**MATRIX p-NORMS ARE NP-HARD TO APPROXIMATE IF p \text{ not equal to 1, 2, infinity}**

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**Detailed Terms**
MATRIX \( p \)-NORMS ARE NP-HARD TO APPROXIMATE IF 
\( p \neq 1, 2, \infty \)

JULIEN M. HENDRICKX† AND ALEX OLSHEVSKY‡

Abstract. We show that, for any rational \( p \in [1, \infty) \) except \( p = 1, 2 \), unless \( P = NP \), there is no polynomial time algorithm which approximates the matrix \( p \)-norm to arbitrary relative precision. We also show that, for any rational \( p \in [1, \infty) \) including \( p = 1, 2 \), unless \( P = NP \), there is no polynomial-time algorithm which approximates the \( \infty, p \) mixed norm to some fixed relative precision.

Key words. matrix norms, complexity, NP-hardness

AMS subject classifications. 68Q17, 15A60

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1. Introduction. The \( p \)-norm of a matrix \( A \) is defined as

\[ \|A\|_p = \max_{\|x\|_p = 1} \|Ax\|_p. \]

We consider the problem of computing the matrix \( p \)-norm to relative error \( \epsilon \), defined as follows: given the inputs (i) a matrix \( A \in \mathbb{R}^{n \times n} \) with rational entries and (ii) an error tolerance \( \epsilon \) which is a positive rational number, output a rational number \( r \) satisfying

\[ |r - \|A\|_p| \leq \epsilon \|A\|_p. \]

We will use the standard bit model of computation. When \( p = \infty \) or \( p = 1 \), the \( p \)-matrix norm is the largest of the row/column sums and thus may be easily computed exactly. When \( p = 2 \), this problem reduces to computing an eigenvalue of \( A^T A \) and thus can be solved in polynomial time in \( n, \log \frac{1}{\epsilon} \) and the bit size of the entries of \( A \). Our main result suggests that the case of \( p \in \{1, 2, \infty\} \) may be different.

Theorem 1.1. For any rational \( p \in [1, \infty) \) except \( p = 1, 2 \), unless \( P = NP \), there is no algorithm which computes the \( p \)-norm of a matrix with entries in \( \{-1, 0, 1\} \) to relative error \( \epsilon \) with running time polynomial in \( n, \frac{1}{\epsilon} \), and the bit size of the entries of \( A \).

On the way to our result, we also slightly improve the NP-hardness result for the mixed norm \( \|A\|_{\infty,p} = \max \|x\|_\infty, \|Ax\|_p \) from [5]. Specifically, we show that, for every rational \( p \geq 1 \), there exists an error tolerance \( \epsilon(p) \) such that, unless \( P = NP \), there is no polynomial time algorithm approximating \( \|A\|_{\infty,p} \) with a relative error smaller than \( \epsilon(p) \).

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†Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA. Current address: Institute of Information and Communication Technologies, Electronics and Applied Mathematics, Université catholique de Louvain, 4 avenue George Lemaître, B-1348 Louvain-la-Neuve, Belgium (julien.hendrickx@uclouvain.be).

‡Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA. Current address: Department of Mechanical and Aerospace Engineering, Princeton University, Princeton, NJ (aolshevs@princeton.edu).
1.1. Previous work. When $p$ is an integer, computing the matrix norm can be recast as solving a polynomial optimization problem. These are known to be hard to solve in general [3]; however, because the matrix norm problem has a special structure, one cannot immediately rule out the possibility of a polynomial time solution. A few hardness results are available in the literature for mixed matrix norms $||A||_{p,q} = \max_{||x||_p \leq 1} ||Ax||_q$. Rohn has shown in [4] that computing the $||A||_{\infty,1}$ norm is NP-hard. In her thesis, Steinberg [5] proved more generally that computing $||A||_{p,q}$ is NP-hard when $1 \leq q < p \leq \infty$. We refer the reader to [5] for a discussion of applications of the mixed matrix norm problems to robust optimization.

It is conjectured in [5] that there are only three cases in which mixed norms are computable in polynomial time: First, $p = 1$, and $q$ is any rational number larger than or equal to 1. Second, $q = \infty$, and $p$ is any rational number larger than or equal to 1. Third, $p = q = 2$. Our work makes progress on this question by settling the “diagonal” case of $p = q$; however, the case of $p < q$, as far as the authors are aware, is open.

1.2. Outline. We begin in section 2 by providing a proof of the NP-hardness of approximating the mixed norm $|| \cdot ||_{\infty,p}$ within some fixed relative error for any rational $p \geq 1$. The proof may be summarized as follows: observe that, for any matrix $M$,

$\max_{||x||_\infty = 1} ||Mx||_p$ is always attained at one of the $2^n$ points of $\{-1,1\}^n$. So by appropriately choosing $M$, one can encode an NP-hard problem of maximization over the latter set. This argument will prove that computing the $|| \cdot ||_{\infty,p}$ norm is NP-hard.

Next, in section 3, we exhibit a class of matrices $A$ such that $\max_{||x||_p = 1} ||Ax||_p$ is attained at each of the $2^n$ points of $\{-1,1\}^n$ (up to scaling) and nowhere else. These two elements are combined in section 4 to prove Theorem 1.1. More precisely, we define the matrix $Z = (M^T \alpha A^T)^T$, where we will pick $\alpha$ to be a large number depending on $n,p$ ensuring that the maximum of $||Zx||_p/||x||_p$ occurs very close to vectors $x \in \{-1,1\}^n$. As mentioned several sentences ago, the value of $||Ax||_p$ is the same for every vector $x \in \{-1,1\}^n$; as a result, the maximum of $||Zx||_p/||x||_p$ is determined by the maximum of $||Mx||_p$ on $\{-1,1\}^n$, which is proved in section 2 to be hard to compute. We conclude with some remarks on the proof in section 5.

2. The $|| \cdot ||_{\infty,p}$ norm. We now describe a simple construction which relates the $\infty,p$ norm to the maximum cut in a graph.

Suppose $G = (\{1, \ldots, n\}, E)$ is an undirected, connected graph. We will use $M(G)$ to denote the edge-vertex incidence matrix of $G$; that is, $M(G) \in \mathbb{R}^{|E| \times n}$. We will think of columns of $M(G)$ as corresponding to nodes of $G$ and of rows of $M(G)$ as corresponding to the edges of $G$. The entries of $M(G)$ are as follows: orient the edges of $G$ arbitrarily, and let the ith row of $M(G)$ have +1 in the column corresponding to the origin of the ith edge, −1 in the column corresponding to the endpoint of the ith edge, and 0 at all other columns.

Given any partition of $\{1, \ldots, n\} = S \cup S^c$, we define $\text{cut}(G,S)$ to be the number of edges with exactly one endpoint in $S$. Furthermore, we define $\text{maxcut}(G) = \max_{S \subseteq \{1, \ldots, n\}} \text{cut}(G,S)$. The indicator vector of a cut $(S,S^c)$ is the vector $x$ with $x_i = 1$ when $i \in S$ and $x_i = -1$ when $i \in S^c$. We will use $\text{cut}(x)$ for vectors $x \in \{-1,1\}^n$ to denote the value of the cut whose indicator vector is $x$.

**Proposition 2.1.** For any $p \geq 1$,

$$\max_{||x||_\infty \leq 1} ||M(G)x||_p = 2^{\text{maxcut}(G)}^{1/p}.$$
Proof. Observe that $||M(G)x||_p$ is a convex function of $x$, so that the maximum is achieved at the extreme points of the set $||x||_{\infty} \leq 1$, i.e., vectors $x$ satisfying $x_i = \pm 1$. Suppose we are given such a vector $x$; define $S = \{i \mid x_i = 1\}$. Clearly, $||M(G)x||_p^p = 2^p \text{cut}(G,S)$. From this the proposition immediately follows.

Next, we introduce an error term into this proposition. Define $f^*$ to be the optimal value $f^* = \max_{||x||_{\infty} \leq 1} ||M(G)x||_p$; the above proposition implies that $(f^*/2)^p = \text{maxcut}(G)$. We want to argue that if $f_{\text{approx}}$ is close enough to $f^*$, then $(f_{\text{approx}}/2)^p$ is close to $\text{maxcut}(G)$.

**Proposition 2.2.** If $p \geq 1$, $|f^* - f_{\text{approx}}| < \epsilon f^*$ with $\epsilon < 1$, then

$$\left| \left( \frac{f_{\text{approx}}}{2} \right)^p - \text{maxcut}(G) \right| \leq \frac{1}{2} |f^* - f_{\text{approx}}| p \max \left( \frac{f^*}{2}, \frac{f_{\text{approx}}}{2} \right)^{p-1}. $$

It follows from $\epsilon < 1$ that $f_{\text{approx}} \leq 2f^*$. We have therefore

$$\left| \left( \frac{f_{\text{approx}}}{2} \right)^p - \text{maxcut}(G) \right| \leq \frac{1}{2} |f^* - f_{\text{approx}}| \cdot p \cdot (f^*)^{p-1} \leq \frac{\epsilon}{2} p(f^*)^p,$$

where we have used the assumption that $|f^* - f_{\text{approx}}| \leq \epsilon f^*$. The result follows then from $\text{maxcut}(G) = (f^*/2)^p$.

We now put together the previous two propositions to prove that approximating the $|| \cdot ||_{\infty,p}$ norm within some fixed relative error is NP-hard.

**Theorem 2.3.** For any rational $p \geq 1$ and $\delta > 0$, unless $P = NP$, there is no algorithm which, given a matrix with entries in $\{-1,0,1\}$, computes its $p$-norm to relative error $\epsilon = ((33 + \delta)p2^{p-1})^{-1}$ with running time polynomial in the dimensions of the matrix.

Proof. Suppose there was such an algorithm. Call $f^*$ its output on the $|E| \times n$ matrix $M(G)$ for a given connected graph $G$ on $n$ vertices. It follows from Proposition 2.2 that

$$\left| \left( \frac{f_{\text{approx}}}{2} \right)^p - \text{maxcut}(G) \right| \leq \frac{2^{p-1}p}{(33 + \delta)p2^{p-1}} \text{maxcut}(G) = \frac{1}{33 + \delta} \text{maxcut}(G).$$

Observing that

$$\frac{32 + \delta}{34 + \delta} \text{maxcut}(G) = \frac{33 + \delta}{34 + \delta} \left( \text{maxcut}(G) - \frac{1}{33 + \delta} \text{maxcut}(G) \right),$$

the former inequality implies

$$\frac{32 + \delta}{34 + \delta} \text{maxcut}(G) \leq \frac{33 + \delta}{34 + \delta} \left( \frac{f_{\text{approx}}}{2} \right)^p \leq \text{maxcut}(G).$$

Since $p$ is rational, one can compute in polynomial time a lower bound $V$ for $\frac{33 + \delta}{34 + \delta} (f_{\text{approx}}/2)^p$ sufficiently accurate so that $V > \frac{32 + \delta/2}{34 + \delta/2} \text{maxcut}(G) > \frac{16}{17} \text{maxcut}(G)$.  

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However, it has been established in [2] that, unless \( P = NP \), for any \( \delta' > 0 \), there is no algorithm producing a quantity \( V \) in polynomial time in \( n \) such that

\[
\left( \frac{16}{17} + \delta' \right) \text{maxcut}(G) \leq V \leq \text{maxcut}(G). \tag*{\Box}
\]

Remark. Observe that the matrix \( M(G) \) is not square. If one desires to prove hardness of computing the \( \infty, p \)-norm for square matrices, one can simply add \( |E| - n \) zeros to every row of \( M(G) \). The resulting matrix has the same \( \infty, p \)-norm as \( M(G) \) and is square, and its dimensions are at most \( n^2 \times n^2 \).

3. A discrete set of exponential size. Let us now fix \( n \) and a rational \( p > 2 \). We denote by \( X \) the set \( \{-1, 1\}^n \) and use \( S(a, r) = \{ x \in \mathbb{R}^n \mid ||x - a||_p = r \} \) to stand for the sphere of radius \( r \) around \( a \) in the \( p \)-norm. We consider the following matrix in \( R^{2n \times n} \):

\[
A = \begin{pmatrix}
1 & -1 \\
1 & 1 \\
1 & -1 \\
\vdots & \ddots \\
1 & -1 \\
-1 & 1 \\
1 & 1
\end{pmatrix}
\]

and show that the maximum of \( ||Ax||_p \) for \( x \in S(0, n^{1/p}) \) is attained at the \( 2^n \) vectors in \( X \) and at no other points. For this, we will need the following lemma.

Lemma 3.1. For any real numbers \( x, y \) and \( p \geq 2 \)

\[
|x + y|^p + |x - y|^p \leq 2^{p-1} \left( |x|^p + |y|^p \right).
\]

In fact, \( |x + y|^p + |x - y|^p \) is upper bounded by

\[
2^{p-1} \left( |x|^p + |y|^p \right) - \frac{(|x| - |y|)^2}{4} \left( p(p - 1) ||x||^p + ||y||^p - 2||x||^p - 2||y||^p \right),
\]

where the last term on the right is always nonnegative.

Proof. By symmetry we can assume that \( x \geq y \geq 0 \). In that case, we need to prove

\[
(x + y)^p + (x - y)^p \leq 2^{p-1} (x^p + y^p) - \frac{(x - y)^2}{4} \left( p(p - 1) (x^p + y^p) - 2(x^p) - 2(y^p) \right).
\]

Divide both sides by \( (x + y)^p \), and change the variables to \( z = (x - y)/(x + y) \):

\[
1 + z^p \leq \frac{(1 + z)^p + (1 - z)^p}{2} - \left( \frac{p(p - 1)}{4} z^2 - \frac{1}{2} z^p \right).
\]

The original inequality holds if this inequality holds for \( z \in [0, 1] \). Let’s simplify:

\[
2 + z^p \leq (1 + z)^p + (1 - z)^p - \frac{p(p - 1)}{2} z^2.
\]
Observe that we have equality when \( z = 0 \), so it suffices to show that the right-hand side grows faster than the left-hand side, namely,

\[
z^{p-1} \leq (1 + z)^{p-1} - (1 - z)^{p-1} - (p - 1)z,
\]

and this follows from

\[
(1 + z)^{p-1} \geq 1 + (p - 1)z \geq (1 - z)^{p-1} + z^{p-1} + (p - 1)z,
\]

where we have used the convexity of \( f(a) = a^{p-1} \). \( \square \)

Now we prove that every vector of \( X \) optimizes \( \|Ax\|_p/\|x\|_p \) or, equivalently, optimizes \( \|Ax\|^p_p \) over the sphere \( S(0, n^{1/p}) \).

**Lemma 3.2.** For any \( p \geq 2 \), the supremum of \( \|Ax\|^p_p \) over \( S(0, n^{1/p}) \) is achieved by any vector in \( X \).

**Proof.** Observe that \( \|Ax\|^p_p = n2^p \) for any \( x \in X \). To prove that this is the largest possible value, we write

\[
\|Ax\|^p_p = \sum_{i=1}^{n} |x_i - x_{i+1}|^p + |x_i + x_{i+1}|^p,
\]

using the convention \( n + 1 = 1 \) for the indices. Lemma 3.1 implies that

\[
|x_i - x_{i+1}|^p + |x_i + x_{i+1}|^p \leq 2^{p-1} (|x_i|^p + |x_{i+1}|^p).
\]

By applying this inequality to each term of (3.1) and by using \( \|x\|^p_p = n \), we obtain

\[
\|Ax\|^p_p \leq \sum_{i=1}^{n} 2^{p-1} (|x_i|^p + |x_{i+1}|^p) = 2^p \sum_{i=1}^{n} |x_i|^p = 2^pn. \quad \square
\]

Next we refine the previous lemma by including a bound on how fast \( \|Ax\|^p_p \) decreases as we move a little bit away from the set \( X \) while staying on \( S(0, n^{1/p}) \).

**Lemma 3.3.** Let \( p \geq 2 \), \( c \in (0, 1/2] \), and suppose \( y \in S(0, n^{1/p}) \) has the property that

\[
\min_{x \in X} \|y - x\|_\infty \geq c.
\]

Then

\[
\|Ay\|^p_p \leq n2^p - 3(p - 2) \frac{2pn^2}{c^2}.
\]

**Proof.** We proceed as before in the proof of Lemma 3.2 until the time comes to apply Lemma 3.1, when we include the error term which we had previously ignored:

\[
\|Ay\|^p_p \leq n2^p - \frac{1}{4} \sum_i (|y_i| - |y_{i+1}|)^2 \left( p(p - 1) \|y_i| + |y_{i+1}| \right)^{p-2} - 2 \|y_i| - |y_{i+1}| \right)^{p-2},
\]

Note that on the right-hand side, we are subtracting a sum of nonnegative terms. The upper bound will still hold if we subtract only one of these terms, so we conclude that, for each \( k \),

\[
\|Ay\|^p_p \leq n2^p - \frac{1}{4} (|y_k| - |y_{k+1}|)^2 \left( p(p - 1) \|y_k| + |y_{k+1}| \right)^{p-2} - 2 \|y_i| - |y_{i+1}| \right)^{p-2}.
\]
By assumption, there is at least one $y_k$ with $|y_k| - 1 \geq c$. Suppose first that $|y_k| > 1$. Then we have $|y_k| > 1 + c$, and there must be a $y_j$ with $|y_j| < 1$, for otherwise $y$ would not be in $S(0,n^{1/p})$. Similarly, if $|y_k| < 1$, then $|y_k| < 1 - c$ and there is a $j$ for which $|y_j| > 1$. In both cases, this implies the existence of an index $m$ with $|y_m|$ and $|y_m+1|$ differing by at least $c/n$ and such that at least one of $|y_m|$ and $|y_m+1|$ is larger than or equal to $1 - c$. Therefore,

$$||Ay||_p^p \leq n2^p - \frac{1}{4}c^2 \left[ p(p-1) ||y_m| + |y_{m+1}||^{p-2} - 2 ||y_m| - |y_{m+1}||^{p-2} \right].$$

Now observe that $||y_m| - |y_{m+1}|| \leq |y_m| + |y_{m+1}|$ and that $|y_m| + |y_{m+1}| \geq (1-c) \geq 1/2$ because $c \in (0,1/2]$. These two inequalities suffice to establish that the term in square brackets is at least $(1/2)^{p-2}(p(p-1) - 2) \geq (3/2p)(p-2)$ so that

$$||Ay||_p^p \leq n2^p - \frac{3(p-2)}{2p^2}c^2. \Box$$

4. **Proof of Theorem 1.1.** We now relate the results of the last two sections to the problem of the $p$-norm. For a suitably defined matrix $Z$ combining $A$ and $M(G)$, we want to argue that the optimizer of $||Zx||_p/||x||_p$ is very close to satisfying $|x_i| = |x_j|$ for every $i,j$.

**Proposition 4.1.** Let $p > 2$ and $G$ be a graph on $n$ vertices. Consider the matrix

$$\tilde{Z} = \left( \frac{p-2}{64pn^8} A \right) M(G)$$

with $M(G)$ and $A$ as in sections 2 and 3, respectively. If $x^*$ is the vector at which the optimization problem $\max_{x \in S(0,n^{1/p})} ||\tilde{Z}x||_p$ achieves its supremum, then

$$\min_{x \in \Lambda} ||x^* - x||_{\infty} \leq \frac{1}{4pn^6}.$$ 

**Proof.** Suppose the conclusion is false. Then using Lemma 3.3 with $c = 1/4pn^6$, we obtain

$$||Ax^*||_p^p \leq n2^p - \frac{3(p-2)}{2p^2n^{14}} = n2^p - \frac{3(p-2)}{32p^{14}}.$$ 

It follows from Proposition 2.1 that

$$||Mx^*||_p^p \leq 2^p \maxcut(G) \leq 2^p n^2$$

so that

$$||\tilde{Z}x^*||_p^p = ||Ax^*||_p^p + \left( \frac{p-2}{64pn^8} \right)^p ||Mx^*||_p^p \leq 2^p n^2 - \frac{3(p-2)}{32p^{14}} + \frac{2^p(p-2)n^2}{64pn^8}.$$ 

Observe that the last term in this inequality is smaller than the previous one (in absolute value). Indeed, for $p > 2$, we have that $3/32p > (2/64)^p$, $p-2 > [(p-2)/p]^p$, and $1/n^{14} > n^2/n^{8p}$. We therefore have $||Zx^*||_p^p < 2^p n$. By contrast, let $x$ be any vector in $\{-1,1\}^n$. Then $x \in S(0,n^{1/p})$ and

$$||\tilde{Z}x||_p^p \geq ||Ax||_p^p \geq 2^p n,$$
which contradicts the optimality of $x^*$. \hfill \square

Next we seek to translate the fact that the optimizer $x^*$ is close to $X$ to the fact that the objective value $||Zx||_p/||x||_p$ is close to the largest objective value at $X$.

**Proposition 4.2.** Let $p > 2$, $G$ be a graph on $n$ vertices, and 

$$
Z = \left( \begin{array}{c}
\frac{64pn^8}{p-2} A \\
\frac{M(G)}{p-2}
\end{array} \right).
$$

If $x^*$ is the vector at which the optimization problem

$$
\max_{x \in S(0,n^{1/p})} ||Zx||_p
$$

achieves its supremum and $x_r$ is the rounded version of $x^*$ in which every component is rounded to the closest of $\{-1, 1\}$, then

$$
||Zx^*||_p - ||Zx_r||_p \leq \frac{1}{n^2}.
$$

**Proof.** Observe that $x^*$ is the same as the extremizer of the corresponding problem with $\tilde{Z}$ instead of $Z$ so that $x$ satisfies the conclusion of Proposition 4.1. Consequently every component of $x^*$ is closer to one of $\pm 1$ than to the other, and so $x_r$ is well defined. We have,

$$
||Zx^*||_p - ||Zx_r||_p = \left( \frac{64pn^8}{p-2} \right)^p (||Ax^*||_p^p - ||Ax_r||_p^p) + (||Mx^*||_p - ||Mx_r||_p^p).
$$

This entire quantity is nonnegative since $x^*$ is the maximum of $||Zx||$ on $S(0,n^{1/p})$. Moreover, $||Ax^*||_p^p - ||Ax_r||_p^p$ is nonpositive since, by Proposition 3.2, $||Ax||_p$ achieves its maximum over $S(0,n^{1/p})$ on all the elements of $X$. Consequently,

$$
||Zx^*||_p - ||Zx_r||_p^p \leq ||Mx^*||_p - ||Mx_r||_p^p
$$

$$
\leq \left( ||Mx^*||_p - ||Mx_r||_p \right) \max(||Mx^*||_p, ||Mx_r||_p)^{p-1}.
$$

We now bound all the terms in the last equation. First

$$
||Mx^*||_p - ||Mx_r||_p \leq ||M||_2 ||x^* - x_r||_2 \leq ||M||_F \sqrt{n} ||x^* - x_r||_\infty = \frac{n\sqrt{n}}{4pn^6},
$$

where we have used $||M(G)||_F = \sqrt{2|E|} < n$ and Proposition 4.1 for the last inequality. Now that we have a bound on the first term in (4.1), we proceed to the last term. It follows from the definition of $M$ that

$$
||Mx_r||_p^p \leq 2^p \cdot \left( \frac{n}{2} \right) \leq 2^pn^2.
$$

Next we bound $||Mx^*||_p^p$. Observe that a particular case of (4.2) is

$$
||Mx^*||_p < ||Mx_r||_p + 1.
$$

Moreover, observe that $||Mx_r||_p \geq 1$. (The only way this does not hold is if every entry of $x_r$ is the same, i.e., $||Mx_r||_p = 0$. But then (4.3) implies that $||Mx^*||_p < 1$, which is
impossible since $G$ has at least one edge.), So (4.3) implies that $\|Mx^*\|_p \leq 2\|Mx_1\|_p$, and so
\[ \|Mx^*\|_p^p \leq 4^n n^2. \]
Thus
\[ \max(\|Mx^*\|_p, \|Mx_1\|_p) \leq 4^n n^2, \]
and therefore $\max(\|Mx^*\|_p, \|Mx_1\|_p)^{p-1} \leq 4^n n^2$. Indeed, this bound is trivially valid if $\max(\|Mx^*\|_p, \|Mx_1\|_p)^p \leq 1$ and follows from $a^{p-1} < a^p$ for $a \geq 1$ otherwise. Using this bound and the inequality (4.2), we finally obtain
\[ \|Zx^*\|_p^p - \|Zx_1\|_p^p \leq n^{1.5} \frac{n^5}{4^n} p \cdot 4^n n^2 \leq \frac{1}{n^2}. \]
\[ \square \]

Finally let us bring it all together by arguing that if we can approximately compute the $p$-norm of $Z$, we can approximately compute the maximum cut.

**Proposition 4.3.** Let $p > 2$. Consider a graph $G$ on $n > 2$ vertices and the matrix
\[ Z = \left( \frac{64pn^8}{p-2} \right) \]
and let $f^* = \|Z\|_p$. If
\[ \|f_{\text{approx}} - f^*\| \leq \frac{(p-2)^p}{132^p p^n n^{8p+3}}, \]
then
\[ \left| \left( \frac{n}{2^n} f_{\text{approx}}^p - n \left( \frac{64pn^8}{p-2} \right)^p \right) - \maxcut(G) \right| \leq \frac{1}{n}. \]

**Proof.** Observe that $n \frac{1}{p} f^* = \max_{x \in S(0,n^{-1/p})} \|Zx\|_p$. It follows thus from Proposition 4.2 that
\[ \left| n f^{*p} - \max_{x \in X} \|Zx\|_p^p \right| < \frac{1}{n^2}. \]
Recall that $\|Zx\|_p^p = \|Mx\|_p^p + \left( \frac{64pn^8}{p-2} \right)^p \|Ax\|_p^p$ and that $\|Ax\|_p^p = n2^p$ for every $x \in X$. Therefore,
\[ \max_{x \in X} \|Zx\|_p^p = \left( \frac{64pn^8}{p-2} \right)^p n2^p + \max_{x \in X} \|Mx\|_p^p = \left( \frac{64pn^8}{p-2} \right)^p n2^p + 2\maxcut(G), \]
and by combining the last two equations, we have
\[ (4.4) \quad \left| \left( \frac{n}{2^n} f_{\text{approx}}^p - n \left( \frac{64pn^8}{p-2} \right)^p \right) - \maxcut(G) \right| \leq \frac{1}{2^n n^2}. \]
Let us now evaluate the error introduced by the approximation $f_{\text{approx}}$:
\[ \left| \left( \frac{n}{2^n} f_{\text{approx}}^p - n \left( \frac{64pn^8}{p-2} \right)^p \right) - \maxcut(G) \right| \leq \frac{1}{2^n n^2} + \frac{n}{2^n} \|f_{\text{approx}}^p - f^p\| \]
\[ \leq \frac{1}{2^n n^2} + \frac{n}{2^n} (f_{\text{approx}} - f^p) \|\max(f^*, f_{\text{approx}})^{p-1}. \]
It remains to bound the last term of this inequality. First we use the fact that $f^* \geq 1$ and (4.4) to argue

\[(4.5) \quad f^{(p-1)} \leq f^p p \leq 2^p \left( \frac{64pn^8}{p-2} \right)^p + \frac{2^p}{n} \maxcut(G) + \frac{1}{n^2} \leq 2^p \left( \frac{66pn^8}{p-2} \right)^p, \]

where we have used $\maxcut(G) < n^2$ and $1 \leq p/(p-2)$ for the last inequality. By assumption, $|f_{\text{approx}} - f^*| \leq 1$, and since $f^* \geq 1$,

$$f_{\text{approx}}^{(p-1)} \leq (2f^*)^{p-1} \leq (2f^*)^p \leq 4^p \left( \frac{66pn^8}{p-2} \right)^p.$$ 

Putting it all together and using the bound on $|f_{\text{approx}} - f^*|$, we obtain (assuming $n > 1$)

$$\left| \frac{n}{2^p} f_{\text{approx}}^p - n \left( \frac{64pn^8}{p-2} \right)^p - \maxcut(G) \right| \leq \frac{1}{2p^2} + \frac{(p-2)^p}{132^p} \cdot \frac{1}{n^2} \cdot \left( \frac{66pn^8}{p-2} \right)^p \leq \frac{1}{n^2} \leq \frac{1}{n}. \quad \square$$

**Proposition 4.4.** Fix a rational $p \in [1, \infty)$ with $p \neq 1, 2$. Unless $P = NP$, there is no algorithm which, given input $\epsilon > 0$ and a matrix $Z$, computes $||Z||_p$ to a relative accuracy $\epsilon$, in time which is polynomial in $1/\epsilon$, the dimensions of $Z$, and the bit size of the entries of $Z$.

**Proof.** Suppose first that $p > 2$. We show that such an algorithm could be used to build a polynomial time algorithm solving the maximum cut problem. For a graph $G$ on $n$ vertices, fix

$$\epsilon = \left( \frac{132}{p} \right)^p \left( \frac{n^{8p+3}}{p} \right)^{-1} \cdot \left( \frac{132}{p} \right)^{-1} \left( \frac{n^8}{p-2} \right)^{-1},$$

build the matrix $Z$ as in Proposition 4.3, and compute the norm of $Z$; let $f_{\text{approx}}$ be the output of the algorithm. Observe that, by (4.5),

$$||Z||_p \leq \frac{132pn^8}{p-2},$$

so

$$f_{\text{approx}} - ||Z||_p \leq \epsilon ||Z||_p \leq \epsilon \left( \frac{132}{p} \right) n^8 \leq \left( \frac{132}{p} \right)^p \left( \frac{n^{8p+3}}{p} \right)^{-1}.$$

It follows then from Proposition 4.3 that

$$n \left( \frac{f_{\text{approx}}}{2} \right)^p - n \left( \frac{64}{p-2} \right) \cdot n^8 \right)^p$$

is an approximation of the maximum cut with an additive error at most $1/n$. Once we have $f_{\text{approx}}$, we can approximate this number in polynomial time to an additive accuracy of $1/4$. This gives an additive error $1/4 + 1/n$ approximation algorithm for
maximum cut, and since the maximum cut is always an integer, this means we can compute it exactly when \( n > 4 \). However, maximum cut is an NP-hard problem [1].

For the case of \( p \in (1, 2) \), NP-hardness follows from the analysis of the case of \( p > 2 \) since, for any matrix \( Z \), \( ||Z||_p = ||Z^T||_{p'} \), where \( 1/p + 1/p' = 1 \).

Remark. In contrast to Theorem 2.3 which proves the NP-hardness of computing the matrix \( \infty \), \( k \)-norm to relative accuracy \( \epsilon = 1/C(p) \), for some function \( C(p) \), Proposition 4.4 proves the NP-hardness of computing the \( p \)-norm to accuracy \( 1/C'(p)n^{8p+11} \) for some function \( C'(p) \). In the latter case, \( \epsilon \) depends on \( n \).

Our final theorem demonstrates that the \( p \)-norm is still hard to compute when restricted to matrices with entries in \( \{-1, 0, 1\} \).

**Theorem 4.5.** Fix a rational \( p \in [1, \infty) \) with \( p \neq 1, 2 \). Unless \( P = NP \), there is no algorithm which, given input \( \epsilon \) and a matrix \( M \) with entries in \( \{-1, 0, 1\} \), computes \( ||M||_p \) to relative accuracy \( \epsilon \), in time which is polynomial in \( \epsilon^{-1} \) and the dimensions of the matrix.

Proof. As before, it suffices to prove the theorem for the case of \( p > 2 \); the case of \( p \in (1, 2) \) follows because \( ||Z||_p = ||Z^T||_{p'} \), where \( 1/p + 1/p' = 1 \).

Define

\[
Z^* = \left( \left\lfloor \frac{64}{p} n^8 \right\rfloor \right) A,
\]

where \( \lfloor \cdot \rfloor \) refers to rounding up to the closest integer. Observe that, by an argument similar to the proof of the previous proposition, computing \( ||Z^*||_p \) to an accuracy \( \epsilon = (C(p)n^{8p+11})^{-1} \) is NP-hard for some function \( C(p) \). But if we define

\[
Z^{**} = \begin{pmatrix} A \\ A \\ \vdots \\ A \\ M \end{pmatrix},
\]

where \( A \) is repeated \( \left\lfloor \frac{64}{p} n^8 \right\rfloor \) times, then

\[
||Z^{**}||_p = ||Z^*||_p.
\]

The matrix \( Z^{**} \) has entries in \( \{-1, 0, 1\} \), and its size is polynomial in \( n \), so it follows that it is NP-hard to compute \( ||Z^{**}||_p \) within the same \( \epsilon \). \( \square \)

Remark. Observe that the argument also suffices to show that computing the \( p \)-norm of square matrices with entries in \( \{-1, 0, 1\} \) is NP-hard: simply pad each row of \( Z^{**} \) with enough zeros to make it square. Note that this trick was also used in section 2.

5. **Concluding remarks.** We have proved the NP-hardness of computing the matrix \( p \)-norm approximately with relative error \( \epsilon = 1/C(p)n^{8p+11} \), where \( C(p) \) is some function of \( p \), and the NP-hardness of computing the matrix \( \infty \), \( p \)-norm to some fixed relative accuracy depending on \( p \). We finish with some technical remarks about various possible extensions of the theorem:

- Due to the linear property of the norm \( ||\alpha A|| = ||\alpha||||A|| \), our results also imply the NP-hardness of approximating the matrix \( p \)-norm with any fixed or polynomially growing additive error.
Our construction also implies the hardness of computing the matrix $p$-norm for any irrational number $p > 1$ for which a polynomial time algorithm to approximate $x^p$ is available.

Our construction may also be used to provide a new proof of the NP-hardness of the $\|\cdot\|_{p,q}$ norm when $p > q$, which has been established in [5]. Indeed, it rests on the matrix $A$ with the property that $\max \|Ax\|_p/\|x\|_p$ occurs at the vectors $x \in \{-1, 1\}^n$. We use this matrix $A$ to construct the matrix $Z = (\alpha A M)^T$ for large $\alpha$ and argue that $\max \|Zx\|_p/\|x\|_p$ occurs close to the vectors $x \in \{-1, 1\}^n$. At these vectors, it happens $Ax$ is a constant, so we are effectively maximizing $\|Mx\|_p$, which is hard as shown in section 2. If one could come up with such a matrix for the case of the mixed $\|\cdot\|_{p,q}$ norm, one could prove NP-hardness by following the same argument. However, when $p > q$, actually the same matrix $A$ works. Indeed, one could simply argue that

$$\|A\|_{p,q} = \max_{x \neq 0} \frac{\|Ax\|_q}{\|x\|_p} = \max_{x \neq 0} \frac{\|Ax\|_q}{\|x\|_q} \frac{\|x\|_q}{\|x\|_p},$$

and since the maximum of $\|x\|_q/\|x\|_p$ when $1 \leq q < p \leq \infty$ occurs at the vectors $x \in \{-1, 1\}^n$, we have that both terms on the right are maximized at $x \in \{-1, 1\}^n$, and that is where $\|Ax\|_q/\|x\|_p$ is maximized.

Finally we note that our goal was only to show the existence of a polynomial time reduction from the maximum cut problem to the problem of matrix $p$-norm computation. It is possible that more economical reductions which scale more gracefully with $n$ and $p$ exist.

REFERENCES


