \) vortices can be identified with the non-Abelian quasiparticles with quantum dimension \( \sqrt{n} \). We can in fact obtain the formula for the degeneracy more precisely and find that it agrees exactly, for \( l=3 \), with results from the \( Z_2 \) parafermion FQH states.

The basic idea is that a sphere with \( n \) pairs of \( Z_2 \) vortices can be related to a \( U(1) \) Chern-Simons theory on a genus \( g=n-1 \) Riemann surface. We will find that the \( Z_2 \) invariant subspace of this theory has \((|l|-1)/2 \) states while the \( Z_2 \) noninvariant subspace has \((|l|+1)/2 \) states when \( l \) is odd.

We may define a pair of \( Z_2 \) vortices more precisely as a one-dimensional closed submanifold \( \gamma \) of our spatial two-manifold \( M_0 \). The two boundary points of \( \gamma \) are thought of as the location of the \( Z_2 \) vortices. The gauge field \( A_\mu \) is defined on \( M=\overline{M_0} \setminus \gamma \), with the following boundary conditions along \( \gamma \):

\[
\gamma = \{(x, y)|x_1 \leq x \leq x_2\}. \tag{70}
\]

The two \( Z_2 \) vortices are located at the two ends of \( \gamma \) and \( f \) maps the neighborhood of these \( Z_2 \) vortices to the end of a cylinder; the boundary \( M \) in this neighborhood gets mapped to a circle. In terms of the new coordinates \( x'^\mu \), this neighborhood of \( M \) gets mapped to

\[
\{(x', y)|y \in \mathbb{R} \% 2\pi\}. \tag{71}
\]

The location of the \( Z_2 \) vortices in the new coordinates is taken to be at \((0, y_0)\) and \((\pi, y_0)\). Fix some small \( \epsilon > 0 \). Let us choose an \( f \) that takes

\[
(x_0, y_0) \pm \epsilon \rightarrow (\mp x_0, y_0 - \epsilon) \tag{72}
\]

for \( x_1 < x_0 < x_2 \). It is easy to see that as \( \epsilon \) is taken to zero, we have
Applying Eq. (69), we can immediately see that the boundary conditions for $\text{A}^{t}_{\mu}$ acquire an additional minus sign,

$$\text{A}^{t}_{\mu}(\pm x',y_{0}) = - \sigma_{1}[\text{A}^{t}_{\mu}(\pm x',y_{0})]_{\sigma_{1}}.$$  

(74)

Let us now study the cases $n=1$ and $n=2$ for $M_{g}=S^{2}$ before attempting to generalize to arbitrary $n$. We begin by considering the case $n=2$, the case of two pairs of $Z_{2}$ vortices on a sphere. Consider also the diffeomorphism $f$ shown in Fig. 5. Clearly, the situation with two pairs of $Z_{2}$ vortices on a sphere is equivalent to having the gauge field $\text{A}_{\mu}$ defined on the space

$$M = \{(x,y)|0 \leq y \leq L, x \in \mathbb{R}% L\}$$  

(75)

for any $L$ with the following periodicity/boundary conditions:

$$\text{A}_{\mu}(x+L,y) = \text{A}_{\mu}(x,y),$$

$$\text{A}_{\mu}(x,L) = - \sigma_{1}[\text{A}_{\mu}(-x,L)]_{\sigma_{1}},$$

$$\text{A}_{\mu}(x,0) = - \sigma_{1}[\text{A}_{\mu}(-x,0)]_{\sigma_{1}},$$  

(76)

and with the action of Eq. (3). We can now define a new continuous field $c_{\mu}$ defined on

$$\tilde{M} = \{(x,y)|x \in \mathbb{R}% L, y \in \mathbb{R}% 2L\}$$

(77)

as follows:

$$c_{\mu}(x,y) = \begin{cases} a_{\mu}(x,y), & 0 \leq y \leq L \\ \tilde{a}_{\mu}(-x,2L-y), & L \leq y \leq 2L, \end{cases}$$

(78)

where now $c_{\mu}$ is doubly periodic,

$$c_{\mu}(x,y) = c_{\mu}(x + L,y) = c_{\mu}(x,y + 2L).$$

(79)

Recall that the $U(1) \times U(1)$ gauge transformations on $A_{\mu}$ are of the form

$$U = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\phi} \end{pmatrix}.$$  

(80)

These gauge transformations must preserve the boundary conditions (76) on $A_{\mu}$. This implies that $U$ obeys the following boundary conditions:

$$U(x + L,y) = U(x,y),$$

$$U(x,L) = \sigma_{1}U^{-1}(-x,L)\sigma_{1},$$

$$U(x,0) = \sigma_{1}U^{-1}(-x,0)\sigma_{1}. $$  

(81)

Just as we defined $c_{\mu}$ from $A_{\mu}$, we can define the gauge transformation that acts on $c_{\mu}$ in the following way:

$$c_{\mu} \rightarrow c_{\mu} - \partial_{\mu}h.$$  

(82)

So we see that $c_{\mu}$ behaves like a typical $U(1)$ gauge field defined on a torus. In particular, the only condition on $h(x,y)$ is that $e^{ih(x,y)}$ be doubly periodic, which allows for the possibility of large gauge transformations along the two noncontractible loops of the torus.

In the $A_{0}=0$ gauge, the Lagrangian can be written as

$$L = e^{i\theta}\int_{0}^{L}dx\int_{0}^{L}dy\left[\frac{k}{4\pi}(a_{\mu}\dot{a}_{\mu} + \tilde{a}_{\mu}\dot{\tilde{a}}_{\mu}) + \frac{k-1}{4\pi}(\dot{a}_{\mu}\dot{\tilde{a}}_{\mu} + \tilde{a}_{\mu}\dot{a}_{\mu}) - \frac{1}{4\pi}\epsilon_{J}\epsilon_{K}\partial_{J}\partial_{K}c_{\mu}\right].$$

(83)

where the integration is over the region $0 \leq x,y \leq L$. In terms of $c_{\mu}$,

$$\int_{0}^{L}dx\int_{0}^{L}dy(a_{\mu}\dot{a}_{\mu} + \tilde{a}_{\mu}\dot{\tilde{a}}_{\mu}) = \int_{0}^{L}dx\int_{0}^{2L}dy\epsilon_{J}\epsilon_{K}\partial_{J}\partial_{K}c_{\mu}.$$  

(84)

Using $\tilde{a}_{\mu}(x,y) = -c_{\mu}(-x,2L-y)$, we see

$$\int_{0}^{L}dx\int_{0}^{L}dy\epsilon_{J}\epsilon_{K}\partial_{J}\partial_{K}c_{\mu} = -\int_{0}^{L}dx\int_{0}^{L}dy\epsilon_{J}\epsilon_{K}\partial_{J}\partial_{K}c_{\mu}(-x,2L-y).$$

(85)

Therefore we can write the action in terms of $c_{\mu}$ as

$$L = e^{i\theta}\int_{0}^{L}dx\int_{0}^{2L}dy\left[\frac{k}{4\pi}c_{\mu}\dot{c}_{\mu} - \frac{k-1}{4\pi}c_{\mu}(x,y)c_{\mu}(-x,2L-y)\right].$$

(86)

The equation of motion for $c_{\mu}$ serves as a constraint for zero field strength, which implies that we can parameterize $c_{\mu}$ as

$$c_{\mu}(x,y,t) = \frac{2\pi}{L_{\mu}}X_{\mu}(t) + \tilde{c}_{\mu}(x,y,t).$$

(87)

The large gauge transformations take $X_{\mu} \rightarrow X_{\mu} + \text{integer}$. The topological degeneracy is given by the degeneracy of this
FIG. 6. A single pair of $Z_2$ vortices on a sphere. This sequence of diffeomorphisms illustrates that this situation is equivalent to $M$ being a hemisphere, but with a different set of boundary conditions on $A_{\mu}$.

The zero-mode sector. The action of the zero-mode sector is found upon substituting Eq. (88) into action (87),

$$L = 2\pi X_2 X_1.$$  \hfill (89)

Now we must make sure that we project onto the $Z_2$ invariant sector. The $Z_2$ exchanges $a$ and $\bar{a}$, so it takes $c(x, y) \rightarrow -c(x, y + L)$ if $y \leq L$ and $c(x, y) \rightarrow -c(x, y - L)$ if $y \geq L$. Thus, the action of the $Z_2$ is to take the zero modes to minus them: $X_i \rightarrow -X_i$. The states can be labeled by $|n\rangle$, where $n$ is an integer and with the identifications $|n\rangle = |n + l\rangle$. Thus, before the projection, there are $|l|/2 + 1$ $Z_2$ invariant states. If $l$ is even, then there are two fixed points of the $Z_2$ action, so in all there are $|l|/2 + 1$ $Z_2$ invariant states.

Consider now the case of a single pair of $Z_2$ vortices on a sphere and the diffeomorphism $f$ shown in Fig. 6. Clearly, the situation with a single pair of $Z_2$ vortices is equivalent to having the gauge field $A_{\mu}$ defined on a hemisphere, but with modified boundary conditions on $A_{\mu}$. Let the angular coordinates $(\theta, \varphi)$ be defined so that the locations of the two $Z_2$ vortices are $(\pi/2, 0)$ and $(\pi/2, \pi)$ for the left and right vortices, respectively. The south pole is at $\theta = \pi$. As in the previous case with two $Z_2$ vortices, the boundary conditions on $A_{\mu}$ at $\theta = \pi/2$ are as follows:

$$A_{\mu}(\pi/2, \varphi) = -\alpha_1 (\pi/2, -\varphi) \alpha_1.$$  \hfill (90)

As a result, we can define a new continuous gauge field $c_{\mu}$ on a sphere as follows:

$$c_{\mu}(\theta, \varphi) = \begin{cases} a_{\mu}(\theta, \varphi), & \pi/2 \leq \theta \leq \pi \\ -\bar{a}_{\mu}(\varphi - \theta, -\varphi), & 0 \leq \theta \leq \pi/2. \end{cases}$$  \hfill (91)

It is easy to see that in this case, there is no possibility for large gauge transformations or holonomies around noncontractible loops. The Lagrangian will be given by an expression similar to Eq. (87), but this time the degeneracy will be 1.

We now tackle the case for general $n$. Suppose that there are $n$ pairs of $Z_2$ vortices on a sphere. We will define the new gauge field $c_{\mu}$ on a genus $g = (n - 1)$ surface in the following way. From Fig. 7, we can clearly see that the situation with four pairs of vortices is equivalent to having a gauge field $A_{\mu}$ defined on the surface shown in the lower left of Figs. 7(a) and 7(b) with modified boundary conditions. The generalization from four to $n$ is obvious. Consider the space shown in Fig. 7(c), which contains two copies of the original space. Parametrize this doubled space with the coordinates $i = (x, y)$. We will refer to the copy on the left side, which has $x = 0$, as $M_1$; the copy on the right side, which has $x = 0$, will be referred to as $M_2$. Suppose that the length in the $x$ direction of each copy is $L_x$, so that the total horizontal length of the doubled space is $2L_x$.

Consider a map $R$ defined on this doubled space with the following properties: $R$ takes $M_1$ to $M_2$ and $M_2$ into $M_1$ in such a way that $R\circ R = 1$, it has unit Jacobian, and it maps the boundaries of $M_1$ and $M_2$ into each other. The way it maps $\partial M_1$ and $\partial M_2$ into each other is illustrated in Fig. 7(d); if we...
identify $\partial M_1$ and $\partial M_2$ using the map $R$, then we obtain a surface of genus $g=n-1$, which we call $M$. In the coordinates illustrated in Fig. 7(c), this way of mapping $\partial M_1$ and $\partial M_2$ results in the following boundary conditions on $A_\mu$:

$$A_\mu(x,y) = -\sigma_l A_\mu[R^1(x,y) - L_\mu, R^2(x,y)]\sigma_l$$  

(92)

for $(x,y) \in \partial M_1$ and where $R^I(x,y)$ is the $i$th coordinate of $R$ [note that $A_\mu(x,y)$ is only defined for $-L_\mu = x \leq 0$]. This allows us to define a continuous gauge field $c_\mu$, defined on the doubled space $\tilde{M}$, in the following way:

$$c_\mu(x,y) = \begin{cases} 
  a_\mu(x,y), & x \leq 0 \\
  -\tilde{a}_\mu[R(x,y)], & x \geq 0. 
\end{cases}$$

(93)

We now rewrite the various terms in the action in terms of $c_\mu$.

$$\int_{M_1} d^2x \tilde{a}_i \tilde{a}_j = \int_{M_2} d^2x \tilde{a}_i \tilde{a}_j [R(x,y)] \tilde{a}_j [R(x,y)] = \int_{M_2} d^2xc_\mu \tilde{c}_j.$$  

(94)

The cross terms give a nonlocal term in the action,

$$\int_{M_1} d^2x a_\mu a_\nu = -\int_{M_1} d^2xc_\mu(x,y)c_\nu[R(x,y)].$$

(95)

Thus the Lagrangian is

$$L = e^{i\frac{k}{4\pi} c_\mu \tilde{c}_\nu - \frac{k-l}{4\pi} c_\mu(x,y)c_\nu[R(x,y)]].}$$  

(96)

As usual in pure Chern-Simons theory, the equation of motion for $c_\mu$ implies that the gauge field must be flat. It is therefore characterized by the value of $\int e^{i\phi} d\mathbf{l}$ along its noncontractible loops. To parametrize the gauge field, as is typical we introduce a canonical homology basis $\alpha_i$ and $\beta_j$ such that $\alpha_i$ ($\beta_j$) do not intersect while $\alpha_i$ and $\beta_j$ intersect if $i=j$. Then we introduce the dual basis $\omega_i$ and $\eta_j$, which satisfy

$$\int_{\alpha_i} \omega_j = \delta_{ij}, \quad \int_{\beta_j} \omega_j = 0,$$

$$\int_{\alpha_i} \eta_j = 0, \quad \int_{\beta_j} \eta_j = \delta_{ij}.$$  

(97)

Since $c$ must be a flat connection, we can parametrize it as

$$c = c_1 dx^1 + c_2 dx^2 = 2\pi i \xi_1 \omega_1 + i \eta_1.$$  

(98)

Two connections $c$ and $c'$ are gauge equivalent if

$$x'^i - x^i = \text{integer}, \quad y'^i - y^i = \text{integer}.$$  

(99)

Notice that here the action of $R$ is trivial on the canonical homology cycles. This is because of the way the genus $n-1$ surface was glued together from its pieces [see Fig. 7(d)]. This is in contrast to Eq. (41), which we obtained when we were analyzing the ground-state degeneracy on higher genus surfaces. Therefore, the action in terms of the $x^i$ and $y^i$ becomes simply

$$L = 2\pi i x^i y^i$$  

(100)

for $i=1, \ldots, n-1$. However, the $Z_2$ action here is not exactly the same as the action of the sheet exchange map $R$. This is because the $Z_2$ exchanges $a$ and $\tilde{a}$, so it takes $c(x,y) \rightarrow -c[R(x,y)]$. Thus, the action of the $Z_2$ is to change the sign of the $x^i$ and $y^i$, $x^i \rightarrow x^i$, $y^i \rightarrow -y^i$ for every $i$, under the action of the $Z_2$. Before projection, it is clear that we have $|l|^{n-1}$ states. These can be labeled in the following way:

$$\Phi_{m_1}^{\pm \frac{n-1}{n}}(m_1),$$  

(101)

where $m_1$ is an integer and $m_i \sim m_i + l$. The $Z_2$ action takes $m_i \rightarrow -m_i$. So if $l$ is odd, there is one state that is already $Z_2$ invariant: the state with $m_1 = 0$ for all $i$. There are $|l|^{n-1}$ remaining states, and exactly half of them are $Z_2$ invariant. Thus if $l$ is odd, the degeneracy of $(Z_2)$-invariant) states in the presence of $n$ pairs of $Z_2$ vortices on a sphere is $(|l|^{n-1} + 1)/2$. For $l=2$, $|m_1| = 0$ or $1$, which are both $Z_2$ invariant, so for $l=2$ the degeneracy in the presence of $n$ pairs of $Z_2$ vortices on a sphere is $2^{n-1}$. One may ask also about the number of states that are not $Z_2$ invariant. These may correspond to a different set of quasiparticle states that carry $Z_2$ charge. We see that there are $(3^{n-1} - 1)/2$ $Z_2$ noninvariant states for $l=3$ if there are $n$ pairs of $Z_2$ vortices on a sphere.

### B. Comparison to quasiparticles in $Z_2$ parafermion and Pfaffian FQH states

Let us now compare the results from the previous section to the quasiparticles in the Pfaffian and $Z_2$ parafermion FQH states. The topological properties of the quasiparticles in FQH states can be computed through the pattern of zeros approach or through their connection to conformal field theory. In the Pfaffian quantum Hall state, there are two main types of quasiparticles, corresponding to two different representations of a magnetic translation algebra.

These two classes of quasiparticles are commonly labeled in the following way:

$$\psi \sigma^{Q(1/\sqrt{7})}\psi, \quad \sigma^{Q(1/\sqrt{7})}\psi,$$

(102)

where $\psi$ is the Majorana fermion and $\sigma$ is the spin field of the Ising CFT. $Q$ is the charge of the quasiparticle and $\nu$ is the filling fraction of the quantum Hall state. The ones made of $\psi$ are Abelian; there are $2q$ of them when the filling fraction is $\nu = 1/q$. The ones made of $\sigma$ are non-Abelian; there are $q$ of them and their quantum dimension is $\sqrt{2}$. In the presence of $n$ pairs of the $\sigma$ quasiparticles, the Pfaffian state has a degeneracy of $2^{n-1}$ on a sphere. This follows from the fusion rules of the conformal primary fields in the Ising CFT:

$$\psi \psi = 1,$$

$$\sigma \sigma = 1 + \psi,$$

$$\psi \sigma = \sigma.$$  

(103)
TABLE I. Some values of the $Z_2$ vortex degeneracy for $l=3$ for the $Z_2$ invariant states, given by $(3^{n-1}+1)/2$, and for the $Z_2$ noninvariant states given by $(3^{n-1}-1)/2$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>No. $Z_2$ vortex pairs</th>
<th>No. $Z_2$ inv. states</th>
<th>No. $Z_2$ noninv. states</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
<td>14</td>
<td>13</td>
</tr>
<tr>
<td>5</td>
<td>41</td>
<td>41</td>
<td>40</td>
</tr>
<tr>
<td>6</td>
<td>122</td>
<td>122</td>
<td>121</td>
</tr>
</tbody>
</table>

The parafermionic primary fields in the $Z_4$ parafermion CFT have the fusion rules,

\[ \Phi_m^0 \times \Phi_{m'}^0 = \Phi_{m+m'}^0, \]
\[ \Phi_1^0 \times \Phi_1^0 = \Phi_2^0 + \Phi_0^0, \]
\[ \Phi_2^0 \times \Phi_2^0 = \Phi_0^0 + \Phi_0^0 + \Phi_0^0, \]
\[ \Phi_1^0 \times \Phi_2^0 = \Phi_1^0 + \Phi_1^0. \]

The fusion rules imply

\[ \Phi_1^0 \Phi_1^0 = \Phi_2^0 + \Phi_0^0, \]
\[ (\Phi_1^0 \Phi_1^0)^2 = \Phi_0^0 + 2 \Phi_0^0 + 3 \Phi_0^0, \]
\[ (\Phi_1^0 \Phi_1^0)^3 = 9 \Phi_0^0 + 4 \Phi_0^0 + 5 \Phi_0^0, \]
\[ (\Phi_1^0 \Phi_1^0)^4 = 27 \Phi_0^0 + 13 \Phi_0^0 + 14 \Phi_0^0, \]
\[ (\Phi_1^0 \Phi_1^0)^5 = 81 \Phi_0^0 + 40 \Phi_0^0 + 41 \Phi_0^0. \]

\[ (\Phi_1^0 \Phi_1^0)^6 = 243 \Phi_0^0 + 122 \Phi_0^0 + 121 \Phi_0^0. \]

There appears to be a connection between the $\Phi_1^0$ quasiparticles and the $Z_2$ vortices. First, notice that one member of a pair of $Z_2$ vortices should be conjugate to the other member. This is because a pair of $Z_2$ vortices can be created out of the vacuum on a sphere. Suppose that we identify one member of a pair with the operator $V_{a} = \Phi_1^0 e^{iQ/(1/2)}$ and the other member with its conjugate $V_{a} = \Phi_1^0 e^{-iQ/(1/2)}$. From Eq. (106), we see that the number of ways to fuse to the identity for $(V_a V_{a})^n = \Phi_0^0 (\Phi_1^0)^n$ is as displayed in Table II. Notice that this agrees exactly with the number of $Z_2$ invariant states for $n Z_2$ vortices on a sphere (see Table I).

Notice that the number of ways for $(V_a V_{a})^n$ to fuse to the quasiparticle $\Phi_0^0$ is exactly equal to the number of $Z_2$ noninvariant states that we obtain from $n$ pairs of $Z_2$ vortices (see Table II and I). This shows that the $Z_2$ noninvariant states have a meaning as well. These states carry nontrivial $Z_2$ charge, so we interpret this as a situation in which there are $n$ pairs of $Z_2$ vortices and an extra $Z_2$ charged quasiparticle. The above fusion indicates that we should associate this $Z_2$ charged quasiparticle to the operator $\Phi_1^0$.

Based on this quantitative agreement between the properties of the $Z_2$ vortices and results from the $Z_4$ parafermion FQH state, we conclude that for a pair of $Z_2$ vortices, one of them should be associated with an operator of the form $\Phi_1^0 e^{iQ/(1/2)}$ and the one to which it is connected by a branch cut should be associated with $\Phi_1^0 e^{-iQ/(1/2)}$. Furthermore, the possibility of $Z_2$ noninvariant states should be interpreted as the possibility for the $Z_2$ vortices to fuse to an electromagnetically neutral $Z_2$ charged quasiparticle, which we associate with the operator $\Phi_1^0$.

We have not seen how to understand the quantization of electromagnetic charge, $Q$, for the $Z_2$ vortices. The external electromagnetic field couples to the field $a' = a + \tilde{a}$, so we expect electromagnetically charged quasiparticles to carry flux of the $a'$ field. The quantization of charge for the quasiparticles generally arises from the constraint that quasiparticles are mutually local with respect to electrons. We should be able to see how the $Z_2$ vortices must carry certain quantized units of $a'$ flux, but we have not performed this analysis.

VI. CONCLUSION

In this paper, we have computed several topological properties of $U(1) \times U(1) \times Z_2$ Chern-Simons theory and discussed its relation to the Pfaffian and $Z_2$ parafermion FQH states. For the $l=3 U(1) \times U(1) \times Z_2$ Chern-Simons theory, many topological properties agree with those of the $Z_4$ parafermion state, which strongly suggests that the Chern-Simons theory correctly describes all of the topological properties of this state. This identification also suggests that the phase transition between the $(k,k,k-3)$ bilayer state and the $Z_4$ parafermion FQH state can be continuous and may, for instance, be described by a $Z_2$ transition in 2 + 1 dimensions.
In the simplest case, for \( k=3 \), this would be a continuous \( Z_2 \) transition at \( \nu=2/3 \) between the \((3,3,0)\) state and the non-Abelian \( Z_4 \) parafermion state. We leave a study of the phase transition itself for future work.

More generally, the methods in this paper may be extended to compute topological properties of Chern-Simons theories with disconnected gauge groups of the form \( G \times H \), where \( G \) is a connected Lie group and \( H \) is a discrete automorphism group of \( G \). There may be other situations also in which an \( n \)-layer FQH state passes through a phase transition to an \( m \)-layer FQH state, where the Chern-Simons gauge theories for each of the phases will be \( G \times H_a \) and \( G \times H_m \), respectively, and the phase transition will be described by a discrete gauge symmetry breaking of \( H_a \) to \( H_m \). We expect that such a scenario may be possible if the central charges of the corresponding edge theories are the same for the two phases. In this paper, for example, we found that even though there is a phase transition between the \((k,k,k-2)\) bilayer states and the Pfaffian states as the interlayer tunneling is increased, the \( l=2 \) \( U(1) \times U(1) \times Z_2 \) theory does not describe the Pfaffian state. In contrast, there is a possible phase transition between the \((k,k,k-3)\) bilayer states and the \( Z_2 \) parafermion states, and in this case the \( l=3 \) \( U(1) \times U(1) \times Z_2 \) theory does correctly describe the \( Z_4 \) parafermion state. One way to understand why simply gauging a \( Z_2 \) symmetry does not describe the Pfaffian state is that the central charges of the edge theory changes as the interlayer tunneling is tuned through a phase transition from the bilayer \((k,k,k-2)\) phase to the Pfaffian state, which indicates there is additional physics taking place that this approach does not capture here. The parent bilayer Abelian phase has \( c=2 \), as does the edge theory of the \( Z_2 \) parafermion state, while the edge theory of the Pfaffian state has \( c=3/2 \).

**ACKNOWLEDGMENTS**

We would like to thank Brian Swingle for helpful discussions. This research is supported by NSF Grant No. DMR-0706078.

**APPENDIX A: \( Z_4 \) PARAFLMION FQH STATES AND PROJECTIVE CONSTRUCTION**

Here we explain, from the point of view of a procedure called projective construction, \(^12\) how to understand that the \((k,k,k-3)\) bilayer wave function, upon symmetrization, yields the \( Z_2 \) parafermion wave function at \( \nu=2/3 \), and a different explanation for why we expect that the corresponding Chern-Simons theory should have the gauge group \( U(1) \times U(1) \times Z_2 \).

In the projective construction approach, one writes the electron operator (which is either bosonic or fermionic, depending on whether we are interested in FQH states of bosons or fermions) in terms of several other fermionic fields, \( \psi_1, \ldots, \psi_n \), referred to as “partons,”

\[
\Psi_\psi = \psi_{a_1} \cdots \psi_{a_n} C_{a_1 \cdots a_n},
\]  

(A1)

where \( C_{a_1 \cdots a_n} \) are constant coefficients. The continuum field theory that describes interacting electrons in an external magnetic field can then be rewritten in terms of the partons and a gauge field. The introduction of the partons expands the Hilbert space, so the gauge field is included in order to project the states onto the physical Hilbert space, which is generated by the electron operator. If the partons form a state \( |\Phi_{\text{parton}}\rangle \), the electron wave function is the projection onto the physical electronic Hilbert space,

\[
\Phi_\psi(z_1, \ldots, z_N) = \langle 0 | \prod_i \Psi_\psi(z_i) |\Phi_{\text{parton}}\rangle.
\]  

(A2)

If \( G \) is the group of transformations on the partons that keeps the electron operator invariant, then the continuum field theory description will be partons interacting with a gauge field with gauge group \( G \), which ensures that physical excitations, which are created by electron operators, will be singlets of the group \( G \). Since the partons are assumed to form a gapped integer quantum Hall state, they can be integrated out to obtain a Chern-Simons theory with gauge group \( G \).

For example, if we choose the electron operator to be

\[
\Psi_{e,1/3} = \psi_1 \psi_2 \psi_3,
\]  

(A3)

then \( \Psi_{e,1/3} \) is a \( SU(3) \) singlet. If we assume that the partons each form a \( \nu=1 \) integer quantum Hall state, then the electron wave function is

\[
\Phi_{e,1/3}(z_1, \ldots, z_N) = \prod_{i<j} (z_i - z_j)^3,
\]  

(A4)

which is the Laughlin \( \nu=1/3 \) wave function. The continuum field theory is a theory of three fermions coupled to an \( SU(3) \) gauge field. Integrating out the partons will yield a \( SU(3)_1 \) Chern-Simons theory. This theory is equivalent to the \( U(1)_3 \) Chern-Simons theory, which is the topological field theory for the \( \nu=1/3 \) Laughlin state.

If we choose the electron operator to be

\[
\Psi_{e,\psi} = \psi_1 \psi_2 + \psi_3 \psi_4
\]  

(A5)

and assume the partons form a \( \nu=1 \) IQH state, we can obtain the wave function after projection by using the following observation. The \( \nu=1 \) wave functions are equal to free chiral fermion correlators of a \( 1+1 \)-dimensional CFT,

\[
\Phi_{\nu=1} = \langle 0 | \prod_i \psi(z_i) | \nu=1 \rangle = \prod_{i<j} (z_i - z_j)
\]  

\[
\sim e^{-N \phi(z_0)} \prod_{i=1}^N \psi(z_i),
\]  

(A6)

where in the first line, \( \psi(z_i) \) is a free fermion operator that annihilates a fermion at position \( z_i \) and \( | \nu=1 \rangle \) is the \( \nu=1 \) integer quantum Hall state for the fermion \( \psi_0 \); in the second line, \( \psi(z_i) \) is interpreted as a free chiral fermion operator in a \( 1+1 \)-dimensional CFT and \( \frac{1}{\pi} \partial_\phi \partial_\psi = \psi^\dagger \psi \) is the density of the fermions. From this, it follows that wave function (A2) with the electron operator \( \Psi_{e,\psi} \) can be obtained by taking the correlator
where $\Psi_{\text{c,p}} = \psi_1 \psi_2 + \psi_3 \psi_4$ and $\psi_i(z)$ now interpreted as a free complex chiral fermion in a 1+1-dimensional CFT. The operator product algebra generated by the electron operator in this case can be checked to be reproduced if we instead write the electron operator as

$$\Psi_{\text{e,pf}} = \psi \eta,$$

where $\eta$ is a Majorana fermion and $\psi$ is a free complex chiral fermion. The correlation function with $N$ insertions of this operator is known to yield the $\nu=1$ Pfaffian wave function. The gauge group that keeps $\Psi_{\text{c,pf}}$ invariant is $SO(5)$, and thus the Chern-Simons theory for $\nu=1$ is a $SO(5)_1$ Chern-Simons theory.

Now consider a bilayer wave function, where we have two electron operators, one for each layer, and the wave function is given by

$$\Phi(z_1, z_2) = S(\Phi_{\text{c,1}}(z_1), \Phi_{\text{c,2}}(z_2))$$

$$\sim \left( e^{-iN\Phi(z_0)} \prod_{i=1}^{2N} \Psi_{\text{c,1}}(z_i) \Psi_{\text{c,2}}(z_i) \right),$$

where we have set $z_{2i} = w_i$.

In the case of the Pfaffian, this shows us that the $(2, 2, 0)$ state, when symmetrized, yields the Pfaffian wave function. If we instead consider $\Psi_{\text{c,1}} = \psi_1 \psi_2 \psi_3 \psi_4$ and $\Psi_{\text{c,2}} = \psi_1 \psi_2 \psi_3 \psi_4$, we obtain the $(3, 3, 0)$ state. The $(3, 3, 0)$ state, when symmetrized, will therefore be given by

$$\Phi(z_1, z_2) \sim \left( e^{-iN\Phi(z_0)} \prod_{i=1}^{2N} \Psi_{\text{c,1}}(z_i) \right),$$

with $\Psi_{\text{c,1}} = \psi_1 \psi_2 \psi_3 \psi_4$. It can be checked that the operator product algebra generated by this electron operator is also generated by the operator $\Psi_{\text{e,1}} = \Psi_{\text{e,1}}^{(0)} e^{i\theta \phi}$, where $\Phi_{\text{c,1}}^{(0)}$ is a simple-current operator in the $Z_4$ parafermion CFT and $\phi$ is a scalar boson. Thus, this wave function is the wave function of the $Z_4$ parafermion FQH state at $\nu=2/3$. Furthermore, the gauge group that keeps the electron operator invariant is $SU(3) \times SU(3) \times Z_2$, so we expect that the corresponding Chern-Simons theory for this phase should be $SU(3)_1 \times SU(3)_2 \times Z_2$ Chern-Simons theory, which we expect to be equivalent to $U(1)_1 \times U(1)_2 \times Z_2$ Chern-Simons theory. One would then guess that the $k$-construction degeneracy of the $(k,k,k-3)$ states and the $\nu = \frac{2k}{3}$ $Z_4$ parafermion states is the $U(1) \times U(1) \times Z_2$ Chern-Simons theory described in this paper.

### Appendix B: More Detailed Discussion of the Ground-State Degeneracy

Here we like to discuss the ground-state degeneracy of the $U(1) \times U(1) \times Z_2$ Chern-Simons theory in more detail. For $l=2$, the filling fraction is $\nu = \frac{1}{k-1}$ and formula (32) gives $9(k-1)$ states on a torus. Compare this to the torus degeneracy of the $\nu = \frac{1}{k-1}$ Pfaffian state, which is $3(k-1)$. We see that the $U(1) \times U(1) \times Z_2$ Chern-Simons theory for $l=2$ has a three ground-state degeneracy that is three times that of the Pfaffian state. The origin of this factor of 3 can be thought of in the following way. It is known that $O(2)_2$ Chern-Simons theory has $l+7$ ground states and the Chern-Simons theory for this phase should be $SU(2)_1 \times O(2)_2$ of the $Z_2$ Chern-Simons theory. One $O(2)_2$ Chern-Simons theory at level 2 is known to correspond to the $Z_2$ rational orbifold conformal field theory at level 2l, where for $l=2$, is known to be dual to two copies of the Ising CFT. The Ising CFT has three primary fields, and the CFT corresponding to the Pfaffian is one that contains an Ising CFT and a $U(1)$ CFT. In this sense our theory has an extra copy of the Ising model, which accounts for the extra factor of 3 in the torus degeneracy. We can see this another way by noticing that the central charge of the Ising CFT is $1/2$ and the central charge of the CFT that corresponds to the Pfaffian state is $c=3/2$. Meanwhile, the CFT corresponding to the $U(1) \times U(1) \times Z_2$ Chern-Simons theory has $c=2$, which corroborates the fact that it has an extra copy of the Ising model.

For $l=3$, the filling fraction is $\nu = \frac{2}{k-3}$ and Eq. (32) gives $5(k-3)$ ground states on a torus. Compare this to the $\nu = \frac{2}{k-3}$ $Z_4$ parafermion state, which also has a torus degeneracy of $5(k-3)$. This might be expected from the fact that $O(2)_2$ Chern-Simons theory corresponds to the $Z_4$ parafermion CFT when $l=3$. However, there is a crucial issue that needs to be addressed here. In the case $l=2$, we could see that $U(1)_{k=1} \times O(2)_2$ Chern-Simons theory gives the same number of ground states on a torus as the $U(1) \times U(1) \times Z_2$ theory did, implying that we could perhaps think of the $U(1)$ sector of the theory as separate from the $O(2)$ sector. This fails in the $l=3$ case. We would be tempted to write $U(1)_{k=3/2} \times O(2)_2$ because $(k-3/2) \times (3+7)$ gives the right ground-state degeneracy. This fails because the ground state degeneracy of $U(1)_{k=3/2}$ Chern-Simons theory is not $(k-3/2) \times (3+7)$ but the right ground-state degeneracy. This fails because the ground state degeneracy of $U(1)_{k=3/2}$ Chern-Simons theory is not $(k-3/2) \times (3+7)$ and the quantum states do not transform as a one-dimensional representation under large gauge transformations. One may wish to reject a theory in which the quantum states are not gauge invariant, in which case $U(1)_q$ is not defined for noninteger $q$. On the other hand, if these situations are allowed, then it can be shown that $U(1)_q$ Chern-Simons theory, for $q=p/p'$ (where $p$ and $p'$ are coprime), has a torus degeneracy of $pp'$. Therefore, $U(1)_{k=3/2}$ Chern-
Simons theory, to the extent that it is well defined, has degeneracy $2(2k-3)$. In either case, it is clear that the $U(1)$ and $O(2)$ sectors cannot be disentangled and that the correct definition of the theory is the $U(1) \times U(1) \times Z_2$ Chern-Simons theory presented here. To summarize, for $l=2$, $U(1) \times U(1) \times Z_2$ Chern-Simons theory describes the Pfaffian state but with an extra copy of the Ising model, while for $l=3$, the $U(1) \times U(1) \times Z_2$ theory gives the same ground-state degeneracy as the $Z_4$ parafermion quantum Hall state.

**APPENDIX C: O(2) CHERN-SIMONS THEORY AND Z₂ RATIONAL ORBIFOLD CONFORMAL FIELD THEORIES**

Here we summarize previously known results from $O(2)$ Chern-Simons theory and the $Z_2$ orbifold CFT and apply the $Z_2$ vortex analysis of this paper to the $O(2)$ Chern-Simons theory. Moore and Seiberg\(^2\) first discussed Chern-Simons theories with disconnected gauge groups of the form $G \times P$, where $G$ is a connected group with a discrete automorphism group $P$, and the connection of these Chern-Simons theories to $G/P$ orbifold conformal field theories. As a special example, they discussed the case where $G=U(1)$ and $P=Z_2$. In the two-dimensional conformal field theory, this is known as the $Z_2$ orbifold and it was explicitly analyzed in Ref. 20. It is the theory of a scalar boson $\varphi$ compactified at a radius $R$ so that $\varphi \sim \varphi + 2\pi R$, and with an additional $Z_2$ gauge symmetry: $\varphi \sim -\varphi$.

### 1. Z₂ rational orbifold CFT

When $1/2 R^2$ is rational, i.e., $1/2 R^2 = p/p'$, with $p$ and $p'$ coprime, then it is useful to consider an algebra generated by the fields $j = i \varphi$, and $e^{\pm i \sqrt{2N} \varphi}$ for $N=pp'$. This algebra is referred to as an extended chiral algebra. The infinite number of Virasoro primary fields in the $U(1)$ CFT can now be organized into a finite number of representations of this extended algebra $A$. There are $2N$ of these representations, and the primary fields are written as $V_i = e^{i\varphi/\sqrt{2N}}$, with $l = 0, 1, \ldots, 2N-1$.

In the $Z_2$ orbifold, one now considers representations of the smaller algebra $A/Z_2$. This includes the $Z_2$ invariant combinations of the original primary fields, which are of the form $\cos(l\varphi/\sqrt{2N})$; there are $N+1$ of these. In addition, there are six new operators. The gauging of the $Z_2$ allows for twist operators that are not local with respect to the fields in the algebra $A/Z_2$, but rather local up to an element of $Z_2$. It turns out that there are two of these twisted sectors, and each sector contains one field that lies in the trivial representation of the $Z_2$, and one field that lies in the nontrivial representation of $Z_2$. These twist fields are labeled $\sigma_1$, $\tau_1$, $\sigma_2$, and $\tau_2$. In addition to these, an in-depth analysis\(^2\) shows that the fixed points of the $Z_2$ action in the original $U(1)$ theory split into a $Z_2$ invariant and a noninvariant field. We have already counted the invariant ones in our $N+1$ invariant fields, which leaves two new fields. In total, there are $N+7$ fields in the $Z_2$ rational orbifold at “level” $2N$.

The dimension of the space of conformal blocks on a genus $g$ surface is given by the following formula:\(^2\)

### TABLE III. Primary fields in the $Z_2$ orbifold for $N=2$, their scaling dimensions, and the fields from Ising$^2$ to which they correspond.

<table>
<thead>
<tr>
<th>$Z_2$ orb. field</th>
<th>Scaling dimension, $h$</th>
<th>Ising$^2$ fields</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$0$</td>
<td>$1 \otimes 1$</td>
</tr>
<tr>
<td>$j$</td>
<td>$1$</td>
<td>$\varphi \otimes \varphi$</td>
</tr>
<tr>
<td>$\phi_{ij}$</td>
<td>$1/2$</td>
<td>$1 \otimes \psi$</td>
</tr>
<tr>
<td>$\phi'_{ij}$</td>
<td>$1/2$</td>
<td>$\psi \otimes 1$</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>$1/8$</td>
<td>$\sigma \otimes \sigma$</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>$1/16$</td>
<td>$\sigma \otimes 1$</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>$1/16$</td>
<td>$1 \otimes \sigma$</td>
</tr>
<tr>
<td>$\tau_1$</td>
<td>$9/16$</td>
<td>$\sigma \otimes \varphi$</td>
</tr>
<tr>
<td>$\tau_2$</td>
<td>$9/16$</td>
<td>$\psi \otimes \sigma$</td>
</tr>
</tbody>
</table>

$$\dim V_g = \text{Tr} \left( \sum_{j=0}^{N-1} N_j^+ \right)^{g-1} = \sum_{n=0}^{N-1} S_{2(n-1)}^{2g-1}.$$  \hfill (C1)

The $S$ matrix was computed for the $Z_2$ orbifold in Ref. 20, so we can immediately calculate the above quantity in this case. The result is

$$\dim V_g = 2^{g-1} \left[ (2^g + (2^g - 1)N^{g-1} + 2N^g) \right].$$  \hfill (C2)

For $N=2$, it was observed that the $Z_2$ orbifold is equivalent to two copies of the Ising CFT. For $N=3$, it was observed that the $Z_2$ orbifold is equivalent to the $Z_4$ parafermion CFT of Zamolodchikov and Fateev.\(^2\)

In Tables III and IV we list the fields from the $Z_2$ orbifold for $N=2$ and $N=3$, their scaling dimensions, and the fields in the Ising$^2$ or $Z_4$ parafermion CFTs that they correspond to.

### 2. O(2) Chern-Simons theory on a torus

The claim of Moore and Seiberg was that $O(2)$ Chern-Simons theory at level $2N$ corresponds to the $Z_2$ rational orbifold CFT at level $2N$. The first step in showing this is to show that the degeneracy of this theory on a torus is $N+7$.

### TABLE IV. Primary fields in the $Z_2$ orbifold for $N=3$, their scaling dimensions, and the $Z_4$ parafermion fields that they correspond to.

<table>
<thead>
<tr>
<th>$Z_2$ orb. field</th>
<th>Scaling dimension, $h$</th>
<th>$Z_4$ parafermion field</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$0$</td>
<td>$\Phi_0^0$</td>
</tr>
<tr>
<td>$j$</td>
<td>$1$</td>
<td>$\Phi_0^0$</td>
</tr>
<tr>
<td>$\phi_{ij}$</td>
<td>$3/4$</td>
<td>$\Phi_0^0$</td>
</tr>
<tr>
<td>$\phi'_{ij}$</td>
<td>$3/4$</td>
<td>$\Phi_0^0$</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>$1/12$</td>
<td>$\Phi_0^1$</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>$1/3$</td>
<td>$\Phi_0^2$</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>$1/16$</td>
<td>$\Phi_1^0$</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>$1/16$</td>
<td>$\Phi_1^1$</td>
</tr>
<tr>
<td>$\tau_1$</td>
<td>$9/16$</td>
<td>$\Phi_1^1$</td>
</tr>
<tr>
<td>$\tau_2$</td>
<td>$9/16$</td>
<td>$\Phi_2^0$</td>
</tr>
</tbody>
</table>
This is done in the following way. The classical configuration space of pure Chern-Simons theory with gauge group $G$ consists of flat $G$ bundles on a torus. Flat $O(2)$ bundles can be split into two classes, those that can be considered to be $SO(2) = U(1)$ bundles, and those that cannot. In the first case, we simply need to take the space of states in $U(1)_{2N}$ Chern-Simons theory and keep the $Z_2$ invariant states. This leaves $N+1$ states.

In addition to these, there are flat, twisted bundles. Flat bundles are classified by $\text{hom}(\pi_1(M) \rightarrow G)/G$. This is the space of homomorphisms of the fundamental group of the manifold $M$ into the gauge group $G$, modulo $G$. Let us study the space of flat twisted $O(2)$ bundles. We first write the gauge field as

$$A_\mu = \begin{pmatrix} a_\mu & 0 \\ 0 & -a_\mu \end{pmatrix}. \quad (C3)$$

The group is composed of $U(1)$ elements, which we write in terms of the Pauli matrix $\sigma_3$: $e^{i\sigma_3}$. We write the $Z_2$ element as the Pauli matrix $\sigma_1$. The $Z_2$ action is therefore $A_\mu \rightarrow \sigma_1 A_\mu \sigma_1 = -A_\mu$. We can write a Lagrangian for this theory,

$$L = \frac{2N}{4\pi} \int_M d^2x a \partial_\mu a. \quad (C4)$$

We are concerned with the case where $M$ is the torus, $T^2$. $\pi_1(T^2)$ is generated by two elements, $a$ and $b$, the two non-contractible loops of the torus. We must study the homomorphism $h: \pi_1(T^2) \rightarrow G$. $\pi_1(T^2)$ is an Abelian group, and since $h$ is a homomorphism, we must have

$$h(aa + \beta b) = \alpha h(a) h(b) = \alpha \beta h(b) h(a). \quad (C5)$$

Suppose we are twisted in the $a$ direction only. Then, we have

$$h(a) = \sigma_1 e^{i\theta a}, \quad h(b) = e^{i\phi a}. \quad (C6)$$

Modding out by the group $O(2)$, we find that $\theta \sim -\theta + 2\pi m \sim \theta + 2\pi a$ for any $\alpha$. The first equivalence comes from modding out by the $Z_2$ element, while the second element comes from modding out by the $U(1)$ element. Similarly, $\phi \sim -\phi + 2\pi n$. $n$ and $m$ are integers. The constraint $h(a) h(b) = h(b) h(a)$ further implies that $\phi = 0$ or $\pi$. The distinct solutions to these relations are therefore that

$$(\theta, \phi) = (0, \pi) \text{ or } (0, 0). \quad (C7)$$

A similar analysis shows that the cases in which the bundle is twisted in the $b$ direction only or along both $a$ and $b$ also each admit only two distinct bundles. Therefore, there are a total of six distinct twisted flat $O(2)$ bundles. Each corresponds to a single quantum state for a total of $N+7$ states in the $O(2)$ Chern-Simons theory on a torus.

3. $Z_2$ vortices in $O(2)$ Chern-Simons theory

This section is essentially an application of the analysis of $Z_2$ vortices in the case of $G = U(1) \times U(1)/Z_2$ to the case $G = O(2)$. In this case, a $Z_2$ vortex takes the gauge field to minus itself. With $n$ pairs of $Z_2$ vortices, we again deform the manifold on which the gauge field $A_\mu$ is defined, consider a double copy, and glue the two copies together to obtain a genus $g = n - 1$ surface.

The analog of Eq. (93) in this case is

$$c_\mu(x,y) = \begin{cases} a_\mu(x,y), & x \leq 0 \\ a_\mu(y,x), & y \geq 0, \end{cases} \quad (C8)$$

and in terms of $c_\mu$ we immediately see that the action is that of a $U(1)$ Chern-Simons theory at level $N$,

$$L = \frac{2N}{4\pi} \int_M d^2x a \partial_\mu a. \quad (C9)$$

On a genus $g = n - 1$ surface, there are $|N|^n$ states. But we need to project onto the $Z_2$ invariant sector. The action of the $Z_2$ is to take $c \rightarrow -c$. We count $(|N|^n + 1)/2$ $Z_2$ invariant states when $N$ is odd. If $N = 2$, all of the $N^n$ states are $Z_2$ invariant.

How does this relate to the corresponding conformal field theory, which is the $Z_2$ rational orbifold at level $2N$? Let us examine a few cases. When $N = 1$, this theory is the same as $U(1)_8$ CFT, which is abelian and which therefore should have degeneracy 1 for all $n$.

When $N = 2$, the orbifold CFT is the same as two copies of the Ising CFT. The Ising CFT has a single non-Abelian field, $\sigma$. The space of conformal blocks corresponding to $2n$ $\sigma$ fields on a sphere in the Ising CFT is $2^{n-1}$, which agrees with our above analysis for $N = 2$. However, a theory with two copies of the Ising CFT would have many non-abelian fields:

$$\sigma \otimes \mathbb{I}, \quad \sigma \otimes \psi, \quad \mathbb{I} \otimes \sigma, \quad \psi \otimes \sigma, \quad \sigma \otimes \sigma. \quad (C10)$$

The space of conformal blocks corresponding to $2n$ of either $\sigma \otimes \mathbb{I}, \sigma \otimes \psi, \mathbb{I} \otimes \sigma, \psi \otimes \sigma$ will have dimension $2^{n-1}$. However, the dimension of the space of conformal blocks corresponding to $2n$ $\sigma \otimes \sigma$ fields will be different. Thus $Z_2$ vortices in the $O(2)$ Chern-Simons with $N = 2$ are closely related to the fields $\sigma \otimes \mathbb{I}, \sigma \otimes \psi, \mathbb{I} \otimes \sigma, \text{ and } \psi \otimes \sigma$.

When $N = 3$, the orbifold CFT is dual to the $Z_4$ parafermion CFT of Zamolodchikov and Fateev. We expect the $Z_2$ vortices to correspond to the $\Phi^4$ fields, and in fact we obtain the correct number of states in the presence of $n$ pairs of $Z_2$ vortices, as discussed earlier.