Characterization of Majorization Monotone Quantum Dynamics

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(v) the second smallest eigenvalue of $L(A)$, which is called the algebraic connectivity of $G(A)$ and denoted by $\lambda_2(L(A))$, is larger than zero;
(vi) the algebraic connectivity of $G(A)$ is equal to $\min_{\omega \in \mathbb{R}^n} \xi^T L(A) \xi / \xi^T \xi$, and thus, if $\xi^T \xi = 0$, then $\xi^T L(A) \xi \geq \lambda_2(L(A)) \xi^T \xi$.

**Corollary 1:** Suppose $G(A)$ is strongly connected and $\omega$ is positive column vector such that $w^T L(A) = 0$. Then $\text{diag}(\omega)L(A) + (L(A)^T \text{diag}(\omega))$ is the graph Laplacian of the undirected weighted graph $G(\text{diag}(\omega)A + A^T \text{diag}(\omega))$. And it is positive semi-definite, 0 is its algebraically simple eigenvalue and 1 is the associated eigenvector.

**Corollary 2:** For any nonnegative column vector $b$ with compatible dimensions, if $b \neq 0$ and $G(A)$ is undirected and connected, then $L(A) + \text{diag}(b)$ is positive definite.

**Lemma 3 ([14], Lemma 1):** If $\xi_1, \xi_2, \ldots, \xi_n \geq 0$ and $0 < p \leq 1$, then

$$\left( \sum_{i=1}^{n} \xi_i \right)^p \leq \sum_{i=1}^{n} \xi_i^p.$$

**Lemma 3 (cf. [6], Theorem 1):** Suppose that function $V(t) : [0, \infty) \rightarrow [0, \infty)$ is differentiable (the derivative of $V(t)$ at 0 is in fact its right derivative) and

$$\frac{dV(t)}{dt} \leq -K V(t)^\alpha,$$

where $K > 0$ and $0 < \alpha < 1$. Then $V(t)$ will reach zero at finite time $t^* \leq \sqrt{V(0)^{-1/\alpha} / (K(1-\alpha))}$ and $V(t) = 0$ for all $t \geq t^*$.

**REFERENCES**


**Characterization of Majorization**

**Monotone Quantum Dynamics**

Haidong Yuan

**Abstract**—In this technical note, the author studies the dynamics of open quantum system in Markovian environment. The author gives necessary and sufficient conditions for such dynamics to be majorization monotone, which are those dynamics always mixing the states.

**Index Terms**—Majorization, open quantum dynamics.

**I. INTRODUCTION**

In the last two decades, control theory has been applied to an increasingly wide number of problems in physics and chemistry whose dynamics are governed by the time-dependent Schrödinger equation (TDSE), including control of chemical reactions [1]–[8], state-to-state population transfer [9]–[13], shaped wavepackets [14], NMR spin dynamics [15]–[19], Bose-Einstein condensation [20]–[22], quantum computing [23]–[27], oriented rotational wavepackets [28]–[30], etc. More recently, there has been vigorous effort in studying the control of open quantum systems which are governed by Lindblad equations, where the central object is the density matrix, rather than the wave function [31]–[37]. The Lindblad equation is an extension of the TDSE that allows for the inclusion of dissipative processes. In this article, the author will study those dynamics governed by Lindblad equations and give necessary and sufficient conditions for the dynamics to be majorization monotone, which are those dynamics always mixing the states. This study suggests that majorization may serve as time arrow under these dynamics in analog to entropy in second law of thermal dynamics.

The article is organized as follows: Section II gives a brief introduction to majorization; Section III gives the definition of majorization monotone quantum dynamics; then in Section IV, necessary and sufficient conditions for majorization monotone quantum dynamics are given.

**II. BRIEF INTRODUCTION TO MAJORIZATION**

In this section, the author gives a brief introduction on majorization, most stuff in this section can be found in the second chapter of Bhattacharyya’s book [42]. For a vector $x = (x_1, \ldots, x_n)^T$ in $\mathbb{R}^n$, we denote by $x^\dagger = (x_1^\dagger, \ldots, x_n^\dagger)^T$ a permutation of $x$ so that $x_i^\dagger \geq x_j^\dagger$ if $i < j$, where $1 \leq i, j \leq n$.

**Definition 1 (Majorization):** A vector $x \in \mathbb{R}^n$ is majorized by a vector $y \in \mathbb{R}^n$ (denoted by $x \prec y$), if

$$\sum_{j=1}^{d} x_{j}^{\dagger} \leq \sum_{j=1}^{d} y_{j}^{\dagger}$$

for $d = 1, \ldots, n-1$, and the inequality holds with equality when $d = n$.

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Proposition 1: $x \prec y$ iff $x$ lies in the convex hull of all $P_y y$, where $P_y$ are permutation matrices.

Proposition 2: $x \prec y$ if and only if $x = Dy$ where $D$ is a doubly stochastic matrix.

Remark 1: A doubly stochastic matrix $D$ is a matrix with non-negative entries and every column and row sum to 1, i.e., $d_{i,j} \geq 0, \sum_i d_{i,j} = 1, \sum_j d_{i,j} = 1$.

Proposition 3: Suppose $f$ is a convex function on $\mathbb{R}$, and $x \prec y$ in $\mathbb{R}^n$, then
\[
\sum_{i}^n f(x_i) \leq \sum_{i}^n f(y_i).
\]

Proposition 4: For a vector $\lambda = (\lambda_1, \ldots, \lambda_n)^T$, denote $D_\lambda$ a diagonal matrix with $(\lambda_1, \ldots, \lambda_n)$ as its diagonal entries, let $a = (a_1, \ldots, a_n)^T$ be the diagonal entries of matrix $A = K^T D_\lambda K$, where $K \in SO(n)$. Then $a \prec \lambda$. Conversely for any vector $a \prec \lambda$, there exists a $K \in SO(n)$ such that $(a_1, \ldots, a_n)^T$ are the diagonal entries of $A = K^T D_\lambda K$.

Remark 2: $SO(n)$ is the group of special orthogonal matrices, $K \in SO(n)$ means $K^T K = I$ and $\det(K) = 1$.

### III. MAJORIZATION IN OPEN QUANTUM DYNAMICS

The state of an open quantum system of N-level can be represented by a $N \times N$ positive semi-definite, trace 1 matrix, called density matrix. Let $\rho$ denote the density matrix of a quantum system, its dynamics in markovian environment is governed by the Lindblad equation, which takes the form
\[
\dot{\rho} = -i[H, \rho] + L(\rho)
\] (2)
where $-i[H, \rho]$ is the unitary evolution of the quantum system and $L(\rho)$ is the dissipative part of the evolution. The term $L(\rho)$ is linear in $\rho$ and is given by the Lindblad form [38, 40]
\[
L(\rho) = \sum_{\alpha, \beta} a_{\alpha, \beta} \left( F_{\alpha} \rho F_{\beta}^\dagger - \frac{1}{2} \left( F_{\beta}^\dagger F_{\alpha} \rho + F_{\alpha} \rho F_{\beta}^\dagger \right) \right)
\]
where $F_{\alpha}, F_{\beta}$ are the Lindblad operators, which form a basis of a $N \times N$ trace 0 matrices (we have $N^2 - 1$ of them) and $[A, B] = AB - BA$. If we put the coefficient $a_{\alpha, \beta}$ into a $(N^2 - 1) \times (N^2 - 1)$ matrix $A = (a_{\alpha, \beta})$, it is known as the Gorini, Kossakowski, and Sudarshan (GKS) matrix [39], which needs to be positive semi-definite.

Equation (2) has the following three well-known properties: 1) $\Tr(\dot{\rho})$ remains unity for all time; 2) $\rho$ remains a Hermitian matrix; and 3) $\rho$ stays positive semi-definite, i.e., $\rho$ never develops nonpositive eigenvalues.

Definition 2: Suppose $\rho_1$ and $\rho_2$ are two states of a quantum system, we say $\rho_1$ is majorized by $\rho_2$ ($\rho_1 \prec \rho_2$) if the eigenvalues of $\rho_1$ are majorized by the eigenvalues of $\rho_2$ ($\lambda(\rho_1) \prec \lambda(\rho_2)$).

Basically majorization gives an order of mixed-ness of quantum states, i.e., if $\rho_1 \prec \rho_2$, then $\rho_1$ is more mixed than $\rho_2$, which can be seen from the following propositions.

Definition 3 (von Neumann Entropy): The von Neumann entropy of a density matrix is given by
\[
S(\rho) = -\Tr[\rho \log(\rho)].
\]

Proposition 5: If $\rho_1 \prec \rho_2$, then $S(\rho_1) \geq S(\rho_2)$.

Proposition 6: If $\rho_1 \prec \rho_2$, then $\Tr(\rho_1^2) \leq \Tr(\rho_2^2)$.

Remark 3: The above two propositions can be easily derived from Proposition 3.

The entropy and trace norm are usually used to quantify how mixed quantum states are. But majorization is a more strong condition than these two functions, and in some sense it gives a more proper order of mixedness as we can see from the following proposition.

Proposition 7 [44]: $\rho_1 \prec \rho_2$ if and only if $\rho_1$ can be obtained by mixing the unitary conjugations of $\rho_2$, i.e., $\rho_1 = \sum p_i U_i \rho_2 U_i^\dagger$, where $p_i > 0, \sum p_i = 1$ and $U_i$ are unitary operators.

### IV. NECESSARY AND SUFFICIENT CONDITION OF MAJORIZATION MONOTONE QUANTUM DYNAMICS

Definition 4 (Majorization Monotone Dynamics): An open quantum dynamics governed by (2) is majorization monotone if and only if $\rho(t_2) \prec \rho(t_1)$ when $t_2 > t_1, \forall t_1, t_2$.

Intuitively majorization monotone dynamics are those kind of dynamics which always mixing the states. As we can see from Proposition 5, these kind of dynamics always increase the entropy of the system. One can immediately see a necessary condition for a dynamics to be majorization monotone: the state $\rho = (1/N)I$ has to be a steady state of such dynamics, where $I$ is identity matrix. As $(1/N)I$ is the most mixed state, any state $\rho \prec (1/N)I$ would imply $\rho = (1/N)I$. The question now is whether this condition is also sufficient.

Let us first look at a simple system: a single spin in a Markovian environment.

### A. An Example on Single Spin

Take the general expression of the master equation
\[
\dot{\rho} = -i[H, \rho] + L(\rho)
\]
where
\[
L(\rho) = \sum_{\alpha, \beta} a_{\alpha, \beta} \left( F_{\alpha} \rho F_{\beta}^\dagger - \frac{1}{2} \left( F_{\beta}^\dagger F_{\alpha} \rho + F_{\alpha} \rho F_{\beta}^\dagger \right) \right).
\]

For the single spin, we can take the basis $\{F_\sigma\}$ as normalized Pauli spin operators $(1/\sqrt{2})\{\sigma_x, \sigma_y, \sigma_z\}$, where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The coefficient matrix
\[
A = \begin{pmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{pmatrix}
\]
is positive semi-definite.

If identity state is a steady state, the right-hand side of (3) should be 0 when $\rho = (1/N)I$. As $-i[H, (1/N)I] = 0$, so the condition reduces to $\dot{L}(I) = 0$ (since $L(\rho)$ is a linear map, we can ignore the constant $1/N$). This is equivalent to
\[
\sum_{\alpha, \beta} a_{\alpha, \beta} \left[ F_{\alpha}, F_{\beta}^\dagger \right] = 0.
\]

In the single-spin case, substitute $F$ by Pauli matrices and it is easy to see that the above condition reduces to
\[
a_{\alpha, \beta} = a_{\beta, \alpha}
\]
i.e., the GKS matrix should be real symmetric, positive semi-definite matrix [41], while the general GKS matrix is Hermitian, positive definite. We want to see whether the dynamics of single spin under this condition is majorization monotone.

As majorization monotone is defined by the eigenvalues of density matrices, we are going to focus on the dynamics of the eigenvalues of density matrix. Let $\Lambda$ be its associated diagonal form of eigenvalues of density matrix $\rho$, i.e., $\Lambda$ is a diagonal matrix with eigenvalues of $\rho$ as its diagonal entries. At each instant of time, we can diagonalize the density matrix $\rho(t) = U(t)\Lambda(t)U(t)^\dagger$ by a unitary matrix $U(t)$. 
Substitute \( \rho(t) = U(t)\Lambda(t)U^{-1}(t) \) into (2), we get
\[
\dot{\rho}(t) = (U(t)\Lambda(t)U^{-1}(t) + U(t)\Lambda(t)U^{-1}(t) + U(t)\Lambda(t)U^{-1}(t) + \rho(t)U(t)\Lambda(t)U^{-1}(t)
\]
\[
- \frac{i}{2} \left[ H(t), U(t)\Lambda(t)U^{-1}(t) \right] + L \left[ U(t)\Lambda(t)U^{-1}(t) \right]
\]
where \( H(t) \) is defined by \( U(t) = -iH'(t)U(t) \), which is Hermitian. We obtain
\[
\dot{\Lambda}(t) = U^{-1}(t) \left\{ -i \left[ H(t) + \frac{1}{2} \left( U(t)\Lambda(t)U^{-1}(t) \right) \right] + L \left[ U(t)\Lambda(t)U^{-1}(t) \right] \right\}
\]
where \( \dot{\Lambda}(t) = \text{diag} \left( U^{-1}(t) U \Lambda(t) U^{-1}(t) \right) \) (8)
where we use \( \text{diag}(M) \) to denote a diagonal matrix whose diagonal entries are the same as matrix \( M \)
\[
\dot{\Lambda}(t) = \text{diag} \left( U(t) \Lambda(t) U^{-1}(t) \right)
\]
where
\[
U^{-1} U \Lambda U^{-1} = \text{diag} \left( \sum_{\alpha} a_{\alpha} \right)
\]
where
\[
\dot{\Lambda}(t) = \text{diag} \left( \sum_{\alpha} a_{\alpha} \right)
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\[ \rho(t_2) = \Psi(\rho(t_1)) = \sum_i K_i \rho(t_1) K_i^\dagger \]  
(11)

where in our case \( \{ K_i \} \) are \( N \times N \) matrices, which depend on the dynamical (10) and the time difference between \( t_1 \) and \( t_2 \). Also the Kraus operator sum has to be trace preserving as the trace of density matrix is always 1, which implies that

\[ \sum_i K_i K_i^\dagger = I. \]

If we have additional condition that identity state is a steady state of this dynamics, which means if \( \rho(t_1) = (1/N)I \) then \( \rho(t_2) \) remains at \( (1/N)I \), substitute them into the Kraus operator sum representation, we will get an extra condition

\[ \sum_i K_i = I. \]

We will show these two conditions are enough to ensure the dynamics to be majorization monotone. First, let us diagonalize \( \rho(t_1) \) and \( \rho(t_2) \)

\[ \rho(t_1) = U_1 \Lambda(\rho(t_1)) U_1^\dagger \]
\[ \rho(t_2) = U_2 \Lambda(\rho(t_2)) U_2^\dagger \]

where \( \Lambda(\rho) \) are diagonal matrix with eigenvalues of \( \rho \) as its diagonal entries, substitute them into (11), we get

\[ U_2 \Lambda(\rho(t_2)) U_2^\dagger = \sum_i K_i U_1 \Lambda(\rho(t_1)) U_1^\dagger K_i^\dagger, \]
\[ \Lambda(\rho(t_2)) = \sum_i U_2^\dagger K_i U_1 \Lambda(\rho(t_1)) U_1^\dagger K_i U_2. \]

(12)

Let \( V_i = U_2^\dagger K_i U_1 \), then

\[ \Lambda(\rho(t_2)) = \sum_i V_i \Lambda(\rho(t_1)) V_i^\dagger \]

(13)

and it is easy to check that

\[ \sum_i V_i V_i^\dagger = U_2 \left( \sum_i K_i K_i^\dagger \right) U_2 = I \]
\[ \sum_i V_i^\dagger V_i = U_1 \left( \sum_i K_i K_i^\dagger \right) U_1 = I. \]

(14)

It is a linear map from the eigenvalues of \( \rho(t_1) \) to eigenvalues of \( \rho(t_2) \), so we can find a matrix \( D \), such that

\[ \lambda(\rho(t_2)) = D \lambda(\rho(t_1)) \]

(15)

where \( \lambda(\rho) \) is a vector in \( \mathbb{R}^N \) with eigenvalues of \( \rho \) as its entries, which is arranged in the same order as the diagonal entries of \( \Lambda(\rho) \). The matrix \( D \) can be computed from (13) as

\[ D_{\alpha\beta} = \sum_i |(V_i)_{\alpha\beta}|^2 \]

where \( D_{\alpha\beta} \) and \( (V_i)_{\alpha\beta} \) are the \( \alpha\beta \) entry of \( D \) and \( V_i \), respectively. It is straightforward to show that, by using the two conditions in (14)

\[ \sum_\alpha D_{\alpha\beta} = 1, \]
\[ \sum_\beta D_{\alpha\beta} = 1 \]

i.e., \( D \) is a doubly stochastic matrix. From Proposition 2, we get

\[ \lambda(\rho(t_2)) < \lambda(\rho(t_1)), \]

so

\[ \rho(t_2) < \rho(t_1), \quad \forall t_2 > t_1 \]

i.e. it is majorization monotone.

From Proposition 5 and 6, it is easy to see that majorization monotone implies entropy monotone and trace norm monotone, and they share the same necessary and sufficient condition: \( I(I) = 0 \).

V. CONCLUSION

Understanding open quantum systems is an important problem for a wide variety of physics, chemistry, and engineering applications. This technical note analyzed the dynamics of open quantum systems and gives necessary and sufficient condition on majorization monotone dynamics, which are those dynamics always mixing the states. This suggests that for this class of dynamics, majorization defines an evolution arrow, which begs for the connection to the entropy arrow in the second law of thermal dynamics. The author hopes further investigation will reveal more on this connection.

REFERENCES

Abstract—In this technical note, we fully characterize the dynamic response of a saturating static feedback-controlled single integrator driven by Gaussian white noise by solving the derived Falkner-Planck equation.

Index Terms—Actuator saturation, dynamic response, white noise.

I. INTRODUCTION

Our effort here is motivated by the need for understanding the external (disturbance) responses of nonlinear and in particular saturating feedback systems. Actuator saturation nonlinearities, as well as numerous other static nonlinearities, are ubiquitous in control systems, and so characterizing both the internal stability and the disturbance response of control systems with such nonlinearities is of fundamental importance in numerous application areas. While the internal stability of feedback systems with actuator saturation is very well understood, much remains to be done in characterizing the responses of saturating control systems to disturbances and noise, see [1] and [2] for background. With this motivation in mind, several works aim to find the steady-state probability distribution and/or statistics of some special nonlinear stochastic systems driven by noise (e.g., [3] and [5]), including of feedback-controlled saturating systems (e.g., [6]). However, to the best of our knowledge, there have been no works so far that explicitly characterize the the transient response of feedback-controlled dynamic systems with actuator saturation. With the aim of better understanding the transient response, we here fully characterize the joint probability density functions (pdfs) of a canonical feedback-controlled saturating dynamic system, namely a single integrator with linear static feedback.

Specifically, we characterize the dynamic solution of the stochastic differential equation

\[
\Sigma : \begin{cases} 
    x(t) = \sigma \left[ -Kx(t) \right] + \omega(t) \\
    x(h_0) = x_0
\end{cases}
\]

where \( t \in \mathbb{R}^+ \), \( x(t) \) is the state, the feedback gain \( K \) is a positive real number, \( \omega(t) \) is a Gaussian white noise process with zero mean and autocorrelation \( \mathbb{E}(\omega(t)\omega(t')) = \delta(t-t') \), the initial state \( x_0 \) is a random

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