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THE SMALL QUANTUM GROUP AS A QUANTUM DOUBLE

PAVEL Etingof and Shlomo Gelaki

Abstract. We prove that the quantum double of the quasi-Hopf algebra $A_q(g)$ of dimension $n^{\dim g}$ attached in [EG] to a simple complex Lie algebra $g$ and a primitive root of unity $q$ of order $n^2$ is equivalent to Lusztig’s small quantum group $u_q(g)$ (under some conditions on $n$). We also give a conceptual construction of $A_q(g)$ using the notion of de-equivariantization of tensor categories.

1. Introduction

It is well known from the work of Drinfeld [D] that the quantum group $U_q(g)$ attached to a simple complex Lie algebra $g$ can be produced by the quantum double construction. Namely, the quantum double of the quantized Borel subalgebra $U_q(b)$ is the product of $U_q(g)$ with an extra copy of the Cartan subgroup $U_q(h)$, which one can quotient out and get the pure $U_q(g)$. This principle applies not only to quantum groups with generic $q$, but also to Lusztig’s small quantum groups at roots of unity, $u_q(g)$ ([L1, L2]). However, $u_q(g)$ itself (without an additional Cartan) is not, in general, a quantum double of anything: indeed, its dimension is $d = m^{\dim g}$ (where $m$ is the order of $q$), which is not always a square.

However, in the case when $m = n^2$ (so that the dimension $d$ is a square), we have introduced in [EG], Section 4, a quasi-Hopf algebra $A_q = A_q(g)$ of dimension $d^{1/2}$, constructed out of a Borel subalgebra $b$ of $g$. So one might suspect that the quantum double of $A_q(g)$ is twist equivalent to $u_q(g)$. This indeed turns out to be the case (under some conditions on $n$), and is the main result of this note. In other words, our main result is that the Drinfeld center $Z(\text{Rep}(A_q(g)))$ of the category of representations of $A_q(g)$ is $\text{Rep}(u_q(g))$.

We prove our main result by showing that the category $\text{Rep}(u_q(b))$ of representations of the quantum Borel subalgebra $u_q(b)$ is the equivariantization of the category $\text{Rep}(A_q(g))$ with respect to an action of a certain finite abelian group. Thus, $\text{Rep}(A_q(g))$ can be conceptually defined as a de-equivariantization of $\text{Rep}(u_q(g))$. So, one may say that the main outcome of this paper is a demystification of the quasi-Hopf algebra $A_q(g)$ constructed “by hand” in [EG].

The structure of the paper is as follows. In Section 2 we recall the theory of equivariantization and de-equivariantization of tensor categories. In Section 3 we recall the construction of the quasi-Hopf algebra $A_q(g)$ from the paper [EG]. In Section 4 we state the main results. Finally, Section 5 contains proofs.

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2. Equivariantization and de-equivariantization

The theory of equivariantization and de-equivariantization of tensor categories was developed in [B, M] in the setting of fusion categories; it is now a standard technique in the theory of fusion categories, and has also been used in the setting of the Langlands program [F]. A detailed description of this theory is given in [DGNO] (see also [ENO], Sections 2.6 and 2.11). This theory extends without major changes to the case of finite tensor categories (as defined in [EO]), i.e., even if the semisimplicity assumption is dropped. Let us review the main definitions and results of this theory.

2.1. Group actions. Let \( \mathcal{C} \) be a finite tensor category (all categories and algebras in this paper are over \( \mathbb{C} \)). Consider the category \( \text{Aut}(\mathcal{C}) \), whose objects are tensor auto-equivalences of \( \mathcal{C} \) and whose morphisms are isomorphisms of tensor functors. The category \( \text{Aut}(\mathcal{C}) \) has an obvious structure of a monoidal category, in which the tensor product is the composition of tensor functors.

Let \( G \) be a group, and let \( G \) denote the category whose objects are elements of \( G \), the only morphisms are the identities and the tensor product is given by multiplication in \( G \).

Definition 2.1. An action of a group \( G \) on a finite tensor category \( \mathcal{C} \) is a monoidal functor \( G \to \text{Aut}(\mathcal{C}) \).

If \( \mathcal{C} \) is equipped with a braided structure we say that an action \( G \to \text{Aut}(\mathcal{C}) \) respects the braided structure if the image of \( G \) lies in \( \text{Aut}^{br}(\mathcal{C}) \), where \( \text{Aut}^{br}(\mathcal{C}) \) is the full subcategory of \( \text{Aut}(\mathcal{C}) \) consisting of braided equivalences.

2.2. Equivariantization. Let a finite group \( G \) act on a finite tensor category \( \mathcal{C} \). For any \( g \in G \) let \( F_g \in \text{Aut}(\mathcal{C}) \) be the corresponding functor and for any \( g, h \in G \) let \( \gamma_{g,h} \) be the isomorphism \( F_g \circ F_h \simeq F_{gh} \) that defines the tensor structure on the functor \( G \to \text{Aut}(\mathcal{C}) \). A \( G \)-equivariant object of \( \mathcal{C} \) is an object \( X \in \mathcal{C} \) together with isomorphisms \( u_g : F_g(X) \simeq X \) such that the diagram

\[
\begin{array}{ccc}
F_g(F_h(X)) & \xrightarrow{F_g(u_h)} & F_g(X) \\
\gamma_{g,h}(X) \downarrow & & \downarrow u_g \\
F_{gh}(X) & \xrightarrow{u_{gh}} & X
\end{array}
\]

commutes for all \( g, h \in G \). One defines morphisms of equivariant objects to be morphisms in \( \mathcal{C} \) commuting with \( u_g, g \in G \). The category of \( G \)-equivariant objects of \( \mathcal{C} \) will be denoted by \( \mathcal{C}^G \). It is called the equivariantization of \( \mathcal{C} \).

Note that \( \text{Vec}^G = \text{Rep}(G) \), so there is a natural inclusion \( \iota : \text{Rep}(G) \to \mathcal{C}^G \).

One of the main results about equivariantization is the following theorem (see [ENO], Proposition 2.10 for the semisimple case; in the non-semisimple situation, the proof is parallel).

Theorem 2.2. Let \( G \) be a finite group acting on a finite tensor category \( \mathcal{C} \). Then \( \text{Rep}(G) \) is a Tannakian subcategory of the Drinfeld center \( Z(\mathcal{C}^G) \) (i.e., the braiding of \( Z(\mathcal{C}^G) \) restricts to the usual symmetric braiding of \( \text{Rep}(G) \)), and the composition

\[ \text{Rep}(G) \to Z(\mathcal{C}^G) \to \mathcal{C}^G \]

(where the last arrow is the forgetful functor) is the natural inclusion \( \iota \).
If \( \mathcal{C} \) is a braided category, and the \( G \)-action preserves the braided structure, then \( \mathcal{C}^G \) is also braided. Thus \( \mathcal{C}^G \) is a full subcategory of \( Z(\mathcal{C}^G) \), and the inclusion \( \iota \) factors through \( \mathcal{C}^G \). Thus in this case \( \operatorname{Rep}(G) \) is a Tannakian subcategory of \( \mathcal{C}^G \).

2.3. **De-equivariantization.** Let \( \mathcal{D} \) be a finite tensor category such that the Drinfeld center \( Z(\mathcal{D}) \) contains a Tannakian subcategory \( \operatorname{Rep}(G) \), and the composition \( \operatorname{Rep}(G) \rightarrow Z(\mathcal{D}) \rightarrow \mathcal{D} \) is an inclusion. Let \( A := \operatorname{Fun}(G) \) be the algebra of functions \( G \rightarrow \mathbb{C} \). The group \( G \) acts on \( A \) by left translations, so \( A \) can be considered as an algebra in the tensor category \( \operatorname{Rep}(G) \), and thus as an algebra in the braided tensor category \( Z(\mathcal{D}) \). As such, the algebra \( A \) is braided commutative. Therefore, the category of \( A \)-modules in \( \mathcal{D} \) is a tensor category, which is called the **de-equivariantization** of \( \mathcal{D} \) and denoted by \( \mathcal{D}_G \).

Let us now separately consider de-equivariantization of braided categories. Namely, let \( \mathcal{D} \) be a finite braided tensor category, and \( \operatorname{Rep}(G) \subset \mathcal{D} \) a Tannakian subcategory. In this case \( \operatorname{Rep}(G) \) is also a Tannakian subcategory of the Drinfeld center \( Z(\mathcal{D}) \) (as \( \mathcal{D} \subset Z(\mathcal{D}) \)), so we can define the de-equivariantization \( \mathcal{D}_G \). It is easy to see that \( \mathcal{D}_G \) inherits the braided structure from \( \mathcal{D} \), so it is a braided tensor category.

We will need the following result (see [ENO], Section 2.6 and Proposition 2.10 for the semisimple case; in the non-semisimple situation, the proof is parallel).

**Theorem 2.3.** (i) The procedures of equivariantization and de-equivariantization are inverse to each other.

(ii) Let \( \mathcal{C} \) be a finite tensor category with an action of a finite group \( G \). Let \( \mathcal{E}' \) be the M"uger centralizer of \( \mathcal{E} = \operatorname{Rep}(G) \) in \( Z(\mathcal{C}^G) \) (i.e., the category of objects \( X \in Z(\mathcal{C}^G) \) such that the squared braiding is the identity on \( X \otimes Y \) for all \( Y \in \mathcal{E} \)). Then the category \( \mathcal{E}'_G \) is naturally equivalent to \( Z(\mathcal{C}) \) as a braided category.

3. **The quasi-Hopf algebra** \( A_q = A_q(\mathfrak{g}) \)

In this section we recall the construction of the finite dimensional basic quasi-Hopf algebras \( A_q = A_q(\mathfrak{g}) \), given in [EG], Section 4.

Let \( \mathfrak{g} \) be a finite dimensional simple Lie algebra of rank \( r \), and let \( \mathfrak{b} \) be a Borel subalgebra of \( \mathfrak{g} \).

Let \( n \geq 2 \) be an odd integer, not divisible by 3 if \( \mathfrak{g} = \mathfrak{g}_2 \), and let \( q \) be a primitive root of 1 of order \( n^2 \). We will also assume, throughout the rest of the paper, that \( n \) is relatively prime to the determinant \( \det(a_{ij}) \) of the Cartan matrix of \( \mathfrak{g} \).

Let \( u_q(\mathfrak{b}) \) be the Frobenius-Lusztig kernel associated to \( \mathfrak{b} \) (\[1\] \[2\]): it is a finite dimensional Hopf algebra generated by grouplike elements \( g_i \) and skew-primitive elements \( e_i \), \( i = 1, \ldots, r \), such that

\[
\begin{align*}
g_i^2 &= 1, \\
g_i g_j &= g_j g_i, \\
g_i e_j g_i^{-1} &= q^{\delta_{ij}} e_j,
\end{align*}
\]

\( e_i \) satisfy the quantum Serre relations, and

\[
\Delta(e_i) = e_i \otimes K_i + 1 \otimes e_i, \quad K_i := \prod_j g_j^{a_{ij}}.
\]

The algebra \( u_q(\mathfrak{b}) \) has a projection onto \( \mathbb{C}[[\mathbb{Z}/n^2\mathbb{Z}]] \), \( g_i \mapsto g_i \) and \( e_i \mapsto 0 \). Let \( B \subset u_q(\mathfrak{b}) \) be the subalgebra generated by \( \{e_i\} \). Then by Radford’s theorem \( \mathbb{R} \), the multiplication map \( \mathbb{C}[[\mathbb{Z}/n^2\mathbb{Z}]] \otimes B \rightarrow u_q(\mathfrak{b}) \) is an isomorphism of vector spaces. Therefore, \( A_q := \mathbb{C}[[\mathbb{Z}/n\mathbb{Z}]] B \subset u_q(\mathfrak{b}) \) is a subalgebra. It is generated by \( g_i^n \) and \( e_i, \ 1 \leq i \leq r \).
Let \( \{1_z|z = (z_1, \ldots, z_r) \in (\mathbb{Z}/n^2\mathbb{Z})^r \} \) be the set of primitive idempotents of \( \mathbb{C}[(\mathbb{Z}/n^2\mathbb{Z})^r] \) (i.e., \( 1_z g_i = q^{z_i} 1_z \)).

Following [G], for \( z, y \in \mathbb{Z}/n^2\mathbb{Z} \) let \( c(z, y) = q^{-zy}y' \), where \( y' \) denotes the remainder of division of \( y \) by \( n \).

Let 
\[
\mathcal{J} := \sum_{z,y \in (\mathbb{Z}/n^2\mathbb{Z})^r} \prod_{i,j=1}^r c(z_i, y_j)^{a_{ij}} 1_z \otimes 1_y.
\]

It is clear that it is invertible and \( (\varepsilon \otimes \text{id})(\mathcal{J}) = (\text{id} \otimes \varepsilon)(\mathcal{J}) = 1 \). Define a new coproduct
\[
\Delta_\mathcal{J}(z) = \mathcal{J} \Delta(z) \mathcal{J}^{-1}.
\]

**Lemma 3.1.** The elements \( \Delta_\mathcal{J}(e_i) \) belong to \( \mathbb{A}_q \otimes \mathbb{A}_q \).

**Lemma 3.2.** The associator \( \Phi := d\mathcal{J} \) obtained by twisting the trivial associator by \( \mathcal{J} \) is given by the formula
\[
\Phi = \sum_{\beta, \gamma, \delta \in (\mathbb{Z}/n\mathbb{Z})^r} \left( \prod_{i,j=1}^r q^{a_{ij} \beta_i((\gamma_i + \delta_i)' - \gamma_i - \delta_i)} \right) 1_\beta \otimes 1_\gamma \otimes 1_\delta,
\]
where \( 1_\beta \) are the primitive idempotents of \( \mathbb{C}[(\mathbb{Z}/n\mathbb{Z})^r] \), \( 1_\beta g_i^n = q^{n\beta_i} 1_\beta \), and we regard the components of \( \beta, \gamma, \delta \) as elements of \( \mathbb{Z}^r \).

Thus \( \Phi \) belongs to \( \mathbb{A}_q \otimes \mathbb{A}_q \otimes \mathbb{A}_q \).

**Theorem 3.3.** The algebra \( \mathbb{A}_q \) is a quasi-Hopf subalgebra of \( \mathbb{U}_q(\mathfrak{b})^2 \), which has coproduct \( \Delta_\Phi \) and associator \( \Phi \). It is of dimension \( n^{\dim \mathfrak{g}} \).

**Remark 3.4.** The quasi-Hopf algebra \( \mathbb{A}_q \) is not twist equivalent to a Hopf algebra. Indeed, the associator \( \Phi \) is non-trivial since the \( 3 \)--cocycle corresponding to \( \Phi \) restricts to a non-trivial \( 3 \)--cocycle on the cyclic group \( \mathbb{Z}/n\mathbb{Z} \) consisting of all tuples whose coordinates equal 0, except for the \( i \)th coordinate. Since \( \mathbb{A}_q \) projects onto \( (\mathbb{C}[(\mathbb{Z}/n\mathbb{Z})^r], \Phi) \) with non-trivial \( \Phi \), \( \mathbb{A}_q \) is not twist equivalent to a Hopf algebra.

### 4. Main results

Let \( T := (\mathbb{Z}/n^2\mathbb{Z})^r \). We have the following well known result.

**Theorem 4.1.** The quantum double \( D(\mathbb{U}_q(\mathfrak{b})) \) of \( \mathbb{U}_q(\mathfrak{b}) \) is twist equivalent, as a quasitriangular Hopf algebra, to \( \mathbb{U}_q(\mathfrak{g}) \otimes \mathbb{C}[T] \). Therefore,
\[
Z(\mathbb{Rep}(\mathbb{U}_q(\mathfrak{b}))) = \mathbb{Rep}(\mathbb{U}_q(\mathfrak{g})) \boxtimes \text{Vec}_T
\]
as a braided tensor category, where the braiding on \( \mathbb{Rep}(\mathbb{U}_q(\mathfrak{g})) \) is the standard one, and \( \text{Vec}_T \) is the category of \( T \)--graded vector spaces with the braiding coming from the quadratic form on \( T \) defined by the Cartan matrix of \( \mathfrak{g} \).

**Proof.** It is well known ([D], [CP]) that \( D(\mathbb{U}_q(\mathfrak{b})) \) is isomorphic as a Hopf algebra to \( H := \mathbb{U}_q(\mathfrak{g}) \otimes \mathbb{C}[T] \), with standard generators \( e_i, f_i, K_i \in \mathbb{U}_q(\mathfrak{g}) \) and \( K'_i \in \mathbb{C}[T] \), and comultiplication
\[
\Delta_e(e_i) = e_i \otimes K_i K'_i + 1 \otimes e_i, \quad \Delta_f(f_i) = f_i \otimes K'_i^{-1} + K_i^{-1} \otimes f_i
\]
(in fact, this is not hard to check by a direct computation). Note that the group algebra \( \mathbb{C}[T \times T] \) is contained in \( H \) as a Hopf subalgebra (with the two copies of \( T \))

\[\sum \text{1_{1_{ij}}} \text{ should not be confused with 1_{z} that appeared above.}\]

\[\text{Actually, the quadratic form gives the inverse braiding, but this is not important for our considerations.}\]
generated by $K_i$ and $K_i'$, respectively). Consider the bicharacter of $T \times T$ given by the formula
\[
\langle (a, b), (c, d) \rangle = \langle a, d \rangle,
\]
where $\langle , \rangle : T \times T \to \mathbb{C}^*$ is the pairing given by the Cartan matrix. Consider the twist $J \in \mathbb{C}[T \times T]^\otimes 2$ corresponding to this bicharacter. It is easy to compute directly that twisting by $J$ transforms the above comultiplication $\Delta_\ast$ to the usual “tensor product” comultiplication of $H$:
\[
\Delta(e_i) = e_i \otimes K_i + 1 \otimes e_i,
\]
\[
\Delta(f_i) = f_i \otimes 1 + K_i^{-1} \otimes f_i,
\]
and the same holds for the universal R-matrix (this computation uses that $K_i'$ are central elements). This implies the theorem. \hfill \Box

Let $\Gamma \cong (\mathbb{Z}/n\mathbb{Z})^r$ be the $n$–torsion subgroup of $T$.

Our first main result is the following.

**Theorem 4.2.** The group $\Gamma$ acts on the category $\mathcal{C} = \text{Rep}(A_q)$, and the equivariantization $\mathcal{C}^\Gamma$ is tensor equivalent to $\text{Rep}(u_q(b))$.

The proof of Theorem 4.2 will be given in the next section.

By Theorem 4.2(i), Theorem 4.2 implies that the category $\text{Rep}(A_q)$ can be conceptually defined as the de-equivariantization of $\text{Rep}(u_q(b))$.

Our second main result is the following.

**Theorem 4.3.** The Drinfeld center $\mathcal{Z}(\text{Rep}(A_q))$ of $\text{Rep}(A_q)$ is braided equivalent to $\text{Rep}(u_q(g))$. Equivalently, the quantum double $D(A_q)$ of the quasi-Hopf algebra $A_q$ is twist equivalent (as a quasitriangular quasi-Hopf algebra) to the small quantum group $u_q(g)$.

**Proof.** Since $\mathcal{Z}(\text{Rep}(u_q(b))) = \text{Rep}(u_q(g)) \boxtimes \text{Vec}_T$ as a braided category, and $\text{Rep}\Gamma \subset \text{Vec}_T$ is a Tannakian subcategory, we have that $\text{Rep}(\Gamma) \subset \mathcal{Z}(\text{Rep}(u_q(b)))$ is a Tannakian subcategory. Moreover, $\text{Rep}\Gamma \subset \text{Vec}_T$ is a Lagrangian subcategory (i.e. it coincides with its Müger centralizer in $\text{Vec}_T$), so the Müger centralizer $D$ of $\text{Rep}\Gamma$ in $\mathcal{Z}(\text{Rep}(u_q(b)))$ is equal to $\text{Rep}(u_q(g)) \boxtimes \text{Rep}(\Gamma)$. This implies that the de-equivariantization $D_T$ is $\text{Rep}(u_q(g))$. On the other hand, by Theorem 4.2, $\text{Rep}(u_q(b)) = \text{Rep}(A_q)^\Gamma$, so by Theorem 4.2(ii) we conclude that $\mathcal{Z}(\text{Rep}(A_q)) = \text{Rep}(u_q(g))$, as desired. \hfill \Box

5. **Proof of Theorem 4.2**

Let us first define an action of $\Gamma$ on $\mathcal{C} = \text{Rep}(A_q)$.

For $j = 0, \ldots, n - 1$, $i = 1, \ldots, r$, let $F_{ij} : \text{Rep}(A_q) \to \text{Rep}(A_q)$ be the functor defined as follows. For an object $(V, \pi_V)$ in $\text{Rep}(A_q)$, $F_{ij}(V) = V$ as a vector space, and $\pi_{F_{ij}(V)}(a) = \pi_V(g_i^a g_i^{-j})$, $a \in A_q$.

The isomorphism $\gamma_{ij_1, ij_2} : F_{ij_1}(F_{ij_2}(V)) \to F_{i, (j_1 + j_2)^r}(V)$ is given by the action of
\[
(g_i^a)^{i j_1 + j_2 + j_1 + j_2}_{i j_1 - j_1 - j_2} \in A_q,
\]
and $\gamma_{ii_1, ii_2} = 1$ for $i_1 \neq i_2$.

Let us now consider the equivariantization $\mathcal{C}^\Gamma$. By definition, an object of $\mathcal{C}^\Gamma$ is a representation $V$ of $A_q$ together with a collection of linear isomorphisms $p_{ij} : V \to V$, $j = 0, \ldots, n - 1$, $i = 1, \ldots, r$, such that
\[
p_{ij}(av) = g_i^a g_i^{-j} p_{ij}(v), \quad a \in A_q, \quad v \in V,
\]
and
\[ p_{i,j_1} p_{i,j_2} = p_{i,(j_1+j_2)'} (g^n_{i'})^{-\frac{(j_1+j_2)'+j_1+j_2}{2}}.\]

It is now straightforward to verify that this is the same as a representation of \( u_q(b) \), because \( u_q(b) \) is generated by \( A_q \) and the \( p_{i,j} := g_i^j \) with exactly the same relations. Moreover, the tensor product of representations is the same as for \( u_q(b)^J \). Thus \( C^\Gamma \) is naturally equivalent to \( \text{Rep}(u_q(b)) \), as claimed.

This completes the proof of Theorem 4.2.

References


Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

E-mail address: etingof@math.mit.edu

Department of Mathematics, Technion-Israel Institute of Technology, Haifa 32000, Israel

E-mail address: gelaki@math.technion.ac.il