Low-Order Spectral Analysis of the Kirchhoff Matrix for a Probabilistic Graph With a Prescribed Expected Degree Sequence

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Low-Order Spectral Analysis of the Kirchhoff Matrix for a Probabilistic Graph With a Prescribed Expected Degree Sequence

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Abstract—We study the eigenvalue distribution of the Kirchhoff matrix of a large-scale probabilistic network with a prescribed expected degree sequence. This spectrum plays a key role in many dynamical and structural network problems such as synchronization of a network of oscillators. We introduce analytical expressions for the first three moments of the eigenvalue distribution of the Kirchhoff matrix, as well as a probabilistic asymptotic analysis of these moments for a graph with a prescribed expected degree sequence. These results are applied to the analysis of synchronization in a large-scale probabilistic network of oscillators.

Index Terms—Complex network, Kirchhoff matrix, random graph, spectral graph theory, synchronization.

I. INTRODUCTION

In recent years, systems of dynamical nodes interconnected through a complex network have attracted a good deal of attention [1]. Biological and chemical networks, neural networks, social and economic networks [2], the power grid, the Internet and the World Wide Web [3] are examples of the wide range of applications that motivate this interest (see also [4], [5] and references therein). Several modeling approaches can be found in the literature [3], [6], [7]. One common approach is to measure a real network property and use this to constrain a probabilistic graph ensemble [8], [9]. For example, many probabilistic models replicate the degree sequence of the graph, as this is often the simplest to retrieve and the most widely reported measurement [9]–[11]. We focus our attention on a probabilistic model proposed by Chung and Lu [12], in which a given expected degree sequence is prescribed on the graph.

Once the network is modeled, one is usually interested in two types of problems. The first involves structural properties of the model. The second involves the performance of dynamical processes run on those networks. In the latter direction, the performance of random walks [13], Markov processes [14], gossip algorithms [15], consensus in a network of agents [16], [17], or synchronization of oscillators [18], [19] are very well reported in the literature. These dynamical processes are mostly studied in the traditional context of deterministic networks of relatively small size and/or regular structure. Even though many noteworthy results have been achieved for large-scale probabilistic networks [20]–[24], there is substantial reliance on numerical simulations. The authors of [20] study the ability of a small-world network to synchronize, as well as its robustness under attacks. In [21] the synchronization properties of networks with prescribed degree sequences are examined. In [22], chaos synchronization in time-invariant, time-varying, and switching topologies is studied. The work in [23] focuses on the problem of synchronization of a set of identical oscillators coupled via a time-varying stochastic network, modeled as a weighted directed random graph that switches at a given rate within a set of possible graphs. In [24], the effects of degree correlation on the synchronization of networks of coupled identical nonlinear oscillators are investigated.

The eigenvalue spectrum of an undirected graph contains a great deal of information about structural and dynamical properties [25]. In particular, we focus our attention on the spectrum of the so-called Kirchhoff matrix uniquely associated with an undirected graph [26]. This spectrum contains useful information about, for example, the number of spanning trees, or the stability of synchronization of a network of oscillators. We analyze the low-order moments of the Kirchhoff matrix spectrum corresponding to the probabilistic Chung-Lu model.

The paper is organized as follows. In Section II, we derive closed-form expressions for the first three moments of the Kirchhoff eigenvalue distribution associated with a given network. In Section III, we compute the expected values of the first three Kirchhoff moments in the probabilistic Chung-Lu ensemble. Our expressions are valid for networks of asymptotically large size, under realistic constraints on the degree sequence. We devote Section IV to illustrating the application of the results and techniques developed in Sections II and III to computing the expected values of the first three Kirchhoff moments of a random small-world network. Section V applies our spectral results to the problem of synchronization of a probabilistic network of oscillators. The numerical results in this section corroborate our predictions.

II. SPECTRAL MOMENTS OF THE KIRCHHOFF MATRIX

We first deduce closed-form expressions for the first three moments of the Kirchhoff eigenvalue spectrum associated with any undirected graph. We use algebraic graph theory concepts to deduce expressions based on the degree sequence and the
number of triangles in the graph. We start by providing some needed graph theory background.

A. Spectral Graph Theory Background

In the case of a network with symmetrical connections, undirected graphs provide a proper description of the network topology. An undirected graph \( G \) consists of a set of \( N \) nodes or vertices, denoted by \( V \), and a set of edges \( E \), where \( E \subseteq V \times V \). In our case, \( (u, w) \in E \) implies \( (w, u) \in E \), and this pair corresponds to a single edge with no direction; the vertices \( u \) and \( w \) are called adjacent vertices (denoted by \( u \sim w \)) and the edge \( (u, w) \) is termed incident to vertices \( u \) and \( w \). We only consider simple graphs (i.e., undirected graphs that have no self-loops, so \( u \not= w \) for an edge \( (u, w) \), and no more than one edge between any two different vertices). A walk on \( G \) of length \( k \) from \( v_0 \) to \( v_k \) is an ordered set of vertices \( (v_0, v_1, \ldots, v_k) \) such that \( (v_i, v_{i+1}) \in E \), for \( i = 0, 1, \ldots, k - 1 \); if \( v_k = v_0 \) the walk is said to be closed.

The degree \( d_i \) of a vertex \( v_i \) is the number of edges incident to it. The degree sequence of \( G \) is the list of degrees, usually given in nonincreasing order. A degree sequence is graphic, or realizable, if there exists some graph with such a degree sequence.

The clustering coefficient, introduced in [6], is a measure of the number of triangles in a given graph, where a triangle is defined by a set of edges \( \{(i, j), (j, k), (k, i)\} \) such that \( i \sim j \sim k \sim i \). Specifically, this coefficient is defined as the total number of triangles in a graph, \( T(G) \), divided by the total number of triangles that could exist in a graph with \( N \) vertices, i.e., the number of triangles in a complete (all-to-all) graph. Thus, the coefficient is given by \( T(G) / (\binom{N}{3}) \).

It is often convenient to represent graphs via matrices. There are several choices for such a representation. For example, the adjacency matrix of a directed graph \( G \), denoted by \( A(G) = [a_{ij}] \), is defined entry-wise by \( a_{ij} = 1 \) if nodes \( i \) and \( j \) are adjacent, and \( a_{ij} = 0 \) otherwise, so it is a symmetric matrix. (Note that \( a_{ij} = 0 \) for simple graphs.) Notice also that the degree \( d_i \) can be written as \( d_i = \sum_j a_{ij} \). We can arrange the degrees on the diagonal of a diagonal matrix to yield the degree matrix, \( D = \text{diag}(d_i) \).

The (normalized) Laplacian matrix is defined in terms of the degree and adjacency matrices as \( L = I - D^{-1/2} A D^{-1/2} \). (Notice that for graphs with isolated nodes, the diagonal matrix \( D \) is not invertible, and the normalized Laplacian matrix is not defined.) The spectrum of this matrix arises in multiple applications (see [25] for a thorough treatment). The Kirchhoff matrix \( K \), also known as the combinatorial Laplacian, is defined as \( K = D - A \).

For undirected graphs, both \( L \) and \( K \) are symmetric positive semidefinite matrices [26]. Consequently, both matrices have a full set of \( N \) real and orthogonal eigenvectors with real non-negative eigenvalues. Since all rows of \( K \) sum to zero, it always admits a trivial eigenvalue \( \lambda_1 = 0 \), with corresponding eigenvector \( v_1 = (1, 1, \ldots, 1)^T \).

The moments of the Kirchhoff eigenvalue spectrum are central to our paper. Denote the eigenvalues of our \( N \times N \) symmetric Kirchhoff matrix \( K \) by \( 0 = \lambda_1 \leq \cdots \leq \lambda_N \). The empirical spectral density (ESD) of \( K \) is defined as

\[
\rho(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i)
\]

where \( \delta(\cdot) \) is the Dirac delta function. The \( k \)th-order moment of the ESD of \( K \) is defined as

\[
q_k = \frac{1}{N} \sum_{i=1}^{N} \lambda_i^k = \frac{1}{N} \text{tr} K^k.
\]

This identity is derived from the fact that trace is conserved under diagonalization (in general, under any similarity transformation). In the case of the first spectral moment, we obtain

\[
q_1 = \frac{1}{N} \text{tr}(D - A) = \frac{1}{N} \sum_{i=1}^{N} d_i = \bar{d}
\]

where \( \bar{d} \) is the average degree of the graph. For analytical and numerical reasons, we define the normalized Kirchhoff moment as

\[
\tilde{q}_k = \frac{1}{N} \sum_{i=1}^{N} (\lambda_i / \bar{d})^k = \frac{1}{N \bar{d}^k} \text{tr}(D - A)^k.
\]

The fact that \( D \) and \( A \) do not commute forecloses the possibility of using Newton’s binomial expansion on \( (D - A)^k \). On the other hand, the trace operator allows us to cyclically permute multiplicative chains of matrices. For example, \( \text{tr}(A A D) = \text{tr}(A D A) = \text{tr}(D A A) \). Thus, for words of length \( k \leq 3 \), one can cyclically arrange all binary words in the expansion of (2) into the standard binomial expression:

\[
\tilde{q}_k = \frac{k}{\alpha} \sum_{\alpha=0}^{k} \beta^k \frac{d_\alpha}{\alpha!} \text{tr}(\alpha^k D^\alpha), \quad k \leq 3.
\]

Also, we can make use of the identity \( \text{tr}(\alpha^k D^\alpha) = \sum_{i=1}^{N} (\alpha^k)_{\alpha_{ii}} \) to write

\[
\tilde{q}_k = \frac{k}{\alpha} \sum_{\alpha=0}^{k} \beta^k \sum_{i=1}^{N} \alpha^k_{\alpha_{ii}} (\alpha^k)_{ii}, \quad k \leq 3.
\]

Given that our interest is in networks of growing size (i.e., growing number of nodes \( N \)), a more explicit notation for \( \rho \) and \( q_k \) would have been \( \rho(N) \) and \( q_k^N \). However, for notational simplicity, we shall omit reference to \( N \) in these and other quantities in this paper.
Note that this expression is not valid for $k \geq 4$. For example, for $k = 4$, we have that $\text{tr}(A^{4}D^{2}) \neq \text{tr}(D^{2}A^{4})$.

We now analyze each summand in expression (4) from a graph-theoretical point of view. Specifically, we find a closed-form expression for each term $\text{tr}(A^{i}D^{j})$, for all pairs $1 \leq i + j \leq 3$, as a function of the degree sequence and the number of triangles in the network. In our analysis, we make use of the following results from [26]:

**Lemma 1:** The number of closed walks of length $\alpha$ joining node $i$ to itself is given by the $i$th diagonal entry of the matrix $A^{\alpha}$, where $A$ is the adjacency matrix of the graph.

**Corollary 2:** Let $G$ be a simple graph. Denote by $t_i$ the number of triangles touching node $i$. Then

$$
(A)_{ii} = 0, (A^2)_{ii} = d_i, \text{ and } (A^3)_{ii} = 2t_i.
$$

After substituting (5) into (4) and straightforward algebraic simplifications, we obtain the following exact expression for the low-order normalized spectral moments of a given Kirchhoff matrix $K$:

$$
\bar{q}_k = \left\{ \begin{array}{ll}
\frac{1}{N^3} \sum_{i=1}^{N} d_i^2 + \sum_{i=1}^{N} d_i, & \text{for } k = 1, \\
\frac{1}{N^3} \left( \sum_{i=1}^{N} d_i^3 + 3 \sum_{i=1}^{N} d_i^2 \right) - 2T, & \text{for } k = 3 \\
\end{array} \right.
$$

where $T$ is the total number of triangles in the network.

We illustrate the use and validity of this result in the following example.

**Example 3:** In this example we compute the low-order Kirchhoff moments of a complete (all-to-all) graph of $N$ nodes. A complete graph of $N$ nodes has $N - 1$ Kirchhoff eigenvalues located at $N$, and one trivial eigenvalue at 0. Thus, the first-, second-, and third-order moments are $N - 1, (N - 1)N, (N - 1)N^2$, respectively. Equation (6) allows us to compute these moments without an explicit eigenvalue decomposition. In a complete graph the degree sequence presents the uniform value $d_i = \bar{d} = N - 1$. Also, the total number of triangles $T$ is $\binom{N}{3}$.

Therefore, substituting in (6) and simple algebraic simplifications result in $q_1 = (N - 1), q_2 = N - 1, q_3 = N^3 - N^2$ (in agreement with the results from the eigenvalue decomposition).

It is worth noting how our spectral results are written in terms of two widely reported measurements [4]: the degree sequence and the clustering coefficient (which provides us with the total number of triangles.) This allows us to compute low-order spectral moments of many real-world networks without performing an explicit eigenvalue decomposition.

### III. Probabilistic Analysis of Kirchhoff Moments for Chung–Lu Graphs

We first analyze the Kirchhoff eigenvalue distribution of a probabilistic graph with a prescribed expected degree sequence. After describing the probabilistic graph, we deduce approximate expressions for the low-order expected Kirchhoff moments, accurate in the limit of large networks. Results on concentration of the spectral moments around their mean values are also provided.

### A. Description of the Chung–Lu Model

Many probabilistic graphs have been proposed to generate complex topologies since Erdős and Rényi published their seminal work on random graphs in 1959 [8]. Even though this model exhibits serious limitations in its ability to replicate degree distributions of interest (it only presents binomial or Poisson distributions), it started a whole field of active research. More recently, availability of extensive databases has triggered the appearance of models incorporating empirical measurements (see [3] for a thorough exposition).

The degree sequence is the most widely reported statistic. One can find in the literature several probabilistic graphs able to replicate a given degree sequence [10], [11]. In this paper we study a graph model recently proposed by Chung and Lu [12] presenting a given expected degree sequence. Many interesting mathematical results concerning graph-theoretical properties of this model have already been reported [27]–[29].

The Chung–Lu model is a probabilistic graph ensemble with a prescribed expected degree sequence $\mathbf{w} = (w_1, w_2, \ldots, w_N)$. We assume, without loss of generality, that the sequence is given in nonincreasing order. This random graph is constructed by independently assigning edges between each pair of nodes $(i, j)$, $1 \leq i < j \leq N$ with corresponding probabilities

$$
p_{ij} = \Pr(a_{ij} = 1) = \rho w_i w_j, \text{ where } \rho = \left( \sum_k w_k \right)^{-1}.
$$

For the expected degree sequence to be graphic, or realizable, we need the condition $w_i^2 < \sum_k w_k$, which ensures $p_{ij} \leq 1$. As a particular case, the classical Erdős-Rényi graph is recovered from a constant expected degree sequence, i.e., $w_i = \overline{d}$ for all $i$.

Important results concerning the distribution of connected components [12], average distance and diameter [27], and several spectral properties [29] of the Chung–Lu model can be found in the literature. In [29], it is shown that the maximum eigenvalue of the adjacency matrix converges almost surely to a simple explicit expression under some constraints on the expected degree sequence. Also, for graphs with relatively large minimum degree, the Laplacian eigenvalue spectrum follows a semicircle law for asymptotically large graphs. The main drawback of these results for many potential applications is the assumption that the expected nodal degrees grow as a function of the network size (for example, that the maximum expected degree grows to infinity for asymptotically large networks). Since in many real networks the maximum degree is bounded independently of $N$, the available results are not always applicable.

In this paper, we analyze the asymptotic Kirchhoff spectrum for networks with bounded expected degree sequences. Even though our analysis is far from complete, in that only low-order moments are provided, valuable information regarding spectral properties can be retrieved from our results. Some preliminary definitions and results are needed.

### B. Definitions and Useful Results

We first define some concepts related to the Chung–Lu random graph model with expected degree sequence
The sample space of this model consists of all graphs with $N$ nodes. We define the $k$-th normalized expected moment over this probability space as

$$\bar{m}_k = \frac{1}{N} \bar{d}^k \mathbb{E} \left[ \sum_{i,j=1}^{N} \lambda_i \right]$$

where $\lambda_i$ denotes the $i$-th Kirchoff eigenvalue of the graph, and $\bar{d}$ is the mean expected degree. In this section we derive closed-form expressions for the low-order normalized expected moments of asymptotically large graphs. Our approach consists of an asymptotic probabilistic analysis of the expressions in (6).

Several useful results are needed in our derivations. The following lemma can be found in [30]:

**Lemma 4:** If $X$ is the number of successes in $N$ nonidentical binomial trials, with probability of success $p_i$ on the $i$-th trial, $i = 1, 2, \ldots, N$, satisfying both $\sum_{i=1}^{N} p_i = w$ and $\max p_i \to 0$ as $N \to \infty$, then the distribution of $X$ has a Poisson limit with mean $w$, $\text{Poi}(w)$.

Since $\text{Pr}(a_{ij} = 1) = p_{ij} = p_i w_i w_j$ for our Chung–Lu graph, we obtain the following corollary.

**Corollary 5:** Consider the Chung–Lu model (7) with bounded expected degree sequence, i.e., $w_i \leq W < \infty$ for $i = 1, \ldots, N$, where $W$ does not depend on $N$. Then the degree of node $i, d_i = \sum_{j=1}^{N} a_{ij}$, behaves as a Poisson random variable, $\text{Poi}(w_i)$ for $N \to \infty$.

In our derivations, we will make use of the first three asymptotic moments of the (Poisson) degrees

$$\begin{align*}
\mathbb{E}[d_i] &\to w_i \\
\mathbb{E}[d_i^2] &\to w_i^2 + w_i \\
\mathbb{E}[d_i^3] &\to w_i^3 + 3w_i^2 + w_i.
\end{align*}$$

We also need to compute the expected number of triangles, $T$, in the Chung–Lu model. The next lemma states that the size-normalized expected number of triangles goes to zero as $O(N^{-1})$.

**Lemma 6:** Consider the random graph model (7) with $w_i \leq W < \infty$, for all $i$. The size-normalized expected number of triangles satisfies

$$\mathbb{E}[T/N] = O(N^{-1}).$$

**Proof:** The number of triangles can be written in terms of the adjacency matrix entries as

$$T = \sum_{1 \leq i < j < k \leq N} a_{ij}a_{jk}a_{ki}.$$  

From (7), the expectation can be written as

$$\mathbb{E}[T/N] = \frac{1}{\bar{d}^4 N^4} \sum_{1 \leq i < j < k \leq N} w_i^2 w_j^2 w_k^2 \leq \frac{w_i^4}{\bar{d}^4 N^4} \left( \frac{N}{3} \right)^3 = O(N^{-1}).$$

(Remember that the constraint $w_i \leq W < \infty$ implies that $w_1 = O(1)$ and $\bar{d} = O(1)$.)

### C. Asymptotic Probabilistic Analysis

In this section, we deduce the asymptotic values of the normalized expected low-order moments. We deduce this result by taking the term by term limit in (6).

**Theorem 7:** Consider the random graph ensemble defined in (7). For bounded expected degrees, i.e., $w_i \leq W < \infty$, $i = 1, \ldots, N$, the limiting normalized expected moments of the eigenvalue distribution corresponding to the associated Kirchoff matrix ensemble are given by

$$\bar{m}_k \to \frac{1}{\bar{d}^k N} \left( \sum_{i=1}^{N} u_i^2 + 2 \sum_{i=1}^{N} w_i \right), \quad \text{for } k = 1$$

$$\frac{1}{\bar{d}^k N} \left( \sum_{i=1}^{N} u_i^3 + 6 \sum_{i=1}^{N} w_i^2 \right), \quad \text{for } k = 2$$

$$+4 \sum_{i=1}^{N} w_i^4, \quad \text{for } k = 3$$

for $N \to \infty$.

**Proof:** We must compute the asymptotic expected values of (6) for $N \to \infty$. Since we fulfill the requirements for Lemma 4, we can apply Corollary 5 to the degree sequence. Thus, the moments of the degrees, $E[d_i^k]$, can be written as (8). Also, from Lemma 6, the size-averaged number of triangles goes to zero asymptotically in network size, i.e., $E[T/N] \to 0$ for $N \to \infty$. Substituting in (6), we reach (9) after simple algebraic manipulations.

We validate this result with the following example.

**Example 8:** In this example, we analyze the spectral moments of a power-law random graph. We follow the definition of the power-law degree sequence given in [29]

$$w_i = c i^{-\beta}$$

for $i_0 \leq i \leq i_0 + N, \beta > 1$. (10)

Also, coefficients $c$ and $i_0$ are defined as $c = (\beta - 2)/(\beta - 1)\bar{d}N^{(1/\beta - 1)}$, $i_0 = N((\bar{d}/(\beta - 2))/(W(\beta - 1)))^{1/\beta - 1}$, where $\bar{d}$ and $W$ are prescribed averaged and maximum expected degrees, respectively. The Kirchoff spectrum of a single realization of a network with 500 nodes and parameters $\bar{d} = 50, W = 100$, and $\beta = 3, 4$ can be observed in Fig. 1. In this case, we can arrive at analytical solutions of the summations in (9) using the approximation

$$\sum_{j=v_0}^{i_0+N} u_j^p \approx c^p \left( \frac{\beta - 1}{\beta - 1 - p} \right) \left( i_0 + N \right)^{\frac{p+1}{\beta - 1 - p}}$$

for $\beta > 2$ and under the assumption $N \gg i_0$. The numerical moments from eigenvalue decomposition and the analytical predictions from (9) are compared in Table I, where it is important to point out that the indicated numerical values are obtained for one realization only, with no benefit from averaging.
No statement about the concentration of the moments around their expected values has been given so far. In fact, the distributions of the various moments asymptotically concentrate around the respective expectations. Results on quadratic convergence of the moments to their expectations are stated in the following section.

D. Concentration Results

In [31], we give a detailed proof of quadratic convergence of the moments to their means. In this section, we state the relevant result and also present a numerical illustration.

Theorem 9: Consider the random graph ensemble defined via (7). For bounded expected degree sequence, \( w_i \leq W < \infty, i = 1, \ldots, N \), the low-order normalized moments of the Kirchhoff eigenvalue spectrum, \( \tilde{m}_k \), tend quadratically to the normalized expected moments, \( m_k \), i.e.,

\[
\lim_{N \to \infty} \mathbb{E}[(\tilde{m}_k - m_k)^2] = 0, \quad \text{for } 1 \leq k \leq 3.
\]

In Fig. 2 we show a numerical illustration of quadratic convergence of the moments in the ensemble. In this experiment we obtained the empirical mean and variance of the moments for different network sizes. Specifically, the network sizes go from \( N = 10 \) to 500, with increments of 10 nodes. For each network size, we take 10 random realizations with a prescribed linear expected degree sequence, with \( w_1 = 10 \) and \( w_N = 3 \). In the upper part of Fig. 2, we show the evolution of the empirical mean values for the second and third normalized moments\(^2\) for growing network size. We can observe a clear convergence of the means to the theoretical asymptotic limits indicated by the dashed horizontal lines. In the lower part of Fig. 2, we show the evolution of the empirical variance for the second and third normalized moments. It can be observed that, in conformity with Theorem 9, the variances tend to zero for growing network size.

This result indicates that typical random graph realizations of even moderate size display moments close to the expectations in (9). Similar concentration results regarding other spectral properties were reported in [32]. In the next section, we apply our spectral technique to the spectral analysis of small-world networks.

\(^2\)We use the normalized definition of spectral moments; therefore, the first moment is always equal to 1.

IV. MOMENT ANALYSIS FOR A SMALL-WORLD NETWORK

The results and techniques developed in Sections II and III are also applicable to other random graph ensembles. In particular, we dedicate this section to studying the asymptotic low-order moments of the Kirchhoff matrix associated with a canonical small-world graph, [4]. Equation (6) allows us to compute the first three expected Kirchhoff moments of any given random graph ensemble as a function of the moments of the degrees, \( \mathbb{E}[d_i], \mathbb{E}[d_i^2], \) and \( \mathbb{E}[d_i^3] \) and the expected number of triangles, \( \mathbb{E}[T] \). In the following, we present closed-form expressions for the expected Kirchhoff moments of the following small-world network.

1) Start with a regular four-neighbors ring (see Fig. 3, in which \( N = 8 \), and where edges inside the octagon are not part of this regular ring).

2) Add random edges with probability \( p/N \), for \( p \in \mathbb{N}_c \) (in Fig. 3, we draw these random edges inside the octagon.)

For \( N \to \infty \), one can prove the following:

a) The expected number of triangles in this random graph is

\[
\mathbb{E}[T] = (1 + o(1))N.
\]

In other words, the number of triangles in the small-world graph is dominated by the number of triangles in the regular four-neighbors ring.

b) The degrees, \( d_i \), are governed by the following “shifted” Poisson distribution:

\[
\Pr(d_i = k) = \begin{cases} 
\frac{\lambda^{k-4} e^{-\lambda}}{(k-4)!}, & \text{for } k \geq 4 \\
0, & \text{for } k < 4
\end{cases}
\]

where the Poisson distribution is related to the inclusion of random edges inside the regular ring, and the shifting by...
is contained in a certain region on the real line. This region of synchronization is exclusively defined by the dynamics of each isolated oscillator and the type of coupling [19]. This simplifies the problem of synchronization to the problem of locating the Kirchhoff eigenvalue spectrum.

We exploit the fact that the first three moments of the spectral distribution can be exactly matched using a triangular distribution function; this triangular function is used to estimate the support of the spectral distribution. We illustrate this method with several examples, and use it to predict whether or not the eigenvalue spectrum is located in the region of synchronization. Numerical simulations support our predictions.

A. Synchronization as a Spectral Graph Problem

Several techniques have been proposed to analyze the synchronization of coupled identical oscillators. In [33], well-known results in control theory, such as the passivity criterion, the circle criterion, and a result on observer design, are used to derive synchronization criteria for an array of identical nonlinear systems. In [34], the authors use contraction theory to derive sufficient conditions for global synchronization in a network of nonlinear oscillators. Belykh et al. developed in [35] a method, called the Connection Graph Stability method, that combines Lyapunov functions with graph theoretical reasoning for proving synchronization in a network of identical nonlinear oscillators. We pay special attention to the master stability function approach, [19]. This approach provides us with a criterion for local stability of synchronization based on the numerical computation of Lyapunov exponents. Even though quite different in nature, the mentioned techniques emphasize the key role played by the graph eigenvalue spectrum.

In this paper we consider a time-invariant network of N identical oscillators, one located at each node, linked with ‘diffusive’ linear coupling. The state equations modeling the dynamics of the network are

\[ \dot{x}_i = f(x_i) + \gamma \sum_{j=1}^{N} a_{ij} I(x_j - x_i), \quad i = 1, \ldots, N \]  

(14)

where \( x_i \) represents an \( n \)-dimensional state vector corresponding to the \( i \)-th oscillator. The nonlinear function \( f(\cdot) \) describes the (identical) dynamics of the isolated nodes. The positive scalar \( \gamma \) can be interpreted as a global coupling strength parameter. The \( n \times n \) matrix \( I \) represents how states in neighboring oscillators couple, and \( a_{ij} \) are the entries of the adjacency matrix. By simple algebraic manipulations, one can write down (14) in terms of the Kirchhoff entries, \( K = [k_{ij}] \), as

\[ \dot{x}_i = f(x_i) - \gamma \sum_{j=1}^{N} k_{ij} x_j, \quad \text{for } i = 1, \ldots, N. \]  

(15)

We say that the network of oscillators is at a synchronous equilibrium if \( x_1(t) = x_2(t) = \cdots = x_N(t) = \phi(t) \), where \( \phi(t) \) represents a solution for \( \dot{x} = f(x) \). In [19], the authors studied the local stability of the synchronous equilibrium. Specifically, they considered a sufficiently small perturbation, denoted by \( \epsilon_i(t) \), from the synchronous equilibrium, i.e.,

\[ x_i(t) = \phi(t) + \epsilon_i(t). \]
After appropriate linearization, one can derive the following equations to approximately describe the evolution of the perturbations:

$$\dot{\epsilon}_i = \mathbf{Df}(t)\epsilon_i(t) - \gamma \sum_{j=1}^{N} k_{ij}\Gamma_2\epsilon_j(t), \quad \text{for } i = 1, \ldots, N, \quad (16)$$

where $\mathbf{Df}(t)$ is the Jacobian of $f(\cdot)$ evaluated along the trajectory $\phi(t)$. This Jacobian is an $n \times n$ matrix with time-variant entries. Following the methodology introduced in [19], (16) can be similarity transformed into a set of linear time-variant (LTV) ODEs of the form

$$\dot{\xi}_i = [\mathbf{Df}(t) + (\gamma \lambda_i)\Gamma_2]\xi_i, \quad \text{for } i = 1, \ldots, N \quad (17)$$

where $\{\lambda_i\}_{1 \leq i \leq N}$ is the set of eigenvalues of $K$. Based on the stability analysis introduced in [19], the network of oscillators in (14) presents a locally stable synchronous equilibrium if the corresponding maximal Lyapunov exponent of (17) is negative for $i = 2, \ldots, N$, i.e., for the nontrivial $\lambda_i$.

Inspired by (17), Pecora and Carroll studied in [19] the stability of the following parametric LTV ODE in the parameter $\sigma$:

$$\dot{\xi} = [\mathbf{Df}(t) + \sigma \Gamma]\xi \quad (18)$$

where $\mathbf{Df}(t)$ is the linear time-variant Jacobian in (16). The master stability function (MSF), denoted by $F(\sigma)$, is defined as the value of the maximal Lyapunov exponent of (18) as a function of $\sigma$. Note that this function depends exclusively on $\mathbf{Df}(t)$ and $\Gamma$, and is independent of the coupling topology, i.e., independent of $K$. The region of (local) synchronization is, therefore, defined by the range of $\sigma \geq 0$ for which $F(\sigma) < 0$. For a broad class of systems, the MSF is negative in a bounded range, $\sigma \in [\sigma_{\text{min}}, \sigma_{\text{max}}] \equiv S$ (we assume this is the case in subsequent derivations). In order to achieve synchronization, the set of scaled nontrivial Kirchhoff eigenvalues, $\{\gamma \lambda_i\}_{2 \leq i \leq N}$, must be located inside the region of synchronization, $S$. This condition is equivalent to: $\gamma \lambda_2 > \sigma_{\text{min}}$ and $\gamma N < \sigma_{\text{max}}$.

In the following example, we illustrate the use of the above methodology.

**Example 10:** Consider a ring of 6 coupled Rössler oscillators [36], [37]. The dynamics of the coupled network is

$$\dot{x}_i = -(y_i + z_i) + \gamma \sum_{j \in \mathcal{R}(i)} (x_j - x_i),$$
$$\dot{y}_i = x_i + a y_i,$$
$$\dot{z}_i = b + z_i(x_i - c),$$

where $i = 1, \ldots, 6$, and $\mathcal{R}(i) \equiv \{i + 1 \mod 6, i - 1 \mod 6\}$. We choose parameter values $a = 0.2$, $b = 0.2$, and $c = 7.0$.

In Fig. 4, we represent the numerically computed MSF for this oscillator. One can observe how the MSF divides the positive real line into two regions: $S \equiv \{\sigma \in \mathbb{R} \mid F(\sigma) < 0\}$, and $U \equiv \{\sigma \in \mathbb{R} \mid F(\sigma) \geq 0\}$. In our case, the numerically-computed region of synchronization is $S \approx (0.08, 4.96)$. According to [19], the ring of coupled Rössler oscillators synchronizes locally if $\{\gamma \lambda_i\}_{i=2,\ldots,N} \subset S$, where $\{\lambda_i\}_{i=2,\ldots,N}$ are the nontrivial eigenvalues of the Kirchhoff matrix $K$ corresponding to the ring, and $\gamma$ is the global coupling strength. The nontrivial eigenvalues of the ring are $\{1, 1, 3, 3, 4\}$; therefore, we achieve stability under the condition $0.08 < \gamma < 4.96/4 = 1.24$. Fig. 5 represents the temporal evolution of the $x_i$ states of the 6-ring for $\gamma = 1.0$.

In the next subsection, we propose an approach to estimating the support of the eigenvalue distribution of large-scale probabilistic networks from low-order spectral moments. This allows us to predict local synchronization in a large-scale Chung-Lu network of identical oscillators.

**B. Triangular Reconstruction of the Kirchhoff Spectrum**

Our approach, described more fully in [31], approximates the spectral distribution with a triangular function that exactly preserves the first three moments.

We define a triangular distribution $t(\lambda)$ based on a set of abscissae $\{x_1, x_2, x_3\}$ as

$$t(\lambda) := \begin{cases} \frac{h}{x_2 - x_1}(\lambda - x_1), & \text{for } \lambda \in [x_1, x_2], \\ \frac{h}{x_3 - x_2}(\lambda - x_3), & \text{for } \lambda \in [x_2, x_3], \\ 0, & \text{otherwise} \end{cases}$$

where $h = 2/(x_3 - x_1)$. The first three moments of this distribution, as a function of the abscissae, are given by

$$M_1 = \frac{1}{3}(x_1 + x_2 + x_3),$$
$$M_2 = \frac{1}{6}(x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3),$$
$$M_3 = \frac{1}{16}(x_1^3 + x_2^3 + x_3^3 + x_1x_3^2 + x_2x_3^2 + x_1x_2x_3). \quad (19)$$
Our task is to find the set of values \( \{x_1, x_2, x_3\} \) in order to fit a given set of moments. The resulting system of algebraic equations is amenable to analysis, based on the observation that the moments are symmetric polynomials\(^3\). Following the methodology in [31], we can find the abscissae \( \{x_1, x_2, x_3\} \) as roots of the polynomial:

\[
x^3 - \Pi_1 x^2 + \Pi_2 x - \Pi_3 = 0
\]  
\((20)\)

where

\[
\Pi_1 = 3M_1
\]
\[
\Pi_2 = 9M_1^2 - 6M_2
\]
\[
\Pi_3 = 27M_1^3 - 36M_1M_2 + 10M_3.
\]  
\((21)\)

The numerical results in the following two examples show how this technique provides a reasonable estimation of the Kirchhoff spectrum.

**Example 11:** We fit a triangular distribution to the Kirchhoff spectrum of a Chung–Lu network with \( N = 200 \) nodes and \( E = 1000 \) expected edges. The expected degree sequence is chosen to be affine, with a maximum expected degree of \( W = 13 \). From the data in the statement, we can determine the average degree \( \bar{\omega} = 2E/N = 10 \), and the expected degree sequence

\[
w_i = \frac{7}{199} (i-1) - \frac{13}{199} (i-200), \quad i = 1, 2, \ldots, 200.
\]  
\((22)\)

One can then use (9) to compute the expected moments. These moments are \( m_1 = 10 \), \( m_2 = 122 \) and \( m_3 = 1740 \). Using the moment-preserving reconstruction above, setting \( M_k = m_k \) for \( k = 1, 2, 3 \), we fit a triangular function with abscissae \( x_i = \{1.798, 4.732, 23.470\} \). In Fig. 6, we compare the triangular function with a histogram of the Kirchhoff eigenvalues (notice the position of the trivial eigenvalue at the origin).

**Example 12:** We now fit a triangular function to the analytical Kirchhoff moments obtained in Section IV for the small-world network described there. Using the values of the analytical expectations in Table II, we obtain the following abscissae for the triangular function: \( x_1 = -0.76 \), \( x_2 = 5.45 \), and \( x_3 = 13.10 \). In Fig. 7, we represent one empirical realization of the Kirchhoff eigenvalues histogram and its triangular fitting. We observe how the triangular function provides a reasonable estimation of the eigenvalue histogram for the small-world network.

As shown in these examples, our methodology allows us to estimate the support of the Kirchhoff spectrum. This estimation can be used, for example, to predict synchronization in a large-scale probabilistic network. On the other hand, it is not always possible to fit a triangular function to any given three moments. In the Appendix we provide detailed conditions under which such a triangular construction is possible.

### C. Estimation of Synchronization

We use our moment expressions (9) to predict synchronization in a large-dimensional Chung–Lu random graph. Specifically, we study a large random network of Rössler oscillators, as those in Example 10. The expected degree sequence is the same as the one analyzed in Example 8. Based on the moment-preserving triangular reconstruction, we will predict the range of coupling strength \( \gamma \) to achieve synchronization. The steps in our methodology are the following:

1) We compute the region of synchronization following the technique presented in Subsection V-A. As illustrated in Example 10, we note that the set of scaled eigenvalues \( \{\gamma \lambda_i\}_{i=2,\ldots,N} \) must lie in the region of (local) synchronization, \( S \), to achieve (local) synchronization. Recall that in our example \( S = (0.08, 4.96) \).

2) We estimate the support of the nontrivial Kirchhoff eigenvalue spectrum, \( \{\lambda_i\}_{i=2,\ldots,N} \), associated with a random graph of given expected degree sequence. This step uses the triangular heuristic from Subsection V-B and the low-order moment expressions in (9). From Example 11, we see that \( s_l = 1.798 \) and \( s_u = 23.470 \) are good estimates of the lower and upper extremes of the nontrivial spectral support, respectively.

3) Compute the interval of coupling strength \( \gamma \) to locate the scaled set of eigenvalues, \( \{\gamma \lambda_i\}_{i=2,\ldots,N} \), in the region of synchronization. Specifically, the minimum value of coupling strength that keeps the scaled set \( \{\gamma \lambda_i\}_{i=2,\ldots,N} \) above 0.08 is given by \( \gamma_{\text{min}} = 0.08/1.798 = 0.0444 \). On the other hand, the maximum value of coupling that keeps the set below 4.96, is \( \gamma_{\text{max}} = 4.96/23.470 = 0.2113 \).

Therefore, a random network of coupled Rössler oscillators with the linear expected degree sequence in (22) is predicted...
to synchronize whenever the global coupling strength satisfies \( \gamma \in (0.0444, 0.2113) \).

We now present numerical simulations supporting our conclusions. Consider again a 200-node Chung-Lu network of Rössler oscillators with the linear expected degree sequence in (22). We run simulations of the dynamics of the oscillators for different values of the global coupling strength \( \gamma \). For each coupling strength, we plot figures with the superposition of 200 plots representing the time evolution of all the \( x \) states. Also, we represent the superposition of 199 plots representing the errors between the \( x \) state of each oscillator and the one with the highest expected degree, i.e., \( x_i(t) - x_1(t), i = 2, 3, \ldots, 200 \) (these errors are scaled in the plot for better discernment).

In Fig. 8(a), we use a coupling strength \( \gamma_a = 0.01 \). Since the coupling strength is below 0.0444, the eigenvalue spectrum invades the unstable regions. It can be observed in Fig. 8(a) how synchronization is clearly not achieved (in the scaled error plots, we observe an exponential divergence in time). In Fig. 8(b) we present the temporal evolution for a coupling strength \( \gamma_b = 0.1 \), which is inside the region of stable coupling strength. In this case, we observe an exponential convergence of the errors. In Fig. 8(c) we use a strength \( \gamma_c = 0.35 \), outside the region of stable coupling strength. In the time evolution, one can observe an initial period of convergence followed by divergence.

VI. CONCLUSION AND FUTURE RESEARCH

In this paper, we have studied the eigenvalue distribution of the Kirchhoff matrix of a large-scale probabilistic network with a prescribed expected degree sequence. We have focused our attention on the low-order moments of the spectral distribution. For the case of a general graph, we have deduced expressions for these moments as functions of the degree sequence and the number of triangles. For the case of a large-scale probabilistic Chung-Lu network, as well as a specific small-world graph, we have analyzed the asymptotic behavior of the moments, giving closed-form expressions. As an example of the role of this spectrum in many dynamical problems, we have examined the problem of synchronization of a network of oscillators. Using our analytical expressions, we have studied a large-scale probabilistic network of oscillators described by their expected degree sequence. Our approach is based on performing a triangular reconstruction matching the first three moments of the unknown spectral measure. Our numerical results match our predictions rather well.

Several questions remain open. The most important extension would be to deduce expressions for higher order moments of the Kirchhoff spectrum. A more detailed reconstruction of the spectral measure can be done based on more moments. As a consequence, a better precision in our estimations of the spectral support would also be expected.

APPENDIX

REALIZABILITY OF THE TRIANGULAR APPROXIMATION

In this appendix, we address the problem of deciding whether or not a given moment sequence corresponds to a triangular distribution. Our solution is based on the following observation: A sequence \( \{M_1, M_2, M_3\} \) corresponds to a triangular distribution if and only if the roots of the polynomial in (20) are all real. Hence, the discriminant of this polynomial [38] provides us with an algebraic criterion for triangular realizability.

Using the expressions in (21), we can write the discriminant of (20) as a function of the sequence of moments, \( \{M_1, M_2, M_3\} \). After elementary algebraic simplifications, we derive the following algebraic criterion:

\[
8 \left( M_2^3 - M_2 \right)^3 + 25 \left( 2 M_2^3 - 3 M_1 M_2 + M_3 \right)^2 < 0. \tag{23}
\]

Similarly, we can use the criterion in (23) to decide whether or not there exists a triangular function matching the sequence of Kirchhoff moments associated with a random graph with a given expected degree sequence, \( \{w_i \}_{1 \leq i \leq N} \). In particular, we substitute the expressions for the Kirchhoff moments (9) in the criterion (23) to derive the following realizability condition:

\[
8 \left( -2 W_1 + W_2^2 - W_2 \right)^3 + 25 \left[ -6 W_1^2 + 2 W_1^3 + W_1 (4 - 3 W_2) + 6 W_2 + W_3 \right]^2 < 0
\]

where \( W_k = \sum_{i=1}^N w_i^k \).

REFERENCES


