Topological superconductors as nonrelativistic limits of Jackiw-Rossi and Jackiw-Rebbi models

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Topological superconductors as nonrelativistic limits of Jackiw-Rossi and Jackiw-Rebbi models

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We argue that the nonrelativistic Hamiltonian of $p_s+ip_y$ superconductor in two dimensions can be derived from the relativistic Jackiw-Rossi model by taking the limit of large Zeeman magnetic field and chemical potential. In particular, the existence of a fermion zero mode bound to a vortex in the $p_s+ip_y$ superconductor can be understood as a remnant of that in the Jackiw-Rossi model. In three dimensions, the nonrelativistic limit of the Jackiw-Rebbi model leads to a “$p+is$” superconductor in which spin-triplet $p$-wave and spin-singlet $s$-wave pairings coexist. The resulting Hamiltonian supports a fermion zero mode when the pairing gaps form a hedgehog-like structure. Our findings provide a unified view of fermion zero modes in relativistic (Dirac-type) and nonrelativistic (Schrödinger-type) superconductors.

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I. INTRODUCTION

Fermion zero modes bound to topological defects have been discovered by Jackiw and Rebbi in 1976 (Ref. 1) and recently received renewed interest in condensed matter physics (see, for example, Ref. 2). Vortices in a certain class of superconductors in two dimensions (2D) support zero-energy Majorana bound states and obey non-Abelian statistics,3 which can be potentially used for topological quantum computation.4 Although vortices in the ordinary nonrelativistic $s$-wave superconductor do not support Majorana zero modes, the weakly paired phase of the $p_s+ip_y$ superconductor, which is believed to be realized in Sr$_2$RuO$_4$,5 does support Majorana zero modes bound to vortex cores.6,7

It is also known from the pioneering work by Jackiw and Rossi that the relativistic $s$-wave superconductor in 2D (Jackiw-Rossi model) has similar properties.8 Remarkably it has been shown that such a system can be realized on the surface of the three-dimensional (3D) topological insulator in contact with the $s$-wave superconductor.9 Besides these examples, there is a number of proposals to realize Majorana zero modes using heterostructures of semiconductor and superconductor,10–12 superconductor and ferromagnet,13 and quantum (anomalous) Hall state and superconductor.14

Although the nonrelativistic $p_s+ip_y$ superconductor and the relativistic Jackiw-Rossi model share similar properties, the existence of a fermion zero mode bound to a vortex has been discussed separately in the two systems.9,15–19 In this paper (Sec. II), we argue that they are actually linked by the relativistic descendant both in 2D and 3D.

Then in Sec. III, we turn to the relativistic Jackiw-Rebbi model in 3D, which is known to exhibit a fermion zero mode associated with a pointlike topological defect (hedgehog).11,18,20,21 The limit of large mass and chemical potential (nonrelativistic limit) leads to a “$p+is$” superconductor in which spin-triplet $p$-wave and spin-singlet $s$-wave pairings coexist. We show that the resulting nonrelativistic Hamiltonian supports a fermion zero mode when the pairing gaps form a hedgehog-like structure.

We note that the analysis presented in this paper is largely motivated by the recent paper by Silaev and Volovik.22 The nonrelativistic Hamiltonian of the Balian-Werthamer (BW) state of the superfluid $^3$He was derived from the relativistic superconductor with the odd parity pairing and their topological properties were studied. In this paper, we shall broadly use “relativistic” to indicate Dirac-type Hamiltonians and “nonrelativistic” to indicate Schrödinger-type Hamiltonians. For readers’ convenience, references to the main results are summarized in Table I.

II. JACKIW-ROSSI MODEL IN 2D AND ITS NONRELATIVISTIC LIMIT

A. Jackiw-Rossi model and fermion zero mode at a vortex

We start with the Hamiltonian describing 2D Dirac fermions coupled with an $s$-wave pairing gap (Jackiw-Rossi8 or Fu-Kane9 model)

$$H = \frac{1}{2} \int dx \Psi^\dagger \mathcal{H} \Psi$$

with $\Psi^\dagger=(\psi^\dagger,-i\psi^\dagger \sigma_2)$ and

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\[ \mathcal{H} = \left( \sigma \cdot p + \sigma \cdot h - \mu \right) \frac{\Delta}{\Delta^+} - \sigma \cdot p + \sigma \cdot h + \mu \). \tag{2} \]

This Hamiltonian can be realized on the surface of the 3D topological insulator in contact with the s-wave superconductor.\(^9\) \(h\) is the Zeeman magnetic field and \(\mu\) is the chemical potential. When the pairing gap \(\Delta\) is spatially dependent, \(p = (p_x, p_y, p_z)\) has to be regarded as derivative operators \((-i\partial_z, -i\partial_y, -i\partial_x)\). The energy eigenvalue problem is

\[ e \begin{pmatrix} u_1 \\ u_2 \\ v_2 \\ v_1 \end{pmatrix} = \mathcal{H} \begin{pmatrix} u_1 \\ u_2 \\ v_2 \\ v_1 \end{pmatrix}. \tag{3} \]

When \(h\) and \(\mu\) are both zero, the number of fermion zero modes \((\varepsilon = 0)\) bound to a vortex formed by \(\Delta(x, y) = \Delta_1 + \Delta_2\) is determined by the winding number of the two scalar fields.\(^8,15,18\)

\[ \text{Index } \mathcal{H} = \frac{1}{2\pi} \int dld\sigma \hat{\Delta}_a \hat{\Delta}_b = N_w, \tag{4} \]

where \(\hat{\Delta}_a = \Delta_a / \sqrt{\Delta_1^2 + \Delta_2^2}\) and the line integral is taken at spatial infinity. However, in the presence of \(h\) and \(\mu\), the index theorem is no longer valid: \(h\) and \(\mu\) terms in the Hamiltonian can couple zero modes and they become nonzero energy states so that two states form a pair with opposite energies. Therefore, in general, only one zero mode survives for odd \(N_w\) while no zero mode survives for even \(N_w.\)\(^{2, 18, 24}\)

If we work in polar coordinates \((r, \theta)\) with the gap function given by the vortex form

\[ \Delta(x, y) = |\Delta(r)| e^{i\theta} \text{ with } |\Delta(\infty)| > 0, \tag{5} \]

it is easy to find the explicit zero-energy solution for odd \(N_w = n\) (Ref. 25)

\[ \begin{bmatrix} u_1 \\ u_2 \\ v_2 \\ v_1 \end{bmatrix} = \begin{bmatrix} \sqrt{\mu + hJ}/(\sqrt{\mu^2 + h^2}) e^{i(\pi/4)} \\ \sqrt{\mu - hJ}/(\sqrt{\mu^2 + h^2}) e^{i(3\pi/4)} \end{bmatrix} \times e^{i(a-b)r - \lambda|\Delta(r)|}, \tag{6} \]

and \(v_1 = -u_1^*, \ v_2 = u_2^*\) with an integer \(l = (n-1)/2.\) We note that the zero-energy solution, Eq. (6), is normalizable as long as

\[ \mu^2 + |\Delta(\infty)|^2 > h^2 \tag{7} \]

is satisfied and there is a topological phase transition at \(\mu^2 + |\Delta|^2 = h^2\) [see also Eq. (24) below].

**B. Derivation of \(p_x + ip_y\) superconductor and fermion zero mode**

We now derive a clear connection between the Jackiw-Rossi model and the nonrelativistic \(p_x + ip_y\) superconductor. Suppose we are interested in the low-energy spectrum of Hamiltonian (2) in the limit where both \(h > 0\) and \(\mu > 0\) are equally large

\[ \varepsilon, |\mu^2 + |\Delta|^2 - h| \ll \mu - \mu. \tag{8} \]

The low-energy spectrum in such a limit can be obtained by eliminating small components \(u_2\) and \(v_2.\)\(^{26}\) Substituting the following two equations from Eq. (3):

\[ (\varepsilon + h + \mu) u_2 = p_x u_1 + \Delta v_1 \]

\[ (\varepsilon - h - \mu) v_2 = -p_y u_1 + \Delta^* u_1 \tag{9} \]

into the remaining two equations, we obtain

\[ (\varepsilon + h + \mu) u_1 = \frac{p_x^2 u_1 + p_x \Delta u_1}{\varepsilon + h + \mu} - \frac{\Delta p_x u_1 + |\Delta|^2 u_1}{\varepsilon - h + \mu} \]

\[ (\varepsilon - h - \mu) v_1 = \frac{p_x^2 v_1 - p_x \Delta^* u_1}{\varepsilon - h - \mu} + \frac{\Delta^* p_x u_1 + |\Delta|^2 v_1}{\varepsilon + h + \mu}. \tag{10} \]

Here we introduced \(p_x = p_x \pm ip_y.\)

In the limit under consideration, Eq. (8), we can neglect \(\varepsilon\) compared to \(h + \mu\) and approximate \(\sqrt{\mu^2 + |\Delta|^2}\) by \(h.\) The remaining components \(u_1\) and \(v_1\) obey the new energy eigenvalue problem

\[ \varepsilon \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} \frac{p_x^2}{2m} - \mu \omega \frac{1}{2} \frac{p_x \Delta}{\mu} \\ \frac{1}{2} \frac{p_x + \Delta^*}{p_x - \Delta^*} - \frac{p_y^2}{2m} + \mu \omega \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \tag{11} \]

where we defined the nonrelativistic mass, chemical potential, and pairing gap as

\[ m = h, \quad \mu \omega = \sqrt{\mu^2 + |\Delta|^2 - h}, \quad \Delta_m = \frac{\Delta}{h}. \tag{12} \]

The resulting Hamiltonian

\[ \mathcal{H}_{\mu m} = \begin{pmatrix} \frac{p_x^2}{2m} - \mu \omega \frac{1}{2} \frac{p_x \Delta}{\mu} \\ \frac{1}{2} \frac{p_x + \Delta^*}{p_x - \Delta^*} - \frac{p_y^2}{2m} + \mu \omega \end{pmatrix} \tag{13} \]

describes the nonrelativistic \(p_x + ip_y\) superconductor. We note that when \(h < 0,\) one obtains the Hamiltonian of the \(p_x - ip_y\) superconductor where \(p_x\) and \(p_y\) are exchanged in Eq. (13).

The first nontrivial check of this correspondence is the comparison of spectrum in a uniform space where \(\Delta\) is constant. The relativistic Hamiltonian (2) has the energy eigenvalues

\[ \varepsilon^2 = p_x^2 + \mu^2 + |\Delta|^2 \pm 2 \sqrt{p_x^2 \mu^2 + h^2 (\mu^2 + |\Delta|^2)}. \tag{14} \]

Its low-energy branch (lower sign) at small \(p\) is correctly reproduced by the energy eigenvalue of the nonrelativistic Hamiltonian (13)

\[ \varepsilon_{m}^2 = \left( \frac{p_x^2}{2m} - \mu \omega \right)^2 + p_y^2 |\Delta|^2, \tag{15} \]

under the assumptions in Eq. (8).

Because the above “nonrelativistic limit” does not rely on the spatial independence of \(\Delta,\) the fermion zero mode found
in Eq. (6) persists into the \(p_x + ip_y\) superconductor, Eq. (13). In order to demonstrate it, we consider the simplified vortex configuration with a constant \(|\Delta_n| > 0\)
\[
\Delta_n(x, y) = e^{in\theta} |\Delta_n|.
\]
When \(n\) is odd, we can find the explicit zero-energy solution \((e=0)\) to Eq. (11) (Ref. 17)
\[
u_1 = J_0\sqrt{2m|\Delta_n| - (m|\Delta_n|)^2} \exp(-4\pi^2 \theta - 4i\theta m|\Delta_n|/r)
\]
and \(v_1 = -v_1^*\). One can see that this zero-energy solution is the direct consequence of that in Eq. (6) because Eqs. (8) and (12) lead to
\[
\mu_n > 0
\]
which coincides with the well-known topological phase transition in the \(p_x + ip_y\) superconductor existing at \(\mu_n=0\) (Refs. 3, 17, and 27) [see also Eq. (25) below]. Our finding also clarifies why a vortex with winding number \(N_w\) in the \(p_x + ip_y\) superconductor cannot support \(|N_w|\) zero modes in contrast to in the Jackiw-Rossi model with \(h = \mu = 0\).16,17 In order to derive the \(p_x + ip_y\) superconductor as a nonrelativistic limit of the Jackiw-Rossi model, one needs to introduce \(h\) and \(\mu\) which split an even number of zero modes into positive- and negative-energy states. Therefore, only one zero mode survives for odd \(N_w\) in the \(p_x + ip_y\) superconductor.

**C. Altland-Zirnbauer symmetry class (Refs. 28 and 29) and topological invariant**

Finally, we note the Altland-Zirnbauer symmetry class of the Hamiltonians that we have investigated in this section. Hamiltonian (2) with spatially dependent \(\Delta_{1,2} \neq 0\) has the charge conjugation symmetry
\[
C^{-1} \mathcal{H} C = -\mathcal{H}^* \quad \text{with} \quad C = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}.
\]
The properties of each term under the time-reversal operator \(T\) (\(T^{-1}\mathcal{H}T = \mathcal{H}^*\) at \(h = \mu = \Delta_{1,2} = 0\)) are summarized in Table II. In particular, the so-called chiral symmetry,
\[
\chi^{-1} \mathcal{H} \chi = -\mathcal{H} \quad \text{with} \quad \chi = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix},
\]
is present only if \(h = \mu = 0\) and essential for the index theorem, Eq. (4). Therefore, the relativistic Hamiltonian (2) with \(h, \mu \neq 0\) and thus the resulting nonrelativistic Hamiltonian (13) belong to the symmetry class D.

According to Refs. 30 and 31, the class D Hamiltonians defined in compact 2D momentum spaces can be classified by an integer-valued topological invariant, which is the first Chern number32-34
\[
C_1 = \frac{-i}{2\pi} \oint dp \left( \frac{\partial a_x}{\partial p_y} - \frac{\partial a_y}{\partial p_x} \right)
\]
with
\[
a_i(p) = \sum_{\varepsilon_i < 0} \langle \varepsilon_i | p \frac{\partial}{\partial p_i} | \varepsilon_i p \rangle.
\]
We shall use Eqs. (22) and (23) as the definition of the topological invariant \(C_1\) even for relativistic (Dirac-type) Hamiltonians while \(C_1\) in this case can be a half integer. However, for superconductors, \(C_1\) is always an integer because of the Nambu-Gor’kov doubling. We find that the topological invariant for the relativistic Hamiltonian (2) is given by
\[
C_1 = \begin{cases} 0 & \text{for } \mu^2 + |\Delta|^2 > h^2 \\ -\text{sgn}(h) & \text{for } \mu^2 + |\Delta|^2 < h^2 \end{cases}
\]
while the topological invariant for the nonrelativistic Hamiltonian (13) is given by
\[
C_1 = \begin{cases} 1 & \text{for } \mu_n > 0 \\ 0 & \text{for } \mu_n < 0 \end{cases}
\]
Therefore, in general, the topological invariant of the momentum space Hamiltonian is not preserved by the nonrelativistic limit.35

Nevertheless, both values of \(C_1\) computed for the relativistic and nonrelativistic Hamiltonians are consistent with recent conjectures relating the topological invariant of a momentum space Hamiltonian to the number of fermion zero modes bound to a vortex.24,36 For class D superconductors defined in compact momentum spaces (as is the case for nonrelativistic Hamiltonians), Teo and Kane in Ref. 2 conjecture that the number of fermion zero modes is
\[
\nu = C_1 N_w \mod 2.
\]
This formula gives \(\nu=1\) for \(\mu_n > 0\) and \(\nu=0\) for \(\mu_n < 0\) for an odd winding number \(N_w\). On the other hand, Santos et al. in Ref. 36 do not constrain Hamiltonians to be defined in compact momentum spaces, allowing for relativistic (Dirac-
type) Hamiltonians, and conjecture that the number of fermion zero modes is
\begin{equation}
\nu = (C_1 + N_T) N_w \mod 2, \tag{27}
\end{equation}
where \(N_T\) is the number of Dirac flavors \([N_T=1\) for the Jackiw-Rossi model, Eq. (2), and \(N_T=0\) for the \(p_x + ip_y\) superconductor, Eq. (13)]. For an odd winding number \(N_w\), their formula gives \(\nu \neq 1\) for \(\mu^2 + |\Delta|^2 > h^2\) and \(\mu_w > 0\) and \(\nu = 0\) for \(\mu^2 + |\Delta|^2 < h^2\) and \(\mu_w < 0\). Therefore, the conjectured counting of fermion zero modes in terms of the momentum space topological invariant works both in the relativistic and nonrelativistic Hamiltonians, even though the value of \(C_1\) is not preserved by the nonrelativistic limit.

III. JACKIW-REBBI MODEL IN 3D AND ITS NONRELATIVISTIC LIMIT

A. Jackiw-Rebbi model and fermion zero mode at a hedgehog

In this section, we extend the above developed analysis to three dimensions. For this purpose, we consider the following Hamiltonian describing 3D Dirac fermions coupled with three real scalar fields (\(\Delta = \Delta_1 + i \Delta_2 + \Delta_3\)):
\begin{equation}
H = \frac{1}{2} \int dx \Psi^\dagger \mathcal{H} \Psi \tag{28}
\end{equation}
with \(\Psi^\dagger = (\psi^\dagger, -i \psi^\dagger \alpha_2)\) and
\begin{equation}
\mathcal{H} = \left( \begin{array}{cc}
\alpha \cdot p + \beta m - \mu - i \gamma^5 \beta \Delta_3 & \Delta \\
\Delta^* & - \alpha \cdot p + \beta m + \mu + i \gamma^5 \beta \Delta_3
\end{array} \right), \tag{29}
\end{equation}
where \(\Delta_{1,2,3}\) are spatially dependent, \(p = (p_x, p_y, p_z)\), and \(\psi^\dagger = (\psi^\dagger_x, -i \psi^\dagger_y, -i \psi^\dagger_z)\). The energy eigenvalue problem is
\begin{equation}
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4
\end{pmatrix} = \mathcal{H} \begin{pmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4
\end{pmatrix}, \tag{30}
\end{equation}
where \(\psi_{1,2}\) are two-component fields. Here we employ the standard representation of Dirac matrices
\begin{equation}
\alpha = \begin{pmatrix}
0 & \sigma \\
\sigma & 0
\end{pmatrix}, \quad \beta = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}, \quad \gamma^5 = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \tag{31}
\end{equation}
and hence
\begin{equation}
i \gamma^5 \beta = \begin{pmatrix}
0 & -i l \\
-i l & 0
\end{pmatrix}. \tag{32}
\end{equation}

When \(\mu\) and \(\mu_w\) are both zero, the number of fermion zero modes \((\nu = 0)\) bound to a hedgehog formed by \(\Delta_{1,2,3}(x,y,z)\) is determined by the winding number of the three scalar fields\(^{1,8,20}\)
\begin{equation}
\text{Index } \mathcal{H} = \frac{1}{8 \pi} \int dS \epsilon_{ijk} \epsilon_{abc} \Delta_{ij} \delta a b c = N_w, \tag{33}
\end{equation}
where \(\hat{a} = \Delta / \sqrt{\mu^2 + |\Delta|^2} + \mu_1^2 + \mu_2^2 + \mu_3^2\) and the surface integral is taken at spatial infinity. However, in the presence of \(m\) and \(\mu\), the index theorem is no longer valid: \(m, \mu\) terms in the Hamiltonian can couple zero modes and they become nonzero energy states so that two states form a pair with opposite energies. Therefore, in general, only one zero mode survives for odd \(N_w\) while no zero mode survives for even \(N_w\).\(^{2,18,19}\)

Here, instead of the symmetric hedgehog \((\Delta_1, \Delta_5, \Delta_5)\), we assume the hedgehoglike configuration in which \(\Delta_{1,2}\) depend only on \((x, y)\) and form a vortex and \(\Delta_3\) depends only on \(z\) and forms a kink. They have the same winding number but the latter has the advantage that an analytic solution can be found even with \(m, \mu \neq 0\). If we work in cylindrical coordinates \((r, \theta, z)\) with the gap functions given by the forms
\begin{equation}
\Delta(r, z) = |\Delta(r)| e^{i m \theta} \text{ with } |\Delta(z)| > 0 \tag{34}
\end{equation}
and
\begin{equation}
\Delta_3(z \to \pm \infty) \to \mp |\Delta_3|, \tag{35}
\end{equation}
it is easy to find the explicit zero-energy solution for odd \(N_w = n\) (Ref. 38)
\begin{equation}
\begin{pmatrix}
u_1 \\
\nu_2
\end{pmatrix} = \begin{pmatrix}
\sqrt{\mu + m J_1} (\sqrt{\mu^2 + m^2 r}) e^{-i (\pi/4)} \\
0 \\
0 \\
\sqrt{\mu - m J_1} (\sqrt{\mu^2 + m^2 r}) e^{i (\pi/4) + \theta}
\end{pmatrix} \times e^{i \theta \int \int r dr dz \cdot \Delta_3(c)} \tag{36}
\end{equation}
and \(v_1 = i \sigma_2 v_1, v_2 = i \sigma_2 v_2\) with an integer \(l = (n-1)/2\). On the other hand, when
\begin{equation}
\Delta_3(z \to \pm \infty) \to \mp |\Delta_3| \tag{37}
\end{equation}
with the same \(\Delta(x,y)\) in Eq. (34), we have \(N_w = -n\) and the zero-energy solution in Eq. (36) is replaced by
\begin{equation}
\begin{pmatrix}
u_1 \\
\nu_2
\end{pmatrix} = \begin{pmatrix}
0 \\
\sqrt{\mu - m J_1} (\sqrt{\mu^2 + m^2 r}) e^{-i (\pi/4)} \\
0 \\
\sqrt{\mu + m J_1} (\sqrt{\mu^2 + m^2 r}) e^{i (\pi/4) + \theta}
\end{pmatrix} \times e^{i \theta \int \int r dr dz \cdot \Delta_3(c)} \tag{38}
\end{equation}
We note that the zero-energy solution, Eq. (36) or (38), is normalizable as long as
\begin{equation}
\mu^2 + |\Delta(z)|^2 > m^2 \tag{39}
\end{equation}
is satisfied.

B. Derivation of \(p + is\) superconductor and fermion zero mode

We now study the nonrelativistic limit of the above Jackiw-Rebbi model. Suppose we are interested in the low-
energy spectrum of Hamiltonian (29) in the limit where both $m > 0$ and $\mu > 0$ are equally large

$$\epsilon_{i}\sqrt{\mu^{2} + |\Delta|^{2} - m} \ll m - \mu. \quad (40)$$

The low-energy spectrum in such a limit can be obtained by eliminating small components $u_{2}$ and $v_{2}$. Substituting the following two equations from Eq. (30):

$$(\epsilon + m + \mu)u_{2} = (\textbf{\sigma} \cdot \textbf{p} - i\Delta_{3})u_{1} + \Delta v_{1}$$

$$(\epsilon - m - \mu)v_{2} = -(\textbf{\sigma} \cdot \textbf{p} + i\Delta_{3})v_{1} + \Delta' u_{1} \quad (41)$$

into the remaining two equations, we obtain

$$(\epsilon - m - \mu)u_{1} = \left[ p^{2} + \Delta_{3}^{2} - \textbf{\sigma} \cdot (\partial \Delta_{3}) \right] u_{1} + (\textbf{\sigma} \cdot \textbf{p} + i\Delta_{3})\Delta v_{1}$$

$$- \Delta(\textbf{\sigma} \cdot \textbf{p} + i\Delta_{3})v_{1} + |\Delta|^{2}u_{1}$$

$$+ \Delta' (\textbf{\sigma} \cdot \textbf{p} - i\Delta_{3})u_{1} + |\Delta'|^{2}v_{1} \quad (42)$$

Here the derivative operator $\partial$ in $\textbf{\sigma} \cdot (\partial \Delta_{3})$ acts only on $\Delta_{3}$.

In the limit under consideration, Eq. (40), we can neglect $\epsilon$ compared to $m + \mu$ and approximate $\sqrt{\mu^{2} + |\Delta|^{2}}$ by $m$. The remaining components $u_{1}$ and $v_{1}$ obey the new energy eigenvalue problem

$$\epsilon \left( \begin{array}{c} u_{1} \\ v_{1} \end{array} \right) = \left[ \begin{array}{cc} \frac{p^{2}}{2m} - \mu_{\text{ne}} - \frac{\textbf{\sigma} \cdot (\partial \Delta_{3})}{2m} & \frac{1}{2}(\textbf{\sigma} \cdot \textbf{p} \cdot \Delta_{3}) + i\Delta_{3} \\ \frac{1}{2}(\textbf{\sigma} \cdot \textbf{p} \cdot \Delta_{3}^{*}) - i\Delta_{3}^{*} & -\frac{p^{2}}{2m} + \mu_{\text{ne}} - \frac{\textbf{\sigma} \cdot (\partial \Delta_{3})}{2m} \end{array} \right] \left( \begin{array}{c} u_{1} \\ v_{1} \end{array} \right), \quad (43)$$

where we defined the nonrelativistic chemical potential as

$$\mu_{\text{ne}} = \sqrt{\mu^{2} + |\Delta|^{2}} - m - \frac{\Delta_{3}^{2}}{2m} \quad (44)$$

and the spin-triplet $p$-wave and spin-singlet $s$-wave pairing gaps as

$$\Delta_{t} = \frac{\Delta}{m} \quad \text{and} \quad \Delta_{s} = \frac{\Delta_{3}\Delta}{m}. \quad (45)$$

The resulting Hamiltonian

$$\mathcal{H}_{\text{ne}} = \left[ \begin{array}{cc} \frac{p^{2}}{2m} - \mu_{\text{ne}} - \frac{\textbf{\sigma} \cdot (\partial \Delta_{3})}{2m} & \frac{1}{2}(\textbf{\sigma} \cdot \textbf{p} \cdot \Delta_{3}) + i\Delta_{3} \\ \frac{1}{2}(\textbf{\sigma} \cdot \textbf{p} \cdot \Delta_{3}^{*}) - i\Delta_{3}^{*} & -\frac{p^{2}}{2m} + \mu_{\text{ne}} - \frac{\textbf{\sigma} \cdot (\partial \Delta_{3})}{2m} \end{array} \right] \quad (46)$$

describes the $p+is$ superconductor in which spin-triplet $p$-wave and spin-singlet $s$-wave pairings coexist. $\Delta_{t}$ can be complex but its phase is locked to the phase of $\Delta$ [see Eq. (45)] and thus there are three independent degrees of freedom. The last term in the diagonal elements resembles the Zeeman coupling $\textbf{\sigma} \cdot \textbf{B}$ with “magnetic field” $B_{z} = -\partial \Delta_{3}/(2m)$ generated by the gradient of $\Delta_{3} = \Delta_{3}/\Delta$. We note that the nonrelativistic Hamiltonian (46) in the absence of $\Delta_{3}$ is the BW state of the superfluid $^4\text{He}$ and studied in Ref. 22.

The first nontrivial check of this correspondence is the comparison of spectrum in a uniform space where $\Delta$ and $\Delta_{3}$ are constant. The relativistic Hamiltonian (29) has the energy eigenvalues

$$\epsilon^{2} = p^{2} + m^{2} + \mu^{2} + |\Delta|^{2} + |\Delta_{3}|^{2} \quad \pm 2\sqrt{p^{2}m^{2} + m^{2}(\mu^{2} + |\Delta|^{2}) + \mu^{2}|\Delta_{3}|^{2}}, \quad (47)$$

Its low-energy branch (lower sign) at small $p$ and $\Delta_{3}$ is correctly reproduced by the energy eigenvalue of the nonrelativistic Hamiltonian (46)

$$\epsilon_{\text{ne}}^{2} = \left( \frac{p^{2}}{2m} - \mu_{\text{ne}} \right)^{2} + p^{2}|\Delta|^{2} + |\Delta_{3}|^{2} \quad (48)$$

under the assumptions in Eq. (40).

Because the above nonrelativistic limit does not rely on the spatial independence of $\Delta$ and $\Delta_{3}$, the fermion zero mode found in Eq. (36) or (38) persists into the $p+is$ superconductor, Eq. (46). In order to demonstrate it, we consider the simplified hedgehog-like configuration resulting from Eqs. (34), (35), and (37) with constant $|\Delta_{t}| > 0$ and $|\Delta_{s}| > 0$:

$$\Delta_{t}(x, y, z) = e^{i\theta(x, y)}|\Delta_{t}| \quad \text{and} \quad \Delta_{s}(x, y, z) = \pm e^{i\theta(x, y)}\text{sgn}(z)|\Delta_{s}|. \quad (49)$$

When $n$ is odd, we can find the explicit zero-energy solution ($\epsilon = 0$) to Eq. (43)

$$u_{1} = \left( \begin{array}{c} J_{t}\left[ \sqrt{2m\mu_{\text{ne}} - (m|\Delta|^{2})^{2} + \frac{|\Delta_{3}|^{2}}{|\Delta_{3}|^{2}} r} \\ 0 \end{array} \right] \\ 0 \right) \times e^{(-\pi/4) + i(\theta - m|\Delta|^{2})/|\Delta|} \quad (50)$$

corresponding to the upper sign in Eq. (49), or

$$u_{1} = \left( \begin{array}{c} J_{t+1}\left[ \sqrt{2m\mu_{\text{ne}} - (m|\Delta|^{2})^{2} + \frac{|\Delta_{3}|^{2}}{|\Delta_{3}|^{2}} r} \\ 0 \end{array} \right] \\ 0 \right) \times e^{(\pi/4) + i(\theta + m|\Delta|^{2})/|\Delta|} \quad (51)$$

corresponding to the lower sign in Eq. (49), and $v_{1} = i\sigma_{2}u_{1}^{*}$. One can see that this zero-energy solution is the direct consequence of that in Eq. (36) or (38) because Eqs. (40), (44), and (45) lead to

$$\mu^{2} - m^{2} \approx 2m\left[ \mu_{\text{ne}} - \frac{m|\Delta|^{2}}{2} + \frac{1}{2m} \frac{|\Delta_{3}|^{2}}{|\Delta_{3}|} \right]. \quad (52)$$

Thus we have established that the existence of a fermion zero mode bound to the hedgehog-like structure, Eq. (49), formed by $\Delta_{t}$ and $\Delta_{s}/\Delta$ in the $p+is$ superconductor, Eq. (46), is a remnant of that in the Jackiw-Rebbi model, Eq.
TABLE III. Properties under the time-reversal operator $T$. $\tau$-matrices act on the particle-hole space and $\otimes$ ($\times$) indicates even (odd) under $T$. Replacement of $\tau_0$ by $\tau_3$ exchanges the roles of $\Delta_1$ and $\Delta_2$.

<table>
<thead>
<tr>
<th>$T$</th>
<th>$T^\dagger/T$</th>
<th>$\mu$</th>
<th>$m$</th>
<th>$\Delta_3$</th>
<th>$\Delta_1$</th>
<th>$\Delta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_2 \otimes \tau_0$</td>
<td>$-1$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$\gamma^0 \alpha_2 \otimes \tau_0$</td>
<td>$-1$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$\beta \alpha_2 \otimes \tau_1$</td>
<td>$+1$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\gamma^0 \beta \alpha_2 \otimes \tau_1$</td>
<td>$-1$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

(29). In particular, the condition for the normalizability of the zero-energy solution, Eq. (39), is translated into

$$\mu_m + \frac{1}{2m} \left( \frac{\Delta_1}{\Delta_r} \right)^2 > 0.$$ (53)

C. Altland-Zirnbauer symmetry class (Refs. 28 and 29)

Finally, we note the Altland-Zirnbauer symmetry class of the Hamiltonians that we have investigated in this section. Hamiltonian (29) with spatially dependent $\Delta_{1,2,3} \neq 0$ has the charge conjugation symmetry

$$\mathcal{C}^{-1} \mathcal{H} \mathcal{C} = -\mathcal{H}^*$$ with

$$\mathcal{C} = \begin{pmatrix} 0 & -i \alpha_2 \\ i \alpha_2 & 0 \end{pmatrix}.$$ (54)

The properties of each term under the time-reversal operator $T (T^\dagger \mathcal{H} T = \mathcal{H}^*)$ at $m = \mu = 0$ are summarized in Table III. In particular, the so-called chiral symmetry

$$\chi^{-1} \mathcal{H} \chi = -\mathcal{H}$$ with

$$\chi = \begin{pmatrix} \beta & 0 \\ 0 & -\beta \end{pmatrix}.$$ (55)

is present only if $m = \mu = 0$ and essential for the index theorem, Eq. (33). Therefore, the relativistic Hamiltonian (29) with $m, \mu \neq 0$ and thus the resulting nonrelativistic Hamiltonian (46) belong to the symmetry class D. There is no topological classification of class D Hamiltonians in 3D momentum spaces.30,31

IV. SUMMARY

We have studied the nonrelativistic limit of the Jackiw-Rossi model in 2D and the Jackiw-Rebbi model in 3D, both of which are known to exhibit fermion zero modes associated with pointlike topological defects (vortex and hedgehog). We showed that the nonrelativistic limit of the 2D Jackiw-Rossi model leads to the $p_\perp + ip_\parallel$ superconductor. Because the fermion zero mode persists under taking this limit, we obtain a clear understanding of the existence of a fermion zero mode bound to a vortex in the $p_\perp + ip_\parallel$ superconductor as a remnant of that in the Jackiw-Rossi model. Similarly, the nonrelativistic limit of the 3D Jackiw-Rebbi model leads to the $p + is$ superconductor in which the spin-triplet $p$-wave pairing gap $\Delta_p$ and the spin-singlet $s$-wave pairing gap $\Delta_s$ coexist. We showed that the resulting Hamiltonian supports a fermion zero mode when $\Delta_p$ and $\Delta_s / \Delta_p$ form a hedgehoglike structure. Fermion zero modes in the superconductors studied in this paper correspond to Majorana fermions and the associated pointlike defects obey non-Abelian statistics both in 2D (Refs. 3 and 7) and 3D.31,39

Our findings provide a unified view of Majorana zero modes in relativistic (Dirac-type) and nonrelativistic (Schrödinger-type) superconductors. It should be possible to generalize our analysis to other interesting cases and find new examples of nonrelativistic Hamiltonians, which are more common in condensed matter systems, with topological properties that descend from Dirac-type Hamiltonians, which are generally easier to analyze.

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TOPOLOGICAL SUPERCONDUCTORS AS

24. Two exceptional cases are \( h = \pm \mu \). When \( N_m > 0 (<0) \), the zero-energy solutions at \( h = \mu = 0 \) still solve Eq. (3) with \( h = (+(-) \mu \neq 0 \) and thus there are \( |N_m| \) zero modes. This can be easily seen if one rewrites the Hamiltonian in the basis where the chiral operator defined in Eq. (21) has the form \( \chi = \text{diag}(1, 1, -1, -1) \) and recognizes that the \( |N_m| \) zero-energy solutions are eigenstates of \( \chi \) with the eigenvalue \(+1(-1)\) for \( N_m > 0 (<0) \).
25. The same solution with \( n=1 \) was obtained independently in I. F. Herbut and C.-K. Lu, Phys. Rev. B 82, 125402 (2010).
35. The topological invariant can be matched if we properly regularize the large \( p \) behavior of the relativistic Hamiltonian: Replacing \( h \) in Eq. (2) by \( h(1 + |p|^2) \), the Chern number becomes \( \text{sgn}(h) \) for \( \mu^2 + |\Delta|^2 > h^2 \) and 0 for \( \mu^2 + |\Delta|^2 < h^2 \), which coincides with that of Eq. (13).
37. Two exceptional cases are \( m = \pm \mu \). When \( N_m > 0 (<0) \), the zero-energy solutions at \( m = \mu = 0 \) still solve Eq. (30) with \( m = (+(-) \mu \neq 0 \) and thus there are \( |N_m| \) zero modes. This can be easily seen if one rewrites the Hamiltonian in the basis where the chiral operator defined in Eq. (55) has the form \( \chi = \text{diag}(1, 1, -1, -1) \) and recognizes that the \( |N_m| \) zero-energy solutions are eigenstates of \( \chi \) with the eigenvalue \(+1(-1)\) for \( N_m > 0 (<0) \).
38. The same problem with \( m=0 \) was studied in T. Fukui, Phys. Rev. B 81, 214516 (2010).