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An Extremum Solution of the Monin–Obukhov Similarity Equations

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ABSTRACT

An extremum hypothesis of turbulent transport in the atmospheric surface layer is postulated. The hypothesis has led to a unique solution of Monin–Obukhov similarity equations in terms of simple expressions linking shear stress (momentum flux) and heat flux to mean wind shear and temperature gradient. The extremum solution is consistent with the well-known asymptotic properties of the surface layer. Validation of the extremum solution has been made by comparison to field measurements of momentum and heat fluxes. Furthermore, a modeling test of predicting surface heat fluxes using the results of this work is presented. A critical reexamination of the interpretation of the Obukhov length is given.

1. Introduction

The Monin–Obukhov similarity theory (MOST) (Obukhov 1946; Monin and Obukhov 1954) is considered the most successful theory of atmospheric turbulence, leading to innumerable publications over five decades of active development and application. However, as Obukhov cautioned in his seminal paper, MOST is semiempirical. “Empirical” is understood to imply that the mean wind and temperature profiles are formulated (or fitted) based on dimensional analysis (Bridgman 1931) rather than derived from more fundamental physical laws (underlying the dynamics of turbulence); “semi” indicates that some physics of turbulence does enter the formalism through the selection of the key dimensional parameters. Although not a physical law per se, MOST elements may have physical interpretation. For example, the Obukhov length scale, the most important parameter in the MOST, was interpreted by Monin and Obukhov as the “thickness of the sublayer of dynamic turbulence.”

In the classical MOST formalism, four dimensionless state variables (i.e., momentum flux, heat flux, mean wind shear, and temperature gradient) are expressed as functions of a single dimensionless independent variable (i.e., vertical coordinate) and related through two dimensionless governing equations according to the Buckingham theorem (Buckingham 1914). Therefore, any two of the state variables are expected to be uniquely determined from the other two state variables. For instance, the heat flux can be determined by specifying the temperature gradient and momentum flux. It turns out that given wind shear and heat flux under stable conditions do not always correspond to a unique value of momentum flux, and neither do temperature gradient and heat flux under unstable conditions always yield a unique value of momentum flux according to the widely used MOST equations. There are no known rules for selecting a unique physical solution when that happens. Since the two governing equations include two nonlinear functions characterizing the atmospheric instability, the heat flux must be obtained numerically using an iterative procedure. In practice, we sometimes face the problem of a nonconverging iteration. An arbitrary decision has to be made (to stop the nonconverging iterations) when this situation occurs, resulting in inaccurate, even potentially incorrect, solution(s). One main motivation of this study is to find a rationale leading to a unique solution to avoid the iterative procedure in the theoretical
and modeling applications of the MOST where the parameterization of eddy diffusivity becomes much simplified. We explore the possibility of an extremum principle.

Basic physical laws, such as Newton’s law in classical mechanics, are often equivalent to extremum principles (e.g., Lanczos 1970). Busse (1978) developed an optimality theory of turbulence with some success. Whether there exists general simple extremum principle(s) for turbulence is unclear; nevertheless, extremum hypotheses have been proposed. Cheng et al. (2005) argued that the MOST holds once the turbulence “became homogeneous and stationary” when the wind shear and buoyancy are in equilibrium. It is well known that some physical quantities are extremized for a system at equilibrium. The principle of minimum potential energy in dynamics (Goodman and Warner 2001) and the second law in thermodynamics (Kondepudi and Prigogine 1998) are two good examples. The clue pointing to the proposed hypothesis comes from the fact that the multiple solutions allowed by the MOST equations become unique when some of the state variables reach extrema, referred to herein as the extremum solution of the Monin–Obukhov similarity equations. The proposed extremum hypothesis may be justified qualitatively with the emerging theory of maximum entropy production (MEP) (Dewar 2003, 2005). Theoretical justification of the hypothesis will also be offered by confirming that the extremum solution has the desired asymptotic properties. We present two kinds of experimental evidence in support of the hypothesis: direct validation of a newly derived extremum principle selects the unique solution will be further elaborated below.

Here we follow the convention in hydrometeorology that a flux is positive when it points away from the land surface and an entity increasing with height is defined as positive. Note that minimizing a negative variable is equivalent to maximizing its absolute value. According to this definition, minimum (downward) heat flux under stable conditions implies its magnitude is maximum, and minimum (negative) temperature gradient under unstable condition implies its magnitude being maximum. Some clarification is needed to avoid ambiguity and confusion. Turbulence is a dynamic system with a large number of degrees of freedom. There are many possible scenarios of turbulent flow that conserve mass, momentum, and energy and have other physical and/or mathematical constraints. For the stationary and homogeneous turbulence in the ASL, a variety of combinations of fluxes and gradients described by the (two) dimensionless equations in the MOST are physically possible. Theoretically, there are an infinite number of combinations of the four unknowns (i.e., wind shear, temperature gradient, heat flux, and momentum flux) that satisfy the two dimensionless equations. The proposed extremum principle selects the unique extremum solution among the possible ones allowed by the dimensionless equations in the MOST. The physical significance of the extremum solution will be further elaborated below.

2. An extremum hypothesis

The idea of the proposed hypothesis results from reasoning based on some physical and mathematical arguments. Based on daily experience, it is reasonable to state that mechanical mixing (forced convection) is arguably more effective than buoyancy (free convection) in transporting heat and momentum; “effectiveness” may be measured by the relative magnitude of fluxes to the corresponding scalar gradients. The effectiveness of a transport mechanism implies that the corresponding flux(es) would be extremum (under certain conditions or constraints). In modeling turbulence in the atmospheric surface layer (ASL) using the MOST, we expect to obtain a unique solution of unknown quantities (such as fluxes of momentum and heat), given the input information (such as temperature gradient and wind shear).

By inspection of the similarity equations of the MOST (see below), we realized that unique solution of the similarity equations coincides with extremum fluxes. The exact conditions under which this (i.e., extremum and uniqueness) occurs may be expressed in terms of the following extremum hypothesis about the turbulent structure of the ASL described by the MOST:

- Within the atmospheric surface layer in an environment for which the MOST applies, momentum flux would reach such values as to minimize heat flux and wind shear under stable conditions and to minimize heat flux and temperature gradient under unstable conditions.

3. Extremum solution

In the framework of the MOST, the turbulence in the ASL is characterized by four independent variables: the distance from the surface \(z\), shear stress \(\tau\) (equivalent to a velocity scale \(u_{*}\)), kinematic heat flux \(H/\rho C_p\) (equivalent to a temperature scale \(\theta_{*}\)), and the buoyancy \(g/T_0\) related through

\[
\frac{\tau}{\rho} = u_{*}^2; \tag{1}
\]

\[
\frac{H}{\rho C_p} = -u_{*} \theta_{*}, \tag{2}
\]
where $\rho$ is the (constant) density of air, $c_p$ is the heat capacity of the air at constant pressure, and $T_0$ is a reference temperature. The well-known Obukhov length $L$ is defined as

$$L = -u^3 / \kappa g H / T_0 \rho c_p,$$

(3)

where $g$ is the gravitational acceleration and $\kappa$ is the von Kármán constant.

According to the Buckingham $\pi$ theorem (Buckingham 1914), dimensionless wind shear and temperature gradient may be expressed in terms of a dimensionless composition of the independent variables

$$\frac{\kappa z U}{u_*} = \phi_m \left( \frac{z}{L} \right),$$

(4)

$$\frac{\kappa z \Theta}{\theta_*} = \phi_h \left( \frac{z}{L} \right),$$

(5)

where the subscript $z$ of $U$ and $\Theta$ stands for partial derivative with respect to $z$. In Eqs. (4) and (5) $\phi_m$ and $\phi_h$ are empirical functions introduced by Monin and Obukhov to represent the effect of the stability on the mean profiles of wind speed and temperature. The most popular functional forms of $\phi_m$ and $\phi_h$ (used in this study) were proposed by Businger et al. (1971):

$$\phi_m = 1 + \beta \frac{z}{L};$$

(6)

$$\phi_h = \alpha + \beta \frac{z}{L};$$

(7)

under stable conditions, $L > 0$, and

$$\phi_m = \left( 1 - \gamma_1 \frac{z}{L} \right)^{-1/4};$$

(8)

$$\phi_h = \alpha \left( 1 - \gamma_2 \frac{z}{L} \right)^{-1/2}$$

(9)

under unstable conditions, $L < 0$. The universal empirical constants in the above expressions are estimated as

$$\alpha \sim 0.75, \quad \beta \sim 4.7, \quad \gamma_1 \sim 15, \quad \gamma_2 \sim 9,$$

where the approximation symbol emphasizes that these values have appreciable uncertainties. Alternative forms of $\phi_m$ and $\phi_h$ have been suggested by other authors (e.g., Beljaars and Holtslag 1991). They do not differ qualitatively from those of Businger et al. and will not affect our analysis below.

\section*{a. Stable layer}

The wind profile described by Eqs. (4) and (6) leads to two equivalent expressions for $H$ and $U_z$,

$$\frac{H}{\rho c_p} = \frac{1}{2} \frac{T_0}{\beta g k z} u_*^2 (u_* - \kappa z U_z);$$

(10)

$$U_z = \frac{u_*}{\kappa z} \left( 1 - \beta \frac{g k z}{\alpha^2} \frac{H}{T_0} \frac{1}{u_*^2} \rho c_p \right);$$

(11)

as functions of $u_*$. These relationships are illustrated in Figs. 1a and 1b. Here $H$ and $U_z$ are not monotonic functions of $u_*$, opening the possibility of an extremum hypothesis applying to the wind profile to yield a unique solution.

The temperature profile described by Eqs. (5) and (7) leads to two equivalent expressions,

$$\frac{H}{\rho c_p} = \frac{\alpha}{2 \beta g k z} \frac{T_0}{u_*^3} \left( 1 - \sqrt{1 + \frac{4 \beta g k z^2 \Theta}{\alpha^2} \frac{T_0}{u_*^2}} \right);$$

(12)

$$\Theta_z = \beta \frac{g}{T_0 \rho c_p} \frac{1}{u_*^2} \left( \frac{H}{\rho c_p} \Theta - \alpha \frac{T_0}{u_*^2} \right);$$

(13)

$H$ and $\Theta_z$ as functions of $u_*$ are illustrated in Figs. 2a and 2b and indicate that $H$ and $\Theta_z$ are monotonic functions of $u_*$, assuring a unique solution. No extremum hypothesis would be associated with the temperature profile.

As shown in Fig. 1, a given value of $U_z$ ($H$) leads to two solutions of $u_*$ for a given $H$ ($U_z$) except when $U_z$ ($H$) reaches its minimum. It is straightforward to verify that $H$ in Eq. (10) is minimized (or $-H$ is maximized) when

$$U_z = \frac{3 u_*}{2 \kappa z}$$

(14)

and $U_z$ in Eq. (11) is minimized when

$$\frac{H}{\rho c_p} = \frac{1}{2} \frac{T_0}{\beta g k z} u_*^2$$

(15)

Since $\Theta_z$ is a monotonic function of $H$ and $u_*$ according to Eq. (13), substituting $H$ in Eq. (15) into Eq. (13) leads to

$$\Theta_z = \left( \alpha + \frac{1}{2} \right) \frac{T_0}{2 \beta g k z} \left( \frac{u_*}{\kappa z} \right)^2.$$

(16)

Equations (14)–(16) are the extremum solution under stable condition.
Substituting the extremum solution of $U_z$ given in Eq. (14) into Eq. (11) leads to Eq. (15), while substituting the extremum solution of $H$ in Eq. (15) into Eq. (10) leads to Eq. (14). This is not surprising since Eqs. (10) and (11) are identical. This consistency implies that the momentum flux that minimizes heat flux also minimizes wind shear. Therefore, a minimum (negative) heat flux coincides with a minimum wind shear. The fact that the extremum solution of $u_*$ is determined by the wind profile suggests that dynamic mixing is responsible for the heat transfer in a stable layer where thermal convection is suppressed. The extremum hypothesis implies that dynamic mixing not only dominates the heat transfer, but also does so effectively. The kinetic energy drawn from the mean flow is used in such an efficient way (with a minimum wind shear) that the atmosphere takes an optimal path toward a potential thermal equilibrium by maximizing the heat flux.

**FIG. 1.** Plot of (a) $H$ vs $u_*$ according to Eq. (10) for a given $U_z$ and (b) $U_z$ vs $u_*$ according to Eq. (11) for a given $H$ in a stable layer. The dashed line represents the wind profile in a neutral layer. The graph is for illustration purposes and is not drawn to scale and has arbitrary units.

**FIG. 2.** As in Fig. 1 but of (a) $H$ vs $u_*$ according to Eq. (12) for a given $U_z$ and (b) $\Theta_z$ vs $u_*$ according to Eq. (13) for a given $H$ in a stable layer.
b. Unstable layer

The wind profile under unstable condition described by Eqs. (4) and (8) has two equivalent expressions,

\[
\frac{H}{\rho C_p} = \frac{1}{\gamma_1 g \kappa z} u_0^3 \left[ \left( \frac{u_0}{\kappa z U_z} \right) - 1 \right]^{-1}; \quad (17)
\]

\[
U_z = \frac{u_0^{7/4}}{\kappa z} \left[ u_0^3 + \frac{g \kappa z H}{T_0 \rho C_p} \right]^{-1/4}; \quad (18)
\]

\( H \) and \( U_z \), given the other, in Eqs. (17) and (18) are illustrated in Figs. 3a and 3b. Contrary to the case of a stable layer, \( H \) and \( U_z \) are monotonic functions of \( u_0 \).

The temperature profile described by Eqs. (5) and (9) can also be written as two equivalent forms,

\[
\frac{H}{\rho C_p} = \frac{1}{2 \gamma_2 g \kappa z} u_0^3 \left( \frac{g \kappa z^2 H}{\alpha T_0 u_0^3} \right)^2 \times \left[ 1 + \left( \frac{\alpha T_0 u_0^3}{g \kappa z^2 H} \right)^{2} \right]; \quad (19)
\]

\[
\Theta_z = -\alpha \frac{H u_0^{1/2}}{\rho C_p \kappa z} \left( u_0^3 + \frac{g \kappa z H}{T_0 \rho C_p} \right)^{-1/2}; \quad (20)
\]

\( H \) and \( \Theta_z \) as functions of \( u_0 \) are illustrated in Figs. 4a and 4b. Contrary to the case of stable layer, \( H \) and \( \Theta_z \) are not monotonic functions of \( u_0 \), opening the possibility of an extremum hypothesis applying to the temperature profile to yield a unique solution.

As shown in Fig. 4, a certain \( \Theta_z (H) \) corresponds to two values of \( u_0 \) unless \( \Theta_z (H) \) reaches a minimum to have a unique solution of \( u_0 \). It is straightforward to show that minimizing \( \Theta_z \) (or maximizing \( \Theta_z \)) in Eq. (20) leads to

\[
\frac{H}{\rho C_p} = \frac{2}{\gamma_2 g \kappa z} T_0 u_0^3 \quad (21)
\]

Meanwhile, minimizing \( H \) in Eq. (19) leads to

\[
\Theta_z = -\frac{2}{\sqrt{3}} \frac{\alpha T_0}{\gamma_2 g \kappa z} \left( u_0^3 \right)^{2}. \quad (22)
\]

Since \( U_z \) is a monotonic function of \( H \) and \( u_0 \) according to Eq. (18), substituting \( H \) in Eq. (21) into Eq. (18) leads to

\[
U_z = \left( \frac{\gamma_2}{\gamma_2 + 2 \gamma_1} \right)^{1/4} \frac{u_0}{\kappa z}. \quad (23)
\]

Equations (21)–(23) are the extremum solution under unstable condition.

The extremum solutions under unstable condition are also consistent. Substituting the extremum solution of \( \Theta_z \) in Eq. (22) into Eq. (19) leads to Eq. (21), while substituting the extremum solution of \( H \) in Eq. (21) into Eq. (20) leads to Eq. (22). This is again expected since Eq. (19) and Eq. (20) are identical. The consistency implies that momentum flux that minimizes heat flux also minimizes temperature gradient. Therefore, a minimum
temperature profile instead of wind shear suggests that buoyancy is what limits the heat transfer in an unstable layer. It also implies that thermally driven convection is less efficient than mechanical mixing as a heat transfer mechanism because the maximum temperature gradient corresponds to a minimum heat flux. It is true that heat flux in an unstable layer is usually higher than that in a stable layer. However, this is due to the greater daytime heat supply at the surface rather than the effectiveness of buoyancy-driven turbulent transport.

c. Neutral layer

The wind and temperature profiles under neutral condition have a trivial solution,

\[ H = 0, \]
\[ \Theta_z = 0, \]
\[ U_z = \frac{u_*}{\kappa z}. \]

It is important to point out that Eq. (26) results directly from the dimensional analysis, although it can also be formally derived from Eq. (4) as \( H \to 0 \). Theoretically, \( L \) does not exist for a neutral layer even though \( L \to \infty \) as \( H \to 0 \) according to Eq. (3). Based on the dimensional argument, the only length scale for the case of a neutral layer is \( z \). Hence, \( L \) is not continuous in \( H \) at \( H = 0 \); that is,

\[ \lim_{H \to 0} L = \infty \neq L(H = 0), \]

where \( L \) on the right-hand side is understood as a length scale, if it exists. This intricate point will be elaborated further in section 5.

d. Eddy diffusivity for heat transfer

The extremum solution derived above allows \( H \) to be formally expressed in terms of the temperature gradient,

\[ \frac{H}{\rho c_p} = -C_k \kappa z u_* \frac{\partial \Theta}{\partial z}, \]

where \( C_k \) is a constant,

\[ C_k = \begin{cases} \sqrt{\frac{3}{\alpha}}, & \text{unstable} \\ \frac{2}{1 + 2\alpha}, & \text{stable}. \end{cases} \]
Then, the eddy diffusivity for heat transfer, $K_h$, is parameterized as

$$K_h = C_u k z u_a,$$  \hspace{1cm} (30)

referred to as the extremum solution of the eddy diffusivity. Later in the paper we will show that $K_h$ in Eq. (30) plays an important role in modeling the energy balance over a land surface.

e. Properties of the extremum solution

The existence of the extremum solution (summarized in Table 1) based on the MOST results from two (not very restrictive) assumptions: 1) $\phi_u(\zeta)$ and $\phi_h(\zeta)$ are well-behaved functions of $\zeta$ that do not even have to be monotonic and 2) any two of the states variables (i.e., $U_z$, $\Theta_z$, $u_a$, and $H$) are uniquely determined given the other two. The basic features of $U_z$, $\Theta_z$, and $H$ in terms of $u_a$, as shown in Figs. 1–4, are independent of the functional forms of $\phi_u$ and $\phi_h$ as long as they are well-behaved in terms of $\zeta = z/L$.

A remarkable feature of the extremum solution is that $U_z$, $\Theta_z$, and $H$ expressed in terms of $u_a$ and $z$ have exactly the same functional forms for unstable and stable conditions except for the different proportionality coefficients. This is somewhat surprising, but does make qualitative sense. For example, with the same $u_a$, heat flux in an unstable layer is about twice as strong as in a stable layer. Also for the same $u_a$, the temperature gradient in a stable layer is greater than that of an unstable layer, and wind shear in a stable layer is greater than that of an unstable layer. These results do agree with our experience and intuition. A second notable feature is that heat flux is nonlinearly dependent on temperature gradient. Eliminating $u_a$ in the expressions of $H$ and $\Theta_z$ leads to an explicit nonlinear equation linking heat flux to temperature gradient. This is consistent with the classical flux-gradient equation in which the nonlinearity is through the diffusivity parameter that is a nonlinear function of $H$. The eddy diffusivity parameterized using the extremum solution (see section 3d) is also a nonlinear function of $H$. An application of this parameterization will be presented in section 4b(2), which also serves as a validation of the extremum solution.

We note that the assumption of height invariant fluxes of heat and momentum is not required in the formulation of the governing equations in the MOST unless explicit analytical expressions of wind and temperature distributions are desirable when integrating the dimensionless gradient equations, that is, Eqs. (4) and (5). The derivation of the extremum solution does not require that assumption either. As a result, the wind speed will deviate from the log profile when $u_a$ varies with $z$ according to the extremum solution, regardless of heat flux. Hence, the extremum solution is general and not limited to the case of height-independent fluxes of momentum and heat. Significant variation of heat flux with height has been observed (Elliott 1964).

4. Validation of the extremum solutions

The extremum solution may be justified theoretically and experimentally. The theoretical justification has two components: 1) the extremum solution is consistent with the recent development in nonequilibrium thermodynamics, that is, the emerging theory of maximum entropy production, and 2) the extremum solution has the correct asymptotic properties. The experimental justification comes from two tests: 1) direct confirmation of the extremum solution using field observations and 2) successful prediction of land surface energy budget by a model formulated using the extremum solution.

a. Theoretical justification

1) NONEQUILIBRIUM THERMODYNAMICS

The recent advances in nonequilibrium thermodynamics, that is, the emerging theory of maximum entropy production (MEP) (Dewar 2003, 2005), offer a new tool for characterizing and modeling atmospheric turbulent transfer. The MEP theory is a derivative of the theory of maximum entropy (MaxEnt) first formulated as a general method to assign probability distribution in statistical mechanics (Jaynes 1957). The theoretical foundation of MaxEnt is Bayesian probability theory (e.g., Jaynes 2003) in which the concept of entropy is defined as a quantitative measure of information (Shannon and Weaver 1949) for any systems that need to be described probabilistically for making statistical inferences. The MaxEnt distribution is interpreted as the most probable and macroscopically reproducible state among all physically possible states. In the formalism of

| Table 1. A summary of the extremum solution based on the Monin–Obukhov similarity theory. The constants in the equations, $\alpha \sim 0.75$ or 1, $\beta \sim 4.7$, $\gamma_1 \sim 15$, and $\gamma_2 \sim 9$, were reported by Businger et al. (1971). |
|---|---|---|
| Stable | Unstable | Neutral |
| $U_z$ | $\frac{3 u_a}{2 k z}$ | $\left(\frac{\gamma_2}{\gamma_2 + 2 \gamma_1}\right)^{1/4} u_a$ |
| $\Theta_z$ | $\left(\alpha + \frac{1}{2}\right) \frac{1}{2 \beta} \frac{T_g (u_a)}{g k z}$ | $-\frac{2 T_g (u_a)}{\sqrt[3]{8 \gamma_1 g / k z}}$ |
| $H$ | $\frac{1}{2 \beta} \frac{T_g u_a}{g k z}$ | $\frac{2 T_g u_a}{\gamma_2 g / k z}$ |


MEP, the microscopic configurations of a nonequilibrium system are characterized by a small number of observable parameters such as macroscopic fluxes and the corresponding scalar gradients that are related functionally. It remains uncertain at the moment whether the proposed extremum hypothesis can be "proved" using the MEP theory. Nonetheless, the extremum hypothesis appears to be consistent with the MEP theory, implying that the extremum hypothesis may result from some fundamental laws of nonequilibrium thermodynamics, an ongoing research subject.

According to the MEP theory (Dewar 2005, p. L378), imposed gradients of temperature and wind speed, for example, would correspond to maximum (or minimum) fluxes of heat and momentum and vice versa. Under a stable condition, the dominant transport mechanism in a boundary layer is forced convection driven by the synoptic wind where the effect of thermal forcing on the transport is relatively weak. This is equivalent to the situation of imposing a temperature gradient. Then the MEP predicts that (downward) heat flux would be maximized. This is the same property of the extremum solution (Fig. 1a) where the actual friction velocity corresponds to minimum (negative) heat flux. Under an unstable condition, the turbulence in the ASL is driven by combined free and forced convection. This is the situation of imposed heat flux due to the fact that sensible heat flux into the atmosphere is determined by maximum evaporation (Wang et al. 2004). Then MEP predicts a maximum temperature gradient in agreement with the extremum solution that corresponds to a maximum (inverse) temperature gradient (Fig. 4b). Several recent studies (Ozawa et al. 2001; Lorenz and McKay 2003; Kleidon et al. 2006) have suggested that various maximum transport properties of the atmospheric flows may be explained by the MEP under specific boundary conditions, although these studies did not directly use the formalism of Dewar (2005). We acknowledge that we have not been able to derive the entire extremum solution directly using the MEP formalism. It is still unclear whether the extremum solution of wind shear and momentum flux agrees with the MEP prediction, a subject of ongoing research. Nonetheless, the consistency of the extremum solution with the MEP theory demonstrated here is the first step in that direction.

2) ASYMPTOTIC PROPERTIES

Here we demonstrate that the extremum solution has the correct asymptotic properties of the MOST associated with a pure convective regime in which the mean wind shear vanishes. These well-known asymptotic properties have been obtained from various arguments independent of those behind the extremum solution.

(i) Unstable layer

One situation of large \( z/L \) (for a fixed \( L \)) is the "purely thermal turbulence with zero wind".\(^1\) Under the condition of zero (mean) wind, mechanical mixing plays no role, hence \( u_a \) drops out of the similarity equations in the MOST. Eliminating \( u_a \) in the extremum solution of \( H \) and \( \Theta_z \) (Table 1) leads to

\[
\Theta_z = -\frac{2\alpha}{\sqrt{3}} \left( \frac{2}{\gamma_2} \right)^{1/3} \left( \frac{H}{\rho c_p} \right)^{2/3} \left( \frac{g}{T_0} \right)^{-1/3} (\kappa z)^{-4/3}
\]

Also

\[
= -\frac{C_h}{\frac{\theta_z}{5}} \left( \frac{z}{L} \right)^{-1/3},
\]

where \( \theta_z \) is defined as in Eq. (2). Equation (31) is identical to that obtained by Monin and Obukhov (1954) [see Eqs. (2.182) and (2.184) in the English translation] based on the argument that the purely thermal turbulence is self-similar, and \( C_h \) (unspecified in their paper) is given as

\[
C_h = 2\alpha \sqrt{3} \left( \frac{2}{\gamma_2} \right)^{1/3} \approx 2.0.
\]

The asymptotic extremum solution of \( U_z \) (Table 1) can be expressed in two different forms. The first one is an expression where \( u_a \) remains\(^2\) in the equation,

\[
U_z = C_m u_a^2 \left( \frac{H}{\rho c_p} \right)^{-1/3} \left( \frac{g}{T_0} \right)^{-1/3} \left( \frac{z}{L} \right)^{-1/3} = -\frac{C_m}{\kappa} u_a \left( \frac{z}{L} \right)^{-1/3},
\]

where the constant \( C_m \) is

\[
C_m = \left( \frac{\gamma_2}{\gamma_2 + 2\gamma_1} \right)^{1/4} \left( \frac{2}{\gamma_2} \right)^{1/3} \kappa^{-4/3} \approx 1.4.
\]

This is the solution given in Monin and Obukhov (1954) [Eqs. (2.180) and (2.185) in the English translation] as the limiting case of purely thermal turbulence. However, it is inconsistent with the definition of the free convection; that is, \( u_a \to 0 \). They did not explain why \( u_a \) vanishes in the case of purely thermal turbulence but

\(^1\) Monin and Obukhov (1954) defined the purely thermal turbulence as the case of zero wind and zero friction velocity. Theoretically, zero friction velocity only requires zero mean wind shear, while wind speed is not necessarily zero.

\(^2\) In Monin and Obukhov (1954), the authors argued that wind and temperature will have similar profiles, following the \(-1/3\)-power law. Then \( u_a \) will not drop out of the equation as required by the definition of free convection, an obvious inconsistency in their analysis.
remains in the equation. Later, Kader and Yaglom (1990) derived the same solution using a new argument called “directional dimensional analysis.” Kader and Yaglom showed that Eq. (33) is the wind profile of a dynamic–convective layer using a three-sublayer model of an unstable surface layer (dynamic, dynamic–convective, and free convective).

The second form is obtained using the extremum solution of $H$ and $U_z$ to eliminate $u_b$,

$$
U_z = \left(\frac{\gamma_2}{2}\right)^{1/3} \left(\frac{\gamma_2}{\gamma_2 + 2 \gamma_1}\right)^{1/4} \left(\frac{H}{pc_p}\right)^{1/3} \left(g \frac{T_0}{T} \right)^{1/3} \left(\kappa z\right)^{-2/3}
$$

$$
= -\frac{C_m^2 u_b}{\kappa z^2},
$$

(35)

where the constant $C_m^2$ is

$$
C_m^2 = 3 \left(\frac{\gamma_2}{2}\right)^{1/3} \left(\frac{\gamma_2}{\gamma_2 + 2 \gamma_1}\right)^{1/4} \simeq 3.4.
$$

Equation (35) is identical to the wind profile of the free convective sublayer given by Kader and Yaglom (1990) based on the directional dimensional analysis. Curiously, Eq. (35) implies that there could be nonzero wind shear in the case of purely thermal turbulence where $u_b$ vanishes when $L$ is fixed.

(ii) Stable layer

Turbulence would degenerate were it not for dynamic mixing. Hence, $u_b$ remains important for a stable layer. Monin and Obukhov argued that “turbulence characteristics must not explicitly depend on the distance $z$ from the underlying surface” for large $z/L$ (for a fixed $L$) in a stable layer. Eliminating $z$ from the extremum solutions of $H$, $U_z$, and $\Theta_z$ leads to

$$
U_z = -3\beta g \frac{H}{T_0} \rho c_p u_b
$$

(37)

$$
\Theta_z = (1 + 2\alpha)\beta g \frac{H}{T_0} \rho c_p \left(\frac{1}{u_b}\right).
$$

(38)

They are identical to those given by Monin and Obukhov [1954, Eqs. (2.188) and (2.189)] with the stationary Richardson number, $R$,

$$
R = \frac{1}{3\beta} \simeq \frac{1}{14}.
$$

(39)

Monin and Obukhov did not give its value except for stating $R$ to be no greater than the critical value $R_{cr}$. The extremum solution not only retrieves the Monin and Obukhov result but also specifies the parameter that they were unable to find.

When $H$ and $u_b$ are independent of $z$ as commonly assumed for the ASL, Eq. (28) implies $\Theta_z \propto z^{-1}$, which is what Priestley (1955) and Taylor (1956) called the “forced convection regime.” Dyer (1964, p. 153) summarized four cases (i.e., near-neutral, Priestley regime, and two Townsend regimes) in which $\Theta_z$ was expressed in terms of various power laws of $z$, all of which can be obtained from the extremum solutions. For example the two Townsend regimes are identical according to the extremum solution.

b. Experimental justification

1) DIRECT VALIDATION OF THE EXTREMUM SOLUTION

Direct validation of the extremum solution requires independent measurements of mean wind shear, temperature gradient, momentum, and heat flux. The fluxes are routinely measured using eddy-covariance devices in field campaigns. However, most field experiments do not have multiple-level measurements of wind velocity and temperature suitable for accurate retrieval of the mean wind shear and temperature gradient. Here we focus on testing the relationship of $H \sim u_b^3$ predicted by the extremum solution.

The data products from two field experiments are used in this study: at Owens Lake, California, during 20 June–2 July 1993 and at Lucky Hill near Tombstone, Arizona, during 2–17 June 2008. The Owens Lake site has a bare soil surface, whereas the Lucky Hill site was covered with sparse dry shrubs. Eddy-covariance systems were employed at both locations to measure turbulent variables as well as soil variables including ground heat fluxes and skin temperature. Further details about the two experiments can be found in Katul (1994) and Wang and Bras (2009).

The measured $H$ versus $u_b$ at Owens Lake and Lucky Hill are shown in Figs. 5 and 6, respectively. It is evident that the data points tend to follow a straight line with a 3:1 slope according to the extremum solution. Relatively large scatter in the $H$ versus $u_b$ plots does not allow accurate estimates of the coefficients in the regression equations of $H - u_b^3$. Nonetheless, Figs. 5 and 6 indicate that they are comparable to the theoretical value. The extremum solution is supported, at least qualitatively, by the field observations.

Additional observational validation was given by Dyer (1967) nearly forty years ago. Dyer estimated the dimensionless heat flux,

$$
H^* = \frac{H}{\rho c_p (g/T_0)^{1/2} \Theta_z^{3/2} \kappa z^2}.
$$

(40)
using field measurements. The estimated $H^*$ ranges from 1.01 to 1.40 with a mean 1.15. According to the extremum solution, $H^*$ is obtained as

$$H^* = \kappa^2 \left( \frac{\gamma^2}{2} \right)^{1/2} \left( \frac{\sqrt{3}}{\alpha} \right)^{3/2} \simeq 1.19,$$

which is in good agreement with the observation. In addition, Taylor (1959) also showed that $H^* \approx \kappa^2$ with values ranging from 0.76 to 2.28.

2) MODELING OF LAND SURFACE ENERGY BUDGET

The extremum solution may be quantitatively validated through a model of energy balance over a dry land surface based on the MEP theory (Wang and Bras 2009) briefly described in the appendix. In the MEP model of surface energy balance, the extreme of a so-called dissipation function of sensible and ground heat fluxes is found under the constraint of conservation of energy. One key component of the dissipation function is the “thermal inertia” of turbulent diffusion defined in analogy to that for conduction where the molecular diffusivity is replaced by eddy diffusivity $K_H$. The extremum solution offers a mathematically more tractable parameterization of $K_H$ expressed in Eq. (30). Compared to the common MOST-based parameterization of the eddy diffusivity (e.g., Arya 1988, p. 161), the extremum solution allows $K_H$ to be expressed as an explicit nonlinear function of sensible heat flux alone. This new parameterization of $K_H$ using the extremum solution leads to a simple solution of sensible and ground heat flux $H$ and $G$ [see Eqs. (A1) and (A2)]. Figure 7 compares $H$ and $G$ predicted by the MEP model with the observed fluxes at the Owens Lake site. Close agreement is evident between the modeled and observed fluxes under both stable (nighttime) and unstable (daytime) conditions.

The success of the MEP model in which the extremum solution plays an important role clearly demonstrates the potential applications of the extremum solution. This case study is a validation of both the extremum solution and the MEP theory, which have been developed independently so far. Furthermore, the MEP model sheds light on the link between them as 1) the extremum solution may be derived directly from the MEP theory and 2) the correction prediction of heat fluxes by the MEP model uses the extremum solution while the basic idea of the MEP theory is independent of the extremum solution. Therefore, this type of modeling test, complementary to the direct confirmation of the scaling relations as in Table 1 against measurements of fluxes and mean gradients, offers strong support of the extremum solution as well.

c. Implications of the extremum solution

The extremum solution suggests a possible simplification of the classical MOST formalism: reducing the two empirical functions, $\phi_m$ and $\phi_n$ in Eqs. (4) and (5), to
some empirical constants as $\phi_m$ and $\phi_h$ are obtained from curve fitting assuming constant momentum and heat flux (hence constant $L$) in $z$. Note that $z$-invariant flux profiles are only an approximation. This approximation is acceptable, maybe even necessary in practice, but it has severe theoretical drawbacks: it violates conservation laws and is inconsistent with the asymptotic properties of the surface layer. The extremum solution corresponds to $z = 0.1, 0.2$ (particularly for larger $z$), around which the majority of the data points cluster in $z \sim \phi_m, \phi_h$ empirical observations (see Businger et al. 1971). In addition, the assumption of constant $L$ makes the data points appear more variable than they should on the $z \sim \phi_m, \phi_h$ diagrams, especially for larger $z$. The above suggests the possibility of reducing two empirical functions, $\phi_m$ and $\phi_h$, to some empirical constants in the formalism of the MOST. It may sound like a circular argument as the derivation of the extremum solution uses the prescribed $\phi_m$ and $\phi_h$ functions. Yet, the existence of an extremum solution does not depend on the functional forms of $\phi_m$ and $\phi_h$. It is true that the parameters in the extremum solution are related to those in the prescribed $\phi_m$ and $\phi_h$ functions (i.e., $\alpha, \beta, \gamma_1$, and $\gamma_2$ identified in Businger et al.). They could be obtained by fitting the extremum solution (as in Table 1) to field data directly. Use of the prescribed $\phi_m$ and $\phi_h$ merely facilitates the derivation of the extremum solution. In fact, a number of previous studies have hinted at this possibility, suggesting $\phi_m$ and $\phi_h$ being constant for $z/L \approx 0.5$ (e.g., Handorf et al. 1999; Johansson et al. 2001; Brutsaert 2005, p. 48). Constant $\phi_m$ and $\phi_h$ for larger $z$ implies that 1) the assumption of constant $L$ is only valid for a limited range of $z$ (see section 5) and 2) $L$ would be proportional to $z$ for larger $z$, corresponding to a constant $\zeta$, as predicted by the extremum solution.

FIG. 7. The MEP model prediction (dashed) of (a) $H$ and (b) $G$ vs the observed (solid) fluxes (observed $R_n$ not shown). Data collected at Owens Lake, California, 20 Jun–2 Jul 1993.
The original MOST equations, (4) and (5), do not uniquely determine the flux-gradient relationships, such as that between $H$ and $u_g$, illustrated in Fig. 1a, for almost any nontrivial $\phi_m$ and $\phi_h$ functions. Such nonuniqueness is more than a technical inconvenience. It signals potential loopholes in the classical treatment of the MOST including violation of conservation laws and inconsistancy in asymptotic properties. The extremum solution completely removes this nonuniqueness. Uniqueness is not only a technical advantage of the extremum solution (e.g., to avoid the problem of nonconverging iteration) but, more importantly, a desired property of the MOST since the multiple-valued relationship resulting from a mathematical artifact is physically unrealistic. Specifically for the case of the unstable layer shown in Fig. 4a, the original MOST equations allow two values of $u_g$ corresponding to a given $H$ except at the bottom of the curve. Yet, according to the definition of $L$ in Eq. (3), $u_g$ has only one real root for any given $H$ regardless of $L$ being constant in, or varying with, $z$. Then, the extremum solution must be the only mathematically consistent and physically realistic solution among all possible ones allowed by the MOST equations.

5. On the interpretation of Obukhov length

The asymptotic extremum solution under unstable conditions presented in section 4.2 is known as the "$1/3$-power law", a character of the "free convection regime". Yet, it has been found that (Monin and Yaglom 1971, p. 482, 497), the "$1/3$-power law" holds throughout nearly the entire surface layer: they wrote "...all existing data show that the "$1/3$-power law" (which refers theoretically only to the case of very large negative $\xi$) begin to be valid at unexpectedly small values of $\xi$ of the order of $-0.1$ (or even of several hundredth)." This phenomenon is attributed, according to Monin and Yaglom, to the dominance of buoyancy-driven thermal turbulence as they argued "...convection produces vertical turbulent mixing much more effectively than wind shear. As a result, the thickness of the dynamic sub-layer in fact comprises only a small part of $[L]$" (p. 487). Kader and Yaglom (1990) further argued that the length scale of the dynamic sublayer is actually much smaller than the original Obukhov length according to the directional dimensional analysis. The above arguments imply that the dynamic sublayer is unimportant and the entire boundary layer is essentially thermally driven, assuming $L$ being independent of $z$. Yet, the Obukhov length $L$ has been interpreted as the thickness of the dynamic sublayer in which the thermal factor plays no significant role, a concept first proposed by Obukhov (1946) and promoted by Monin and Obukhov (1954) and Monin and Yaglom (1971). The classical reasoning leading to this view is that the limiting case of $\xi = z/L \rightarrow 0$ corresponding to a neutral layer of $H = 0$ is equivalent to the limiting case of either $z \rightarrow 0$ or $H \rightarrow 0$ with the other parameters (in the expression of $\xi$) fixed. Hence, near the surface the wind profile differs very little from the layer with a uniform temperature. Since $|L|$ is on the order of $\sim 10^2 m$ (e.g., Irwin and Binkowski 1981) and the depth of the dynamic sublayer may be comparable to $L$ (e.g., Carl et al. 1973), the classical interpretation of $L$ is potentially incompatible with the findings of Monin and Yaglom (1971) and the view of Kader and Yaglom (1990). Reconciliation of these opinions opens the possibility of an alternative interpretation of the physical meaning of $L$, which we put forward below.

It appears to us that the classical interpretation of $L$ based on Obukhov’s reasoning (i.e., equating $\xi \rightarrow 0$ to $z \rightarrow 0$ and/or $H \rightarrow 0$) might be due to a mathematical artifact. Note that according to the Monin–Obukhov similarity theory, all state variables are determined by the dimensionless parameter $\xi$ in the similarity equations. Mathematically, various scenarios of limiting cases should be examined under the condition of a given $\xi$. That is, to understand the individual roles of $z$ and $H$ in $\phi_m$ and $\phi_h$ in Eqs. (6) and (7) through $\xi$, one ought to first hold $\xi$ fixed, however small or large, and then find out what will happen as $z$ and/or $H$ tend to the limits. When $\xi$ is fixed, $z \rightarrow 0$ leads to $H \rightarrow \infty$; meanwhile $z \rightarrow \infty$ leads to $H \rightarrow 0$. Then we obtain an opposite conclusion to that of the earlier authors: the buoyancy effect cannot be ignored at small $z$, and mechanical mixing dominates the turbulence away from the surface.

The alternative interpretation of $L$ seems to be physically reasonable. It would be counterintuitive that the buoyancy effect is negligible close to the source of heat (i.e., the land surface where radiative energy is received) and becomes more dominant away from the heat source. This view appears to be consistent with the analysis of fluid turbulence by Ozawa et al. (2001), who have shown that the thickness of a molecular thermal (and viscous) boundary layer is minimum; therefore the entire layer is essentially dominated by turbulent thermal convection. In addition, it has been known that the (turbulent) thermal convection corresponds to the largest temperature gradient across the molecular thermal boundary layer (Malkus 1954). Since a strong temperature gradient exists near the surface owing to the difference between the skin temperature and near-surface air temperature and the largest temperature gradient occurs right next to the surface, it would be difficult to imagine the buoyancy
effect playing no role where the temperature gradient is the strongest regardless of the effect of wind shear. It would be equally counterintuitive that the buoyancy effect dominates away from the surface (i.e., heat source) where the source of kinetic energy driving the mechanical mixing is supplied by the mean flow. Furthermore, the alternative interpretation of \( L \) is compatible with the findings of Monin and Yaglom (1971) and the view of Kader and Yaglom (1990) in the sense that the buoyancy-driven thermal turbulence is important through the entire surface layer. We will return to this point later. It is important to emphasize that the extremum solution, as mathematical functions, is not affected by the interpretation of \( L \), whichever it is.

The alternative interpretation of \( L \), as well as the extremum solution, suggests that the separation of the “forced convection regime” from the “free convection regime” may be unimportant, even impossible, in modeling the turbulent transport since \( L \) could be a redundant length scale parameter. A key assumption in the MOST is that the turbulence is independent of the underlying surface characteristics. Therefore, there should be no other length scale than \( z \) itself because the effects of buoyancy and wind shear are inseparably coupled. It is true that \( L \rightarrow \infty \) when \( H \rightarrow 0 \) with the other parameters fixed. However, the existence of \( L \) results from the dimensionless combination of three parameters \( g/T_0, H/\rho c_p, \) and \( u_\ast \). When \( H/\rho c_p \) (or \( u_\ast \)) is either undefined or zero, \( L \) does not exist and the only length scale is \( z \).

Hence, \( L \) is not continuous in \( H \) at \( H = 0 \) (or in \( u_\ast \) at \( u_\ast = 0 \)); that is, \( L (H = 0) \neq \lim_{H \rightarrow 0} L = \infty \). After all, nonexistence of \( L \) is not the same thing as \( L \rightarrow \infty \). Under this situation, \( L \) becomes undefined and, hence, loses its physical significance.

Another common assumption in the MOST is the height-invariant fluxes of momentum and heat. This assumption makes \( L \) a length scale parameter independent of \( z \). Yet height-invariant fluxes are rare, if at all, in nature. Consider a typical scenario in which the diurnal variation of temperature is \( \sim 20 \) K. The corresponding flux gradient is given by the energy balance equation:

\[
\frac{\partial H}{\partial z} = \rho c_p \frac{\partial T}{\partial t} \sim 10^3 \times \frac{20}{12 \times 3600} \sim 0.5 \text{ W m}^{-2}.
\]

Assuming the depth of a surface layer is \( O(50 \text{ m}) \), the variation of heat flux, \( \Delta H \), over the surface layer would be \( \Delta H \sim 50 \times 0.5 = 25 \text{ W m}^{-2} \), which is about 10% of a typical value of \( H \) (\( \sim 200 \text{ W m}^{-2} \)). If the surface layer is defined as the layer within which heat flux varies no more than, say, 10% according to popular textbooks, practically all surface layers should be treated as a layer with variable fluxes rather than with constant fluxes. Although use of the MOST in estimating fluxes has been critically dependent on the assumption of constant fluxes, this assumption plays no role in the extremum solution proposed here.

The classical interpretation of the physical meaning of \( L \) is closely associated with the assumption of the height-invariant fluxes or \( L \). According to the equation of turbulent kinetic energy (TKE), the relative magnitude of thermal versus mechanical source of TKE is simply \( z/L \). It is true that the thermal effect on TKE is insignificant for small \( z \) and dominant for large \( z \) when \( L \) is independent of \( z \) as assumed in the classical theories. However, momentum and heat fluxes do vary in \( z \) for the reasons explained in the previous paragraph so that \( L \) in general varies with \( z \). The extremum solution predicts the ratio \( z/L \) equal to \( (2\gamma)^{-1} \sim 0.1 \) for a stable layer and \( 2\gamma^{-1} \sim 0.2 \) for an unstable layer, indicating that the mechanical mixing is a much more effective production mechanism than buoyancy throughout the entire surface layer, although the production of TKE due to buoyancy is twice as strong in the unstable layer than in the stable layer. This is another reason that the classical view on the meaning of \( L \) is open to debate.

6. Conclusions

The significance of the extremum solution of the Monin–Obukhov similarity equation may be summarized as 1) it is the only mathematically consistent and physically realistic solution of the Monin–Obukhov similarity equations; 2) it has overcome some technical difficulties (e.g., nonuniqueness and nonconvergence) in applying the MOST in modeling of turbulent transport in the ASL; 3) it opens a possibility of simplifying the MOST formalism by replacing the two empirical stability functions by some empirical constants; 4) it unifies the asymptotic solutions of the ASL derived from various arguments; 5) it has a solid foundation built on the modern nonequilibrium thermodynamics; and 6) it can play a crucial role in successful modeling of the surface heat fluxes based on the emerging theory of maximum entropy production. Our analysis also raises a doubt on the classical interpretation of the Obukhov length, arguably due to a mathematical artifact. Resolution of the issue may offer new opportunities in improving atmospheric turbulence models.

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APPENDIX

A MEP Model of Heat Fluxes over a Dry Land Surface

Based on the theoretical framework of maximum entropy production (Dewar 2005), Wang and Bras (2009) proposed a model of $H$ and ground heat flux $G$ over a dry land surface. Maximizing a dissipation function (or entropy production function) under the constraint of conservation of energy results in the following expression:

$$
G = \frac{I}{I_0} H |H|^{-1/6}.
$$

Combining with energy balance equation,

$$
G + H = R_n,
$$

for a given net radiation input $R_n$ leads to a solution of $H$ and $G$ (as functions of the given $R_n$). In Eq. (A1) $I$ is the thermal inertia parameter for heat conduction in the soil and $I_0$ the coefficient in the “thermal inertia” for turbulent heat transfer in the boundary layer, $I_o$, expressed in terms of the eddy diffusivity $K_H$ given in Eq. (30):

$$
I_o = \rho C_p \sqrt{K_H} = I_0 |H|^{1/6},
$$

where the extremum solution was used to substitute $u_*$ in $K_H$ by $H$. Here $I_0$ is expressed in terms of the coefficients in the stability functions, identified by Businger et al. (1971),

$$
I_0 = \rho C_p \sqrt{C_1 \kappa z \left( \frac{C_2 \frac{\kappa g}{\rho C_p T_0}}{2} \right)^{1/6}},
$$

where

$$
C_1 = \begin{cases} \sqrt{3}, & \text{unstable} \\ \frac{\alpha}{2}, & \text{stable} \end{cases}
$$

and

$$
C_2 = \begin{cases} \frac{\gamma_0}{2}, & \text{unstable} \\ \frac{2 \beta}{2}, & \text{stable}. \end{cases}
$$

Other parameters are given in Table 1.

REFERENCES