Joint estimation and correction of geometric distortions for EPI functional MRI using harmonic retrieval

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Abstract—Magnetic resonance imaging (MRI) uses applied spatial variations in the magnetic field to encode spatial position. Therefore, nonuniformities in the main magnetic field can cause image distortions. In order to correct the image distortions, it is desirable to simultaneously acquire data with a field map in registration. We propose a joint estimation (JE) framework with a fast, noniterative approach using harmonic retrieval (HR) methods, combined with a multi-echo echo-planar imaging (EPI) acquisition. The connection with HR establishes an elegant framework to solve the JE problem through a sequence of 1-D HR problems in which efficient solutions are available. We also derive the condition on the smoothness of the field map in order for HR techniques to recover the image with high signal-to-noise ratio. Compared to other dynamic field mapping methods, this method is not constrained by the absolute level of the field inhomogeneity over the slice, but relies on a generous pixel-to-pixel smoothness. Moreover, this method can recover image, field map, and $T^2*$ map simultaneously.

Index Terms—Echo-planar imaging (EPI), field inhomogeneity, functional magnetic resonance imaging (fMRI), geometric distortions, harmonic retrieval, image reconstruction.

I. INTRODUCTION

A. Motivation

Conventional image reconstruction in magnetic resonance imaging (MRI) relies on a Fourier transform relationship between the acquired data and the image of the object. Given a continuous object with contrast-weighted spin density represented by $f(\mathbf{r})$ and a specific $k$-space trajectory $\mathbf{k}$, which is a function of time $t$ (however for our convenience we will write $t$ as a function of $\mathbf{k}$) and neglecting field inhomogeneity the signal $s(\mathbf{k})$ during the readout can be written as [1]

$$s(\mathbf{k}) = \int f(\mathbf{r}) e^{-\nu(\mathbf{r}) t} e^{-i2\pi\mathbf{k} \mathbf{r}} d\mathbf{r}.$$  (1)

However, in practice the magnetic field across the object being imaged is not homogeneous, which leads to distortions in the resulting image. Due to magnetic susceptibility differences near air/tissue interfaces, an inhomogeneous magnetic field exists in many regions of the body and brain [2], [3]. This in turn leads to an off-resonance in the frequency and the relation (1) between MRI data and object is no longer a Fourier transform [4]–[7]

$$s(\mathbf{k}) = \int f(\mathbf{r}) e^{-\nu(\mathbf{r}) t} e^{-i2\pi\mathbf{k} \mathbf{r}} e^{-i2\pi\mathbf{T}_2^*} d\mathbf{r}$$  (2)

where $\nu(\mathbf{r})$ is a map of the inhomogeneity of the magnetic field, i.e., a field map, and $T^*_2(\mathbf{r})$ term models the destruction of transverse magnetization in the spatial domain. Based on (2) we notice that the long readout time of single-shot acquisitions such as echo-planar imaging (EPI) or spiral imaging leads to a significant diversion from the standard model (1) and causes image distortion [4], [8]. As field inhomogeneity scales with magnetic field, the current trend towards higher field systems makes potential image distortions and correction methods vital to continued accurate imaging. In addition, fast, single-shot acquisitions make scans very sensitive to magnetic susceptibility differences at air/tissue interfaces in the brain [3]. This can cause severe distortions in functional imaging of the brain.

As an example of the impact of magnetic field inhomogeneity, consider functional magnetic resonance imaging (fMRI), which plays an important role in studying localized brain function, both in examining healthy cognitive function and in clinical patient groups. Geometric distortions in EPI and blurring effects in spiral imaging caused by field inhomogeneity severely affect predictions about the subject’s brain [see Fig. 5(c) and Fig. 6(c)]. Many different methods to correct the distortions have been proposed [9]–[15]. Most of these methods involve two steps: estimating magnetic field variation and then compensating for this variation during image reconstruction. The methods differ both in how they estimate the field map and how they use that information to correct the images. A common method measures the field map by acquiring two images (which are called the reference images) with slightly different echo times (TEs), which are times at which the echoes appear, and computing the field map

1In our notations, we assume that the field map is measured in rad/s.

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based on their phase difference divided by difference in TEs [7]. In the case of a rectilinear trajectory, such as echo planar imaging (EPI), once the field map is found, correction of the images can be accomplished by shifting image pixels and correcting their intensities based on the estimated field map [9]. This conventional method for finding the field map from reference images implicitly assumes that magnetic field inhomogeneity is static, an assumption that is violated due to subject motion or physiological changes [4]. To overcome this problem, Nayak and Nishimura [6] used multishot acquisition to generate multiple reference images, each subsequent shot having a delayed TE. Yet, this approach is limited by its temporal resolution (multishot) and by the global range of the field map that can be recovered. In addition, the approach estimates distorted field maps that are used to correct the images. Some other methods try to bypass estimation of the field map. For example, in [2] and [16] two images of the same object are acquired with preparation and readout gradients altered by some ratio \( \alpha \). The corrected image is produced by combining and rectifying these two acquired images pixel by pixel. However, such rectification method requires identification of an initial pair of corresponding points. Another recent approach suggested by [4] and [17] is to combine the two steps together: reconstruct the undistorted image and field map simultaneously from the acquired data. This problem is called the joint estimation (JE) problem. The advantages of the JE approach is that it needs only single shot excitation to estimate the field map dynamically and does not require acquiring reference images, thus bypassing the problem of the displacement error between these images. Moreover, the field map is estimated dynamically, allowing to account for motion or physiological changes; it is also in registration with the data and undistorted by field inhomogeneity effects, so there is no need to realign the field map with the data prior to being able to apply a correction scheme.

### II. Problem Setup

Given the signal during the readout \( s(t) \) with specific \( k \)-space trajectory \( k(t) \), the JE problem recovers the image \( f(\mathbf{r}) \) and the field map \( w(\mathbf{r}) \), both satisfying (2) in the presence of the \( T_2^* \) decay.

We use \( \mathbf{r} = (x, y) \) to denote a location in the image space; \( \mathbf{k} = (k_x, k_y) \) to denote a location in the \( k \)-space; while \( \mathbf{n} = [n_1, n_2] \) and \( \mathbf{m} = [m_1, m_2] \) are indexes for the discrete locations of \( \mathbf{r} \) and \( \mathbf{k} \), respectively.

Assume the object has size \( N_1 \times N_2 \). We expand continuous image \( f(\mathbf{r}) \) by linear expansion using the box function

\[
\varphi_T(\mathbf{r}) = \begin{cases} 
1, & 0 \leq x < T_1, 0 \leq y < T_2 \\
0, & \text{else}
\end{cases}
\]

and its translated versions \( \{\varphi_T(\mathbf{r} - \mathbf{n} \odot T) = \varphi_T(x-n_1 T_1, y-n_2 T_2)\}_{\mathbf{n} \in \mathbb{Z}^2} \), where operator \( \odot \) denotes Schur-Hadamard (element by element) matrix product. That is

\[
f(\mathbf{r}) = \sum_n f(\mathbf{n}) \varphi_T(\mathbf{r} - \mathbf{n} \odot T)
\]

or we have \( f(\mathbf{r}) = f(\mathbf{n}) \) for \( \mathbf{r} \in V_n \), where \( V_n \) denotes the voxel region \( [n_1 T_1, (n_1 + 1) T_1] \times [n_2 T_2, (n_2 + 1) T_2] \) of size \( T_1 \times T_2 \) indexed by \( \mathbf{n} = (n_1, n_2) \). And similarly, we can approximate the continuous field map \( \omega(\mathbf{r}) \) and \( T_2^* \) decay map \( T_2^* \) with discrete signals \( \omega^*(\mathbf{n}) \) and \( T_2^*(\mathbf{n}) \), respectively, using the same box function. Notice that here we assume constant intensity within a voxel for both the image and field map, which ignores the possibility of intravoxel gradients in the field inhomogeneity.

Substituting (4) into (2) yields

\[
s(k) = \sum_n f(\mathbf{n}) \int \varphi_T(\mathbf{r} - \mathbf{T} \odot \mathbf{n}) e^{-i\omega(\mathbf{r})} \delta(\mathbf{k})
\]

\[
\times e^{-2\pi i \mathbf{k} \cdot \mathbf{r}} d\mathbf{r}.
\]

Since \( \varphi_T(\mathbf{r} - \mathbf{n} \odot T) = 1 \) when \( \mathbf{r} \in V_n \) and 0 else, we only need to consider \( \mathbf{r} \in V_n \) in the integral in (5). In that region we have \( \omega(\mathbf{r}) = \omega(\mathbf{n}) \) and \( T_2^*(\mathbf{r}) = T_2^*(\mathbf{n}) \), which yields

\[
s(k) = \sum_n f(\mathbf{n}) e^{-i\omega(\mathbf{n})} \delta(\mathbf{k}) e^{-i(k/T_2\omega(\mathbf{n})}
\]

\[
\times \int \varphi_T(\mathbf{r} - \mathbf{T} \odot \mathbf{n}) e^{-2\pi i \mathbf{k} \cdot \mathbf{r}} d\mathbf{r}
\]

\[
= \sum_n f(\mathbf{n}) e^{-i\omega(\mathbf{n})} \delta(\mathbf{k}) e^{-i(k/T_2\omega(\mathbf{n})}
\]

\[
\times e^{-2\pi i \mathbf{k} \cdot \mathbf{T} \odot \mathbf{n}}.
\]
where \( \varphi_T(\mathbf{r}) \) is the Fourier transform of \( \varphi_T(\mathbf{r}) \). Thus, the discretized form of the JE problem is given by the following:

\[
\mathbf{s}(k) \equiv \mathbf{s}(k) = \sum_n f[n] e^{-i \omega n} \mathbf{t}(k) e^{-i \omega T_2[n]} \times e^{-2\pi k \cdot (T_2[n])}.
\]  

### III. Proposed Approach

In this section, we show the main contribution of this work: realizing and building a bridge between JE problem and another problem, called harmonic retrieval (HR), which can be solved using existing signal processing techniques.

#### A. Connecting JE to HR Problem

We assume the particular case of an EPI trajectory \( \mathbf{k} \) as described in Fig. 1. Denote those \( k \)-space lines along which the data is acquired from left to the right as forward lines and lines along which the data is acquired from right to the left as backward lines.

Let \( \delta_1, \delta_2 \) be the time delays and \( \Delta_1, \Delta_2 \) be the distances in \( k \)-space between two consecutive samples of the EPI trajectory along frequency-encoding \( (k_x) \) and phase-encoding axes \( (k_y) \), respectively, as shown in Fig. 1. The total number of samples along the horizontal and vertical axes is \( M_1 \) and \( M_2 \), respectively. We also denote \( \delta_2 \) to be a time delay between sampling the centers of adjacent horizontal lines, i.e., echo spacing. Hence, the following relation holds

\[
\delta_2 = \delta_1 (M_1 - 1) + \delta_2.
\]  

Next, we need to discretize continuous \( k \)-space trajectory \( \mathbf{k} \) and the related timing function \( t(\mathbf{k}) \). The discrete function \( \mathbf{k}[m] \) has one single form for both forward and backward lines. For simplicity in derivation and without loss of generality through the Fourier shift theorem, we assume that \( k \)-space sampling starts from the origin as it is depicted in Fig. 1

\[
\mathbf{k}[m] = \begin{bmatrix} m_1 \Delta_1 \\ m_2 \Delta_2 \end{bmatrix}.
\]  

The continuous function \( t(\mathbf{k}) \) takes different discrete forms \( t[m] \) depending on the kind of the \( k \)-space line that we are considering. From the figure we see that the timing delay for the samples lying on forward lines \( (m_2 \text{ is even}) \) and backward lines \( (m_2 \text{ is odd}) \) is expressed differently as the following:

\[
t[m] = \begin{cases} 
  m_1 \delta_1 + m_2 \delta_2, & m_2 \text{ even;} \\
  (M_1 - 1 - m_1) \delta_1 + m_2 \delta_2, & m_2 \text{ odd.}
\end{cases}
\]  

**Forward Lines:** Substituting (8) and (9) into (6) while normalizing \( k \)-space locations by \( 1/\text{FOV} \) and spatial locations by \( \text{FOV}, \) which leads to \( T_1 = 1/N_1 \) and \( T_2 = 1/N_2, \) we obtain

\[
\mathbf{S}[m] = \sum_n f[n] e^{-i \omega n} (m_1 \delta_1 + m_2 \delta_2) \\
\times e^{-(m_1 \delta_1 + m_2 \delta_2)/T_2[n]} \\
\times e^{-i \frac{2\pi}{N_1} m_1 \Delta_1/n_1 + \frac{2\pi}{N_2} m_2 \Delta_2/n_2}.
\]  

In practice, the time interval \( \delta_1 \) between two samples along the horizontal \( k \)-space line is two orders of magnitude smaller than the time \( \delta_2 \) needed for the gradient to change its direction and go to the next line, therefore \( \delta_1 \ll \delta_2. \) Hence, we can assume that \( \omega n_1 \delta_1 \approx 0 \) and \( \delta_1/T_2[n] \approx 0. \) The first approximation is widely used for EPI [9] and the second approximation is easily satisfied noting that the typical sampling readout \( \delta_1 \) is about 5 \( \mu \)s, whereas \( T_2[n] \) is usually in the range from 10 to around 100–200 ms. Setting \( \Delta_1 = 1 \) and taking inverse DFT of both sides of the (10) along horizontal \( m_1 \) direction, we get

\[
\mathbf{S}[n_1, m_2] = \sum_{n_2=0}^{N_2-1} a_{n_2} e^{-(M_1-1-m_1)\delta_1 T_2[n_2]} \\
\times e^{-(m_1 \delta_1 + m_2 \delta_2)/T_2[n_2]} \\
\times e^{-i \frac{2\pi}{N_1} m_1 \Delta_1/n_1 + \frac{2\pi}{N_2} m_2 \Delta_2/n_2}.
\]  

**Backward Lines:** Similarly, we get the following equation for backward case:

\[
\mathbf{S}[m] = \sum_n f[n] e^{-i \omega n} (m_1 \delta_1 + m_2 \delta_2) \\
\times e^{-(M_1-1-m_2)\delta_1 T_2[n]} \\
\times e^{-i \frac{2\pi}{N_1} m_1 \Delta_1/n_1 + \frac{2\pi}{N_2} m_2 \Delta_2/n_2}.
\]  

Assuming again that \( \omega n_2 \delta_1 \approx 0 \) and \( \delta_1/T_2[n_2] \approx 0, \) we obtain the same problem formulation as (11) in forward case. As a result, with this assumption, the forward and backward lines can be treated in the same way.

In practice, since \( k \)-space coverage is symmetric, the acquired data is \( \mathbf{s}^*[k_1, k_2], \) where \( -N_1/2 \leq k_1 < N_1/2 \) and \( -N_2/2 \leq k_2 < N_2/2. \) In this case we get the same derivation except that
the shift in k-space, \( s[k_1 - N/2, k_2 - N/2] \) leads to a modulation term in the spatial domain: \( \tilde{f}[n] = f[n]e^{-i\pi n_1}e^{-i\pi n_2} \).

B. Harmonic Retrieval Methods

In (13), \( \{\tau_m\} \) are the given data samples \( \{s[n_1, n_2]\} \), whereas \( \{a_n\} \) and \( \{z_n\} \) are unknowns. Solving (13) is precisely the 1-D harmonic retrieval (HR), which can be solved when \( M \geq 2N \) (\( M \) complex data samples \( \tau_m, 2N \) complex unknowns \( a_n \) and \( z_n \)). Given the complex values \( z_n \), we can recover the field map and \( T_2^* \) map by noting that

\[
|z_n| = e^{-s_2/T_2^*[n_1, n_2]}, \quad \angle z_n = -\omega[n_1, n_2] = \frac{2\pi}{N} \Delta_2 n. \tag{15}
\]

The HR problem (11) can be solved by noting that \( \{z_n\} \) are roots of the characteristic polynomial [19], [20]

\[
z^N + p_{N-1}z^{N-1} + \cdots + p_1 z + p_0 z^0 = 0 \tag{16}
\]

where the polynomial coefficients \( p_n, n = 0, \ldots, N-1 \) are solutions of the following matrix equation:

\[
\begin{bmatrix}
\tau_0 & \tau_1 & \cdots & \tau_{N-1} \\
\tau_1 & \tau_2 & \cdots & \tau_N \\
\vdots & \vdots & \ddots & \vdots \\
\tau_{M-N-1} & \tau_{M-N} & \cdots & \tau_{M-1}
\end{bmatrix}
\begin{bmatrix}
p_0 \\
p_1 \\
p \end{bmatrix}
= \begin{bmatrix}
\tau_N \\
\tau_{N+1} \\
\vdots \\
\tau_{M-1}
\end{bmatrix} - \begin{bmatrix}
p_{N-1} \\
p_{N-2} \\
\vdots \\
p_{M-1}
\end{bmatrix} = \begin{bmatrix}
\tau_0 \\
\tau_1 \\
\vdots \\
\tau_{M-1}
\end{bmatrix} \begin{bmatrix}
0 & \cdots & 0 \\
1 & \cdots & 0 \\
\ddots & \ddots & \ddots \\
0 & \cdots & 1
\end{bmatrix} \begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{N-1}
\end{bmatrix}. \tag{17}
\]

Solving this equation gives coefficients \( \{p_n\} \), from which we can find \( \{z_n\} \) as roots of the polynomial in (16). Once \( \{z_n\} \) are found, the corresponding unknowns \( \{a_n\} \) can be easily computed by solving the following linear equation with a Vandermonde matrix \( Z \):

\[
\begin{bmatrix}
\tau_0 \\
\tau_1 \\
\vdots \\
\tau_{M-1}
\end{bmatrix} = Z \begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{N-1}
\end{bmatrix}. \tag{18}
\]

The total complexity of the proposed approach using HR technique for a fixed \( n_1 \) is \( O(N^2_2) \) by exploiting the Toeplitz and Vandermonde structures of the matrices \( A \) and \( Z \), while the existing method [4] solving JE problem for both the image and field map using iterative CG method requires much larger computation time \( O(N^2_2 N^2_2) \) per each iteration [21]. The described above HR technique is called the LS Prony method. Other techniques exist to solve the HR problem, such as forward-backward (FB), total LS Prony (TLS), Cadzow denoising, and iterated quadratic maximum likelihood (IQML) methods [19], [20], [22]–[24]. FB method is a variation of LS Prony method, where harmonics are assumed to lie on the unit circle. The rest of the listed methods are HR methods which deal with the noise in the data. Particularly, IQML gives a maximum-likelihood solution to the HR problem and it asymptotically reaches the Cramer-Rao lowest bound on the estimated harmonics as the number of data samples approaches infinity. Its first iteration coincides with the TLS Prony method.

C. Condition on the Field Map Smoothness

In the noiseless case for each fixed \( n_1 \) the HR problem (13) solves for each column of \( \tilde{f}[n_1, n_2] \) of the image and a corresponding phase sequence

\[
\theta_{n_2} = -\angle z_{n_2} = \omega[n_1, n_2] \delta_2 + \frac{2\pi}{N} \Delta_2 n_2. \tag{19}
\]

which we call as the harmonics, but the order of \( \theta_{n_2} \) (and hence, \( \tilde{f}[n_1, n_2] \)) in general cannot be preserved, since in the HR problem (13) reordering of \( \{a_n\} \) and \( \{z_n\} \) does not change the data \( \{\tau_m\} \). To resolve this ambiguity, we require the sequence \( \theta_{n_2} = \omega[n_1, n_2] \delta_2 + (2\pi)/(N) \Delta_2 n_2 \) to be strictly increasing with respect to \( n_2 \), i.e.,

\[
\omega[n_1, n_2] \delta_2 + \frac{2\pi}{N} \Delta_2 n_2 < \omega[n_1, n_2 + 1] \delta_2 + \frac{2\pi}{N} \Delta_2 (n_2 + 1), \quad \text{for all } n_1, n_2. \tag{20}
\]

which is equivalent to

\[
\omega[n_1, n_2] - \omega[n_1, n_2 + 1] < -\frac{2\pi \Delta_2}{N \delta_2}, \quad \text{for all } n_1, n_2. \tag{21}
\]

We call this condition the smoothness condition, which requires the field map to be sufficiently smooth. Most standard methods of estimating the field map are limited by the overall field map range that they can afford. Instead, here we require a generous smoothness condition (21) on the field map, which in practice is satisfied. For example, given \( \delta_2 = 0.5 \) ms and \( 64 \times 64 \) data samples, the smoothness condition requires \( \omega[n_1, n_2] - \omega[n_1, n_2 + 1] < 203 \) rad/s or 62.5 Hz. Thus, in this case our approach only requires the variation in field map of adjacent voxels to be bounded by 62.5 Hz, while the whole variation range of the field map can be as much as 63 times larger.

If the smoothness condition is not satisfied, the HR method will retrieve the harmonics with some local swapping of the order. This will lead mostly to a discontinuity in the recovered image and field map. Since there is not any prior information about the correct order of the harmonics, we check for any voxel of the recovered image that has discontinuity and we replace its value by an interpolated intensity of the neighbor voxels. This condition rarely happens in the fMRI data that we have collected so far.

Without the field map term, the \( n_2^{th} \) harmonic is \( (2\pi/N_2) n_2 \Delta_2 \). Thus, the smoothness condition is automatically satisfied and plotting all harmonics over \( n_2 \) gives a straight line ranging from 0 to 2\( \pi \). The presence of the field map can be viewed as fluctuations around this straight line. Noise in the field map \( n_2 \) can further push some harmonic at position \( n_2 \) too close to its neighbor harmonic at \( n_2 + 1 \) or \( n_2 + 1 \), which makes the JE problem ill-conditioned or even ill-posed.

When the data \( s[n_1, n_2] \) are contaminated with noise \( \eta \), which is assumed to be additive complex uncorrelated
Gaussian noise $\mathcal{N}(0, \sigma^2 I)$, then the HR method returns an estimate $\hat{\theta}_{n_2}$ of $\theta_{n_2}$. We denote the noisy observations as

$$\mathbf{y} = \mathbf{s} + \mathbf{\eta}.$$  \hspace{1cm} (22)

where $\mathbf{s}$ is a vector consisting of all signal samples $s[n_1, n_2], n_2 = 0, \ldots, M_2 - 1$. In the case of an unbiased estimator, the harmonic separation $g_{n_2} = \theta_{n_2+1} - \theta_{n_2}$ is the distance $E[\hat{\theta}_{n_2+1}] - E[\hat{\theta}_{n_2}]$ between the means of two distributions. Thus, assuming uniform distribution in the range $[0;2\pi]$ for parameters $\hat{\theta}_{n_2}$, we can impose the following condition on $g_{n_2}$ so that the two harmonics are resolvable [25]:

$$g_{n_2} \geq \left( \sqrt{\text{CRB}_{\theta_{n_2+1}}} + \sqrt{\text{CRB}_{\theta_{n_2}}} \right)$$  \hspace{1cm} (23)

where $\text{CRB}_{\theta_{n_2}}$ is the Cramer-Rao lower bound (CRB) on the variance of the error between estimated $\hat{\theta}_{n_2}$ and exact parameter $\theta_{n_2}$ of the model. This CRB for the HR problem has been studied and derived in the literature (see [26] and other results on CRB in [25] and [27]). This condition is just one reasonable necessary condition for $g_{n_2}$, because if we choose a different resolvability criteria, we will get a different bound on $g_{n_2}$. Using (19) we get

$$g_{n_2} = \theta_{n_2+1} - \theta_{n_2} = \delta_2(\omega[n_1, n_2 + 1] - \omega[n_1, n_2]) + \frac{2\pi}{N_2} \Delta_2.$$  \hspace{1cm} (24)

Substituting this equation into (23) we get the following constraint:

$$\omega[n_1, n_2] - \omega[n_1, n_2 + 1] \leq -\frac{2\pi \Delta_2}{N_2} - \sqrt{\text{CRB}_{\theta_{n_2+1}}} - \sqrt{\text{CRB}_{\theta_{n_2}}},$$  \hspace{1cm} (25)

which is the smoothness condition in the presence of noise. In [28], a noise fluctuation study on five normal subjects was performed to obtain the average value of $\text{SNR}(\mathbf{y})$ at different spatial resolutions and field strengths. For our case, where the resolution is $3 \times 3 \times 3 \text{ mm}^3$ and field strength is $3 \text{ T}$, the listed $\text{SNR}(\mathbf{y})$ was 87.80 or 38.8 dB, where

$$\text{SNR}(\mathbf{y}) = \frac{||\mathbf{s}||_2^2}{||\mathbf{s} - \mathbf{y}||_2^2}.$$  \hspace{1cm} (26)

or

$$\text{SNR}_{\text{dB}}(\mathbf{y}) = 10 \log_{10} \text{SNR}(\mathbf{y}).$$  \hspace{1cm} (27)

At this signal-to-noise ratio and for the same practical values $\delta_2 = 0.5 \text{ ms}$ and $64 \times 64$ data samples as we used in the noiseless case, condition (24) requires the lowest allowable value of $\omega[n_1, n_2] - \omega[n_1, n_2 + 1]$ is approximately 60.6 Hz, as shown in Fig. 2. When SNR of $\mathbf{y}$ is two times smaller (19 dB), the lowest allowable value of $\omega[n_1, n_2] - \omega[n_1, n_2 + 1]$ decreases to 46.1 Hz. In comparison, the lowest allowable value is 62.5 Hz in the noiseless case. The plots in Fig. 2 were produced from a ground truth image and field map of the subject’s brain. We chose a column from a field map low in the brain that ranged from $-53.34 \text{ Hz}$ to $61.77 \text{ Hz}$, avoiding regions outside the brain.

D. Proposed Algorithm

Below is the summary of proposed algorithm (see Fig. 3 for illustration).

1) Given acquisition data samples $\mathbf{s}[n_1, n_2]$, divide correspondingly by the discrete samples of the function $\Phi_T(k)$ and take inverse DFT along $m_1$ axis or 1-D grid if the readout is ramp-sampled. Denote the result by $\mathbf{s}[n_1, m_2]$.

2) For each fixed value of $n_1$ solve the 1-D HR problem described by (11)–(13) to recover $\omega[n_1]$, image $f[n_1]$, and $T_2^*[n_1]$ map.

IV. METHODS

A. Proposed Acquisition Scheme

As it was mentioned, from (13) it can be seen that $2N$ unknowns from the magnitude and phase of the $a_{n_1}$ term, which is the set of all image voxels at a particular column $n_1$, and $2N$ unknowns from the magnitude and phase of the $\pi_{n_1}$ term, which contain $T_2^*$ and field map values at the $n_1^{\text{th}}$ column respectively, lead to the minimum of $2N$ complex data samples $\tau_{n_1}$, which are the desired MRI data samples to be collected. Hence, HR requires size of the collected data to be larger than the final resolution of the image, field map, and $T_2^*$ map by a factor of 2 along the phase-encoding direction, which motivates for a double-echo acquisition. Standard EPI data is not sufficient for the current HR framework. One way to double the number of $k$-space lines would be to decrease the spacing.
Δ2 in k-space along phase-encoding direction twice, thus getting twice more k-space lines. Although this approach indeed works, it reduces the overall range of \( \phi_{n2} \) terms from \([0, 2\pi]\) to \([0, \pi]\), which makes the JE problem more ill-conditioned. Instead, consider k-space trajectory shown in Fig. 4(a), where a standard EPI trajectory is centered about the k-space origin and traversed twice. It can be verified that with this trajectory, after making an assumption that \( \omega[n] \delta_1 \approx 0 \) and \( \delta_1 / T_2[n] \approx 0 \) and taking inverse DFT along \( m_3 \), (6) becomes
\[
\hat{s}_1[n_1, m_2] = \sum_{n_2=0}^{N_2-1} f[n_1, n_2] e^{-j(m_2\delta_2 / T_2[n_2])} e^{-j(\omega[n]m_2\delta_2 + \frac{\pi}{2} \Delta_2 n_2 m_2)}
\]
(26)
for the first time traversing through the k-space center, i.e., \( m_2 = 0, \ldots, M_2/2 - 1 \), and
\[
\hat{s}_1[n_1, m_2] = \sum_{n_2=0}^{N_2-1} f[n_1, n_2] e^{-j(m_2\delta_2 + T_{jump}) / T_2[n_2]}
\times e^{-j(\omega[n](m_2\delta_2 + T_{jump}) + \frac{\pi}{2} \Delta_2 n_2 (m_2 - \frac{M_2}{2}))}
\]
(27)
for the second time, where \( m_2 = M_2/2, \ldots, M_2 - 1 \) and \( T_{jump} \) is the time required to reset the position in k-space between the two passes through the center of k-space. Notice that the desired data by HR, as we discussed, should be
\[
\hat{s}[n_1, m_2] = \sum_{n_2=0}^{N_2-1} f[n_1, n_2] e^{-j(m_2\delta_2 / T_2[n_2])} e^{-j(\omega[n]m_2\delta_2 + \frac{\pi}{2} \Delta_2 n_2 m_2)}
\]
(28)
for both the first and second halves of the data, i.e., \( m_2 = 0, \ldots, M_2 - 1 \). Hence, the first half of the data produced with the considered k-space trajectory is the same as the first half of the data that the HR framework needs. We further notice that the term \((2\pi/N_2)\Delta_2 n_2 (m_2 - M_2/2)\) in (27) can be simplified to \((2\pi/N_2)\Delta_2 n_2 m_2\) when \( M_2 \) is a multiple of \( 2N_2 \). In this case, comparing (26) and (27) with (28), the considered trajectory satisfies the HR framework when \( T_{jump} = 0 \). We keep \( T_{jump} \) short to try to minimize its effect on the estimation scheme. In our sequence \( T_{jump} \) is 420 µs, which is small on the order of one \( \delta_2 \).

We can further extend our proposed k-space trajectory as shown in Fig. 4(b), so that we could also obtain a higher resolution standard EPI data and then use them together with the obtained from HR image and field map of a lower resolution to fit the iterative CG reconstruction proposed in [4]. Note that the magnitude MR image, field map, and \( T_2^* \) map can all be calculated from the sequence illustrated in Fig. 4(b) using conventional postprocessing methods.

The sequence parameters are the following: dwell time \( \delta_1 = 3.4 \mu s \), echo spacing \( \delta_2 = 0.5 \) ms, echo times \( T_E_1 = 25 \) ms and \( T_E_2 = 41 \) ms, \( T_{jump} = 420 \) µs, TR = 2 s, and a stack of 20 2-D slices is acquired every TR. Specifically, to acquire 64 × 64 MRI data, we consider k-space window (k-space is normalized by \((1/(\text{FOV}) \frac{-32 \leq k_x \leq 32}{-32 \leq k_y \leq 32}) \). In that k-space region we cover the first subregion \(-32 \leq k_y \leq 16\) along a standard EPI trajectory (by alternating the polarity of the readout gradient and applying the phase-encoding gradient blips) and then the second subregion \(-16 \leq k_y \leq 32\) also along the same trajectory. Then, the data from both the first and the second subregions that fall within the range \(-16 \leq k_y \leq 16\) are extracted and stacked. By doing this we get the two k-space subregions interleaved and the extracted 64 × 64 data represent the oversampling of k-space center. We should also point out...
that in practice we use ramp-sampling in the $k_x$ direction along with oversampling, providing a more time efficient EPI readout. So instead of getting 64 sampled along $k_x$ direction, we get 128 samples within the range $-32 \leq k_x \leq 32$. Then, we use the gridding procedure [29] to generate 64 uniformly spaced image locations across the field of view in the $x$-direction. At the end of the acquisition window we apply the spoiler to spoil residual magnetization to prepare for the excitation of the next slice.

At the beginning of the EPI readout, we collect three lines (two forward and one backward) at $k_y = 0$ for correcting ghosting artifact, which is caused mainly by eddy currents. Eddy currents cause odd and even echo lines to have an alternating shift in the $k$-space along the frequency encoding direction, which in turn leads to the phase ramps in the image domain, visualized as a ghost effect. Acquiring reference lines with no phase encoding helps to estimate and remove this effect. We used the standard single pair, three reference line method to correct for Nyquist ghost, where the first and third lines were averaged together to obtain an average forward line with the same echo time as the single backward line (refer to [30]).

Along with the mentioned advantages of the proposed acquisition scheme, there are trade-offs. One obvious disadvantage is the reduction of the number of slices that could fit within a TR time interval, since the acquisition time per each slice increases compared with the conventional gradient-echo EPI sequence. For a conventional EPI sequence, the echo time is 25 ms and there is a half of $k$-space left to acquire after the echo, i.e., 16 ms. Given a 2 ms RF pulse (of which half counts towards TE), the total time for a single slice would be $(1 + 25 + 16) = 42$ ms. For our proposed acquisition, recall that the first echo time is 25 ms and the second is 41 ms. With 16 ms of additional data acquisition after the last echo, the total scan time per slice now is $(1 + 41 + 16) = 58$ ms, which is an increase of 16 ms compared to the normal EPI acquisition. Thus, the number of slices that could be scanned will decrease. Second trade-off is that the acquisition will experience a more significant $T_2^*$ decay, especially the second half of the data. This is a main reason for the signal loss in the acquisition. Note that we are modeling voxels as having constant field map, so any gradients would result in a decrease in $T_2^*$ value for that voxel. We may not be able to estimate the signal from the regions of a severe signal decay, which is faster than the echo time of our sequence.

B. Regularization

Recall (12), from which we see that the $T_2^*$ map can be estimated as the following:

$$T_2^* [\mathbf{n}] = -\delta_2 / \ln |\omega_n|.$$  \hspace{1cm} (29)

Due to the characteristics of the logarithmic function in the region when $|\omega_n| \leq 1$, small perturbations in $|\omega_n|$ lead to large perturbations in the $T_2^* [\mathbf{n}]$ value, which implies the need of the regularization in the case of high noise level in the data. In practice the $T_2^* [\mathbf{n}]$ map has values ranging in the order of 10–100 ms, which means that in the case $\delta_2 \approx 0.5$ ms, the magnitude...
of $z_n$ is $e^{-\zeta_n T_2^*} = 0.95$ and 0.995, respectively. This fact is taken as a prior knowledge about $|z_n|$ and incorporated into the JE model.

Suppose that $|z_n| = \alpha$. Using similar arguments as in the Forward-Backward algorithm [19] we can show that in addition to the matrix equation (17) we have the following equation:

$$\begin{bmatrix}
\tau_N^* c_N(\alpha) \\
\tau_{N-1}^* c_N(\alpha) \\
\cdots \\
\tau_{M-N-1}^* c_N(\alpha)
\end{bmatrix}
\begin{bmatrix}
p_0 \\
p_1 \\
\cdots \\
p_{N-1}
\end{bmatrix}
= 
\begin{bmatrix}
\tau_M^* c_N^*(\alpha) \\
\cdots \\
\tau_{M-N-1}^* c_N^*(\alpha)
\end{bmatrix}
\begin{bmatrix}
a_0 \\
\delta_0 \\
\cdots \\
\delta_{N-1}
\end{bmatrix},$$

where

$$c(\alpha) = [(\alpha^N)^*/\alpha^0, (\alpha^{N-1})^*/\alpha^1, \cdots, (\alpha^0)^*/\alpha^N]^T.$$

A least squares solution for (17) and (30) can be formulated as the solution to the optimization problem

$$\min_p \{||Ap - b||^2 + \lambda^2 ||A_\alpha p - b_\alpha||^2\}$$

where parameter $\lambda$ determines the significance of the constraint $|z_n| = \alpha$; the larger $\lambda$, the more significant the constraint is. When $\lambda = 0$, we return to the general joint estimation problem without any constraint. Notice that for a special case when $\alpha = 1$, which means that all $\{z_n\}$ lie on the unit circle, and $\lambda = 1$, the matrix (30) becomes the same as in the Forward-Backward method. The solution to (32) is obtained by solving the least-squares solution of the following matrix equation for the coefficients $\{p_i\}$:

$$\begin{bmatrix}
A \\
\lambda A_\alpha
\end{bmatrix} p = 
\begin{bmatrix}
b \\
\lambda b_\alpha
\end{bmatrix},$$

Having the coefficients $\{p_i\}$, the desired regularized $\{z_n\}$ can be found as the roots of the characteristic polynomial (16).

V. EXPERIMENTAL RESULTS

A. Simulated Data

To quantify performance of the proposed approach we simulate data with realistic field maps. In these experiments, a real MRI image and field map were acquired in accordance with the Institutional Review Board (IRB) of the University of Illinois at Urbana-Champaign. The ground truth field map was obtained by a non-fMRI, multi-echo GRE (not EPI) acquisition with TR = 400 ms, flip angle of 45°, echo times 10 ms and 12.46 ms. The acquisition gives the correct field map, although it takes about 2 min to acquire. In addition, the resulting image from this acquisition does not have the same contrast as the functional image and does not contain function information.

The MRI data is simulated according to the model described in (10) and (14) with the following parameters: $\Delta_1 = \Delta_2 = 1$, $\delta_1 = 5 \mu s$, $\delta_2 = 400 \mu s$ and the $T_2^*$ map is also synthesized. We are interested in the SNR of the reconstructed image with and without compensating for the field map and the SNR of the field map recovered from conventional field map estimation and our joint estimation. The conventional field map estimation that we compare our method to is based on reconstructing the individual images from the oversampled central k-space lines and dividing their phase difference by the difference in echo times, which was 16 ms in this case.

In the first experiment, the input image to the algorithm is an axial slice of the brain and the corresponding field map shows...
a significant off-resonance due to field inhomogeneity in the region of the frontal lobe, as shown in Fig. 5(a) and (f). The field map varies from -10.34 until 40.44 Hz. In this experiment the $T_2^*$ map was modeled as ovals [see Fig. 5(i)]. The $T_2^*$ value for the gray and white matters was taken to be 80 ms (the region outside the two ovals) and 50 ms (the region inside the two ovals) respectively, according to [31].

The results, given in Fig. 5, show that our approach recovers the original image and field map better (SNR = 61.02 dB for the image and SNR = 31.35 dB for the field map) compared to the case where field map is not considered (SNR = 20.48 dB for the image). Note that for an appropriate SNR comparison with the HR image reconstruction, the DFT recovered image does not include $T_2^*$ decay. The total acquisition time along the phase-encoding direction is $N_z \times \delta_z = 64 \times 0.5\text{ ms} = 32\text{ ms}$, which leads to the bandwidth per pixel $BWPP_{pe} = 31.3\text{ Hz}$ along the phase-encoding direction. Since the field map is moderate and has a maximum value of 40.44 Hz, we expect there is about 1–2 off-pixel shifts, which is not noticeable from the image in Fig. 5(c). The distortion along the frequency encoding direction is negligible because $\delta_z$ is very small. For a clear visualization, Fig. 5(d) shows the difference between the HR recovered and original images and Fig. 5(e) shows the difference between the DFT reconstructed image and the original image. Conventional field map estimate results from the distorted image, so its contour is also slightly distorted as can be seen from Fig. 5(h). In addition, the regions of white and grey matter in the $T_2^*$ map recovered by HR, shown in Fig. 5(j), resemble those regions in the ground truth $T_2^*$ map with SNR = 20.93 dB.

In the second simulation experiment, shown in Fig. 6, the phantom is filled with two horizontal plastic structures inside and the ground truth field map is more severe. This field map has a maximum absolute value of 71 Hz, which causes a geometric distortion in the region of field inhomogeneity to a reconstructed image without taking into account the field map. The $T_2^*$ ground truth was simulated to be a 10 ms constant map. For this particular experiment, the field map and the $T_2^*$ decay are more severe than in the first case. We apply the proposed in the Section IV-B regularization approach with the regularization parameters $\alpha = 0.99$ and $\lambda = 1.7$. All scales of field maps are in Hz. (a) DFT reconstructed image. (b) HR reconstructed image. (c) Reference image. (d) HR reconstructed $T_2^*$ map (in sec). (e) Conventional field map estimate (unwrapped). (f) HR reconstructed field map (unwrapped). (g) Ground truth field map.

![Fig. 7. Experimental results with the first subject. HR magnitude image, field map, and $T_2^*$ reconstructions are obtained with the regularization parameters $\alpha = 0.99$ and $\lambda = 1.7$. All scales of field maps are in Hz. (a) DFT reconstructed image. (b) HR reconstructed image. (c) Reference image. (d) HR reconstructed $T_2^*$ map (in sec). (e) Conventional field map estimate (unwrapped). (f) HR reconstructed field map (unwrapped). (g) Ground truth field map.](image-url)
the proposed acquisition scheme. Severeness of field inhomogeneity increases when we go from slice 20 (superior) to slice 1 (inferior). The $T_1$-weighted reference image and ground truth field map were acquired with the non-fMRI GRE acquisition described in the Section V-A ($TE = 10$ ms, 12.46 ms) that is not significantly geometrically distorted by the field inhomogeneity. Acquisition time for this static field map was approximately 2 min. Notice that although the $T_1$-weighted reference image is insensitive to magnetic field inhomogeneity, it is also insensitive to functional changes leading to a different contrast in the image. Thus, we use the reference image only as a ground truth for the overall geometric shape of the object.

As the second simulation study suggests, for the real data case with some noise level, the problem becomes badly ill-conditioned and thus we need to apply regularization in order to solve for the reasonable image, field map, and $T_2^*$ map. Before applying the regularization approach proposed in the Section IV-B, we need to decide which value of the regularization parameters $\alpha$ and $\lambda$ introduced in (32), to take. As already discussed, parameter $\lambda$ is used to enforce the level of significance of the constraint $|z_{n1}| = \alpha$, while parameter $\alpha$ plays a crucial role on the image reconstruction. In fact, the acquired MRI data contain a rough idea about the value of $\alpha$. The data acquired from the proposed gradient-echo sequence always have two data peaks, which we denote as $S_1$ and $S_2$, each corresponding to the time when the echo appears. Since the amplitude of a gradient echo carries a mean $T_2^*$-weighting, we fit an exponential through the signal intensities of these two data peaks. Thus, the mean $T_2^*$ value can be estimated as

$$\hat{T}_2^* = \frac{TE_1 - TE_2}{\ln(S_2/S_1)}$$

(34)

and based on (15) we choose $\alpha$ to be $e^{-\hat{T}_2^*}|n|$. However, note that this is only a rough, poor estimate of $\alpha$. After the HR reconstruction, it is possible to detect the HR failures from the information about $|z_{n1}|$ for each fixed $n_1$. Specifically, if the recovered $|z_{n1}|$ values are out of the possible physical range, i.e., $|z_{n1}| > 1$ (outside of unit circle) or $|z_{n1}| < \epsilon$, where $\epsilon$ is the threshold dictated by the smallest possible $T_2^*$ value, taken from the prior information about the $T_2^*$ range, then at the corresponding recovered HR image pixels we will see the dark or white gaps, respectively. We correct these pixels by setting the value of $|z_{n1}|$ to either 1 or $\epsilon$, which automatically corrects for the $T_2^*$ map. If this still gives an obvious failure in the reconstructed image, we interpolate the nearest neighbor pixels of the image at that location.

Fig. 7 shows the reconstructions of a lower slice covering the medial temporal lobe, where field inhomogeneity begins to have a dramatic effect on the usual DFT image reconstruction (field map varies from $-73.63$ Hz to 35.21 Hz in this slice). As discussed, at those locations that the HR approach fails to recover $|z_{n1}|$ in the reasonable range, we threshold the wrongly recovered $|z_{n1}|$ and apply the interpolation on the neighbor image pixels. Shown in Fig. 7(a) is the DFT image reconstruction, while Fig. 7(b) shows the HR image reconstruction when the regularization parameter $\alpha$ is set to 0.99 and $\lambda = 1.7$, which gives a good image reconstruction with a trade-off on the field map and $T_2^*$ reconstructions. In the middle region of the brain where there is a signal void, indicated by an arrow, the proposed approach succeeds to fill in the image. Fig. 7(d) shows the corresponding $T_2^*$ map reconstruction. The reconstructed $T_2^*$ map is smooth and varies in the range from 20 to 50 ms, with the most common values of $46 \pm 3$ ms. Fig. 7(e) and (f) shows a conventionally estimated field map and a HR recovered field map, both being not phase unwrapped except for a very obvious $2\pi$ wraps, where we use the available information from the neighbor, not-phase wrapped, field map columns to compensate for the wraps. Phase unwrapping for the HR field map reconstruction remains to be the future work and is out of scope of this paper.

Similar to the first experiment, reconstructions from the second real experiment are shown in Fig. 8. Arrows in Fig. 8(a)
Fig. 9. Dynamic imaging study with the first subject: normalized root mean squared error (NRMSE) map for (a) HR image reconstruction; (b) DFT image reconstruction; (c) HR $T_2^*$ map reconstruction.

Fig. 10. Dynamic imaging study with the second subject: normalized root mean squared error (NRMSE) map for (a) HR image reconstruction; (b) DFT image reconstruction; (c) HR $T_2^*$ map reconstruction.

indicate those regions of the brain that are corrected using the HR approach with the regularization parameter $\alpha$ and $\lambda$ being equal to 0.985 and 1, respectively. Recall again from our discussion that in the proposed approach, the term $\left[z_{n1}\right]$ carries a significant amount of information, since it helps to identify those image and $T_2^*$ voxels at which HR fails. This enables a postprocessing step to correct for these failures. Recovered by the HR, the $T_2^*$ map, shown in Fig. 8(d), has the maximum value of 50 ms, with the most common value range from 28 to 34 ms approximately, which is fairly relevant to the estimated value of 33 ms found by fitting an exponential. In the region of the severe signal decay the recovered $T_2^*$ value drops to 2 ms. This results in recovery of signal in the region corresponding to the low $T_2^*$ area, as seen from the reconstructed HR image. Fig. 8(e) and (f) shows the unwrapped field maps, reconstructed from the conventional and HR approaches.

Dynamic imaging study was performed on the same two subjects to assess the stability of the proposed approach. To show the image intensity fluctuations over all voxels, we take the time series of reconstructions, demean whole data with respect to time, and then for each voxel we calculate the normalized root mean squared error (NRMSE) around the corresponding mean to obtain the NRMSE map which quantifies the signal fluctuations over time. Larger fluctuations of the reconstructed signals over time result in a higher NRMSE. Fig. 9(a) and (b) shows the NRMSE maps for the HR and DFT reconstructed images for the first real data experiment. These two maps indicate that both DFT and HR approaches have small signal fluctuations over the time, fairly comparable to each other. The bright spots in Fig. 9(a) shows the locations at which the ill-conditioning of the HR method is more significant than at the other locations. For the second subject, as can be seen from Fig. 10(a) and (b), variations in the HR signal are larger and in this case DFT stability outperforms the HR stability. The $T_2^*$ NRMSE maps for the two subjects are shown in Fig. 9(c) and Fig. 10(c), respectively. These plots suggest that the $T_2^*$ map reconstruction is fairly stable over time and is more stable than the HR image reconstruction. This can be explained from the fact that the $T_2^*$ map relates directly to the recovered from HR $\left[z_{n1}\right]$ according to (29), while the magnitude image is found from $\left[z_{n2}\right]$ by inverting Vandermond matrix, which causes the ill-conditioning in the image reconstruction. It is worth pointing out that the HR method is noniterative: it does not use any information about previous time points nor does it smooth temporally. Overall, the pixel-wise signal fluctuations were not in a large scale, although a further improvements on the stability of the HR method should be investigated.

VI. CONCLUSION

In this work, we propose a new, nonterative method for joint estimation (JE) and correction of susceptibility artifacts problem in the case of EPI functional MRI. We present a method that can dynamically estimate the undistorted field map, magnitude image, and $T_2^*$ map simultaneously from the acquired multi-echo EPI data, noniteratively. The advantages of this approach are: 1) much lower complexity compared with a recent method [4] which views the JE problem as a nonlinear least-squares optimization problem; 2) this approach does not require a prior knowledge of the field map. We convert the JE problem to the HR problem by making a practical approximation that helps to transform the nonlinear 2-D JE problem to a set of 1-D linear problems. To make a bridge between the JE and HR problems, we need a generous condition, which we call the smoothness condition, on the
smootheness voxel-to-voxel along vertical direction of the original field map. In the rare case when the smoothness condition does not hold, we can still reconstruct all details and shape of the image, but with slight voxel shifts caused by incorrect order of harmonics. A further postprocessing to detect and shift back the voxels can be applied. We analyze the noise effect on the performance of the method and derive a bound on the resolution of the image and size of the field map given a fixed noise level in the case of unbiased estimator. In real practice when the problem becomes ill-conditioned, we propose a specific regularization approach on the apriori known possible range of the $T_2^*$ map. Experimental results on the real data show that the reconstructed from HR image has much reduced signal loss artifact in those regions of the brain that correspond to the low $T_2^*$ area and high field inhomogeneity, as the DFT reconstructed image. A validation of whether and how the proposed regularization affects time-domain signal behavior and BOLD detectability is needed to be performed.

REFERENCES


