Research Article

Universal Verma Modules and the Misra-Miwa Fock Space

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1. Introduction

Fock space is an infinite dimensional vector space which is a representation of several important algebras, as described in, for example, [1, Chapter 14]. Here we consider the charge zero part of Fock space, which we denote by $\mathcal{F}$, and its $v$-deformation $\mathcal{F}_v$. The space $\mathcal{F}$ has a standard $\mathbb{Q}$-basis $\{\{\mu\} \mid \lambda$ is a partition\} and $\mathcal{F}_v := \mathcal{F} \otimes_{\mathbb{Q}} \mathcal{Q}(v)$. Following Hayashi [2], Misra and Miwa [3] define an action of the quantized universal enveloping algebra $U_q(\mathfrak{sl}_2)$ on $\mathcal{F}_v$. The only nonzero matrix elements $\langle \mu | F_i | \lambda \rangle$ of the Chevalley generators $F_i$ in terms of the standard basis occur when $\mu$ is obtained by adding a single $i$-colored box to $\lambda$, and these are powers of $v$.

We show that these powers of $v$ also appear naturally in the following context: partitions with at most $N$ parts index polynomial Weyl modules $\Delta(\lambda)$ for the integral quantum group $U_q(\mathfrak{gl}_N)$. Let $V$ be the standard $N$ dimensional representation of $U_q(\mathfrak{gl}_N)$. If the matrix element $\langle \mu | F_i | \lambda \rangle$ is nonzero then, for sufficiently large $N$, $(\Delta^\mu(\lambda) \otimes_{\mathbb{Q}} V) \otimes_{\mathbb{Q}} \mathcal{Q}(q)$ contains the highest weight vector of weight $\mu$. There is a unique such highest weight vector $v_\mu$, which satisfies a certain triangularity condition with respect to an integral basis of $\Delta^\mu(\lambda) \otimes_{\mathbb{Q}} V$. We show that the matrix element $\langle \mu | F_i | \lambda \rangle$ is equal to $v_\mu^{val_{\phi_2}(\langle \mu, v_\mu \rangle)}$, where $\langle \cdot, \cdot \rangle$ is the Shapovalov form and $val_{\phi_2}$ is the valuation at the cyclotomic polynomial $\phi_2$. 

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Our proof is computational, making use of the Shapovalov determinant [4–6]. This is a formula for the determinant of the Shapovalov form on a weight space of a Verma module. The necessary computation is most easily done in terms of the universal Verma modules introduced in the classical case by Kashiwara [7] and studied in the quantum case by Kamita [8]. The statement for Weyl modules is then a straightforward consequence.

Before beginning, let us discuss some related work. In [9], Kleshchev carefully analyzed the $\mathfrak{gl}_{N-1}$ highest weight vectors in a Weyl module for $\mathfrak{gl}_N$ and used this information to give modular branching rules for symmetric group representations. Brundan and Kleshchev [10] have explained that highest weight vectors in the restriction of a Weyl module to $\mathfrak{gl}_{N-1}$ give information about highest weight vectors in a tensor product $\Delta(\lambda) \otimes V$ of a Weyl module with the standard $N$-dimensional representation of $\mathfrak{gl}_N$. Our computations put a new twist on the analysis of the highest weight vectors in $\Delta(\lambda) \otimes V$, as we study them in their “universal” versions and by the use of the Shapovalov determinant. Our techniques can be viewed as an application of the theory of Jantzen [11] as extended to the quantum case by Wiesner [12].

Brundan [13] generalized Kleshchev’s [9] techniques and used this information to give modular branching rules for Hecke algebras. As discussed in [14, 15], these branching rules are reflected in the fundamental representation of $\mathfrak{sl}_p$ and its crystal graph, recovering much of the structure of the Misra-Miwa Fock space. Using Hecke algebras at a root of unity, Ryom-Hansen [16] recovered the full $U_q(\mathfrak{sl}_\ell)$ action on Fock space. To complete the picture, one should construct a graded category, where multiplication by $\nu$ in the $\mathfrak{sl}_\ell$ representation corresponds to a grading shift. Recent work of Brundan-Kleshchev [17] and Ariki [18] explains that one solution to this problem is through the representation theory of Khovanov-Lauda-Rouquier algebras [19, 20]. It would be interesting to explicitly describe the relationship between their category and the present work. Another related construction due to Brundan-Stroppel considers the case when the Fock space is replaced by $\Lambda^m V \otimes \Lambda^n V$, where $V$ is the natural $\mathfrak{gl}_\infty$ module and $m, n$ are fixed natural numbers.

We would also like to mention very recent work of Peng Shan [21] which independently develops a similar story to the one presented here, but using representations of a quantum Schur algebra where we use representations of $U_q(\mathfrak{gl}_N)$. The approach taken there is somewhat different and in particular relies on localization techniques of Bezlinson and Bernstein [22].

This paper is arranged as follows. Sections 2 and 3 are background on the quantum group $U_q(\mathfrak{gl}_N)$ and the Fock space $F_\nu$. Sections 4 and 5 explain universal Verma modules and the Shapovalov determinant. Section 6 contains the statement and proof of our main result relating Fock space and Weyl modules.

### 2. The Quantum Group $U_q(\mathfrak{gl}_N)$ and Its Integral Form $U_q^{\text{dr}}(\mathfrak{gl}_N)$

This is a very brief review, intended mainly to fix notation. With slight modifications, the construction in this section works in the generality of symmetrizable Kac-Moody algebras. See [23, Chapters 6 and 9] for details.

#### 2.1. The Rational Quantum Group

$U_q(\mathfrak{gl}_N)$ is the associative algebra over the field of rational functions $\mathbb{Q}(q)$ generated by

$$X_1, \ldots, X_{N-1}, \quad Y_1, \ldots, Y_{N-1}, \quad L_{-1}^{\pm 1}, \ldots, L_{N}^{\pm 1}$$

(2.1)
with relations

\[ L_i L_j = L_j L_i, \quad L_i L_i^{-1} = L_i^{-1} L_i = 1, \quad X_i Y_j - Y_j X_i = \delta_{ij} \frac{L_i L_{i+1}^{-1} - L_{i+1} L_i^{-1}}{q - q^{-1}}, \]

\[ L_i X_j L_i^{-1} = \begin{cases} qX_j & \text{if } i = j, \\ q^{-1}X_j & \text{if } i = j + 1, \\ X_j & \text{otherwise,} \end{cases} \quad L_i Y_j L_i^{-1} = \begin{cases} q^{-1}Y_j & \text{if } i = j, \\ qY_j & \text{if } i = j + 1, \\ Y_j & \text{otherwise.} \end{cases} \]

\[ X_i X_j = X_j X_i, \quad Y_i Y_j = Y_j Y_i, \quad \text{if } |i - j| \geq 2, \]

\[ X_i^2 X_j - (q + q^{-1}) X_i X_j X_i + X_j X_i^2 = Y_i^2 Y_j - (q + q^{-1}) Y_j Y_i Y_j = 0, \quad \text{if } |i - j| = 1. \]

The algebra \( U_q(\mathfrak{gl}_N) \) is a Hopf algebra with coproduct and antipode given by

\[
\Delta(L_i) = L_i \otimes L_i, \quad S(L_i) = L_i^{-1},
\]

\[
\Delta(X_i) = X_i \otimes L_i L_{i+1}^{-1} + 1 \otimes X_i, \quad S(X_i) = -X_i L_i^{-1} L_{i+1},
\]

\[
\Delta(Y_i) = Y_i \otimes 1 + L_i^{-1} L_{i+1} \otimes Y_i, \quad S(Y_i) = -L_i L_{i+1}^{-1} Y_i,
\]

respectively, (see [23, Section 9.1]).

As a \( \mathbb{Q}(q) \)-vector space, \( U_q(\mathfrak{gl}_N) \) has a triangular decomposition

\[
U_q(\mathfrak{gl}_N) \equiv U_q(\mathfrak{gl}_N)^{\otimes 0} \otimes U_q(\mathfrak{gl}_N)^0 \otimes U_q(\mathfrak{gl}_N)^{>0},
\]

where the inverse isomorphism is given by multiplication (see [23, Proposition 9.1.3]). Here \( U_q(\mathfrak{gl}_N)^{\otimes 0} \) is the subalgebra generated by the \( Y_i \) for \( i = 1, \ldots, N-1 \), \( U_q(\mathfrak{gl}_N)^{>0} \) is the subalgebra generated by the \( X_i \) for \( i = 1, \ldots, N - 1 \), and \( U_q(\mathfrak{gl}_N)^0 \) is the subalgebra generated by the \( L_i^{\pm 1} \) for \( i = 1, \ldots, N \).

### 2.2. The Integral Quantum Group

Let \( \mathcal{A} = \mathbb{Z}[q, q^{-1}] \). For \( n, k \in \mathbb{Z}_{>0} \) and \( c \in \mathbb{Z} \), let

\[
[n] := \frac{q^n - q^{-n}}{q - q^{-1}}, \quad x^{(k)} := \frac{x^k}{[k][k-1] \cdots [2][1]}, \quad \begin{bmatrix} x; c \\ k \end{bmatrix} := \prod_{s=1}^{k} \frac{xq^{s+1-c} - x^{-1}q^{s-1-c}}{q^s - q^{-s}},
\]
in \( \mathbb{Q}(q, x) \). The restricted integral form \( U_q^\mathfrak{sl}(\mathfrak{gl}_N) \) is the \( A \)-subalgebra of \( U_q(\mathfrak{gl}_N) \) generated by \( X_i^{(k)}, Y_i^{(k)}, L_i^{-1} \) and \( \{ L_{i,c}^{-} \} \) for \( 1 \leq i \leq N, c \in \mathbb{Z}, k \in \mathbb{Z}_{>0} \). As discussed in [24, Section 6], this is an integral form in the sense that

\[
U_q^\mathfrak{sl}(\mathfrak{gl}_N) \otimes_A \mathbb{Q}(q) = U_q(\mathfrak{gl}_N).
\] (2.6)

As with \( U_q(\mathfrak{gl}_N) \), the algebra \( U_q^\mathfrak{sl}(\mathfrak{gl}_N) \) has a triangular decomposition

\[
U_q^\mathfrak{sl}(\mathfrak{gl}_N) \cong U_q^\mathfrak{sl}_0(\mathfrak{gl}_N) \otimes U_q^\mathfrak{sl}(\mathfrak{gl}_N)^0 \otimes U_q^\mathfrak{sl}(\mathfrak{gl}_N)^{>0},
\] (2.7)

where the isomorphism is given by multiplication (see [23, Proposition 9.3.3]). In this case, \( U_q^\mathfrak{sl}_0(\mathfrak{gl}_N) \) is the subalgebra generated by the \( X_i^{(k)} \), \( U_q^\mathfrak{sl}(\mathfrak{gl}_N)^{>0} \) is the subalgebra generated by the \( X_i^{(k)} \), and \( U_q^\mathfrak{sl}(\mathfrak{gl}_N)^0 \) is generated by \( L_i^{-1} \) and \( \{ L_{i,c}^{-} \} \) for \( 1 \leq i \leq N, c \in \mathbb{Z}, k \in \mathbb{Z}_{>0} \).

### 2.3. Rational Representations

The Lie algebra \( \mathfrak{gl}_N = M_N(\mathbb{C}) \) of \( N \times N \) matrices has standard basis \( \{ E_{ij} \mid 1 \leq i, j \leq N \} \), where \( E_{ij} \) is the matrix with 1 in position \( (i, j) \) and 0 everywhere else. Let \( \mathfrak{h} = \text{span}\{E_{11}, E_{22}, \ldots, E_{NN}\} \). Let \( \varepsilon_i \in \mathfrak{h}^* \) be the weight of \( \mathfrak{gl}_N \) given by \( \varepsilon_i(E_{ij}) = \delta_{ij} \). Define

\[
\begin{align*}
\mathfrak{h}^*_Z := \{ \lambda = \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \cdots + \lambda_N \varepsilon_N \in \mathfrak{h}^* \mid \lambda_1, \ldots, \lambda_N \in \mathbb{Z} \}, \\
(\mathfrak{h}^*_Z)^+ := \{ \lambda = \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \cdots + \lambda_N \varepsilon_N \in \mathfrak{h}^* \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \}, \\
P^+ := \{ \lambda = \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \cdots + \lambda_N \varepsilon_N \in (\mathfrak{h}^*_Z)^+ \mid \lambda_N \geq 0 \}, \\
R^+ := \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq N \}, \\
Q := \text{span}_{\mathbb{Z}}(R^+), \\
Q^+ := \text{span}_{\mathbb{Z}_{\geq 0}}(R^+), \\
Q^- := \text{span}_{\mathbb{Z}_{< 0}}(R^+)
\end{align*}
\] (2.8)

to be the set of integral weights, the set of dominant integral weights, the set of dominant polynomial weights, the set of positive roots, the root lattice, the positive part of the root lattice, and the negative part of the root lattice, respectively.

For an integral weight \( \lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_N \varepsilon_N \), the Verma module \( M(\lambda) \) for \( U_q(\mathfrak{gl}_N) \) of the highest weight \( \lambda \) is

\[
M(\lambda) := U_q(\mathfrak{gl}_N) \otimes_{U_q(\mathfrak{gl}_N)^{>0}} \mathbb{Q}(q)_{\lambda},
\] (2.9)

where \( \mathbb{Q}(q)_\lambda = \text{span}_{\mathbb{Q}(q)}\{v_\lambda\} \) is the one dimensional vector space over \( \mathbb{Q}(q) \) with \( U_q(\mathfrak{gl}_N)^{>0} \) action given by

\[
X_i \cdot v_\lambda = 0, \quad L_j \cdot v_\lambda = q^{h_i} v_\lambda, \quad \text{for } 1 \leq i \leq N - 1, 1 \leq j \leq N.
\] (2.10)
Theorem 2.1 (see [23, Chapter 10.1]). If $\lambda \in (\mathfrak{h}_\mathbb{Z})^+$ then $M(\lambda)$ has a unique finite dimensional quotient $\Delta(\lambda)$ and the map $\lambda \mapsto \Delta(\lambda)$ is a bijection between $(\mathfrak{h}_\mathbb{Z})^+$, and the set of irreducible finite dimensional $U_q(\mathfrak{gl}_N)$-modules.

A singular vector in a representation of $U_q(\mathfrak{gl}_N)$ is a vector $v$ such that $X_i \cdot v = 0$ for all $i$.

2.4. Integral Representations

The integral Verma module $M^\text{id}(\lambda)$ is the $U_q^\text{id}(\mathfrak{gl}_N)$-submodule of $M(\lambda)$ generated by $v_\lambda$. The integral Weyl module $\Delta^\text{id}(\lambda)$ is the $U_q^\text{id}(\mathfrak{gl}_N)$-submodule of $\Delta(\lambda)$ generated by $v_\lambda$. Using (2.6) and (2.4),

$$M^\text{id}(\lambda) \otimes_{\mathfrak{gl}_N}(q) = M(\lambda), \quad \Delta^\text{id}(\lambda) \otimes_{\mathfrak{gl}_N}(q) = \Delta(\lambda).$$

In general, $\Delta^\text{id}(\lambda)$ is not irreducible as a $U_q^\text{id}(\mathfrak{gl}_N)$ module.

3. Partitions and Fock Space

We now describe the $\nu$-deformed Fock space representation of $U_{\nu}(\hat{\mathfrak{sl}}_\ell)$ constructed by Misra and Miwa [3] following work of Hayashi [2]. Our presentation largely follows [25, Chapter 10].

3.1. Partitions

A partition $\lambda$ is a finite length nonincreasing sequence of positive integers. Associated to a partition is its Ferrers diagram. We draw these diagrams as in Figure 1 so that, if $\lambda = (\lambda_1, \ldots, \lambda_N)$, then $\lambda_i$ is the number of boxes in row $i$ (rows run southeast to northwest "). Say that $\lambda$ is contained in $\mu$ if the diagram for $\lambda$ fits inside the diagram for $\mu$ and let $\mu/\lambda$ be the collection of boxes of $\mu$ that are not in $\lambda$. For each box $b \in \lambda$, the content $c(b)$ is the horizontal position of $b$ and the color $\overline{c}(b)$ is the residue of $c(b)$ modulo $\ell$. In Figure 1, the numbers $c(b)$ are listed below the diagram. The size $|\lambda|$ of a partition $\lambda$ is the total number of boxes in its Ferrers diagram.

The set $P^*$ of dominant polynomial weights from Section 2.3 is naturally identified with partitions with at most $N$ parts. If $\lambda \in P^*$, then

$$\Delta(\lambda) \otimes \Delta(\epsilon_1) \cong \bigoplus_{1 \leq k \leq N, \lambda + \epsilon_k \in P^*} \Delta(\lambda + \epsilon_k)$$

as $U_q(\mathfrak{gl}_N)$-modules. The diagram of $\lambda + \epsilon_k$ is obtained from the diagram of $\lambda$ by adding a box on row $k$, and $\Delta(\lambda + \epsilon_k)$ appears in the sum on the right side of (3.1) if and only if $\lambda + \epsilon_k$ is a partition. See, for example, [26, Section 6.1, Formula 6.8] for the classical statement and [23, Proposition 10.1.16] for the quantum case.
Figure 1: The partition \((7,6,6,5,3,3,1)\) with each box containing its color for \(\ell = 3\). The content \(c(b)\) of a box \(b\) is the horizontal position of \(b\) reading right to left. The contents of boxes are listed beneath the diagram so that \(c(b)\) is aligned with all boxes \(b\) of that content.

### 3.2. The Quantum Affine Algebra

Let \(U'_\ell(\mathfrak{sl}_\ell)\) be the quantized universal enveloping algebra corresponding to the \(\ell\)-node Dynkin diagram

More precisely, \(U'_\ell(\mathfrak{sl}_\ell)\) is the algebra generated by \(E_\ell, F_\ell, K_\ell^\pm 1\), for \(\ell \in \mathbb{Z}/\ell\mathbb{Z}\), with relations

\[
\begin{align*}
K_\ell K_\ell &= K_\ell K_\ell^{-1} = K_\ell^{-1} K_\ell = 1, \\
E_\ell F_\ell - F_\ell E_\ell &= \delta_\ell \frac{K_\ell - K_\ell^{-1}}{v - v^{-1}}, \\
K_\ell E_\ell K_\ell^{-1} &= \begin{cases} v^2 E_\ell, & \text{if } \ell = j, \\
v^{-1} E_\ell, & \text{if } \ell = j \pm 1, \\
E_\ell, & \text{otherwise,} \end{cases} \\
K_\ell F_\ell K_\ell^{-1} &= \begin{cases} v F_\ell, & \text{if } \ell = j, \\
v^{-1} F_\ell, & \text{if } \ell = j \pm 1, \\
F_\ell, & \text{otherwise,} \end{cases} \\
E_\ell E_\ell &= E_\ell, \\
F_\ell F_\ell &= \pm F_\ell, & \text{if } |\ell - j| \geq 2, \\
E_\ell^2 E_\ell - (v + v^{-1})E_\ell F_\ell E_\ell + E_\ell F_\ell^2 E_\ell = F_\ell^2 E_\ell - (v + v^{-1})F_\ell F_\ell^2 + F_\ell E_\ell^2 F_\ell = 0, & \text{if } |\ell - j| = 1.
\end{align*}
\]

(3.2)

See [23, Definition Proposition 9.1.1]. The algebra \(U'_\ell(\mathfrak{sl}_\ell)\) is the quantum group corresponding to the nontrivial central extension \(\mathfrak{sl}'_\ell = \mathfrak{sl}_\ell[t, t^{-1}] \oplus \mathbb{C}\) of the algebra of polynomial loops in \(\mathfrak{sl}_\ell\).
3.3. Fock Space

Define $\nu$-deformed Fock space to be the $\mathbb{Q}(\tau)$ vector space $F_\nu$ with basis $\{|\mu\rangle | \lambda \text{ is a partition}\}$. Our $F_\nu$ is only the charge 0 part of Fock space described in [27]. Fix $\ell \in \mathbb{Z}/\ell\mathbb{Z}$ and partitions $\lambda \subseteq \mu$ such that $\mu/\lambda$ is a single box. Define

$$A_\ell^i(\lambda) := \left\{ \text{boxes } b \mid b \notin \lambda, \ b \text{ has color } \ell \text{ and } \lambda \cup b \text{ is a partition} \right\},$$

$$R_\ell^i(\lambda) := \left\{ \text{boxes } b \mid b \in \lambda, \ b \text{ has color } \ell \text{ and } \lambda \setminus b \text{ is a partition} \right\},$$

$$N_\ell^l(\mu/\lambda) := \left| \left\{ b \in R_\ell^i(\lambda) \mid b \text{ is to the left of } \mu/\lambda \right\} \right| - \left| \left\{ b \in A_\ell^i(\lambda) \mid b \text{ is to the left of } \mu/\lambda \right\} \right|,$$

$$N_\ell^r(\mu/\lambda) := \left| \left\{ b \in R_\ell^i(\lambda) \mid b \text{ is to the right of } \mu/\lambda \right\} \right| - \left| \left\{ b \in A_\ell^i(\lambda) \mid b \text{ is to the right of } \mu/\lambda \right\} \right|,$$

(3.3)

to be the set of addable boxes of color $\ell$, the set of removable boxes of color $\ell$, the left removable-addable difference, and the right removable-addable difference, respectively.

**Theorem 3.1** (see [25, Theorem 10.6]). There is an action of $U'_\nu(\mathfrak{sl}_\ell)$ on $F_\nu$ determined by

$$E_\ell^i(\lambda) := \sum_{\tau(\lambda/\mu)=i} \tau^{N_\ell^l(\lambda/\mu)} |\mu\rangle, \quad F_\ell^i(\lambda) := \sum_{\tau(\mu/\lambda)=i} \tau^{N_\ell^r(\mu/\lambda)} |\mu\rangle,$$

where $\tau(\lambda/\mu)$ denotes the color of $\lambda/\mu$ and the sum is over partitions $\mu$ which differ from $\lambda$ by removing (resp. adding) a single $\ell$-colored box.

As a $U'_\nu(\mathfrak{sl}_\ell)$-module, $F_\nu$ is isomorphic to an infinite direct sum of copies of the basic representation $V(\Lambda_0)$. Using the grading of $F_\nu$, where $|\lambda\rangle$ has degree $|\lambda|$, the highest weight vectors in $F_\nu$ occur in degrees divisible by $\ell$, and the number of the highest weight vectors in degree $\ell k$ is the number of partitions of $k$. Then, $F_\nu \cong V(\Lambda_0) \otimes \mathbb{C}[x_1, x_2, \ldots]$, where $x_k$ has degree $\ell k$, and $U'_\nu(\mathfrak{sl}_\ell)$ acts trivially on the second factor (see [27, Proposition 2.3]). Note that we are working with the “derived” quantum group $U'_\nu(\mathfrak{sl}_\ell)$, not the “full” quantum group $U_\nu(\mathfrak{sl}_\ell)$, which is why there are no $\delta$-shifts in the summands of $F_\nu$.

**Comment 1.** Comparing with [25, Chapter 10], our $N_\ell^l(\mu/\lambda)$ is equal to Ariki’s $-N_\ell^a(\mu/\lambda)$ and our $N_\ell^r(\mu/\lambda)$ is equal to Ariki’s $-N_\ell^b(\mu/\lambda)$. However, these numbers play a slightly different role in Ariki’s work, which is explained by a different choice of conventions.

4. Universal Verma Modules

The purpose of this section is to construct a family of representations which are universal Verma modules in the sense that each can be “evaluated” to obtain any given Verma module. This notion was defined by Kashiwara [7] in the classical case and was studied in the quantum case by Kamita [8].
4.1. Rational Universal Verma Modules

Let \( \mathbb{K} := \mathbb{Q}(q, z_1, z_2, \ldots, z_N) \). This field is isomorphic to the field of fractions of \( \mathcal{U}_q(\mathfrak{gl}_N)^0 \) via the map

\[
\Psi : \mathcal{U}_q(\mathfrak{gl}_N)^0 \to \mathbb{K}, \quad \text{defined by } \Psi \left( L_i^{\pm 1} \right) = z_i^{\pm 1}.
\]  

For each \( \mu \in \mathfrak{h}^*_Z \), define a \( \mathbb{Q}(q) \)-linear automorphism \( \sigma_{\mu} : \mathbb{K} \to \mathbb{K} \) by

\[
\sigma_{\mu}(z_i) := q^{(\mu, \epsilon_i)} z_i, \quad \text{for } 1 \leq i \leq N,
\]

where \( (\cdot, \cdot) \) is the inner product on \( \mathfrak{h}^*_Z \) defined by \( (\epsilon_i, \epsilon_j) = \delta_{i,j} \). Let \( \mathbb{K}_{\mu} = \text{span}_{\mathbb{K}} \{ v_{\mu^+} \} \) be the one-dimensional vector space over \( \mathbb{K} \) with basis vector \( v_{\mu^+} \) and \( \mathcal{U}_q(\mathfrak{gl}_N)^{>0} \) action given by

\[
X_i \cdot v_{\mu^+} = 0, \quad \text{for } 1 \leq i \leq N - 1, \quad a \cdot v_{\mu^+} = \sigma_{\mu}(a) v_{\mu^+}, \quad \text{for } a \in \mathcal{U}_q(\mathfrak{gl}_N)^0.
\]

The \( \mu \)-shifted rational universal Verma module \( \tilde{M} \) is the \( \mathcal{U}_q(\mathfrak{gl}_N) \)-module

\[
\tilde{M} := \mathcal{U}_q(\mathfrak{gl}_N) \otimes_{\mathcal{U}_q(\mathfrak{gl}_N)^0} \mathbb{K}_\mu.
\]

The universal Verma module \( \tilde{M} \) is actually a module over \( \mathcal{U}_q(\mathfrak{gl}_N) \otimes_{\mathcal{U}_q(\mathfrak{gl}_N)^0} \mathcal{U}_q(\mathfrak{gl}_N)^0 \), where \( \mathcal{U}_q(\mathfrak{gl}_N)^0 \) is the field of fractions of \( \mathcal{U}_q(\mathfrak{gl}_N)^0 \). However, if we identify \( \mathcal{U}_q(\mathfrak{gl}_N)^0 \) with \( \mathbb{K} \) using the map \( \Psi \), the action of \( \mathcal{U}_q(\mathfrak{gl}_N)^0 \) on \( \tilde{M} \) is not by multiplication, but rather is twisted by the automorphism \( \sigma_{\mu} \). It is to keep track of the difference between the action of \( \mathcal{U}_q(\mathfrak{gl}_N)^0 \) and multiplication that we use different notation for the generators of \( \mathbb{K} \) and \( \mathcal{U}_q(\mathfrak{gl}_N)^0 \) (i.e., \( z_i \) versus \( L_i \)).

4.2. Integral Universal Verma Modules

The field \( \mathbb{K} \) contains an \( \mathcal{A} \)-subalgebra

\[
\mathcal{R} \text{ generated by } z_i^{\pm 1}, \quad \left[ \begin{array}{c} z_i; c \\ k \end{array} \right], \quad (1 \leq i \leq N, c \in \mathbb{Z}, k \in \mathbb{Z}_{>0}),
\]

which is isomorphic to \( \mathcal{U}_q^{\mathfrak{gl}_N} \) via the restriction of the map \( \Psi \) in (4.1). The integral universal Verma module \( \tilde{M} \) is the \( \mathcal{U}_q^{\mathfrak{gl}_N} \)-submodule of \( \tilde{M} \) generated by \( v_{\mu^+} \). By restricting (4.4),

\[
\tilde{M} = \mathcal{U}_q^{\mathfrak{gl}_N} \otimes_{\mathcal{U}_q^{\mathfrak{gl}_N}} \mathcal{R}_\mu,
\]

where \( \mathcal{R}_\mu \) is the \( \mathcal{R} \)-submodule of \( \mathbb{K}_\mu \) spanned by \( v_{\mu^+} \). In particular, \( \tilde{M} \) is a free \( \mathcal{R} \)-module.
4.3. Evaluation

Let $\ev^R_\lambda : \mathcal{R} \to \mathcal{A}$ be the map defined by

$$
\ev^R_\lambda(z_i) = q^{(\lambda,\varepsilon_i)}, \quad \ev^R_\lambda\left[\frac{z_i}{n}\right] = \left[\frac{q^{(\lambda,\varepsilon_i)}}{n}\right],
$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathfrak{h}^*$ defined by $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$. There is a surjective $U^d_q(\mathfrak{g}_N)$-module homomorphism “evaluation at $\lambda$”

$$
ev^R_{\lambda^\mu}\hat{M}^R \twoheadrightarrow M^d(\mu + \lambda) \text{ defined by } \ev^R_{\lambda^\mu}(a \cdot v_{\mu^+}) := a \cdot v_{\mu^+\lambda}, \quad \forall a \in U^d_q(\mathfrak{g}_N). \quad (4.8)
$$

For fixed $\lambda$, the maps $\ev^R_\lambda$ and $\ev_\lambda$ extend to a map from the subspace of $\mathbb{K}$ and $\hat{M}^R = \hat{M}^R \otimes_\mathbb{K} \mathbb{K}_N$ respectively, where no denominators evaluate to 0. Where it is clear we denote both these extended maps by $\ev_\lambda$.

Example 4.1. Computing the action of $L_i$ on $v_{\mu^+}$ and $v_{\mu^+\lambda}$,

$$L_i \cdot v_{\mu^+} = q^{(\mu,\varepsilon_i)} z_i v_{\mu^+}, \quad L_i \cdot v_{\mu^+\lambda} = \ev^R_{\lambda}(q^{(\mu,\varepsilon_i)} z_i)v_{\mu^+\lambda} = q^{(\mu,\varepsilon_i)} q^{(\lambda,\varepsilon_i)} v_{\mu^+\lambda} = q^{(\mu^+\lambda,\varepsilon_i)} v_{\mu^+\lambda}.
$$

4.4. Weight Decompositions

Let $\tilde{V}$ be a $U_q(\mathfrak{g}_N) \otimes_\mathcal{R} \mathcal{R}$-module. For each $\nu \in \mathfrak{h}_2^*$, we define the $\nu$-weight space of $\tilde{V}$ to be

$$
\tilde{V}_\nu := \left\{ v \in \tilde{V} : L_i \cdot v = q^{(\nu,\varepsilon_i)} z_i v \right\}.
$$

The universal Verma module $\hat{M}^R$ is a $U_q(\mathfrak{g}_N) \otimes_\mathcal{R} \mathcal{R}$-module, where the second factor acts as multiplication. The weight space $\hat{M}^R_{\mu} \neq 0$ if and only if $\eta = \mu - \nu$ with $\nu$ in the positive part $Q^+$ of the root lattice. These nonzero weight spaces and the weight decomposition of $\hat{M}^R$ can be described explicitly by

$$
\hat{M}^R_{\mu} \cap \mathcal{R} = \bigoplus_{\nu \in Q^+} \hat{M}^R_{\mu - \nu}, \quad \hat{M}^R = \bigoplus_{\nu \in Q^+} \hat{M}^R_{\mu - \nu}.
$$

Here, $U_q^d(\mathfrak{g}_N) \cap \mathcal{R}$ is defined using the grading of $U_q(\mathfrak{g}_N) \cap \mathcal{R}$ with $F_i \in U_q(\mathfrak{g}_N) \cap \mathcal{R}$.

4.5. Tensor Products

Let $\tilde{V}$ be a $U^d_q(\mathfrak{g}_N) \otimes_\mathcal{R} \mathcal{R}$-module and $W$ a $U^d_q(\mathfrak{g}_N)$-module. The tensor product $\tilde{V} \otimes_\mathcal{R} W$ is a $U^d_q(\mathfrak{g}_N) \otimes_\mathcal{R} \mathcal{R}$-module, where the first factor acts via the usual coproduct and the second
factor acts by multiplication on $\tilde{V}$. In the case when $\tilde{V}$ and $W$ both have weight space decompositions, the weight spaces of $\tilde{V} \otimes_W W$ are

$$\left(\tilde{V} \otimes_W W\right)_\nu = \bigoplus_{\gamma + \eta = \nu} \tilde{V}_\gamma \otimes_W W_\eta. \quad (4.12)$$

We also need the following.

**Proposition 4.2.** The tensor product of a universal Verma module with a Weyl module satisfies

$$\left(\mu \tilde{M}^R \otimes \Delta^\omega(v)\right) \otimes_K \mathbb{K} \equiv \left(\bigoplus_{\gamma} \left(\mu \gamma \tilde{M}^R\right) \oplus \dim \Delta^\omega(v)\right) \otimes_K \mathbb{K}. \quad (4.13)$$

**Proof.** Fix $\nu \in P^+$. In general, $M(\lambda + \mu) \otimes \Delta(\nu)$ has a Verma filtration (see, e.g., [28, Theorem 2.2]) and if $\lambda + \mu + \gamma$ is dominant for all $\gamma$ such that $\Delta(\nu)\gamma \neq 0$ then

$$M(\lambda + \mu) \otimes \Delta(\nu) \equiv \bigoplus_{\gamma} M(\lambda + \mu + \gamma) \oplus \dim \Delta(\nu)\gamma, \quad (4.14)$$

which can be seen by, for instance, taking central characters. The proposition follows since this is true for a Zariski dense set of weights $\lambda$. \qed

### 5. The Shapovalov Form and the Shapovalov Determinant

#### 5.1. The Shapovalov Form

The Cartan involution $\omega : U_q(\mathfrak{gl}_N) \rightarrow U_q(\mathfrak{gl}_N)$ is the $\mathbb{C}(q)$-algebra anti-involution of $U_q(\mathfrak{gl}_N)$ defined by

$$\omega(L_i^{\pm 1}) = L_i^{\pm 1}, \quad \omega(X_i) = Y_i L_i L_{i+1}^{-1}, \quad \omega(Y_i) = L_i^{-1} L_{i+1} X_i. \quad (5.1)$$

The map $\omega$ is also a coalgebra involution. An $\omega$-contravariant form on a $U_q(\mathfrak{gl}_N)$-module $V$ is a symmetric bilinear form $(\cdot, \cdot)$ such that

$$(u, a \cdot v) = (\omega(a) \cdot u, v), \quad \text{for } u, v \in V, a \in U_q(\mathfrak{gl}_N). \quad (5.2)$$

It follows by the same argument used in the classical case [4] that there is an $\omega$-contravariant form (the Shapovalov form) on each Verma module $M(\lambda)$ and this is unique up to rescaling. The radical of $(\cdot, \cdot)$ is the maximal proper submodule of $M(\lambda)$, so $\Delta(\lambda) = M(\lambda) / \text{Rad}(\cdot, \cdot)$ for all $\lambda \in P^+$. In particular, $(\cdot, \cdot)$ descends to an $\omega$-contravariant form on $\Delta(\lambda)$.

Since $\omega$ fixes $U_q^d(\mathfrak{gl}_N) \subseteq U_q(\mathfrak{gl}_N)$, there is a well-defined notion of an $\omega$-contravariant form on a $U_q^d(\mathfrak{gl}_N)$ module. In particular, the restriction of the Shapovalov form on $\Delta(\lambda)$ to $\Delta^d(\lambda)$ is $\omega$-contravariant.
5.2. Universal Shapovalov Forms

There are surjective maps of $\mathcal{A}$-algebras $p_- : U_q^d (gl_N)^{c_0} \to \mathbb{Q}(q)$ and $p_+ : U_q^d (gl_N)^{>0} \to \mathbb{Q}(q)$ defined by $p_- (F_i) = 0$ and $p_+ (E_i) = 0$, for $1 \leq i \leq N$. Using the triangular decomposition (2.7), there is an $\mathcal{A}$-linear surjection

$$\pi_0 := p_- \circ \text{Id} \circ p_+ : U_q^d (gl_N) \cong U_q^d (gl_N)^{c_0} \otimes_{\mathcal{A}} U_q^d (gl_N)^{>0} \otimes_{\mathcal{A}} U_q^d (gl_N)^{>0} \to U_q^d (gl_N)^{>0}. \quad (5.3)$$

The standard universal Shapovalov form is the $\mathcal{R}$-bilinear form $(\cdot, \cdot)_{\tilde{M}^R}$ on $\tilde{M}^R \otimes \tilde{M}^R \to \mathcal{R}$ defined by

$$(a_1 \cdot \nu_{\mu^+}, a_2 \cdot \nu_{\mu^+})_{\tilde{M}^R} = (\sigma_{\mu} \circ \psi \circ \pi_0) (\omega(a_2) a_1) \quad (5.4)$$

for all $a_1, a_2 \in U_q^d (gl_N)^{c_0}$. Here, $\psi$ and $\sigma_{\mu}$ are as in (4.1) and (4.2). Since

$$(a_1 a_2 \cdot \nu_{\mu^+}, a_3 \cdot \nu_{\mu^+})_{\tilde{M}^R} = (\sigma_{\mu} \circ \psi \circ \pi_0) (\omega(a_2) \omega(a_1) a_3) = (a_2 \cdot \nu_{\mu^+}, \omega(a_1) a_3 \cdot \nu_{\mu^+})_{\tilde{M}^R} \quad (5.5)$$

for $a_1, a_2, a_3 \in U_q^d (gl_N)$, the form $(\cdot, \cdot)_{\tilde{M}^R}$ is $\omega$-contravariant. As with the usual Shapovalov form, distinct weight spaces are orthogonal, where weight spaces are defined as in Section 4.4.

Evaluation at $\lambda$ gives an $\mathcal{A}$-valued $\omega$-contravariant form $(\cdot, \cdot)_{\tilde{M}^R(\mu + \lambda)}$ on $M^d (\mu + \lambda)$ by

$$(ev_1(u_1), ev_1(u_2))_{M^d(\mu + \lambda)} = ev_1((u_1, u_2)_{\tilde{M}^R}) \quad \text{for } u_1, u_2 \in \tilde{M}^R. \quad (5.6)$$

The form $(\cdot, \cdot)_{\tilde{M}^R}$ can be extended by linearity to an $\omega$-contravariant form $(\cdot, \cdot)_{\tilde{M}^R}$ on $\tilde{M}^R$.

5.3. The Shapovalov Determinant

Let $\tilde{V}$ be a $(U_q^d (gl_N) \otimes_{\mathcal{A}} \mathcal{R})$-module with a chosen $\omega$-contravariant form. Let $B_\eta$ be an $\mathcal{R}$ basis for the $\eta$-weight space $\tilde{V}_\eta$ of $\tilde{V}$. Let $\tilde{V}_{\eta}$ be the determinant of the form evaluated on the basis $B_\eta$. Changing the basis $B_\eta$ changes the determinant by a unit in $\mathcal{R}$, and we sometimes write $\det \tilde{V}_\eta$ to mean the determinant calculated on an unspecified basis ($\det \tilde{V}_\eta$ which is only defined up to multiplication by unit in $\mathcal{R}$). The Shapovalov determinant is

$$\det \tilde{M}^R_\eta := \det (b_i, b_j)_{\tilde{M}^R})_{b_i, b_j \in B_\eta}. \quad (5.7)$$

Define the partition function $p : \mathfrak{h}^* \to \mathbb{Z}_{\geq 0}$ by

$$p(\gamma) := \dim M(0)_{\gamma}. \quad (5.8)$$

Then, $p(\gamma) = \dim M(\lambda)_{\gamma + \lambda}$ for any $\lambda$, and $\eta \not\in Q^-$ implies that $p(\eta) = 0$ and $\det \tilde{M}^R_\eta = 1$. 

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Theorem 5.1 (see [5, Proposition 1.9A], [6, Theorem 3.4], [4]). For any weight \( \eta \),

\[
\det \bar{M}_\eta^R = c_\eta \prod_{1 \leq i < j \leq N} \left( z_j z_i^{-1} - q^{2m+2i-2j} z_i^{-1} z_j \right)^{\mu(\eta+me(-m)e)},
\]

where \( c_\eta \) is a unit in \( R \otimes \mathcal{A} \mathbb{Q}(q) = \mathbb{Q}(q)[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \).

Proposition 5.2. Fix \( \mu, \eta \in h^*_C \), with \( \eta - \mu \in Q_\cdot \). Choose an \( \mathcal{A} \)-basis \( B_{\eta, \mu} \) for \( U_q^A(\mathfrak{gl}_N)_{\eta, \mu} \). Consider the \( R \)-bases \( \bar{B}_{\eta, \mu} := \{ b \cdot v_+ \mid b \in B_{\eta, \mu} \} \) for \( \bar{M}_\eta^R \) and \( \mu \bar{B}_{\eta} := \{ b \cdot v_+ \mid b \in B_{\eta, \mu} \} \) for \( \mu \bar{M}_\eta^R \). Then

\[
\det \mu \bar{M}_{(\eta, \mu)}^R = \sigma_{\mu}(\det \bar{M}_{(\eta, \mu)}^R).
\]

Proof. For \( b, b' \in B_{\eta, \mu} \),

\[
( b \cdot v_+, b' \cdot v_+ )_{(\eta, \mu)}^R = \sigma_{\mu} \circ q \circ \pi_0(\omega(b')b) = \sigma_{\mu}( ( b \cdot v_+, b' \cdot v_+ )_{(\eta, \mu)}^R ).
\]

The result follows by taking determinants. \( \square \)

5.4. Contravariant Forms on Tensor Products

If \( V \) and \( W \) are \( U_q^A(\mathfrak{gl}_N) \)-modules with \( \omega \)-contravariant forms \((,)_V\) and \((,)_W\), define an \( \mathcal{A} \)-bilinear form \((,)_W \otimes \mathcal{A}_V\) by \((w_1 \otimes v_1, w_2 \otimes v_2)_W \otimes \mathcal{A}_V = (w_1, w_2)_W(v_1, v_2)_V\). Similarly, for a \( U_q^A(\mathfrak{gl}_N) \otimes \mathcal{A} \) \( R \) module \( W \) with \( R \)-bilinear \( \omega \)-contravariant form \((,)_W\), define a \( \mathcal{A} \)-bilinear form \((,)_W \otimes \mathcal{A}_V\) on \( W \otimes \mathcal{A}_V \) by

\[
(u_1 \otimes v_1, u_2 \otimes v_2)_W \otimes \mathcal{A}_V = (u_1, u_2)_W(v_1, v_2)_V.
\]

Since \( \omega \) is a coalgebra involution (i.e., \( \Delta(\omega(a)) = (\omega \otimes \omega)\Delta(a) \), for \( a \in U_q(\mathfrak{gl}_N) \)), the forms \((,)_W \otimes \mathcal{A}_V\) and \((,)_W \otimes \mathcal{A}_V\) are \( \omega \)-contravariant.

In the case when \( \bar{W} = \mu \bar{M}_R^R \), evaluation of the \( \omega \)-contravariant form \((,)_W \otimes \mathcal{A}_V\) at \( \lambda \) gives an \( \omega \)-contravariant form \((,)_M(\mu, \lambda) \otimes \mathcal{A}_V\):

\[
(u_1 \otimes v_1, u_2 \otimes v_2)_M(\mu, \lambda) \otimes \mathcal{A}_V = \text{ev}_1 \left( (u_1 \otimes v_1, u_2 \otimes v_2)_W \otimes \mathcal{A}_V \right) = \text{ev}_1 (u_1) \otimes v_1, \text{ev}_1 (u_2) \otimes v_2)_M(\mu, \lambda) \otimes \mathcal{A}_V,
\]

for \( u_1, u_2 \in \mu \bar{M} \) and \( v_1, v_2 \in V \). As in Section 4.3, this form can be extended to the \( \mathcal{A} \)-submodule of the rational module where no denominators evaluate to zero.
6. The Misra-Miwa Formula for $F_\ell$ from $U_q^d(gl_N)$

Representation Theory

Let us prepare the setting for our main result (Theorem 6.1). Fix $\ell \geq 2$ and a partition $\lambda$. Let $N$ be a positive integer greater than the number of parts of $\lambda$. All calculations below are in terms of representations of $U_q^d(gl_N)$.

(1) Let $V = \Delta^d(\epsilon_1)$ be the standard $N$-dimensional module. Since $\Delta^d(\lambda) \otimes \mathcal{A}(q) = \Delta(\lambda)$, (3.1) implies

$$
\left( \Delta^d(\lambda) \otimes \mathcal{A} V \right) \otimes \mathcal{A}(q) = \bigoplus \Delta^d(\lambda + \epsilon_k) \otimes \mathcal{A}(q),
$$

(6.1)

where the sum is over those indices $1 = k_1 < k_2 < \cdots < k_m \leq N$ for which $\lambda + \epsilon_{k_i}$ is a partition. For ease of notation, let $\mu^{(i)} = \lambda + \epsilon_{k_i}$.

(2) Fix an $\mathcal{A}$-basis $\{v_1, \ldots, v_N\}$ of $V$ where $v_k$ has weight $\epsilon_k$ and $Y_i(v_k) = \delta_{i,k} v_{k+1}$. Recursively, define singular weight vectors $v_{\mu^{(i)}}$ in $(\Delta^d(\lambda) \otimes V) \otimes \mathcal{A}(q)$ by

(i) $v_{\mu^{(1)}} = v_1 \otimes v_1$

(ii) for each $k$, the submodule $W_k$ of $(\Delta^d(\lambda) \otimes V) \otimes \mathcal{A}(q)$ generated by $\{v_1 \otimes v_1 | 1 \leq i \leq k\}$ contains all weight vectors of $(\Delta^d(\lambda) \otimes V) \otimes \mathcal{A}(q)$ of weight greater than or equal to $\lambda + \epsilon_k$. Thus, using (6.1), for each $1 \leq j \leq m_1$ there is a one-dimensional space of singular vectors of weight $\mu^{(i)}$ in $W_k$, and this is not contained in $W_{k-1}$ (since $k_i > k_{j-1}$). This implies that there unique singular vector $v_{\mu^{(i)}}$ of weight $\mu^{(i)}$ in

$$
v_1 \otimes v_{k_i} + \bigoplus_{1 \leq j < i} U_q(gl_N) v_{\mu^{(i)}} \subseteq \left( \Delta^d(\lambda) \otimes \mathcal{A} V \right) \otimes \mathcal{A}(q),
$$

(6.2)

where we recall that $U_q(gl_N) = U_q^d(gl_N) \otimes \mathcal{A}(q)$.

(3) There is a unique $\omega$-contravariant form on $\Delta^d(\lambda)$ normalized so that $(v_1, v_1) = 1$ and a unique $\omega$-contravariant form on $V$ normalized so that $(v_1, v_1) = 1$. As in Section 5.4, define a $\omega$-contravariant form on $(\Delta^d(\lambda) \otimes \mathcal{A} V) \otimes \mathcal{A}(q)$ by $(u_1 \otimes v_1, u_2 \otimes w_2) = (u_1, w_2)(v_1, w_2)$. For each $1 \leq j \leq m_1$, define an element $r_j(\lambda) \in \mathcal{A}(q)$ by

$$
r_j(\lambda) := \left( v_{\mu^{(i)}}, v_{\mu^{(i)}} \right),
$$

(6.3)

Theorem 6.1. The Misra-Miwa operators $F_\ell$ from Section 3.3 satisfy

$$
F_\ell(\lambda) = \sum_{\tau(v(\mu^{(i)})) = 1} v^{val(\epsilon_2 r_j(\lambda))} \left| \mu^{(j)} \right>,
$$

(6.4)
where $b^{(i)}$ is the box $\mu^{(i)}/\lambda$, $\mathcal{T}(b^{(i)})$ is the color of box $b^{(i)}$ as in Figure 1, $\Phi_{2r}$ is the $2\ell$th cyclotomic polynomial in $q$, and $\val_{\Phi_{2r}}$ is the number of factors of $\Phi_{2r}$ in the numerator of $r$ minus the number of factors of $\Phi_{2r}$ in the denominator of $r$.

The proof of Theorem 6.1 will occupy the rest of this section. We will first prove a similar statement, Proposition 6.6, where the role of the Weyl modules is played by the universal Verma modules from Section 4. For ease of notation, let $\hat{M}^R$ denote the module $0\hat{M}^R$ from Section 4.2.

Definition 6.2. Recursively define singular weight vectors $v_{\varepsilon_1} \in (\hat{M}^R \otimes \mathcal{A}) \otimes_R \mathbb{K}$ and elements $s_k \in \mathbb{K}$ for $1 \leq k \leq N$ by

(i) $v_{\varepsilon_1} = v_+ \otimes v_1$,

(ii) since $\{v_+ \otimes v_j \mid 1 \leq j \leq N\}$ generates $\hat{M}^R \otimes \mathcal{A}$ as a $U_q^{\mathfrak{sl}}(\mathfrak{gl}_N)^{\mathbb{C}_0}$ module, Proposition 4.2 implies that, for each $1 \leq k \leq N$, there is a unique singular vector $v_{\varepsilon_k} \in v_+ \otimes v_k + \oplus_{1 \leq j < k} U_q^{\mathfrak{sl}}(\mathfrak{gl}_N) v_{\varepsilon_j} \subseteq (\hat{M}^R \otimes \mathcal{A}) \otimes_R \mathbb{K}$, where $U_q^{\mathfrak{sl}}(\mathfrak{gl}_N) := U_q^{\mathfrak{sl}}(\mathfrak{gl}_N) \otimes (\mathfrak{g}) \mathbb{K}$ and the factor of $\mathbb{K}$ acts by multiplication on $\hat{M}^R$.

Let $s_k = (v_{\varepsilon_k}, v_{\varepsilon_1})$.

The $s_k$ are quantized versions of the Jantzen numbers first calculated in [11, Section 5] and quantized in [12]. It follows immediately from the definition that $s_1 = 1$.

Lemma 6.3. For any weight $\eta$, up to multiplication by a power of $q$,

$$\prod_{1 \leq k \leq N} s_k^{p(\eta - \varepsilon_k)} = \prod_{1 \leq k \leq N} \frac{\det \hat{M}^R_{\eta - \varepsilon_k}}{\det \hat{M}^R_{\eta - \varepsilon_k}}.$$

(6.5)

where, as in Section 5.3, $\det \hat{M}^R_{\eta - \varepsilon_k}$ is the determinant of the Shapovalov form evaluated on an $\mathcal{R}$-basis for the $\eta - \varepsilon_k$ weight space of $\hat{M}^R$.

Comment 2. In order for Lemma 6.3 to hold as stated, for each $1 \leq k \leq N$, one must calculate the $\det \hat{M}^R_{\eta - \varepsilon_k}$ in the numerator and denominator with respect to the same $\mathcal{R}$-basis. The power of $q$ which appears depends on this choice of $\mathcal{R}$-bases.

Proof of Lemma 6.3. For each $\gamma \in \text{span}_{\mathbb{Z}_{\geq 0}}(R^*)$ fix an $\mathcal{R}$-basis $B_\gamma$ for $U_q^{\mathfrak{sl}}(\mathfrak{gl}_N)^{\mathbb{C}_0}$. Consider the following three $\mathbb{K}$-bases for $((\hat{M}^R \otimes \mathcal{A})_\eta) \otimes_R \mathbb{K}$:

$$A_\eta := \{(b \cdot v_+) \otimes v_k \mid b \in B_{\eta - \varepsilon_k}, 1 \leq k \leq N\},$$

$$C_\eta := \{b \cdot (v_+ \otimes v_k) \mid b \in B_{\eta - \varepsilon_k}, 1 \leq k \leq N\},$$

$$D_\eta := \{b \cdot v_{\varepsilon_k} \mid b \in B_{\eta - \varepsilon_k}, 1 \leq k \leq N\}.$$

Let $\det (\hat{M}^R \otimes \mathcal{A})_B$ denote the determinant of $(\cdot, \cdot)_{\hat{M}^R \otimes \mathcal{A}}$ evaluated on $B$, where $B$ is one of $A_\eta, C_\eta, D_\eta$. Let $\det \hat{M}^R_{\eta - \varepsilon_k}$ denote $\det \hat{M}^R_{\eta - \varepsilon_k}$ calculated with respect to the basis $B_{\eta - \varepsilon_k} \cdot v_{\varepsilon_k}$. 
By the definition of the $\omega$-contravariant form on $\widetilde{M}^R \otimes \mathcal{A}V$ (see Section 4.5),

$$
\det \left( \widetilde{M}^R \otimes V \right)_{A_\eta} = \prod_{k=1}^{N} \left( \det \widetilde{M}^R_{B_{\eta-k}} \right)^{\dim V_{\epsilon_k}} \left( \det V_{\epsilon_k} \right)^{\dim \widetilde{M}^R_{\eta-\epsilon_k}}. \tag{6.7}
$$

For $1 \leq k \leq N$, $V_{\epsilon_k}$ is one dimensional and $\det V_{\epsilon_k}$ is a power of $q$. Hence, up to multiplication by a power of $q$, (6.7) simplifies to

$$
\det \left( \widetilde{M}^R \otimes \mathcal{A}V \right)_{A_\eta} = \prod_{k=1}^{N} \det \widetilde{M}^R_{B_{\eta-k}}. \tag{6.8}
$$

Notice that $U_q^\mathcal{A}(\mathfrak{gl}_N) \cdot v_{\epsilon_{k^+}}$ is isomorphic to $v_{\epsilon_{k^+}}$, and $D_\eta$ is the union of $\mathcal{R}$-bases for each of these submodules. For each $1 \leq k \leq N$, and each $\eta \in \mathfrak{h}_Z^*$ define an $\mathcal{R}$ basis of $v_{\epsilon_{k^+}} \widetilde{M}_\eta$ by

$$
e_{\epsilon_{k^+}} \widetilde{B}_\eta := \{ b \cdot v_{\epsilon_{k^+}} \mid b \in B_{\eta-\epsilon_k} \}. \tag{6.9}
$$

Using $(v_{\epsilon_{k^+}}, v_{\epsilon_{k^+}}) = s_k$,

$$
\det \left( \widetilde{M}^R \otimes V \right)_{D_\eta} = \prod_{k=1}^{N} s_k^{\dim (v_{\epsilon_{k^+}} \widetilde{M}^R_{\eta})} \det e_{\epsilon_{k^+}} \widetilde{M}_{(\epsilon_{k^+}) \widetilde{B}_\eta} = \prod_{k=1}^{N} s_k^{\rho(\eta-\epsilon_k)} \prod_{\sigma \in \epsilon_{\epsilon_{k^+}}} \left( \det \widetilde{M}^R_{B_{\eta-\epsilon_k}} \right), \tag{6.10}
$$

where the last equality uses Proposition 5.2. Here, as in Section 5.3, $\det e_{\epsilon_{k^+}} \widetilde{M}_{(\epsilon_{k^+}) \widetilde{B}_\eta}$ is the Shapovalov determinant calculated with respect to the basis $e_{\epsilon_{k^+}} \widetilde{B}_\eta$.

The change of basis from $A_\eta$ to $C_\eta$ is unitriangular and the change of basis from $C_\eta$ to $D_\eta$ is unitriangular. Thus, $\det(\widetilde{M}^R \otimes \mathcal{A}V)_{A_\eta} = \det(\widetilde{M}^R \otimes \mathcal{A}V)_{D_\eta}$, and so the right sides of (6.8) and (6.10) are equal. The lemma follows from this equality by rearranging.

**Lemma 6.4.** Up to multiplication by a power of $q$,

$$
s_k = \prod_{1 \leq j < k} \left( \frac{z_j z_k^{-1} - q^{2j+2k-2} z_j^{-1} z_k}{\sigma_j \left( z_j z_k^{-1} - q^{2j+2k-2} z_j^{-1} z_k \right)} \right). \tag{6.11}
$$
Proof. Fix $1 \leq k \leq N$. Setting $\eta = \varepsilon_k$ in Lemma 6.3 and applying Theorem 5.1 we see that, up to multiplication by a power of $q$,

$$\prod_{1 \leq x \leq N} s_x^{p(\varepsilon_k - \varepsilon_x)} = \prod_{1 \leq x \leq N} \frac{\det \tilde{M}_{\varepsilon_k - \varepsilon_x}}{\sigma_{\varepsilon_x} \det M_{\varepsilon_k - \varepsilon_x}} = \prod_{1 \leq x \leq N} \prod_{1 \leq i < j \leq N} \left( \frac{c_{\varepsilon_k - \varepsilon_x}(z_i z_j^{-1} - q^{2m+2i-2j} z_i^{-1} z_j)}{\sigma_{\varepsilon_x}(c_{\varepsilon_k - \varepsilon_x})(z_i z_j^{-1} - q^{2m+2i-2j} z_i^{-1} z_j)} \right)^{p(\varepsilon_k - \varepsilon_x + m\varepsilon_i - m\varepsilon_j)}
$$

(6.12)

where, for each $1 \leq x \leq N$, $c_{\varepsilon_k - \varepsilon_x}$ is a unit in $\mathbb{Q}(q)[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$. The value $p(\varepsilon_k - \varepsilon_x + m\varepsilon_i - m\varepsilon_j)$ is 0 unless $m = 1$ and $x \leq i, j \leq k$. If $i > x$, then $\sigma_{\varepsilon_x}$ acts as the identity on $z_i z_j^{-1} - q^{2m+2i-2j} z_i^{-1} z_j$, so the corresponding factors in the numerator and denominator cancel. Hence, we need only consider factors on the right hand side where $m = 1, i = x$, and $x < j \leq k$. If $x > k$, then $\varepsilon_k - \varepsilon_x \notin \mathbb{Q}^\times$, and hence $p(\varepsilon_k - \varepsilon_x) = 0$, so on the left hand side since we only need to consider those factors where $1 \leq x \leq k$. Up to multiplication by a power of $q$, the expression reduces to

$$\prod_{1 \leq x \leq k} s_x^{p(\varepsilon_k - \varepsilon_x)} = \prod_{1 \leq x \leq k} \left( \frac{c_{\varepsilon_k - \varepsilon_x}}{\sigma_{\varepsilon_x}(c_{\varepsilon_k - \varepsilon_x})} \right)^{p(\varepsilon_k - \varepsilon_x)} \prod_{1 \leq x \leq k} \left( \frac{z_x z_j^{-1} - q^{2x+2x-2j} z_x^{-1} z_j}{\sigma_{\varepsilon_x}(z_x z_j^{-1} - q^{2x+2x-2j} z_x^{-1} z_j)} \right)^{p(\varepsilon_k - \varepsilon_x)}
$$

(6.13)

The last two expressions are equal because they are each a product over pairs $(x, j)$ with $1 \leq x < j \leq k$, and the factors of $c_{\varepsilon_k - \varepsilon_x}/(\sigma_{\varepsilon_x}(c_{\varepsilon_k - \varepsilon_x}))$ have been dropped because they are powers of $q$. Using the fact that $s_1 = 1$ and making the change of variables $j \to x$ and $x \to j$ on the right side, (6.13) becomes

$$\prod_{1 \leq x \leq k} s_x^{p(\varepsilon_k - \varepsilon_x)} = \prod_{1 \leq x \leq k} \left( \prod_{1 \leq j < x} \frac{z_j z_x^{-1} - q^{2j+2j-2x} z_j^{-1} z_x}{\sigma_{\varepsilon_x}(z_j z_x^{-1} - q^{2j+2j-2x} z_j^{-1} z_x)} \right)^{p(\varepsilon_k - \varepsilon_x)}.
$$

(6.14)

For $k \geq 2$, the lemma now follows by induction. For $k = 1$, the result simply says that $s_1 = 1$, which we already know. \qed
Figure 2: The partition enclosed by the thick lines is \( \lambda = (10, 10, 8, 8, 8, 6, 6, 6, 1, 1) \). If \( k = 6 \) then \( A(\lambda, < k) = \{a_1, a_3\} \), \( R(\lambda, < k) = \{g_8, g_9\} \), and \( ev_\lambda(s_k) = (\{2\}/[3])\{[3]/[4]\}(\{4]/[5]\)(\{7]/[8]\)(\{8]/[9]) = ([2]/[5])(\{7]/[9]) = ([c(g_3) - c(b)]/[c(g_2) - c(b)])/(\{[c(a_3) - c(b)]/[c(a_1) - c(b)]\}). \) The factors in the numerator of the first expression are displayed. These are the \( q \)-integers corresponding to the hook lengths of the boxes in the same column as the addable box \( b \) in row 6.

**Proposition 6.5.** Let \( \lambda \) be a partition. Let \( A(\lambda, < k) \) (resp. \( R(\lambda, < k) \)) be the set of boxes which can be added to (resp. removed from) \( \lambda \) on rows \( \lambda_j \) with \( j < k \) such that the result is still a partition. Let \( b = (\lambda + \epsilon_k)/\lambda \) and let \( c(\cdot) \) as in Figure 1. Then, up to multiplication by a power of \( q \),

\[
ev_\lambda(s_k) = \begin{cases} 
\prod_{r \in R(\lambda, < k)} [c(r) - c(b)] \\
\prod_{a \in A(\lambda, < k)} [c(a) - c(b)] \\
0,
\end{cases}
\]  

if \( \lambda + \epsilon_k \) is a partition, 
if \( \lambda + \epsilon_k \) is not a partition.

**Proof.** For \( 1 \leq j \leq N \), let \( g_j \) be the last box in row \( j \) of \( \lambda \). By Lemma 6.4, up to multiplication by a power of \( q \),

\[
ev_\lambda(s_k) = \ev_\lambda \left( \prod_{1 \leq j < k} \frac{z_j z_k^{-1} - q^{2j-2k} z_j^{-1} z_k}{\sigma_{e_j} \left( z_j z_k^{-1} - q^{2j-2k} z_j^{-1} z_k \right)} \right) = \prod_{1 \leq j < k} \frac{[c(g_j) - c(b)]}{[c(g_j) - c(b) + 1]},
\]  

where the last equality is a simple calculation from definitions. The denominator on the right side is never zero, and the numerator is zero exactly when \( \lambda_k = \lambda_{k-1} \), so that \( \lambda + \epsilon_k \) is no longer a partition. If \( \lambda_j = \lambda_{j+1} \) for any \( j < k \), then there is cancellation, giving (6.15). See Figure 2.

**Proposition 6.6.** Let \( N^j_\mu(\lambda/\lambda) \) be as in Section 3.3. For any partition \( \lambda \),

\[
val_{\mu, \lambda} ev_\lambda(s_k) = N^j_\mu(\lambda/\lambda), \quad \text{if } \mu = \lambda + \epsilon_k \text{ is a partition, and } \mu/\lambda \text{ is an } i \text{ colored box},
\]  

\[
ev_\lambda(s_k) = 0, \quad \text{otherwise.}
\]  

\[\tag{6.17}\]
Proof. By Proposition 6.5, \( \text{ev}_\lambda(s_k) = 0 \) if \( \lambda + \epsilon_k \) is not a partition. If \( \lambda + \epsilon_k \) is a partition, then
\[
\begin{align*}
\{ b \in A(\lambda, < k) : \overline{\tau}(b) = \overline{\tau}(\mu/\lambda) \} &= \{ b \in A_\gamma(\lambda) \mid b \text{ is to the left of } \mu/\lambda \}, \\
\{ b \in R(\lambda, < k) : \overline{\tau}(b) = \overline{\tau}(\mu/\lambda) \} &= \{ b \in R_\gamma(\lambda) \mid b \text{ is to the left of } \mu/\lambda \},
\end{align*}
\]  
\tag{6.18}
\]
where the notation is as in Section 3.3. Since
\[
[x] = \frac{q^x - q^{-x}}{q - q^{-1}} = q^{-x} (q - q^{-1})^{-1} \prod_{d | 2x} \phi_d,
\]  
\tag{6.19}
\]
\( [x] \) is divisible by \( \phi_{2\ell} \) if and only if \( x \) is divisible by \( \ell \), and \( [x] \) is never divisible by \( \phi_{2\ell}^2 \). The result now follows from Proposition 6.5.
\[ \square \]

Proof of Theorem 6.1. Fix \( \lambda \) and \( 1 \leq k \leq m_1 \). From definitions, \( (\text{ev}_\lambda \otimes 1) v_{\epsilon k_j, +} = v_{\mu^{(j)}} \). Thus, using (5.12),
\[
\begin{align*}
\lambda_j &= \left( v_{\mu^{(j)}}, v_{\mu^{(j)}} \right) = \left( (\text{ev}_\lambda \otimes 1) v_{\epsilon k_j, +} (\text{ev}_\lambda \otimes 1) v_{k_j}, + \right) = \text{ev}_\lambda \left( v_{\epsilon k_j, +}, v_{\epsilon k_j, +} \right) = \text{ev}_\lambda \left( s_k \right).
\end{align*}
\]  
\tag{6.20}
\]
The result now follows from Proposition 6.6.
\[ \square \]

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References