Z\textsubscript{2} spin liquids in the S=1/2 Heisenberg model on the kagome lattice: A projective symmetry-group study of Schwinger fermion mean-field states

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Z$_2$ spin liquids in the $S = \frac{1}{2}$ Heisenberg model on the kagome lattice: A projective symmetry-group study of Schwinger fermion mean-field states

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Due to strong geometric frustration and quantum fluctuation, the $S = 1/2$ quantum Heisenberg antiferromagnet on the kagome lattice has long been considered as an ideal platform to realize a spin liquid (SL), a phase exhibiting fractionalized excitations without any symmetry breaking. A recent numerical study (Yan et al., e-print arXiv:1011.6114) of the Heisenberg $S = 1/2$, kagome lattice model (HKLM) shows, in contrast to earlier results, that the ground state is a singlet-gapped SL with signatures of $Z_2$ topological order. Motivated by this numerical discovery, we use the projective symmetry group to classify all 20 possible Schwinger fermion mean-field states of $Z_2$ SLs on the kagome lattice. Among them we found only one gapped $Z_2$ SL (which we call the $Z_2[0,\pi]\beta$ state) in the neighborhood of the U(1) Dirac SL state. Since its parent state, i.e., the U(1) Dirac SL, was found [Yan et al., Phys. Rev. Lett. 98, 117205 (2007)] to be the lowest among many other candidate U(1) SLs, including the uniform resonating-valence-bond states, we propose this $Z_2[0,\pi]\beta$ state to be the numerically discovered SL ground state of the HKLM.

I. INTRODUCTION

At zero temperature, all degrees of freedom tend to freeze, and usually a variety of different orders, such as superconductivity and magnetism, will develop in different materials. However, in a quantum system with a large zero-point energy, one may expect a liquidlike ground state to exist even at $T = 0$. In a system consisting of localized quantum magnets, we call such a quantum-fluctuation-driven disordered ground state a quantum spin liquid (SL).$^1$ It is an exotic phase with “fractionalized” excitations carrying only a fraction of the electron quantum number, e.g., spinons, which carry spin but no charge. The internal structures of these SLs are so rich that they are beyond the description of Landau’s symmetry-breaking theory$^2$ of conventional ordered phases. Instead they are characterized by long-range quantum entanglement$^{3,4}$ encoded in the ground state, which is called “topological order”$^{5,6}$ in contrast to the conventional symmetry-breaking order.

Geometric frustration in a system of quantum magnets would lead to a huge degeneracy of classical ground-state configurations. The quantum tunneling among these classical ground states provides a mechanism to realize quantum SLs. The quest for quantum SLs in frustrated magnets (for a recent review, see Ref. 7) has been pursued for decades. Among them the Heisenberg $S = 1/2$ kagome lattice model (HKLM)

$$H_{\text{HKLM}} = J \sum_{\langle i,j \rangle} S_i \cdot S_j$$

(1)

has long been thought a promising candidate. Here $\langle i,j \rangle$ denotes $i,j$ being a nearest-neighbor pair. Experimental evidence$^{8-11}$ of SLs has been observed in ZnCu$_3$(OH)$_6$Cl$_2$ (called herbertsmithite), a spin-1/2 antiferromagnet on the kagome lattice. Theoretically, lacking an exact solution of the two-dimensional (2D) quantum Hamiltonian (1) in the thermodynamic limit, in previous studies either a honeycomb valence-bond crystal$^{12-16}$ (HVBC) with an enlarged (6 x 6)-site unit cell, or a gapless SL (Ref. 17) were proposed as the ground state of the HKLM. However, recently an extensive density-matrix-renormalization-group (DMRG) study$^{18}$ on the HKLM revealed the ground state of the HKLM as a gapped SL, which substantially lowers the energy compared to the HVBC. In addition, the authors also observed numerical signatures of $Z_2$ topological order in the SL state.

Motivated by this important numerical discovery, we try to discover the nature of this gapped $Z_2$ SL. Different $Z_2$ SLs on the kagome lattice have been previously studied using the Schwinger boson representation$^{19,20}$ Here we propose candidate states for symmetric $Z_2$ SLs on the kagome lattice by the Schwinger fermion mean-field approach.$^{21-27}$ Following is the summary of our results. First we use the projective symmetry group$^6$ (PSG) to classify all 20 possible Schwinger fermion mean-field Ansätze of $Z_2$ SLs which preserve all the symmetry of the HKLM, as shown in Table I. We analyze these 20 states and rule out some obviously unfavorable ones: e.g., gapless states, and those states whose first-nearest-neighbor (NN) mean-field amplitudes must vanish due to symmetry. Then we focus on those $Z_2$ SLs in the neighborhood of the U(1) Dirac SL.$^{28}$ In Ref. 28 it is shown that the U(1) Dirac SL has a significantly lower energy compared with other candidate U(1) SL states, such as the uniform resonating-valence-bond (RVB) state (or the U(1) SL-[0,0] state in the notation of Ref. 28). We find that there is only one gapped $Z_2$ SL, which we label as $Z_2[0,\pi]\beta$, in the neighborhood of (or continuously connected to) the U(1) Dirac SL. Therefore we propose this $Z_2[0,\pi]\beta$ state as a promising candidate for the ground state of the HKLM. The mean-field ansatz of $Z_2[0,\pi]\beta$ state is shown in Fig. 1(b). Our work also provides guidelines for choosing variational states in future numeric studies of the SL ground state on the kagome lattice.
TABLE I. Mean-field Ansätze of 20 possible $Z_2$ SLs on a kagome lattice. In our notation of mean-field amplitudes $(x,y,z)(0,0,0) \equiv [x,y,z]$, this table summarizes all symmetry-allowed mean-field bonds up to third nearest neighbor, i.e., first-nearest-neighbor bond $u_\alpha = [0,0,v]$, second-nearest-neighbor bond $u_\alpha = [0,1,w]$, and third-nearest-neighbor bonds $u_\alpha = [1,0,u]$ and $u_\beta = [1,−1,u]$ as shown in Fig. 1(a). $\Lambda_{\tau}$ denote the on-site chemical potential terms which enforce the constraint (11). $\tau^{1,2,3}$ are a 2 $\times$ 2 identity matrix while $\tau^{1,2}$ are three Pauli matrices. $\tau^0$ denote hopping while $\tau^{1,2}$ denote pairing terms. 0 means that the coordinates mean-field amplitudes must vanish due to symmetry. The shortest mean-field bonds necessary to realize a $Z_2$ SL are shown in italic type. In other words, the mean-field amplitudes in italics break the U(1) gauge redundancy down to $Z_2$ through the Higgs mechanism. So in states nos. 3, 19, and 7–12 a $Z_2$ SL cannot be realized with up to third-NN mean-field bonds. Note that state no. 15 needs only a third-nearest-neighbor bond $u_\alpha$ to realize a $Z_2$ SL ($\tilde{u}_\alpha$ is not necessary), while state no. 20 needs only $\tilde{u}_\tau$ to realize a $Z_2$ SL ($u_\tau$ is not necessary). Notice that when $\eta_{12} = -1$ the mean-field Ansatz (instead of the SL itself) will break translational symmetry and double the unit cell. There are six $Z_2$ SLs, i.e., nos. 7–12 that do not allow any first-NN mean-field bonds. Among the other 14 $Z_2$ SLs with nonvanishing first-NN mean-field bonds, only five $Z_2$ SL states, i.e., nos. 1, 2, 5, 13, and 15, have gapped spinon spectra. No. (2) in bold type or the $Z_2[0,\pi]\beta$ state in the neighborhood of the U(1) Dirac SL is the most promising candidate of the $Z_2$ SLs for the HKLM ground state.

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II. SCHWINGER FERMION CONSTRUCTION OF SPIN LIQUIDS AND PROJECTIVE SYMMETRY GROUP

A. Schwinger fermion construction of symmetric spin liquids

In the Schwinger fermion construction,21–28 we represent a spin-1/2 operator at site $i$ by fermionic spinors $\{f_{ia},\alpha = \uparrow,\downarrow\}$:

$$\tilde{S}_i = \frac{1}{2} f_{ia}^\dagger \sigma_{\alpha\beta} f_{ib}. \tag{2}$$

FIG. 1. (Color online) (a) Kagome lattice and the elements of its symmetry group. $\tilde{a}_{i\alpha}$ are the translation unit vectors, $C_6$ denotes $\pi/3$ rotation around the honeycomb center, and $\sigma$ represents mirror reflection along the dashed blue line. Here $u_\alpha$ and $u_\beta$ denote first- and second-nearest-neighbor (NN) mean-field bonds while $u_\alpha$ and $\tilde{u}_\alpha$ represent two kinds of independent third-NN mean-field bonds. (b) Mean-field ansatz of the $Z_2[0,\pi]\beta$ state up to second nearest neighbor. Colors in generic denote the sign structure of the mean-field bonds. Dashed lines denote first-NN real hopping terms $t_{ij} \sum_\alpha f_{ja}^\dagger f_{ia} f_{ja} f_{ib} + H.c.$: red ones have $v_{ij} = 1$ and bold black ones have $v_{ij} = -1$. Solid lines stand for second-NN hopping $\chi_{ij} \sum_\alpha f_{ja}^\dagger f_{ja} f_{ia} f_{ja} + H.c.$ and singlet pairing $\sum_\alpha f_{ja}^\dagger f_{ja} f_{ja} f_{ia} + H.c.$: again red ones have $v_{ij} = 1$ and bold blue ones have $v_{ij} = -1$. Here $\chi_{ij}$ and $\Delta_{ij}$ are real parameters after choice of a proper gauge.

The Heisenberg Hamiltonian $H = \sum_{ij} J_{ij} \tilde{S}_i \cdot \tilde{S}_j$ is represented as $H = \sum_{ij} -\frac{1}{2} J_{ij} (f_{ia}^\dagger f_{ja} f_{ja}^\dagger f_{ib} + \frac{1}{2} f_{ia} f_{ia}^\dagger f_{ib} f_{ib} + H.c.)$. This construction enforces the Hilbert space of the original spin system. To obtain the physical spin state from a mean-field state of $f$ spinons, we need to enforce the following one- $f$-spinon-per-site constraint:

$$f_{ia}^\dagger f_{ia} = 1, \quad f_{ia} f_{ib} \epsilon_{\alpha\beta} = 0. \quad (3)$$

The mean-field parameters of symmetric SLs are $\Delta_{ij} \epsilon_{\alpha\beta} = -2(\chi_{ij} \delta_{\alpha\beta} + 2f_{ia}^\dagger f_{ia} + H.c.)$, where $\epsilon_{\alpha\beta}$ is the completely antisymmetric tensor. Both terms are invariant under global SU(2) spin rotations. After a Hubbard-Stratonovich transformation, the Lagrangian of the spin system can be written as:

$$L = \sum_i \psi_i^\dagger \partial_\tau \psi_i + \frac{3}{8} \sum_{(ij)} \left( \frac{1}{2} \right) \tau_i U_{ij} \psi_i \psi_j + H.c. \right) + \sum_i \tilde{d}_i^\dagger(i) \psi_i^\dagger \tau_3 \psi_i. \quad (4)$$

where the two-component fermion notation $\psi_i \equiv (f_{ia}, f_{ia}^\dagger)$ is introduced for reasons that will be explained shortly. We use $\tau^0$ to denote the $2 \times 2$ identity matrix and $\tau^{1,2,3}$ are the three Pauli matrices. $U_{ij}$ is a matrix of mean-field amplitudes:

$$U_{ij} = \left( \begin{array}{cc} \chi_{ij} & \Delta_{ij} \\ \Delta_{ij}^\dagger & -\chi_{ij} \end{array} \right). \quad (5)$$

$\tilde{d}_i(i)$ are the local Lagrangian multipliers that enforce the constraints Eq. (3).

In terms of $\psi$, the Schwinger fermion representation has an explicit SU(2) gauge redundancy: a transformation $\psi_i \rightarrow W_i \psi_i, U_{ij} \rightarrow W_i U_{ij} W_j^\dagger, W_i \in SU(2)$ leaves the action invariant. This redundancy originates from the representation Eq. (2): this local SU(2) transformation leaves the spin
operators invariant and does not change the physical Hilbert space. One can try to solve Eq. (4) by a mean-field (or saddle-point) approximation. At mean-field level, $U_{ij}$ and $a_0^\dagger$ are treated as complex numbers, and $a_0^\dagger$ must be chosen such that the constraints (3) are satisfied at the mean-field level: $\langle \psi_i \tau^l \psi_i \rangle = 0$. The mean-field Ansatz can be written as

$$H_{MF} = - \sum_{(ij)} \bar{\psi}_i^{\dagger} (ij) \psi_j + \sum_i \bar{\psi}_i^0 \psi_i \tau^l \psi_i,$$

where we defined $(ij) = \frac{1}{2} I_{ij} U_{ij}$. Under a local SU(2) gauge transformation $(ij) \rightarrow W_i (ij) W_j$, but the physical spin state described by the mean-field Ansatz $(\{ij\})$ remains the same. By construction the mean-field Ansatz does not break spin rotation symmetry, and the mean-field solutions describe SL states if lattice symmetry is preserved. Different $(\{ij\})$ Ansätze may be in different SL phases. The mathematical language to classify different SL phases is the projective symmetry group (PSG).

### B. Projective symmetry group classification of topological orders in spin liquids

The PSG characterizes the topological order in Schwinger fermion representation: SLs described by different PSGs are treated as complex numbers, and operators invariant and does not change the physical Hilbert space. One can try to solve Eq. (4) by a mean-field (or saddle-point) approximation. At mean-field level, $U_{ij}$ and $a_0^\dagger$ are treated as complex numbers, and $a_0^\dagger$ must be chosen such that the constraints (3) are satisfied at the mean-field level: $\langle \psi_i \tau^l \psi_i \rangle = 0$. The mean-field Ansatz can be written as

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### III. $Z_2$ Spin liquids in the $S = \frac{1}{2}$ ...
TABLE II. A summary of all 20 gauge-invariant PSGs with \( G_T(x,y,s) = \tau_1 \) on the kagome lattice. Notice that there is a free \( Z_2 \) integer \( \eta_{12} = \pm 1 \) in other PSG elements (B1), (B5), and (B7). They correspond to 20 different \( Z_2 \) spin liquids on the kagome lattice.

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<td>18</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( Z_2[0,0] )R</td>
</tr>
<tr>
<td>19</td>
<td>+1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( Z_2[0,0] )S</td>
</tr>
<tr>
<td>20</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( Z_2[0,0] )T</td>
</tr>
</tbody>
</table>

From Table I we can see there are six states, i.e., nos. 7–12, that do not allow nonzero first-NN mean-field amplitudes due to symmetry. Moreover, they cannot realize \( Z_2 \) SLs with up to third-NN mean-field amplitudes. Therefore they are unlikely to be the HKLM ground state. Ruling out these six \( Z_2 \) SLs, we can see that the other 14 \( Z_2 \) SL states fall into four classes. To be specific, they are continuously connected to different parent U(1) gapless SL states on the kagome lattice. These parent U(1) SL states in general have the following mean-field Ansätze:

\[
H_{U(1)SL} = \chi_1 \sum_{ij} v_{ij} (f_{i\alpha}^\dagger f_{j\alpha} + \text{H.c.}), \tag{9}
\]

where \( v_{ij} = \pm 1 \) characterizes the sign structure of hopping terms with \( \chi_1 \in \mathbb{R} \). Different parent U(1) SL states feature the flux of \( f \) spinon hopping phases around the basic plaquette: honeycombs and triangles on the kagome lattice.

The simplest example is the so-called uniform RVB state with \( v_{ij} = +1 \) for all first-NN mean-field bonds. The hopping phase around any plaquette is \( 1 = \exp[0] \), and the corresponding flux is \( [0,0] \) for [triangle, honeycomb] motifs. The four possible \( Z_2 \) spin liquids in the neighborhood\(^{11} \) of uniform RVB states [i.e., the U(1) SL \([0,0]\) state in Ref. 28] are classified in Appendix D. They are nos. 1, 5, 15, and 13 in Tables I and II. We label them as \( Z_2[0,0]A \), \( Z_2[0,0]B \), \( Z_2[0,0]C \), and \( Z_2[0,0]D \) states. They all have gapped spectra of spinons.

The Ansätze of two other parent U(1) SLs are shown in Fig. 2. They both have \( \pi \) flux piercing through a triangle basic plaquette. Following the above notations of the hopping phase in [triangle,honeycomb] motifs, with either \( \pi \) or 0 flux through the honeycomb plaquette, they are called the U(1) SL \([\pi,\pi]\) state and the U(1) SL \([0,\pi]\) state.

Another U(1) SL state is the so-called uniform RVB state and (b) U(1) SL \([0,\pi]\) state, with first-NN real hopping terms \( H_{MF} = \chi_1 \sum_{ij} (v_{ij} f_{i\alpha}^\dagger f_{j\alpha} + \text{H.c.}) \). Colors again denote the sign structure of mean-field bonds: red dashed lines have \( v_{ij} = +1 \) and black dashed lines have \( v_{ij} = -1 \).
an infinitesimal spinon pairing term on top of the U(1) Dirac SL mean-field Ansatz (9) or (C1) to break the IGG from U(1) to $Z_2$ through the Higgs mechanism. Mathematically, we need to find those $Z_2$ SL states whose PSG is a subgroup of the U(1) Dirac SL’s PSG.\(^{31}\) Such $Z_2$ SL states are defined to be in the neighborhood of the U(1) Dirac SL. A similar criterion applies to the neighboring $Z_2$ SL states of any parent U(1) or SU(2) SL state.

We find all four $Z_2$ SLs in the neighborhood of U(1) Dirac SLs in Appendix C. They are states nos. 6, 2, 14, and 16 in Table I, labeled the $Z_2[0, \pi]\alpha, Z_2[0, \pi]\beta, Z_2[0, \pi]\gamma, \text{and } Z_2[0, \pi]\delta$ states, respectively. Since the effective theory of a U(1) Dirac SL is an eight-component Dirac fermion coupled with a dynamical U(1) gauge field,\(^{28,33}\) we can find all symmetry-allowed mass terms that can open up a gap in the Dirac-like spinon spectrum. Following detailed calculations in Appendix C, we can see that among the four $Z_2$ SLs around the U(1) Dirac SL, only one state, i.e., $Z_2[0, \pi]\beta$ (state 2 in Tables I and II), can generate a mass gap in the spinon spectrum. In the other three states the Dirac cone in the spinon spectrum. Following detailed calculations in Appendix C, we can see that among the four $Z_2$ SLs around the U(1) Dirac SL, only one state, i.e., $Z_2[0, \pi]\beta$ (state 2 in Tables I and II), can generate a mass gap in the spinon spectrum. In the other three states the Dirac cone in the spinon spectrum is protected by symmetry. The mean-field Ansatz of the $Z_2[0, \pi]\beta$ SL state up to the second NN is shown in Fig. 1(b):

$$H_{MF} = \sum_i \left( \lambda_3 \sum_{\alpha} \epsilon_{\alpha\alpha} f_{i\alpha}^\dagger f_{i\alpha} + \lambda_1 f_{i1}^\dagger f_{i1} + \text{h.c.} \right)$$

$$+ \chi_{\alpha\beta} \sum_{\langle ij \rangle} v_{ij} (f_{i\alpha}^\dagger f_{j\beta} + \text{h.c.}) + \sum_{\langle ij \rangle} v_{ij} \left( \lambda_2 \sum_{\alpha} f_{i\alpha}^\dagger f_{j\alpha} + \Delta_2 \sum_{\alpha\beta} \epsilon_{\alpha\beta} f_{i\alpha}^\dagger f_{j\beta}^\dagger + \text{h.c.} \right), \tag{10}$$

where $\epsilon_{\alpha\beta}$ is the completely antisymmetric tensor. We only list up to second-NN mean-field amplitudes because, as shown in Table I (see also Appendix C), this $Z_2[0, \pi]\beta$ SL state needs only second-NN pairing terms to realize a $Z_2$ SL. We can always choose a proper gauge so that the mean-field parameters $\chi_{ij}$ and $\lambda_2$ are all real. The sign structure of $v_{ij}$ is determined by the following constraint:

$$\sum_{\langle ij \rangle} \langle f_{i1}^\dagger f_{j1}^\dagger \rangle = \sum_{\langle ij \rangle} \langle f_{i1} f_{j1} \rangle = 0,$$

$$\sum_{\langle ij \rangle} \left( \sum_{\alpha = \uparrow, \downarrow} f_{i\alpha}^\dagger f_{j\alpha} - 1 \right) = 0. \tag{11}$$

For further NN mean-field ansatz see discussions in Appendix C.

\section*{IV. CONCLUSION}

To summarize, motivated by the strong evidence of a $Z_2$ SL as the HKLM ground state in a recent DMRG study,\(^{18}\) we classify all possible $Z_2$ SL states in the Schwinger fermion mean-field approach using the PSG. We found 20 different Schwinger fermion mean-field states of $Z_2$ SLs on the kagome lattice, among which six states are unlikely due to vanishing first-NN mean-field amplitude. Among the other 14 $Z_2$ SLs only five possess a gapped spinon spectrum, which is observed in the DMRG result.\(^{15}\) These five symmetric $Z_2$ SL states are all in the neighborhood of certain parent U(1) gapless SLs. To be precise, four are in the neighborhood of gapless uniform RVB [or U(1) SL $[0, 0]$, state, while the other one, i.e., $Z_2[0, \pi]\beta$, is in the neighborhood of a gapless U(1) Dirac SL for U(1) SL $[0, \pi]$ state. A previous variational Monte Carlo study\(^{28}\) showed that the gapless U(1) Dirac SL has a substantially lower energy in comparison to the uniform RVB state. This suggests that the $Z_2$ SLs in the neighborhood of U(1) Dirac SLs should have lower energy compared to those in the neighborhood of uniform RVB states. Therefore we propose this $Z_2[0, \pi]\beta$ SL state with the mean-field Ansatz (10) shown in Fig. 1(b) as the HKLM ground state numerically detected in Ref. 18. Our work provides important insight for future numeric study, e.g., variational Monte Carlo studies of Gutzwiller-projected wave functions.

\section*{ACKNOWLEDGMENTS}

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\section*{APPENDIX A: SYMMETRY GROUP OF THE KAGOME LATTICE AND ALGEBRAIC CONDITIONS FOR $Z_2$ SPIN LIQUIDS}

As shown in Fig. 1(a), we label the three lattice sites in each unit cell with sublattice indices $(s = u, v, w)$. Choosing the Bravais unit vector as $\vec{a}_1 = a \vec{x}$ and $\vec{a}_2 = \frac{a}{\sqrt{3}}(\vec{x} + \vec{y})$, the positions of the three atoms in a unit cell labeled by indices $i = (x,y,s)$ are

$$\vec{r}(x,y,u) = (x + \frac{1}{2})\vec{a}_1 + (y + \frac{1}{2})\vec{a}_2,$$

$$\vec{r}(x,y,u) = (x + \frac{1}{2})\vec{a}_1 + y\vec{a}_2, \quad (A1)$$

$$\vec{r}(x,y,w) = x\vec{a}_1 + (y + \frac{1}{2})\vec{a}_2.$$}

The symmetry group of such a two-dimensional kagome lattice is generated by the following operations:

$$T_1:(x,y,s) \rightarrow (x + 1,y,s),$$

$$T_2:(x,y,s) \rightarrow (x,y + 1,s),$$

$$\sigma:(x,y,u) \rightarrow (y,x,u),$$

$$(x,y,v) \rightarrow (y,x,w),$$

$$(x,y,w) \rightarrow (y,x,v); \quad (A2)$$

$$C_6:(x,y,u) \rightarrow (-y - 1,x + y + 1,v),$$

$$(x,y,v) \rightarrow (-y,x + y,w),$$

$$(x,y,w) \rightarrow (-y - 1,x + y,u),$$

together with time reversal $T$. 

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The symmetry group of a kagome lattice is defined by the following algebraic relations between its generators:

\[
\begin{align*}
T^2 & = \sigma^2 = (C_6)^6 = e, \\
g^{-1}T^{-1}gT & = e, \quad \forall g = T_{1z}, \sigma, C_6, \\
T_{2z}^{-1}T_{1z}^{-1}T_{2z} & = e, \\
\sigma^{-1}T_1^{-1}\sigma T_1 & = e, \\
C_6^{-1}T_2^{-1}C_6 T_1 & = e, \\
C_6^{-1}T_{2z}^{-1}C_6 T_z & = e, \\
\sigma^{-1}C_6 \sigma C_6 & = e,
\end{align*}
\] (A3)

where \( e \) stands for the identity element in the symmetry group. Therefore the consistent conditions for a generic \( \mathbb{Z}_2 \) PSG on a kagome lattice are written as

\[
\begin{align*}
[G_T(i)]^2 & = \eta_T \tau^0, \\
G_\sigma(\sigma(i))G_\sigma(i) & = \tau^0, \quad \forall \sigma(i) \\
G_T^\dagger(i)G_T^\dagger(i)G_T^\dagger(i)G_T(\sigma(i))G_T(\sigma(i)) & = \eta_T \tau^0, \\
G_T^\dagger(i)G_T^\dagger(i)G_T^\dagger(i)G_T(\sigma(i)) & = \eta_\sigma \tau^0, \\
G_T^\dagger(i)G_T^\dagger(i)G_T^\dagger(i)G_T(\sigma(i))G_T(\sigma(i)) & = \eta_T \tau^0, \\
G_{C_6}(C_6(i)) & = \eta_{C_6} \tau^0, \\
G_{T_2}(T_2(i)) & = \eta_{T_2} \tau^0, \\
G_{T_2}(T_2(i)) & = \eta_{T_2} \tau^0, \\
G_{C_6}(C_6(i))G_{C_6}(C_6(i)) & = \eta_{C_6} \tau^0, \\
G_{T_2}(T_2(i))G_{T_2}(T_2(i))G_{T_2}(T_2(i)) & = \eta_{T_2} \tau^0, \\
G_{C_6}(C_6(i))G_{C_6}(C_6(i)) & = \eta_{C_6} \tau^0.
\end{align*}
\] (A4)–(A13)

for any lattice site \( i = (x,y,s) \). Here all \( \eta \)'s are \( \mathbb{Z}_2 \) integers characterizing different SLs: different (gauge-inequivalent) choices of these \( \mathbb{Z}_2 \) integers (different \( \mathbb{Z}_2 \) PSGs) correspond to different \( \mathbb{Z}_2 \) SLs. Notice that under a local gauge transformation \( W(i) \in SU(2) \) the PSG element \( G_T(i) \) transforms as

\[
G_T(i) \rightarrow W(i)G_T(i)W^{-1}(i).
\] (A17)

**APPENDIX B: CLASSIFICATION OF ALL \( \mathbb{Z}_2 \) SPIN LIQUIDS ON THE KAGOME LATTICE**

1. Classification of \( \mathbb{Z}_2 \) algebraic PSGs on the kagome lattice

In this section we classify all possible \( \mathbb{Z}_2 \) spin liquids on a kagome lattice. Mathematically we need to find all gauge-inequivalent solutions of the algebraic conditions (A4)–(A13) for \( \mathbb{Z}_2 \) PSGs.

First, from condition (A10) we can always choose a proper gauge so that

\[
G_T(x,y,s) = \eta_T^0 \tau^0, \quad G_T(x,y,s) = \tau^0.
\] (B1)

From (A12) and (A13) we can see that \( G_{\sigma}(x,y,s) = \eta_\sigma \eta_T^0 \eta_{T_2}^{x+y} g_{\sigma}(s) \). Condition (A5) further determines \( \eta_{T_2} = \eta_{T_2} \) and therefore we have

\[
G_{\sigma}(x,y,s) = \eta_\sigma^0 \tau^0 \eta_{T_2}^{x+y/2} g_{\sigma}(s),
\]

where the SU(2) matrices \( g_{\sigma}(s) \) satisfy

\[
g_{\sigma}(w)g_{\sigma}(v) = [g_{\sigma}(w)]^2 = \eta_\sigma \tau^0.
\] (B2)

Notice that we can always choose a proper global \( \mathbb{Z}_2 \) gauge on \( G_T(x,y,s) \) (which does not change the mean-field Ansatz) so that \( \eta_{C_6} = 1 \) in (A16). From (A15) and (A16) it is straightforward to show that

\[
G_{C_6}(x,y,u,v) = \eta_{C_6} \eta_{T_2}^{x+y+(x+y+1/2)} g_{C_6}(u,v)
\]

and

\[
G_{C_6}(x,y,u,v) = \eta_{C_6} \eta_{T_2}^{x+y+(x+y+1/2)} g_{C_6}(u,v).
\]

It is condition (A14) that determines \( \eta_{C_6} = \eta_{T_2} \eta_{C_6} \), and finally we have

\[
G_{C_6}(x,y,u,v) = \eta_{C_6} \eta_{T_2}^{x+y+(x+y+1/2)} g_{C_6}(u,v),
\]

\[
G_{C_6}(x,y,u) = \eta_{C_6} \eta_{T_2}^{x+y} g_{C_6}(u).
\]

Now through a gauge transformation \( W(i,y,s) = \eta_T^0 \) we can fix \( \eta_{T_2} = 1 \) and the PSG elements become

\[
G_{\sigma}(x,y,s) = \eta_T^0 \eta_{T_2}^{x+y} g_{\sigma}(s),
\] (B5)

\[
G_{C_6}(x,y,u,v) = \eta_{T_2}^{x+y} g_{C_6}(u,v),
\] (B6)

\[
G_{C_6}(x,y,u) = \eta_{T_2}^{x+y} g_{C_6}(u).
\]

According to (A4), (A6), and (A7) we can see that

\[
G_T(x,y,s) = \eta_T^0 \eta_{T_2}^{x+y} \eta_{T_2}^{x+y} g_T(s),
\] (A9) and (A8) further determine \( \eta_{T_2} = \eta_{T_2} = 1 \) and by choosing a proper gauge we have

\[
G_T(x,y,s) = g_T(s) \equiv \tau^0, \quad \eta_T = 1, \eta_{T_2} = 1, \eta_{T_2} = 1,
\] (B7)

which satisfy

\[
\begin{align*}
g_{\sigma}(u)g_T(u) = \eta_{T_2} g_T(u)g_{\sigma}(u), \\
g_{\sigma}(v)g_T(v) = \eta_{T_2} g_T(v)g_{\sigma}(v), \\
g_{C_6}(u)g_T(u) = \eta_{T_2} g_T(u)g_{C_6}(u), \\
g_{C_6}(v)g_T(v) = \eta_{T_2} g_T(v)g_{C_6}(v), \\
g_{C_6}(u)g_T(u) = \eta_{T_2} g_T(u)g_{C_6}(u), \\
g_{C_6}(v)g_T(v) = \eta_{T_2} g_T(v)g_{C_6}(v),
\end{align*}
\] (B8)–(B9)

according to (A9) and (A8).

In the following we find all the gauge-inequivalent solutions of SU(2) matrices \( g_{\sigma,T_2,C_6}(s) \) satisfying the above conditions. They are summarized in Table II.
(i) $g_{\tau}(s) = \tau^0$ and therefore $g_{\tau} = \eta_{\tau T} = \eta_{G,T} = 1$. Conditions (B8) and (B9) are automatically satisfied.

(ii) $\eta_{\sigma} = 1$. Notice that under a global gauge transformation $W(x,y,s) = W_x \in SU(2)$ the PSG elements transform as

\[
\begin{align*}
g_a(u) & \rightarrow W_u g_a(u) W_u^\dagger, \\
g_v(u) & \rightarrow W_u g_v(u) W_u^\dagger, \\
g_s(u) & \rightarrow W_u g_s(u) W_u^\dagger, \\
g_C(v) & \rightarrow W_v g_C(v) W_v^\dagger, \\
g_C(w) & \rightarrow W_w g_C(w) W_w^\dagger.
\end{align*}
\]

Thus from (B2) and (B4) we can always have $g_a(s) = \tau^0$ and $g_v(u) = \tau^0$, $g_C(v) = \eta_{G,T} \tau^0$ by choosing a proper gauge.

(A) $\eta_{G,T} = \eta_{G,T} = 1$. From (B3) we have $g_C(w) = \tau^0$, $g_C(v) = \eta_{G,T} \tau^0$ by proper gauge fixing.

(B) $\eta_{G,T} = \eta_{G,T} = 1$. From (B3) we have $g_C(w) = \tau^3$ by proper gauge fixing. Also from (B4) we can choose a gauge so that $g_C(v) = \tau^0$ and $g_C(v) = -i \eta_{G,T} \tau^3$.

(A) $\eta_{G,T} = \eta_{G,T} = 1$. From (B3) we have $g_C(w) = \tau^0$ and $g_C(v) = \eta_{G,T} \tau^0$ by proper gauge fixing.

(B) $\eta_{G,T} = \eta_{G,T} = 1$. From (B3) we have $g_C(w) = \tau^0$ and $g_C(v) = \eta_{G,T} \tau^0$ by proper gauge fixing. Also from (B4) we can always choose a proper gauge so that $g_C(v) = \tau^0$ and $g_C(v) = -i \eta_{G,T} \tau^3$.

To summarize, there are $2 \times (2 + 3) = 10$ different algebraic PSGs with $\eta_{G,T} = 1$ and $g_{\tau}(s) = \tau^0$.

(ii) $g_{\tau}(s) = \tau^3$ and $g_{\tau}(s) = \tau^3$.

(A) $\eta_{G,T} = \eta_{G,T} = 1$. From (B3) and (B4) we have $g_C(w) = \tau^0$, $g_C(v) = \eta_{G,T} \tau^0$ by proper gauge fixing.

(B) $\eta_{G,T} = \eta_{G,T} = 1$. From (B3) and (B4) we have $g_C(w) = \tau^3$, $g_C(v) = \eta_{G,T} \tau^3$ by proper gauge fixing. Also from (B4) we can choose a proper gauge so that $g_C(v) = \tau^0$ and $g_C(v) = -i \eta_{G,T} \tau^3$.

To summarize, there are $2 \times (2 + 3) = 10$ different algebraic PSGs with $\eta_{G,T} = 1$ and $g_{\tau}(s) = \tau^3$.

(iii) For $s = u$.

\[
T^1_3 T^2_3 \hat{\sigma}[x,u,v] = \begin{cases} 
-1 & \text{if } x = u,
\end{cases}
\]

(A) $\eta_{G,T} = \eta_{G,T} = 1$. From (B3) and (B4) we have $g_C(w) = \tau^0$ and $g_C(v) = \eta_{G,T} \tau^0$ by proper gauge fixing. Also from (B4) we know that $g_C(v) = -i \eta_{G,T} \tau^3$ and $g_C(v) = \eta_{G,T} \tau^0$ through proper gauge fixing. Also we have $\eta_{G,T} = 1$.

(B) $\eta_{G,T} = 1$. In this case $\eta_{G,T} = 1$ and we can always choose a proper gauge so that $g_C(w) = \tau^0$, $g_C(v) = -i \eta_{G,T} \tau^3$.

(ii) $\eta_{G,T} = 1$. From (B3) and (B4) we have $g_C(w) = \tau^0$, $g_C(v) = \eta_{G,T} \tau^0$ by proper gauge fixing. Also from (B4) we know that $g_C(v) = -i \eta_{G,T} \tau^3$ and $g_C(v) = \eta_{G,T} \tau^0$ through proper gauge fixing. Also we have $\eta_{G,T} = 1$.

(B) $\eta_{G,T} = 1$. In this case $\eta_{G,T} = 1$ and we can always choose a proper gauge so that $g_C(v) = -i \eta_{G,T} \tau^3$, $g_C(w) = \eta_{G,T} \tau^0$.
Now let us consider several very simple examples. First, the on-site chemical potential terms $\Lambda(x,y,z) = \Lambda_x$ satisfy the following consistent conditions:

$$\tau^1 \Lambda_\alpha \tau^1 = -\Lambda_x, \quad g_\sigma(u)\Lambda_\alpha g_\sigma^\dagger(u) = \Lambda_u, \quad g_\sigma(v)\Lambda_\alpha g_\sigma^\dagger(v) = \Lambda_v, \quad g_\sigma(w)\Lambda_\alpha g_\sigma^\dagger(w) = \Lambda_w, \quad \text{for } \alpha = \text{nn}. \quad \text{(B12)}$$

In fact in all 20 $Z_2$ spins on a kagome lattice we have $\Lambda_x = \Lambda_u \equiv \Lambda_v$, with a proper gauge choice.

All the first-NN mean-field bonds can be generated from $u_\alpha \equiv [0,0,v]$. For a generic $Z_2$ spin liquid with PSG elements $G_T(x,y,z) = i \tau^1$ and (B1), (B5), and (B7), the bond $u_\alpha = [0,0,v]$ satisfies the following consistent conditions:

$$\tau^1 u_\alpha \tau^1 = -u_\alpha, \quad g_\sigma(u)g_C_C(u)g_\sigma^\dagger(u)g_C_C^\dagger(w)g_\sigma^\dagger(v) = u_\alpha. \quad \text{(B13)}$$

It follows immediately that for six $Z_2$ spin liquids, i.e., nos. 7–12 in Table II, all NN mean-field bonds must vanish since $u_\alpha = 0$ as required by (B13). Therefore it is unlikely that the $Z_2$ spin liquid realized in the kagome Hubbard model will be one of these six states. In the following we study the remaining 14 $Z_2$ spin liquids on the kagome lattice.

All second-NN mean-field bonds can be generated from $u_\beta \equiv [0,1,w]$, which satisfies the following symmetry conditions:

$$\tau^1 u_\beta \tau^1 = -u_\beta, \quad g_\sigma(u)g_C_C(u)u_\beta g_C_C^\dagger(u)g_\sigma^\dagger(w) = u_\beta. \quad \text{(B14)}$$

There are two kinds of third-NN mean-field bond. The first kind can all be generated by $u_\gamma \equiv [1,0,u]$, which satisfies

$$\tau^1 u_\gamma \tau^1 = -u_\gamma, \quad g_\sigma(u)g_C_C(u)g_\sigma^\dagger(u)g_C_C^\dagger(w)g_C_C^\dagger(v) = u_\gamma. \quad \text{(B15)}$$

The second kind can all be generated by $u_\gamma \equiv [1, -1, u]$, which satisfies

$$\tau^1 u_\gamma \tau^1 = -u_\gamma, \quad g_\sigma(u)u_\gamma g_\sigma^\dagger(u)g_C_C^\dagger(v)g_C_C^\dagger(w)g_\sigma^\dagger(u) = \eta_1 u_\gamma. \quad \text{(B16)}$$

APPENDIX C: $Z_2$ SPIN LIQUIDS IN THE NEIGHBORHOOD OF THE U(1) SL $[0, \pi]$ STATE

1. Mean-field Ansatz

Following the SU(2) Schwinger fermion formulation with $\psi_i \equiv (f_i, f_i^\dagger)^T$, we focus on those $Z_2$ spin liquids (SLs) in the neighborhood of the U(1) SL $[0, \pi]$ state with the following mean-field Ansätze:

$$\langle x,y,u|x,y,v \rangle = -\langle x,y,u|x,y,w \rangle = (1)^2 \chi \tau^3, \quad \text{(C1)}$$

$$\langle x+1,y,w|x,y,u \rangle = \langle x,y|v|x,y,v \rangle = \langle x,y,v|x,y,w \rangle = \langle x+1,y-1,w|x,y,v \rangle = \chi \tau^3,$$

where $\chi$ is a real hopping parameter. We define mean-field bonds $(x,y,z|x',y',z')$ in the following way:

$$H_{MF} = \sum_{i,j} \psi_i^\dagger(i,j) \psi_j + \text{H.c.}. \quad \text{(C2)}$$

For convenience of later calculation we implement the following gauge transformation:

$$\psi_{x,y,u} \rightarrow i \tau^3 \psi_{x,y,u} \quad \text{(C3)}$$

and the original mean-field Ansatz (C1) transforms to

$$\langle x,y,u|x,y,v \rangle = -(x,y,u|x,y,w) = i(-1)^x \chi \tau^0, \quad \text{(C4)}$$

$$\langle x+1,y,w|x,y,v \rangle = \langle x,y+1,v|x,y,u \rangle = -i \chi \tau^0, \quad \text{(C5)}$$

The projected symmetry group corresponding to the above mean-field Ansatz (C4) is

$$G_T(x,y,z) = G_T(x,y,u) = G_T(x,y,v) = G_T(x,y,w) = G_T(x,y,u) = G_T(x,y,v) = G_T(x,y,w) = \tau^3, \quad \text{(C5)}$$

so that the mean-field Ansätze satisfy (8).

2. Classification of $Z_2$ spin liquids around the U(1) SL $[0, \pi]$ state

Plugging (C5) into the algebraic consistency conditions (A4)–(A15), we obtain four algebraic solutions of $Z_2$ PSGs around the U(1) SL $[0, \pi]$ state. With choice of a proper gauge they all satisfy

$$g_T = i \tau_1, \quad \eta = \eta_2 = \eta_3 = -1, \quad \text{(C6)}$$

The four $Z_2$ PSGs near the U(1) SL $[0, \pi]$ state have the following features:

$$\eta_1 = \eta_2 = \eta_3 = 1, \quad \text{(C7)}$$

$$\eta_1 = \eta_2 = \eta_3 = 1; \quad \text{(No. 2)} \quad \text{Z}_2[0, \pi] \beta: \quad g_\sigma = \tau^0, \quad g_C = i \tau^3,$$
\[ \eta_\sigma = \eta_\sigma \tau = 1, \]  
\[ \eta_{\sigma C_\alpha} = \eta_{\sigma C_\alpha \tau} = -\eta_{\sigma C_\alpha} = 1; \quad \text{(No. 14) } Z_2[0,\pi] \]  
\[ \eta_\sigma = \eta_\sigma \tau = 1, \]  
\[ \eta_{\sigma C_\alpha} = \eta_{\sigma C_\alpha \tau} = -\eta_{\sigma C_\alpha} = 1; \quad \text{(No. 16) } Z_2[0,\pi] \]  
\[ \eta_\sigma = \eta_\sigma \tau = 1, \]  
\[ \eta_{\sigma C_\alpha} = \eta_{\sigma C_\alpha \tau} = -\eta_{\sigma C_\alpha} = 1. \]  
\[ \text{Of course they belong to the } 20 \text{ } Z_2 \text{ spin liquids summarized in Table II.} \]

3. Four possible \( Z_2 \) spin liquids around the U(1) SL [0,\pi] state: Mean-field Ansätze

a. Consistent conditions on mean-field bonds

Implementing the generic conditions mentioned earlier on several near-neighbor mean-field bonds with PSG (C6)–(C10), we obtain the following consistent conditions:

(0) For on-site chemical potential terms \( \Lambda_\alpha(x,y,z) = \tilde{\lambda}(x,y,z) \cdot \tau \), the translation operations \( G_{-\vec{r}} T_{1,2} \) in the PSG guarantee that \( \Lambda_\alpha(x,y,z) = \Lambda_\alpha(0,0,0) \equiv \Lambda_\alpha, s = u,v,w \). They satisfy

\[ g_\tau \Lambda_\alpha \hat{g}_\tau = -\Lambda_\alpha, \]
\[ g_\sigma \Lambda_u \hat{g}_\sigma = -\Lambda_u, \quad g_\sigma \Lambda_v \hat{g}_\sigma = -\Lambda_v, \quad g_\sigma \Lambda_w \hat{g}_\sigma = -\Lambda_w; \]
\[ g_{C_a} \Lambda_a \hat{g}_{C_a} = -\Lambda_a, \quad g_{C_a} \Lambda_v \hat{g}_{C_a} = -\Lambda_v, \quad g_{C_a} \Lambda_w \hat{g}_{C_a} = -\Lambda_w. \]  
\[ \text{(C11)} \]

(I) For the first-neighbor mean-field bond \( u_a \equiv [0,0,v]^\dagger \) (there is only one independent mean-field bond, meaning all other first-neighbor bonds can be generated from \([0,0,v]\) through symmetry operations),

\[ g_\tau u_a \hat{g}_\tau = u_a, \]  
\[ (g_{C_a}^2 \tau^3) u_a \hat{g}_{C_a} (g_{C_a}^2 \tau^3)^\dagger = -u_a. \]  
\[ \text{(C12)} \]

(II) For the second-neighbor mean-field bond \( u_b \equiv [0,1,w] \) we have

\[ g_\tau u_b \hat{g}_\tau = u_b, \]  
\[ (g_{C_a} \tau^3) u_a \hat{g}_{C_a} (g_{C_a} \tau^3)^\dagger = -u_b. \]  
\[ \text{(C13)} \]

(II) For second-neighbor mean-field bonds \( u_{c1} \equiv [1,0,u] \) and \( u_{c2} \equiv [-1,1,u] \) we have

\[ g_\tau u_{c1} \hat{g}_\tau = -u_{c1}, \]  
\[ (g_{C_a} \tau^3) u_{c1} \hat{g}_{C_a} (g_{C_a} \tau^3)^\dagger = u_{c1}^\dagger, \]  
\[ \text{(C14)} \]

\[ g_\tau u_{c2} \hat{g}_\tau = -u_{c2}, \]  
\[ g_{C_a} u_{c2} \hat{g}_{C_a} = u_{c2}, \]  
\[ (g_{C_a} \tau^3) u_{c2} \hat{g}_{C_a} (g_{C_a} \tau^3)^\dagger = -u_{c2}^\dagger. \]  
\[ \text{(C15)} \]

b. Mean-field Ansätze of the four \( Z_2 \) spin liquids near the U(1) SL [0,\pi] state

For the \( Z_2[0,\pi] \) state with \( g_\sigma = g_{C_a} = \tau^0 \) the mean-field Ansätze are (up to third-neighbor mean-field bonds)

\[ u_a = i a_0 \tau^0 + a_1 \tau^1, \quad u_b = i b_0 \tau^0, \]
\[ u_{c1} = c_1 \tau^3, \quad u_{c2} = c_2 \tau^2. \]  
\[ \text{(C16)} \]
\[ \Lambda_\alpha = \lambda_3 \tau^3, \quad s = u,v,w. \]

Since we are considering a phase perturbed from the U(1) SL [0,\pi] state, we shall always assume \( a_0 \neq 0 \) (first-neighbor hopping terms) in the following discussion. A \( Z_2[0,\pi] \) spin liquid can be realized by first-neighbor mean-field singlet pairing terms with \( a_1 \neq 0 \).

For the \( Z_2[0,\pi] \) state with \( g_\sigma = \tau^0, g_{C_a} = i \tau^3 \) the mean-field Ansätze are (up to third-neighbor mean-field bonds)

\[ u_a = i a_0 \tau^0 + a_1 \tau^1, \quad u_b = i b_0 \tau^0 + b_1 \tau^1, \]
\[ u_{c1} = c_1 \tau^3, \quad u_{c2} = 0. \]  
\[ \Lambda_\alpha = \lambda_3 \tau^3, \quad s = u,v,w. \]  
\[ \text{(C17)} \]

A \( Z_2[0,\pi] \) spin liquid can be realized by second-neighbor pairing terms with \( a_0 b_1 - a_1 b_0 \neq 0 \).

For the \( Z_2[0,\pi] \) state with \( g_\sigma = \tau^0, g_{C_a} = i \tau^3 \) the mean-field Ansätze are (up to third-neighbor mean-field bonds)

\[ u_a = i a_0 \tau^0, \quad u_b = i b_0 \tau^0 + b_1 \tau^1, \]
\[ u_{c1} = c_1 \tau^3, \quad u_{c2} = 0, \]
\[ \Lambda_\alpha = \lambda_3 \tau^3, \quad s = u,v,w. \]  
\[ \text{(C18)} \]

A \( Z_2[0,\pi] \) spin liquid can be realized by second-neighbor pairing terms with \( b_1 \neq 0 \).

For the \( Z_2[0,\pi] \) state with \( g_\sigma = \tau^0, g_{C_a} = i \tau^3 \) the mean-field Ansätze are (up to third-neighbor mean-field bonds)

\[ u_a = i a_0 \tau^0, \quad u_b = i b_0 \tau^0, \]
\[ u_{c1} = c_1 \tau^3, \quad u_{c2} = 0, \]
\[ \Lambda_\alpha = \lambda_3 \tau^3, \quad s = u,v,w. \]  
\[ \text{(C19)} \]

A \( Z_2[0,\pi] \) spin liquid can be realized by third-neighbor pairing terms with \( c_2 \neq 0 \).

4. Low-energy effective theory

The reciprocal unit vectors (corresponding to unit vectors \( \vec{a}_{1,2} \)) on a kagome lattice are \( \vec{b}_1 = \frac{1}{\sqrt{3}}(\hat{x} - \sqrt{3} \hat{y}) \) and \( \vec{b}_2 = \frac{1}{\sqrt{3}}(\hat{x} + \hat{y}) \), satisfying \( \vec{a}_i \cdot \vec{b}_j = \delta_{i,j} \). In the mean-field Ansatz (C4) of U(1) SL [0,\pi] the unit cell is doubled; the translation unit vectors are \( A_1 = 2 \vec{a}_1 \) and \( A_2 = 2 \vec{a}_2 \). Accordingly, the first BZ for such a mean-field Ansatz is only half of the original first BZ with the new reciprocal unit vectors being \( \vec{B}_1 = \vec{b}_1/2 \) and \( \vec{B}_2 = \vec{b}_2 \). Denoting the momentum as \( k \equiv (k_x, k_y)/a = k_1 B_1 + k_2 B_2 \) with \( |k_{1,2}| \leq \pi \), we have

\[ k_1 = 2 k_x, \quad k_2 = (k_x + \sqrt{3} k_y)/2. \]  
\[ \text{(C20)} \]

The two Dirac cones in the spectra of the U(1) SL [0,\pi] state (C4) are located at \( \pm Q \) with

\[ Q = \left(0, \frac{\pi}{\sqrt{3}}\right) = \frac{\pi}{2} \vec{B}_2. \]  
\[ \text{(C21)} \]
with the proper chemical potential $\Lambda(i) = (i| i) = \chi (\sqrt{3} - 1)^3$ added to the mean-field Ansatz (C4).

For convenience we choose the following basis for the Dirac-like Hamiltonian obtained from expansion around $\pm \mathbf{Q}$:

\[
\begin{align*}
\phi_{+, \uparrow, A} &= \frac{1}{\sqrt{6}} e^{-i \frac{\pi}{2}} \left( e^{-i \frac{\pi}{2}} 0, e^{i \frac{\pi}{2}} 0, 0, 0, e^{-i \frac{\pi}{2}} 0, 0, 0, 0 \right)^T, \\
\phi_{+, \uparrow, B} &= \frac{1}{\sqrt{6}} e^{-i \frac{\pi}{2}} \left( 1, 0, e^{-i \frac{\pi}{2}} 0, 0, 0, e^{-i \frac{\pi}{2}} 0, 0, 0 \right)^T, \\
\phi_{-, \uparrow, b} &= R_T \left( k_1 = 0, k_2 = -\frac{\pi}{2} \right) \phi_{+, \uparrow, b}, \\
\phi_{\pm, \downarrow, b} &= R_T \phi_{\pm, \uparrow, b},
\end{align*}
\]

where $\pm$ are the valley indices for two Dirac cones at $\pm \mathbf{Q}$ with Pauli matrices $\mu$ and $b = A, B$ are band indices with Pauli matrices $v$. The spin indices $\sigma = \uparrow, \downarrow$ are as usual, with Pauli matrices $\sigma$. The corresponding creation operators for these modes are $\Psi^\dagger_{+, \uparrow, \pm} = \Psi^\dagger_{0, \phi_{+, \uparrow, \pm}}$ in the order of $(0,0,u),(0,0,v),(0,0,w),(1,0,u),(1,0,v),(1,0,w)$ for the six sites per doubled new unit cell.

Here

\[
R_T \equiv I_{2 \times 2} \otimes \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \otimes g_T \\ 0 & 0 & 1 \end{bmatrix},
\]

and

\[
R_{T_i}(k) = \begin{bmatrix} 0 & -e^{-ik_1} \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \otimes g_{T_i}
\]

are transformation matrices on 12-component eigenvectors for time reversal $T$ and translation $T_1, T_2$ operations. By definition of the PSG the eigenvectors $\phi_k$ with momentum $k = k_1 \hat{B}_1 + k_2 \hat{B}_2 \equiv (k_1, k_2)$ and energy $E$ have the following symmetric properties:

\[
T: \phi_{k_1, k_2}(E) = R_T \phi_{k_1, k_2}(-E), \\
T_1: \phi_{k_1, k_2}(E) = R_T(k_1, k_2) \phi_{k_1, k_2 + \pi}(E), \\
T_2: \phi_{k_1, k_2}(E) = R_T(k_1, k_2) \phi_{k_1, k_2}(E).
\]

$T$ and $T$ are the bases after and before the symmetry operations.

In such a set of bases the Dirac Hamiltonian obtained by expanding the $U(1)$ SU $[0, \pi]$ mean-field Ansatz (C4) around the two cones at $\pm \mathbf{Q}$ is

\[
H_{\text{Dirac}} = \sum_{k \in \text{basis}} \frac{\chi}{\sqrt{2}} \Psi^\dagger_k \mathbf{\mu} \sigma^0 (-k_x \sigma_1 + k_y \sigma_2) \Psi_k.
\]

$k$ should be understood as small momenta measured from $\pm \mathbf{Q}$. Possible mass terms are $\mu^{0,1,2,3} \sigma_{1,2,3} \sigma_{0,1,2,3} v_{0,1,2,3}$. However, not all of them are allowed by symmetry. Here we enumerate all symmetry operations and associated operator transformations:

- spin rotation along the $\hat{z}$ axis by angle $\theta$:
  \[
  \Psi_k^\dagger \rightarrow \Psi_k^\dagger e^{i\theta};
  \]

- time reversal $T$:
  \[
  \Psi_k^\dagger \rightarrow \Psi_{-k}^\dagger; \quad \Psi_k \rightarrow -\Psi_{-k};
  \]

- translation $T_1$:
  \[
  \Psi_k^\dagger \rightarrow \Psi_{k + \pi}^\dagger; \quad \Psi_k \rightarrow -\Psi_{k + \pi};
  \]

- translation $T_2$:
  \[
  \Psi_k^\dagger \rightarrow \Psi_{k - \pi}^\dagger; \quad \Psi_k \rightarrow -\Psi_{k - \pi};
  \]

Considering the above conditions, the only symmetry-allowed mass terms are $\sum_k \psi^\dagger_k m_{1,2} \psi_k$ with $m_1 = \mu^3 \sigma_1 v_0$ and $m_2 = \mu^3 \sigma_3 v_3$.

The transformation rules for mirror reflection $\sigma$ and $\pi/3$ rotation $\phi$ depend on the choice of $g_{\sigma}, g_{C_6}$ in the PSG. In general we have

\[
\sigma: \Psi_k^\dagger \rightarrow \Psi_{-k}^\dagger M_\sigma(g_{\sigma}), \\
C_6: \Psi_k^\dagger \rightarrow \Psi_{C_6 k}^\dagger M_{C_6}(g_{C_6}).
\]

Using the basis (C22) the $8 \times 8$ matrices $M_{\sigma, C_6}$ can be expressed in terms of Pauli matrices $\mu \otimes \sigma \otimes v$. For the four $Z_2$ spin liquids we have

\[
M_{\sigma}(g_{\sigma} = \tau^0) = \mu^3 \otimes \sigma^0 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
M_{\sigma}(g_{\sigma} = i \tau^3) = \mu^3 \otimes \sigma^3 \otimes \begin{pmatrix} e^{i \pi} & 0 \\ 0 & e^{-i \pi} \end{pmatrix}, \\
M_{C_6}(g_{C_6} = \tau^0) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \otimes \sigma^0 \otimes \begin{pmatrix} e^{i \pi} & 0 \\ 0 & e^{-i \pi} \end{pmatrix}, \\
M_{C_6}(g_{C_6} = i \tau^3) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \otimes \sigma^0 \otimes \begin{pmatrix} e^{i \pi} & 0 \\ 0 & e^{-i \pi} \end{pmatrix}.
\]

It turns out that in the $Z_2[0, \pi] \beta$ state, only the first mass term $m_1 = \mu^3 \sigma_1 v_0$ is invariant under $\sigma$ and $C_6$ operations. In the other three states neither of the mass terms $m_{1,2}$ is symmetry allowed. As a result we only have one gapped $Z_2$ spin liquid, i.e., the $Z_2[0, \pi] \beta$ state in the neighborhood of the U(1) Dirac SU $[0, \pi]$ state.

Let us consider mean-field bonds up to second neighbor for the Ansatz $Z_{2}[0, \pi]$ $\beta$. Perturbations to the two Dirac cones of U(1) SU $[0, \pi]$ with $\lambda_3 = (\sqrt{3} - 1)a_0$ in general have the following form:

\[
\delta H_0 = \lambda_3 (\sqrt{3} - 1) a_0 + (\sqrt{3} + 1) b_1 \mu^0 \sigma_1 v_0 + [(\sqrt{3} + 1) b_1 - \lambda_2 (\sqrt{3} - 1) a_1] \mu^0 \sigma_1 v_0.
\]
Z$_2$ SPIN LIQUIDS IN THE S = $\frac{1}{2}$ \ldots

APPENDIX D: Z$_2$ SPIN LIQUIDS IN THE NEIGHBORHOOD OF THE UNIFORM RVB STATE

The mean-field Ansatz of the uniform RVB state is simple:

$$H_{MF} = \chi \sum_{(ij),\sigma} f_{i\sigma}^\dagger f_{j\sigma},$$  \hspace{1cm} (D1)

where $\chi$ is a real parameter and $(ij)$ represents sites $i, j$ that are nearest neighbors of each other. It is straightforward to show that the PSG elements of such a mean-field Ansatz are

$$G_U(x,y,z) = g_U, \quad U = T_{1,2}, T, \sigma, C_6$$ \hspace{1cm} (D2)

and the SU(2) matrices $g_U$ satisfy

$$g_T T^{\dagger} g_T^\dagger = -T^3, \quad g_U T^{\dagger} g_U^\dagger = T^3, \quad U = T_{1,2}, T, \sigma, C_6.$$ \hspace{1cm} (D3)

It turns out that there are only four gauge-inequivalent $Z_2$ PSGs as solutions to (A4)–(A16) with the form (D2). In other words, there are only four different $Z_2$ in the neighborhood of a uniform RVB state. With choice of a proper gauge they all satisfy $g_T = i T^1$, $g_{T_{1,2}} = \tau_1^0$, and $\eta_{T_1} = \eta_{T_2} = \eta_6 = \eta_{C_1, T_2} = 1, \eta_T = -1$. These four states are characterized by

$$\begin{align*}
(\text{No. 1}) & \quad Z_2[0,0]A: \ g_\sigma = g_{C_6} = \tau_1^0, \\
& \quad \eta_\sigma T = \eta_{C_1, T} = \eta_6 = \eta_{C_6} = 1. \\
(\text{No. 5}) & \quad Z_2[0,0]B: \ g_\sigma = \tau_1^0, \ g_{C_6} = i T^3, \\
& \quad \eta_\sigma T = \eta_6 = 1, \ \eta_{C_1, T} = \eta_{C_6} = -1. \\
(\text{No. 15}) & \quad Z_2[0,0]C: \ g_\sigma = i T^3, \ g_{C_6} = \tau_1^0, \\
& \quad \eta_\sigma T = \eta_6 = -1, \ \eta_{C_1, T} = \eta_{C_6} = \eta_{C_6} = 1. \\
(\text{No. 13}) & \quad Z_2[0,0]D: \ g_\sigma = g_{C_6} = i T^3, \\
& \quad \eta_\sigma T = \eta_{C_1, T} = \eta_6 = \eta_{C_6} = -1. \\
\end{align*}$$

It turns out that these four $Z_2$ SLs around the uniform RVB state are all gapped as shown in Table II.