**Bulk viscosity of a gas of massless pions**

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I. INTRODUCTION

Transport coefficients of quantum chromodynamics (QCD) are of high interest recently. This was triggered by the discovery that quark–gluon plasma (QGP) has a viscosity close to the conjectured universal minimum bound [1], indicating that QGP is close to a “perfect fluid” [2–4] just above the deconfinement temperature. This bound, \( \eta/s \geq 1/4\pi \), being the density entropy, is motivated by the uncertainty principle and is found to be saturated for a large class of strongly interacting quantum field theories whose dual descriptions in string theory involve black holes in anti-de Sitter space [5–8]. There are some debates about whether the minimum bound on \( \eta/s \) is truly universal [9–11] and the Relativistic Heavy Ion Collider data might be better fit with \( \eta/s < 1/4\pi \) [12,13]. In QCD with heavy quarks integrated out and with scattering length limit where conformal symmetry is preserved [25–27], there are some limitations. In the perturbative region, up to logarithmic corrections, \( \xi/s \propto \alpha_s^{-2}(1/3 - \frac{1}{2} \eta/s) \propto \alpha_s^2 \) [28] while \( \eta/s \propto \alpha_s^{-2} \) [29,30]. Thus, \( \xi \) is smaller than \( \eta \) in the perturbative regime. When the temperature is reduced, \( \eta/s \) reaches its minimum near \( T_c \), while \( \xi/s \) rises sharply near \( T_c \) [31–33]. It will be interesting to see whether the maximum of \( \xi/s \) is also reached near \( T_c \) from below, which is the main purpose of this work. We will focus on the case with two flavors of massless quarks such that below \( T_c \) the dominant degrees of freedom are massless pions.

II. LINEARIZED BOLTZMANN EQUATION AND THE GENERALIZED KINEMATIC THEORY

The bulk viscosity of a system is defined by the Kubo formula

\[
\eta = \frac{1}{2} \lim_{\omega \to 0} \frac{1}{\omega} \int_0^\infty dt \int d^3 r e^{i\omega t} \langle \left[ T^{\mu}_\mu(x), T^\nu_\nu(0) \right] \rangle,
\]

where \( T^{\mu}_\mu \) is the trace of the energy-momentum tensor. The Kubo formula involves an infinite number of diagrams at the leading order (LO) even in the weakly coupled \( \phi^4 \) theory [34]. However, it is proven that the summation of LO diagrams in a weakly coupled \( \phi^4 \) theory [34] or in hot QED [35,36] is equivalent to solving the linearized Boltzmann equation with temperature-dependent particle masses and scattering amplitudes. Therefore, the proofs do not use properties restricted to scalar theories, the conclusion is expected to hold for more general theories with weak couplings, including QCD in the perturbative regime [28–30]. Here, we assume the equivalence between the Kubo formula and the Boltzmann equation also applies to massless pions.
The Boltzmann equation describes the evolution of the isospin averaged pion distribution function \( f = f(x, p, t) \equiv f_p(x) \) (a function of space, time, and momentum)

\[
\frac{p^\mu}{E_p} \partial_\mu f_p(x) = \frac{g_\pi}{2} \int \frac{d^3k_i}{(2\pi)^3} \frac{d^3k_f}{(2\pi)^3} \frac{1}{E_p^0} \left\{ f_i f_2 (1 + f_i)(1 + f_p) - (1 + f_i)(1 + f_2) f_p f_i \right\},
\]

where \( E_p = \sqrt{p^2 + m_\pi^2} \), \( p = |p| \) and \( g_\pi = 3 \) is the degeneracy factor for three pions,

\[
d\Gamma_{12,3p} \equiv \frac{1}{2E_p} |T|^2 \prod_{i=1}^{3} \frac{d^3k_i}{(2\pi)^3} (2E_i) \times (2\pi)^4 \delta^4(k_1 + k_2 - k_3 - p),
\]

and where \( T \) is the scattering amplitude for particles with momenta \( 1, 2 \to 3, p \). In chiral perturbation theory (\( \chi PT \)), which is a low-energy effective field theory of QCD, the LO isospin averaged \( \pi \pi \) scattering amplitude in terms of Mandelstam variables \( s, t, \) and \( u \) is

\[
|T|^2 = \frac{1}{9g_\pi^2} (9s^2 + 3(t - u)^2),
\]

where \( f_p = 88.3 \text{ MeV} \) is the pion decay constant in the chiral limit. The pions remain massless below \( \delta T_{\mu\nu} \) where

\[
\frac{\partial^\mu T^{(0)}_{\mu\nu}}{E_p} \partial_\mu f_p(x) = \frac{g_\pi}{2} \int \frac{d^3k_i}{(2\pi)^3} \frac{d^3k_f}{(2\pi)^3} \frac{1}{E_p^0} \left\{ f_i f_2 (1 + f_i)(1 + f_p) - (1 + f_i)(1 + f_2) f_p f_i \right\},
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\]

where \( E_p = \sqrt{p^2 + m_\pi^2} \), \( p = |p| \) and \( g_\pi = 3 \) is the degeneracy factor for three pions,
take into account the pion interaction associate with the loop diagrams. However, one can integrate out the medium effect and sum up the effective one-pion contributions to $T_{\mu\nu}$

$$T_{\mu\nu} = \sum_i \langle \pi_i | \tilde{T}_{\mu\nu} | \pi_i \rangle,$$

(14)

where $\tilde{T}_{\mu\nu}$ is the energy-momentum operator. Note that Eq. (6) is just the leading-order effect of the above equation that takes into account the free pion contribution to $T_{\mu\nu}$ only. Using symmetries, $T_{\mu\nu}$ has the general form:

$$T_{\mu\nu}(x) = g_\pi \int \frac{d^3p}{(2\pi)^3} \frac{f_\pi(x)}{E_p} \left\{ p_\mu p_\nu \{1 + g_1(x)\} + \frac{g_2(x)g_{\mu\nu}}{\beta(x)^2} + \frac{g_3(x) V_\mu V_\nu}{\beta(x)^2} \right\}.$$  

(15)

Here Lorentz symmetry is broken down to O(3) symmetry by the temperature, and $g_1$–$g_3$ are dimensionless functions of $\beta(x)$ and $f_\pi$. In $\chi$PT, $g_1$–$g_3$ are dimensionless functions of $\beta(x)$ and $f_\pi$. The structure ($p^\mu V^\nu + V^\mu p^\nu$) is not allowed because the $\pi^+$ and $\pi^-$ matrix elements should be the same by charge conjugation or isospin symmetry. Thus, $\langle \pi_i | \tilde{T}_{\mu\nu} | \pi_i \rangle$ should be invariant under crossing symmetry ($p^\mu \rightarrow -p^\mu$). In equilibrium, $T_{\mu\nu}^{(0)} = \epsilon - \zeta P$ and

$$c = 4g_2 + 3g_3 = \frac{\epsilon - 3\epsilon P}{\epsilon P} \int \frac{d^3p}{(2\pi)^3} \frac{f_\pi}{E_p}.$$  

(16)

Note that energy-momentum conservation is not a problem with the new terms in Eq. (15). In equilibrium, one just has to replace $v_i^2$ in Eq. (11) by the new value to obtain $\partial^\mu T^{(0)\mu} = 0$. Away from equilibrium, the net effect of $\zeta$ is to replace $P \rightarrow P - \zeta \nabla \cdot V$ in Eq. (9) that will induce second spacial derivative terms in Eq. (11). Thus, as long as Eq. (15) gives the correct $T_{\mu\nu}$, energy-momentum conservation can be satisfied.

Working to the first order in a derivative expansion, $\chi_\rho(x)$ can be parametrized as

$$\chi_\rho(x) = \beta(x) A(p) \nabla \cdot V(x) + \beta(x) B(p) \left( \dot{\rho}_i \dot{\rho}_j - \frac{1}{3} \delta_{ij} \right) \times \left[ \nabla V_i(x) + \nabla V_j(x) - \frac{1}{2} \delta_{ij} \nabla \cdot V(x) \right],$$  

(17)

where $A$ and $B$ are functions of $x$ and $p$. But we have suppressed the $x$ dependence. Substituting (17) into the Boltzmann equation and using Eq. (11), one obtains one linearized equation for $A$ (associated with the $\nabla \cdot V$ structure):

$$\frac{1}{3} \dot{p}^2 - v_i E_p = \frac{g_\pi E_p}{2} \int_{1/2} \frac{d^3x}{(2\pi)^3} \left\{ (1 + n_1) \times (1 + n_2) n_3 (1 + n_1)^{-1} \times [A(p) + A(k_i) - A(k_2) - A(k_3)] \right\}.$$  

(18)

where at point $x$, $f_\pi^{(0)}(x)$ is written as $n_1 = (e^{E_p E_1} - 1)^{-1}$. There is also a linearized equation for $B$ (associated with the $(\nabla V_p + \nabla V_i - \text{trace})$ structure) that is related to the shear viscosity $\eta$. The computation of $\eta$ of the pion gas has been discussed in Ref. [20]. We will focus on solving $\zeta$ in this work.

### III. Variational Calculation

Equation (18) determines only $A(p)$ up to a combination $a_1 + a_2 E_p$, where $a_1$ and $a_2$ are constants [34]. These “zero modes” ($a_1$ and $a_2 E_p$) appear only in the analysis of bulk viscosity but not shear viscosity. We will discuss their effects in this section.

The variation of Eq. (15) yields

$$\delta T_{\mu\nu} = g_\pi \int \frac{d^3p}{(2\pi)^3} E_p \left\{ \delta f_p \left[ p_\mu p_\nu (1 + g_1) + \frac{g_2 g_{\mu\nu}}{\beta^2} + \frac{g_3 V_\mu V_\nu}{\beta^2} \right] \right\}.$$  

(19)

Note that $g_1$–$g_3$ represent loop corrections of the energy-momentum tensor, thus they are functionals of $f_p$. To compute $\zeta$, we need

$$\delta T_{ii} = g_\pi \int \frac{d^3p}{(2\pi)^3} E_p \left\{ \delta f_p \left[ p^2 (1 + g_1) - \frac{3g_2}{\beta^2} \right] + f_p \left[ p^2 \delta g_1 - \frac{3g_2}{\beta^2} \right] \right\}.$$  

(20)

This can be simplified using the constraint,

$$0 = \delta T_{ii} = g_\pi \int \frac{d^3p}{(2\pi)^3} E_p \left\{ \delta f_p \left[ p^2 (1 + g_1) + \frac{g_2 + g_3}{\beta^2} \right] + f_p \left[ p^2 \delta g_1 + \frac{g_2 + g_3}{\beta^2} \right] \right\}.$$  

(21)

After eliminating the $g_2/\beta^2$ term in $\delta T_{ii}$ using the constraint, we have

$$\delta T_{ii} = g_\pi \int \frac{d^3p}{(2\pi)^3} E_p \left\{ \delta f_p \left[ 4p^2 (1 + g_1) + \frac{4g_2 + g_3}{g_2 + g_3} \right] + f_p \left[ p^2 \delta g_1 + \frac{4g_2 + g_3}{g_2 + g_3} + \frac{g_2 \delta g_3 - g_3 \delta g_2}{g_2 + g_3} \right] \right\} \simeq 4g_\pi d \int \frac{d^3p}{(2\pi)^3} f_p^{(0)} \left\{ 1 + O \left( \frac{T^4}{(4\pi f_p)^4} \right) \right\},$$  

(22)

where $d = (4g_2 + g_3)/(g_2 + g_3)$ and the pion remains massless in the chiral limit even at finite $T$, so we have used $p^2 = E_p^2$. The above expression for $\delta T_{ii}$ implies

$$\zeta = \frac{4}{3} g_\pi \beta d \int \frac{d^3p}{(2\pi)^3} E_p n_p (1 + n_p) A(p).$$  

(23)

Then using Eq. (18) and the symmetry property of the scattering amplitude,

$$\zeta = \frac{g_\pi^2 \beta d}{(2\pi)^3 (2E_1)^4} \int \prod_{i=1,2,3} \frac{d^3k_i}{(2\pi)^3} \left| T \right|^2 \left( 2\pi \right)^4 \delta^4(k_1 + k_2 - k_3 - p) \times (1 + n_1)(1 + n_2)n_3 n_p \times [A(p) + A(k_3) - A(k_2) - A(k_1)]^2.$$  

(24)

Note that equating Eqs. (23) and (24) is equivalent to taking a projection of Eq. (18). It can be shown that any ansatz satisfying Eqs. (23) and (24) gives a lower bound on $\zeta$ [38].
Thus, one can solve ζ variationally, i.e., finding an ansatz \( A(p) \) that gives the biggest ζ.

It is known that if one uses the ansatz \( A(p) = a_1 + a_2 E_p \), then it will not contribute to the \( 2 \leftrightarrow 2 \) scattering on the right-hand side of Eq. (18) (the \( a_2 \) terms cancel by energy conservation). In fact, this ansatz will not contribute to all the particle-number-conserving processes but can contribute to particle-number-changing processes, such as \( 2 \leftrightarrow 4 \) scattering, which we have not shown. As we know from Eqs. (18) and (23), ζ is proportional to the size of \( A(p) \) that is inversely proportional to rate of scattering. Thus, if the \( 2 \leftrightarrow 4 \) scattering has a bigger rate than the \( 2 \leftrightarrow 2 \) scattering, then this ansatz gives a bigger ζ by bypassing the faster \( 2 \leftrightarrow 2 \) scattering. In \( \phi^4 \) theory, it was found that ζ is indeed set by the \( 2 \leftrightarrow 4 \) scattering [34]. However, in perturbative QCD (PQCD), the soft particle-number-changing bremsstrahlung is faster than the \( 2 \leftrightarrow 2 \) scattering [28]. Thus, ζ is governed by \( 2 \leftrightarrow 2 \) scattering.

In the case with massless pions, however, \( 2 \leftrightarrow 2 \) scattering is still the dominant process. Although using the ansatz \( A(p) = a_1 + a_2 E_p \), the \( \delta T_{00} = 0 \) constraint in Eq. (21) demands \( a_1/a_2 = 0 \) because \( n_p \propto 1/p \) as \( p \to 0 \). Because \( A(p) \) parametrizes a small deviation of \( f_p \) away from thermal equilibrium, \( a_1/a_2 = 0 \) gives \( a_1 = 0 \) instead of \( a_2 \to \infty \) and \( a_1 \) finite. Thus, to maximize ζ, we use the ansatz \( A(p) = a_1 E_p + a_2 E_p^2 + \cdots \) without the \( a_1 \) term. The point is, \( 2 \leftrightarrow 2 \) scattering cannot be bypassed and it will be the dominant process in our calculation.

To compute \( \delta T_{ii} \), it is easier to eliminate the \((1 + g_1)\) term in Eq. (20) using Eq. (21):

\[
\delta T_{ii} = -g_\pi \int \frac{d^3 p}{(2\pi)^3 E_p} \left[ \delta f_p \left( \frac{4g_2 + g_3}{\beta^2} \right) \right] + f_p \left[ \frac{4\delta g_2 + \delta g_3}{\beta^2} \right].
\]

Note that \( g_2 \) and \( g_3 \) terms at \( \mathcal{O}(T^4/(4\pi f_\pi)^4) \) arise from three-loop diagrams and from two-loop diagrams with insertions of higher-order counterterms and each loop integral has one power of \( f_p \) in the integrand. Thus, we will make an approximation here to assume the \((4\delta g_2 + \delta g_3)\) term is proportional to the \( \delta f_p \) with a proportional constant (\( l - 1 \)), where \( l \) means the power of \( f_p \) (or the number of loops) in \( T_{ii} \). Because \( l \) is between 2 and 3, we take the mean value \( l = 2.5 \) and associate the uncertainty of \( l \) to the error estimation of ζ. Thus,

\[
\zeta = -\frac{g_\pi l c}{3\beta} \int \frac{d^3 p}{(2\pi)^3 E_p} \frac{1}{n_p(1+n_p)} A(p).
\]

Note that \( A(p) \propto g_\pi^{-1}(1/3 - v_\pi^2)f_\pi^4 \) from Eq. (18). Thus, for massless pions,

\[
\zeta = hl (1 - 3\rho) \left( \frac{1}{3} - v_\pi^2 \right) \frac{f_\pi^4}{T^5},
\]

where \( T^5 \) is given by dimensional analysis and \( h \) is a dimensionless constant. To find the numerical solution for \( h \), we neglect the higher-order \( g_{1-3} \) terms in Eq. (21) and use the

\[
\zeta \propto \frac{1}{l} \left( \frac{1}{3} - v_\pi^2 \right) \frac{f_\pi^4}{T^5}.
\]

Using the \( \chi \) PT result of Ref. [37] for \( \epsilon \) and \( P \), we obtain

\[
\zeta \simeq 0.15 \left( \frac{l}{2.5} \right) \left( \ln \frac{\Lambda_p}{T} - \frac{1}{4} \right) \left( \ln \frac{\Lambda_p}{T} - \frac{3}{8} \right) \frac{T^7}{f_\pi^4 T^3}.
\]

The solid line below \( T_c \) is the massless pion gas result [\( T_c \simeq 200 \text{ MeV} \) and \( l = 2.5 \), explained below Eq. (20), are used]. The error on this curve is estimated to be 30–40%. The points are the lattice results for gluon plasma [33]. The solid and dashed lines above \( T_c \) give the central values and the error band from the QGP sum rule result of Ref. [32].

The leading-order contribution for pion entropy density \( s \) is just the result for a free pion gas:

\[
s = \frac{2\pi^2 g_\pi T^3}{45}.
\]
The behavior near the lower temperature peak is similar to where we have used the scaling $f_\pi \propto \sqrt{N_c}$, $g_\pi \propto N_f^2$, $\alpha_s^2 \propto 1/N_c$, and $N_f$ is the number of light quark flavors. Also, for massless pions,

$$\frac{\zeta}{\eta} \approx 180 \left( \frac{1}{2.5} \right) \left( \frac{1 - \alpha_s}{\epsilon} \right) \left( 1 - v_s^2 \right).$$

This is similar to $c/\eta \sim 15(1/3 - v_s^2)^2$, which is obtained for a photon gas coupled to hot matter [46] and is also parametrically correct for PQCD [28]. This is because in those cases, $2 \rightarrow 2$ scattering is the dominant process in both $\zeta$ and $\eta$ computations. It is not the case, however, in $\phi^4$ theory in which $1/(3 - v_s^2)^2$ is large $T$ dependence because $\zeta$ is dominated by $2 \leftrightarrow 4$ scattering while $\eta$ is dominated by $2 \rightarrow 2$ scattering. The scaling is also different from $\zeta/\eta \propto (1/3 - v_s^2)$ for strongly coupled $N = 2^*$ gauge theory using anti-de-Sitter space/conformal field theory [40].

**IV. CONCLUSIONS**

We have computed the bulk viscosity for a gas of massless pions using the Boltzmann equation with the kinetic theory generalized to incorporate the trace anomaly. The resulting $\zeta/s$, together with the corresponding results of gluon plasma [33] and quark gluon plasma [31] indicates $\zeta/s$ reaches its maximum near $T_c$ while $\eta/s$ reaches its minimum near $T_c$. If the $\zeta/s$ behavior is unchanged for massive pions, then the hadronization of the fire ball in heavy-ion collisions would imply large entropy production [31, 33] and slow equilibration. It would be interesting to explore the implications of the possible large bulk viscosity near a phase transition in cosmology if the phase transition above the TeV scale is based on some strongly interacting mechanism.

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