Z\textsubscript{2} spin liquids in the S=1/2 Heisenberg model on the kagome lattice: A projective symmetry-group study of Schwinger fermion mean-field states

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Z₂ spin liquids in the $S = \frac{1}{2}$ Heisenberg model on the kagome lattice: A projective symmetry-group study of Schwinger fermion mean-field states

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Due to strong geometric frustration and quantum fluctuation, the $S = 1/2$ quantum Heisenberg antiferromagnet on the kagome lattice has long been considered as an ideal platform to realize a spin liquid (SL), a phase exhibiting fractionalized excitations without any symmetry breaking. A recent numerical study (Yan et al., e-print arXiv:1011.6114) of the Heisenberg $S = 1/2$, kagome lattice model (HKLM) shows, in contrast to earlier results, that the ground state is a singlet-gapped SL with signatures of $Z₂$ topological order. Motivated by this numerical discovery, we use the projective symmetry group to classify all 20 possible Schwinger fermion mean-field states of $Z₂$ SLs on the kagome lattice. Among them we found only one gapped $Z₂$ SL (which we call the $Z₂[0, \pi |β]$ state) in the neighborhood of the $U(1)$ Dirac SL state. Since its parent state, i.e., the $U(1)$ Dirac SL, was found [Ran et al., Phys. Rev. Lett. 98, 117205 (2007)] to be the lowest among many other candidate $U(1)$ SLs, including the uniform resonating-valence-bond states, we propose this $Z₂[0, \pi |β]$ state to be the numerically discovered SL ground state of the HKLM.

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I. INTRODUCTION

At zero temperature, all degrees of freedom tend to freeze, and usually a variety of different orders, such as superconductivity and magnetism, will develop in different materials. However, in a quantum system with a large zero-point energy, the ground state can be gapped and usually a variety of different orders, such as superconductivity and magnetism, will develop in different materials. The quest for quantum SLs in frustrated magnets (for a recent review, see Ref. 7) has been pursued for decades. Among them the Heisenberg $S = 1/2$ kagome lattice model (HKLM)

$$H_{HKLM} = J \sum_{\langle i,j \rangle} S_i \cdot S_j$$

has long been thought a promising candidate. Here $\langle i,j \rangle$ denotes $i,j$ being a nearest-neighbor pair. Experimental evidence⁸⁻¹¹ of SLs has been observed in ZnCu₃(OH)$_6$Cl₂ (called herbertsmithite), a spin-1/2 antiferromagnet on the kagome lattice. Theoretically, lacking an exact solution of the two-dimensional (2D) quantum Hamiltonian (1) in the thermodynamic limit, in previous studies either a honeycomb valence-bond crystal¹²⁻¹⁶ (HVBC) with an enlarged ($6 \times 6$)-site unit cell, or a gapless SL (Ref. 17) were proposed as the ground state of the HKLM. However, recently an extensive density-matrix-renormalization-group (DMRG) study³⁸ on the HKLM revealed the ground state of the HKLM as a gapped SL, which substantially lowers the energy compared to the HVBC. In addition, the authors also observed numerical signatures of $Z₂$ topological order in the SL state.

Motivated by this important numerical discovery, we try to discover the nature of this gapped $Z₂$ SL. Different $Z₂$ SLs on the kagome lattice have been previously studied using the Schwinger boson representation¹⁹,²⁰ Here we propose candidate states for symmetric $Z₂$ SLs on the kagome lattice by the Schwinger fermion mean-field approach.²¹⁻²⁷ Following is the summary of our results. First we use the projective symmetry group⁵ (PSG) to classify all 20 possible Schwinger fermion mean-field Ansätze of $Z₂$ SLs which preserve all the symmetry of the HKLM, as shown in Table I. We analyze these 20 states and rule out some obviously unfavorable ones: e.g., gapless states, and those states whose first-nearest-neighbor (NN) mean-field amplitudes must vanish due to symmetry. Then we focus on those $Z₂$ SLs in the neighborhood of the $U(1)$ Dirac SL.²⁸ In Ref. 28 it is shown that the $U(1)$ Dirac SL has a significantly lower energy compared with other candidate $U(1)$ SL states, such as the uniform resonating-valence-bond (RVB) state (or the $U(1)$ SL-[0,0] state in the notation of Ref. 28). We find that there is only one gapped $Z₂$ SL, which we label as $Z₂[0, \pi |β]$, in the neighborhood of (or continuously connected to) the $U(1)$ Dirac SL. Therefore we propose this $Z₂[0, \pi |β]$ state as a promising candidate for the ground state of the HKLM. The mean-field ansatz of $Z₂[0, \pi |β]$ state is shown in Fig. 1(b). Our work also provides guidelines for choosing variational states in future numeric studies of the SL ground state on the kagome lattice.
TABLE I. Mean-field Ansätze of 20 possible $Z_2$ SLs on a kagome lattice. In our notation of mean-field amplitudes $(x,y,z)(0,0,u) = [x,y,z]$, this table summarizes all symmetry-allowed mean-field bonds up to third nearest neighbor, i.e., first-nearest-neighbor bond $u = [0,0,u]$, second-nearest-neighbor bond $u = [0,1,u]$, and third-nearest-neighbor bonds $u = [1,0,u]$ and $u = [1,1,u]$ as shown in Fig. 1(a). $A_i$ denote the on-site chemical potential terms which enforce the constraint (11). $t^0$ is a $2 \times 2$ identity matrix while $t^{1,2}$ are three Pauli matrices. $\tilde{t}^{0,3}$ denote hopping while $t^{1,2}$ denote pairing terms. 0 means that the corresponding mean-field amplitudes must vanish due to symmetry. The shortest mean-field bonds necessary to realize a $Z_2$ SL are shown in italic type. In other words, the mean-field amplitudes in italics break the U(1) gauge redundancy down to $Z_2$ through the Higgs mechanism. So in states nos. 3, 19, and 7–12, the $Z_2$ SL can be realized with up to third-NN mean-field bonds. Note that state no. 15 needs only a third-nearest-neighbor bond $u_\tau$ to realize a $Z_2$ SL ($\tilde{u}_\tau$ is not necessary), while state no. 20 needs only $\tilde{u}_\tau$ to realize a $Z_2$ SL ($u_\tau$ is not necessary). Notice that when $n_{\tau_2} = -1$ the mean-field Ansatz (instead of the SL itself) will break translational symmetry and double the unit cell. There are six $Z_2$ SLs, i.e., nos. 7–12 that do not allow any first-NN mean-field bonds. Among the other 14 $Z_2$ SLs with nonvanishing first-NN mean-field bonds, only five $Z_2$ SL states, i.e., nos. 1, 2, 5, 13, and 15, have gapped spinon spectra. No. 2 (in bold type) or the $Z_2[0,\pi/\beta]$ state in the neighborhood of the U(1) Dirac SL is the most promising candidate of the $Z_2$ SLs for the HKLM ground state.

| No. $n_{\tau_2}$ $A_i$ $u_\tau$ $u_\gamma$ $\tilde{u}_\tau$ $\tilde{u}_\gamma$ Label Gapped? |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 1   | +1  | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | Z$[0,0]A$ Yes |
| 2   | −1  | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | Z$[0,0]A$ Yes |
| 3   | +1  | 0        | 0        | 0        | 0        | Z$[0,\pi]A$ No |
| 4   | −1  | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | Z$[0,0]A$ No |
| 5   | +1  | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | Z$[0,0]B$ Yes |
| 6   | −1  | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | Z$[0,\pi]A$ No |
| 7   | +1  | 0        | 0        | 0        | 0        | Z$[0,\pi]A$ No |
| 8   | −1  | 0        | 0        | 0        | 0        | Z$[0,\pi]A$ No |
| 9   | +1  | 0        | 0        | 0        | 0        | Z$[0,\pi]A$ No |
| 10  | −1  | 0        | 0        | 0        | 0        | Z$[0,\pi]A$ No |
| 11  | +1  | 0        | 0        | 0        | 0        | Z$[0,\pi]A$ No |
| 12  | −1  | 0        | 0        | 0        | 0        | Z$[0,\pi]A$ No |
| 13  | +1  | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | Z$[0,0]D$ Yes |
| 14  | −1  | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | Z$[0,0]B$ No |
| 15  | +1  | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | Z$[0,0]C$ Yes |
| 16  | −1  | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | Z$[0,0]D$ No |
| 17  | +1  | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | Z$[0,\pi]B$ No |
| 18  | −1  | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | Z$[0,\pi]B$ No |
| 19  | +1  | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | Z$[0,\pi]C$ No |
| 20  | −1  | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | $t^{0,3}$ | Z$[0,\pi]C$ No |

II. SCHWINGER FERMION CONSTRUCTION OF SPIN LIQUIDS AND PROJECTIVE SYMMETRY GROUP

A. Schwinger fermion construction of symmetric spin liquids

In the Schwinger fermion construction,21–26 we represent a spin-1/2 operator at site $i$ by fermionic spinors $\{ f_{ia}, \alpha = \uparrow, \downarrow \}$:

$$\tilde{S}_i = \frac{1}{2} f_{ia}^{\dagger} \hat{\sigma}_{\alpha\beta} f_{i\beta}. \quad (2)$$

![FIG. 1. (Color online) (a) Kagome lattice and the elements of its symmetry group. $\hat{A}_{i,2}$ are the translation unit vectors, $C_6$ denotes $\pi/3$ rotation around the honeycomb center, and $\sigma$ represents mirror reflection along the dashed blue line. Here $u_\alpha$ and $u_\gamma$ denote first- and second-nearest-neighbor (NN) mean-field bonds while $u_\tau$ and $\tilde{u}_\tau$ represent two kinds of independent third-NN mean-field bonds. (b) Mean-field ansatz of the $Z_2[0,\pi/\beta]$ state up to second nearest neighbor. Colors in general denote the sign structure of the mean-field bonds. Dashed lines denote first-NN real hopping terms $x_1 \sum_{i,j,a} \langle v_{ij} f_{ia}^{\dagger} f_{ja} + H.c. \rangle$: red ones have $v_{ij} = 1$ and bold black ones have $v_{ij} = -1$. Solid lines stand for second-NN hopping $x_2 \sum_{i,j,a,b} \langle v_{ij} f_{ia}^{\dagger} f_{ja} + H.c. \rangle$ and singlet pairing $\sum_{i,j,a,b} \epsilon_{a,b} \langle v_{ij} f_{ia}^{\dagger} f_{ja} + H.c.c. \rangle$: again red ones have $v_{ij} = 1$ and bold blue ones have $v_{ij} = -1$. Here $\chi_1$ and $\Delta_1$ are real parameters after choice of a proper gauge.](Image 313x647 to 435x738)
operators invariant and does not change the physical Hilbert space. One can try to solve Eq. (4) by a mean-field (or saddle-point) approximation. At mean-field level, $U_{ij}$ and $a_0^i$ are treated as complex numbers, and $a_0^i$ must be chosen such that the constraints (3) are satisfied at the mean-field level: $\langle \psi_i^\dagger \tau^l \psi_i \rangle = 0$. The mean-field Ansatz can be written as

$$H_{MF} = -\sum_{\langle ij \rangle} \bar{\psi}_i \hat{T}_{ij} \psi_j + \sum_i \bar{\psi}_i a_0^i \tau^l \psi_i,$$  \hspace{1cm} (6)

where we defined $\langle ij \rangle \equiv \frac{1}{2} \delta_{ij} U_{ij}$. Under a local SU(2) gauge transformation $\langle ij \rangle \to W_i \langle ij \rangle W_j$, but the physical spin state described by the mean-field Ansatz $\langle \hat{T}_{ij} \rangle$ remains the same. By construction the mean-field Ansatz does not break spin rotation symmetry, and the mean-field solutions describe SL states if lattice symmetry is preserved. Different $\{ \langle ij \rangle \}$ Ansätze may be in different SL phases. The mathematical language to classify different SL phases is the projective symmetry group (PSG).  

B. Projective symmetry group classification of topological orders in spin liquids

The PSG characterizes the topological order in Schwinger fermion representation: SLs described by different PSGs are different phases. It is defined as the collection of all combinations of symmetry group and SU(2) gauge transformations that leave mean-field Ansätze $\{ \langle ij \rangle \}$ invariant (as $a_0^i$ are determined self-consistently by $\{ \langle ij \rangle \}$, these transformations also leave $a_0^i$ invariant). The invariance of a mean-field Ansatz $\{ \langle ij \rangle \}$ under an element of PSG $G_U U$ can be written as

$$G_U U \{ \langle ij \rangle \} = \{ \langle ij \rangle \},$$

$$U \{ \langle ij \rangle \} = \{ \langle ij \rangle \},$$

$$G_U \{ \langle ij \rangle \} = \{ \langle ij \rangle \} = \{ \langle ij \rangle \}. \hspace{1cm} (7)$$

$$G_U \{ \langle ij \rangle \} = \{ \langle ij \rangle \}, \hspace{1cm} G_U \{ \langle ij \rangle \} \in SU(2).$$

Here $U \in SG$ is an element of the symmetry group (SG) of the corresponding SL. In our case of symmetric SLs on the kagome lattice, we use $(x,y,s)$ to label a site with sublattice index $s = u,v,w$ and $x,y \in Z$. The Bravis unit vectors are chosen as $\hat{a}_1 = a\hat{x}$ and $\hat{a}_2 = \frac{\sqrt{3}}{2}(\hat{x} + \sqrt{3}\hat{y})$ as shown in Fig. 1(a). The symmetry group is generated by the time-reversal operation $T$, lattice translations $T_{1,2}$ along the $\hat{a}_{1,2}$ directions, a $\pi/3$ rotation $C_6$ around the honeycomb plaquette center, and the mirror reflection $\sigma$ (for details, see Appendix A). For example, if $U = T_1$ is the translation along the $\hat{a}_1$ direction in Fig. 1(a), $T_1(x,y,s) = (x+1, y, s)$. $G_U$ is the gauge transformation associated with $U$ such that $G_U \{ \langle ij \rangle \}$ is invariant. Notice that the condition (7) allows us to generate all symmetry-related mean-field bonds from one by the following relation:

$$\{ \langle ij \rangle \} = G_U \{ \langle U^{-1}(i)U^{-1}(j) \rangle \} G_U^\dagger \{ \langle ij \rangle \}. \hspace{1cm} (8)$$

There is an important subgroup of the PSG, the invariant gauge group (IGG), which is composed of all the pure gauge transformations in the PSG: $IGG \equiv \{ \{ W_i \} \} W_i \{ \langle ij \rangle \} W_j \equiv \{ \langle ij \rangle \}$, $W_i \in SU(2)$. In other words, $W_i = G_e(i)$ is the pure gauge transformation associated with the identity element $e \in SG$ of the symmetry group. One can always choose a gauge in which the elements in IGG are site independent. In this gauge, the IGG can be the global $Z_2$ transformations: $\{ G_e(i) \} \equiv G_e = \pm e^{i\pi}$, the global U(1) transformations: $\{ G_e(i) \} \equiv e^{i\pi^l \tau^l}, \theta \in [0,2\pi]$, or the global SU(2) transformations: $\{ G_e(i) \} \equiv e^{i \theta_k \hat{n}^k}, \theta \in (0,2\pi], \hat{n} \in SU(2)$, and we term them the $Z_2$, U(1), and SU(2) states, respectively.

The importance of the IGG is that it controls the low-energy gauge fluctuations of the corresponding SL states. Beyond the mean-field level, fluctuations of $\langle \langle ij \rangle \rangle$ and $a_0^i$ need to be considered and the mean-field state may or may not be stable. The low-energy effective theory is described by a fermionic spinon band structure coupled with a dynamical gauge field of the IGG. For example, a $Z_2$ state with gapped spinon dispersion can be a stable phase because the low-energy $Z_2$ dynamical gauge field can be in the deconfined phase.  

Notice that the condition $\{ G_e(i) \} \equiv G_e = \pm e^{i\pi}$ for a $Z_2$ SL leads to a series of consistent conditions for the gauge transformations $\{ G_e(i) \} U \in SG$, as shown in Appendix A. Gauge-invariant solutions of these conditions (A4)–(A11) lead to different $Z_2$ SLs. Soon we will show that there are 20 $Z_2$ SLs on the kagome lattice that can be realized by a Schwinger-fermion mean-field Ansatz $\{ \langle ij \rangle \}$.

III. $Z_2$ SPIN LIQUIDS ON THE KAGOME LATTICE AND $Z_2 \{0, 0, 0, \beta \}$ STATE

Following previous discussions, we use the PSG to classify all possible 20 $Z_2$ SL states on the kagome lattice in this section. As will be shown later, among them there is one gapped $Z_2$ SL labeled the $Z_2 \{0, 0, 0, \beta \}$ state in the neighborhood of the U(1) Dirac SL. This $Z_2 \{0, 0, 0, \beta \}$ state is the most promising candidate for the SL ground state of HKLM.

A. PSG classification of $Z_2$ spin liquids on a kagome lattice

Applying the condition $G_e(i) \equiv G_e = \pm e^{i\pi}$ to the kagome lattice with the symmetry group described in Appendix A, we obtain a series of consistent conditions for the gauge transformation $G_e(i)$, i.e., the conditions (A4)–(A11). Solving these conditions, we classify all the 20 different Schwinger fermion mean-field states of $Z_2$ SLs on the kagome lattice, as summarized in Table I. These 20 mean-field states correspond to different $Z_2$ SL phases, which cannot be continuously tuned into each other without a phase transition.

As discussed in Appendix B, 2, from PSG elements $G_e(i)$ one can obtain all other symmetry-related mean-field bonds from one using the symmetry condition (8). Therefore we use $u_\beta = (0,0,0,\beta)$ to represent first-nearest-neighbor mean-field bonds. $u_\beta = (0,1,0,0,0)$ is the representative of second-NN mean-field bonds. There are two kinds of symmetry-unrelated third-NN mean-field bonds, represented by $u_\gamma = (1,0,0,0,0,\gamma)$ and $\tilde{u}_\gamma = (1,1,0,0,0,0,\gamma)$. The symmetry conditions for these mean-field bonds are summarized in (B13)–(B16). In addition, the on-site chemical potential terms $\Lambda(i)$ [which guarantee the physical constraint (3) on the mean-field level] also satisfy symmetry conditions (B12). We can show that $\Lambda(x,y,s) \equiv \Lambda_f$ for these 20 $Z_2$ SL states. The symmetry-allowed mean-field amplitudes and bonds are also summarized in Table I.
TABLE II. A summary of all 20 gauge-inequivalent PSGs with $G_T(x,y,z) = i\pi$ on the kagome lattice. Notice that there is a free $Z_2$ integer $\eta_{ij} = \pm 1$ in other PSG elements (B1), (B5), and (B7). They correspond to 20 different $Z_2$ spin liquids on the kagome lattice.

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<td>19</td>
<td>+1</td>
<td>$\tau^0$</td>
<td>$\tau^0$</td>
<td>$\tau^0$</td>
<td>$\tau^0$</td>
<td>$Z_2[0,\pi \beta]$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>-1</td>
<td>$\tau^0$</td>
<td>$\tau^0$</td>
<td>$\tau^0$</td>
<td>$\tau^0$</td>
<td>$Z_2[0,\pi \beta]$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From Table I we can see there are six states, i.e., nos. 7–12, that do not allow nonzero first-NN mean-field amplitudes due to symmetry. Moreover, they cannot realize $Z_2$ SLs with up to third-NN mean-field amplitudes. Therefore they are unlikely to be the HKLM ground state. Ruling out these six $Z_2$ SLs, we can see that the other 14 $Z_2$ SL states fall into four classes. To be specific, they are continuously connected to different parent U(1) gapless SL states on the kagome lattice. These parent U(1) SL states in general have the following mean-field Ansätze:

$$H_{U(1)SL} = \chi_1 \sum_{ij} v_{ij} (f_{ij}^\dagger f_{ij} + H.c.),$$

(9)

where $v_{ij} = \pm 1$ characterizes the sign structure of hopping terms with $\chi_1 \in \mathbb{R}$. Different parent U(1) SL states feature the flux of $f$ spinon hopping phases around the basic plaquette: honeycombs and triangles on the kagome lattice.

The simplest example is the so-called uniform RVB state with $v_{ij} = +1$ for all first-NN mean-field bonds. The hopping phase around any plaquette is $1 = \exp[i0]$, and the corresponding flux is $[0,0]$ for [triangle, honeycomb] motifs. The four possible $Z_2$ spin liquids in the neighborhood of uniform RVB states [i.e., the U(1) SL [0,0] state in Ref. 28] are classified in Appendix D. They are nos. 1, 5, 15, and 13 in Tables I and II. We label them as $Z_2[0,0]A$, $Z_2[0,0]B$, $Z_2[0,0]C$, and $Z_2[0,0]D$ states. They all have gapped spectra of spinons.

The Ansätze of two other parent U(1) SLs are shown in Fig. 2. They both have $\pi$ flux piercing through a triangle basic plaquette. Following the above notations of the hopping phase in [triangle, honeycomb] motifs, with either $\pi$ or 0 flux through the honeycomb plaquette, they are called the U(1) SL [\[\pi, \pi\]] state and the U(1) SL [\[0,0\]] state. There are three $Z_2$ SLs in the neighborhood of both U(1) SL states, i.e., nos. 3, 17, and 19 around the U(1) SL [\[\pi, \pi\]] state and nos. 4, 18, and 20 around the U(1) SL [\[0,0\]] state. All these six $Z_2$ SLs have gapless spinon spectra, inherited from the two parent gapless U(1) SLs. To be precise, the spinon band structures of these six $Z_2$ SL states feature a doubly degenerate flatband and a Dirac cone at the Brillouin-zone center. This is in contrast to the numerically observed gap in the two-spinon spectrum;28 thus we can also rule out these six $Z_2$ SLs for the HKLM ground state.

Another U(1) SL state is the so-called U(1) Dirac SL or U(1) SL [\[0,0\]] state. Its mean-field Ansätze is shown by the first-NN bonds in Fig. 1(b). Clearly $\pi$ flux pierces through certain triangle plaquettes with no flux through the honeycomb plaquette. According to variational Monte Carlo studies,28,32 this U(1) Dirac SLs have substantially lower energy compared to many other competing phases, including the uniform RVB state. Therefore we shall focus on those $Z_2$ SLs in the neighborhood of the U(1) Dirac SL in our search for the HKLM ground state. We need to mention that, although it is unlikely, the four $Z_2$ SLs in the neighborhood of the uniform RVB state or the U(1) SL [\[0,0\]] state might potentially be the HKLM ground state.

In a previous study using the PSG in Schwinger boson representation,20 it was shown that there are eight different Schwinger boson mean-field Ansätze of $Z_2$ SLs on the kagome lattice which preserve all lattice symmetry. However, these eight $Z_2$ SLs may or may not preserve time-reversal symmetry. One can show that requiring all lattice symmetry and time-reversal symmetry, there are 16 different Schwinger-boson $Z_2$ SLs on the kagome lattice. The relation between the 20 $Z_2$ SLs in Schwinger fermion representation (see Table I) and the 16 $Z_2$ SLs in Schwinger boson representation is not clear. To clarify the relation between SL states in these two different representations, one can compare the neighboring (ordered) phases of the SLs, e.g., by computing the vison quantum numbers,34 of SL states.

B. $Z_2[0,\pi \beta]$ state as a promising candidate for the HKLM ground state

How can we find those $Z_2$ SLs in the neighborhood of (or continuously connected to) the U(1) Dirac SL? Naively, we expect that the mean-field Ansätze of these $Z_2$ SLs can be obtained from that of the U(1) Dirac SL by adding...
an infinitesimal spinon pairing term on top of the U(1) Dirac SL mean-field Ansatz \((9)\) or \((11)\) to break the IGG from U(1) to \(\mathbb{Z}_2\) through the Higgs mechanism. Mathematically, we need to find those \(\mathbb{Z}_2\) SL states whose PSG is a subgroup of the U(1) Dirac SL’s PSG.\(^{31}\) Such \(\mathbb{Z}_2\) SL states are defined to be in the neighborhood of the U(1) Dirac SL. A similar criterion applies to the neighboring \(\mathbb{Z}_2\) SL states of any parent U(1) or SU(2) SL state.

We find all four \(\mathbb{Z}_2\) SLs in the neighborhood of U(1) Dirac SLs in Appendix \(C\). They are states nos. 6, 2, 14, and 16 in Table I, labeled the \(\mathbb{Z}_2[0,\pi]\) beta states, respectively. Since the effective theory of a U(1) Dirac SL is an eight-component Dirac fermion coupled with a dynamical U(1) gauge field,\(^{28,33}\) we can find all symmetry-allowed mass terms that can open up a gap in the Dirac-like spinon spectrum. Following detailed calculations in Appendix \(C\), we can see that among the four \(\mathbb{Z}_2\) SLs around the U(1) Dirac SL, only one state, i.e., \(\mathbb{Z}_2[0,\pi]\) beta states in Tables I and II, can generate a mass gap in the spinon spectrum. In the other three states the Dirac cone in the spinon spectrum is protected by symmetry. The mean-field Ansatz of the \(\mathbb{Z}_2[0,\pi]\) beta SL state up to the second NN is shown in Fig. 1(b):

\[
H_{MF} = \sum_i \left( \lambda_3 \sum_{\alpha} f_{i\alpha}^+ f_{i\alpha} + \lambda_1 f_{i\uparrow}^+ f_{i\downarrow}^+ + \text{H.c.} \right) + \chi \sum_{\langle ij \rangle} \nu_{ij} f_{i\uparrow}^+ f_{j\downarrow}^+ + \sum_{\langle \langle ij \rangle \rangle} \nu_{ij} \left( \chi_2 \sum_{\alpha} f_{i\alpha}^+ f_{j\alpha} + \Delta_2 \sum_{\alpha \beta} \epsilon^{\alpha\beta} f_{i\alpha}^+ f_{j\beta} + \text{H.c.} \right),
\]

where \(\epsilon^{\alpha\beta}\) is the completely antisymmetric tensor. We only list up to second-NN mean-field amplitudes because, as shown in Table I (see also Appendix \(C\)), this \(\mathbb{Z}_2[0,\pi]\) beta state needs only second-NN pairing terms to realize a \(\mathbb{Z}_2\) SL. We can always choose a proper gauge so that the mean-field parameters \(\chi_{1,2}\) and \(\Delta_2\) are all real. The sign structure of \(\nu_{ij}\) is determined by the following constraint:

\[
\sum_i \langle f_{i\uparrow}^+ f_{i\downarrow}^+ \rangle = \sum_i \langle f_{i\uparrow} f_{i\downarrow} \rangle = 0, \quad \sum_i \left( \sum_{\alpha = \uparrow, \downarrow} f_{i\alpha}^+ f_{i\alpha} - 1 \right) = 0.
\]

For further NN mean-field ansatz see discussions in Appendix \(C\).

### IV. CONCLUSION

To summarize, motivated by the strong evidence of a \(\mathbb{Z}_2\) SL as the HKLM ground state in a recent DMRG study,\(^{18}\) we classify all possible \(\mathbb{Z}_2\) SL states in the Schwinger fermion mean-field approach using the PSG. We found 20 different Schwinger fermion mean-field states of \(\mathbb{Z}_2\) SLs on the kagome lattice, among which six states are unlikely due to vanishing first-NN mean-field amplitude. Among the other 14 \(\mathbb{Z}_2\) SLs only five possess a gapped spinon spectrum, which is observed in the DMRG result.\(^{15}\) These five symmetric \(\mathbb{Z}_2\) SL states are all in the neighborhood of certain parent U(1) gapless SLs. To be precise, four are in the neighborhood of gapless uniform RVB [or U(1) SL \([0,0]\)] state, while the other one, i.e., \(\mathbb{Z}_2[0,\pi]\) beta, is in the neighborhood of a gapless U(1) Dirac SL for U(1) SL \([0,\pi]\] state. A previous variational Monte Carlo study\(^{28}\) showed that the gapless U(1) Dirac SL has a substantially lower energy in comparison to the uniform RVB state. This suggests that the \(\mathbb{Z}_2\) SLs in the neighborhood of U(1) Dirac SLs should have lower energy compared to those in the neighborhood of uniform RVB states. Therefore we propose this \(\mathbb{Z}_2[0,\pi]\) beta state with the mean-field Ansatz \((10)\) shown in Fig. 1(b) as the HKLM ground state numerically detected in Ref. 18. Our work provides important insight for future numeric study, e.g., variational Monte Carlo studies of Gutzwiller-projected wave functions.

### ACKNOWLEDGMENTS

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### APPENDIX A: SYMMETRY GROUP OF THE KAGOME LATTICE AND ALGEBRAIC CONDITIONS FOR \(\mathbb{Z}_2\) SPIN LIQUIDS

As shown in Fig. 1(a), we label the three lattice sites in each unit cell with sublattice indices \(s = u, v, w\). Choosing the Bravais unit vector as \(\vec{a}_1 = a\hat{x}\) and \(\vec{a}_2 = \frac{1}{3}(\hat{x} + \sqrt{3}\hat{y})\), the positions of the three atoms in a unit cell labeled by indices \(i = (x,y,s)\) are

\[
\vec{r}(x,y,u) = (x + \frac{1}{2})\vec{a}_1 + (y + \frac{1}{2})\vec{a}_2, \\
\vec{r}(x,y,v) = (x + \frac{1}{2})\vec{a}_1 + y\vec{a}_2, \\
\vec{r}(x,y,w) = x\vec{a}_1 + (y + \frac{1}{2})\vec{a}_2.
\]

The symmetry group of such a two-dimensional kagome lattice is generated by the following operations:

\[
T_1: (x,y,u) \rightarrow (x + 1,y,u), \\
T_2: (x,y,s) \rightarrow (x,y + 1,s), \\
\sigma: (x,y,u) \rightarrow (y,x,u), \\
(x,y,v) \rightarrow (y,x,w), \\
(x,y,w) \rightarrow (y,x,v); \\
C_6: (x,y,u) \rightarrow (-y - 1,x + y + 1,u), \\
(x,y,v) \rightarrow (-y,x + y,u), \\
(x,y,w) \rightarrow (-y - 1,x + y,u),
\]

together with time reversal \(T\).
The symmetry group of a kagome lattice is defined by the following algebraic relations between its generators:

\[
\begin{align*}
T^2 \equiv \sigma^2 &= (C_6)^6 = e, \\
g^{-1}T^\dagger g &= e, \quad \forall \ g = T_{12}\sigma C_6, \\
T_{21}^\dagger T_{12}^\dagger T_{21}^\dagger &= e, \\
\sigma^1T_{12}^\dagger \sigma T_{21}^\dagger &= e, \\
\sigma^2T_{21}^\dagger \sigma T_{12}^\dagger &= e, \\
C_6^{-1}T_{21}^\dagger C_6T_{12} &= e, \\
C_6^{-1}T_{12}^\dagger C_6T_{21} &= e, \\
\sigma^{-1}C_6\sigma C_6 &= e,
\end{align*}
\]

where \(e\) stands for the identity element in the symmetry group. Therefore the consistent conditions for a generic \(\mathbb{Z}_2\) PSG on a kagome lattice are written as

\[
\begin{align*}
[G_T(i)]^2 &= \eta_T \tau^0, \\
G_\sigma(\sigma(i)C_6(i)G_\sigma(i)) &= \eta_\sigma \tau^0, \\
G_\sigma(T_{21}^\dagger(i)G_T(i)G_T(i)G_T(T_{21}^\dagger(i))) &= \eta_T \tau^0, \\
G_\sigma(T_{12}^\dagger(i)G_T(i)G_T(i)G_T(T_{12}^\dagger(i))) &= \eta_T \tau^0,
\end{align*}
\]

First, from condition (A10) we can always choose a proper gauge so that

\[
G_T(x,y,s) = \eta_T \tau^0, \quad G_T(x,y,s) \equiv \tau^0.
\]

From (A12) and (A13) we can see that \(G_\sigma(x,y,s) = \eta_\sigma \eta_T \eta_\tau \tau^0 g_C(s)\). Condition (A5) further determines \(\eta_\sigma \eta_T = \eta_\sigma \eta_T\) and therefore we have

\[
G_\sigma(x,y,s) = \eta_\sigma \eta_T \tau^0 g_C(s),
\]

where the SU(2) matrices \(g_\sigma(s)\) satisfy

\[
[g_\sigma(w)g_\sigma(v)]^2 = g_\sigma(w)g_\sigma(v)g_\sigma(u)g_\sigma(u) = \eta_\sigma \eta_\tau \tau^0.
\]

Notice that we can always choose a proper global \(\mathbb{Z}_2\) gauge on \(G_T(x,y,s)\) (which does not change the mean-field Ansatz) so that \(\eta_C = 1\) in (A16). From (A15) and (A16) it is straightforward to show that

\[
G_C(x,y,u/v) = \eta_2^x \eta_2^y \eta_2^{x+y+(x+y+1)/2} g_C(w).
\]

It is condition (A14) that determines \(\eta_2 \eta_1 \eta_2\), and finally we have

\[
G_C(x,y,u/v) = \eta_2^x \eta_2^y \eta_2^{x+y+(x+y+1)/2} g_C(w),
\]

where the SU(2) matrices \(g_C(s)\) satisfy

\[
[g_C(w)g_C(v)g_C(u)]^2 = \eta_2^x \eta_2^y \tau^0,
\]

\[
[g_\sigma(w)g_\sigma(v)]^2 = g_\sigma(w)g_\sigma(v)g_\sigma(u)g_\sigma(u) = \eta_\sigma \eta_\tau \tau^0.
\]

for any lattice site \(i = (x,y,s)\). Here all \(\eta_\sigma\)'s are \(\mathbb{Z}_2\) integers characterizing different SLs: different (gauge-inequivalent) choices of these \(\mathbb{Z}_2\) integers (different \(\mathbb{Z}_2\) PSGs) correspond to different \(\mathbb{Z}_2\) SLs. Notice that under a local gauge transformation \(W(i) \in SU(2)\) the PSG element \(G_T(i)\) transforms as

\[
G_T(i) \rightarrow W(i)G_T(i)W^\dagger(U^{-1}(i)).
\]

APPENDIX B: CLASSIFICATION OF ALL \(\mathbb{Z}_2\) SPIN LIQUIDS ON THE KAGOME LATTICE

1. Classification of \(\mathbb{Z}_2\) algebraic PSGs on the kagome lattice

In this section we classify all possible \(\mathbb{Z}_2\) spin liquids on a kagome lattice. Mathematically we need to find all gauge-inequivalent solutions of the algebraic conditions (A4)–(A15) for \(\mathbb{Z}_2\) PSGs.
(i) $g_T(s) = t^3$ and therefore $\eta_T = \eta_{CT} = \eta_{C,T} = 1$. Conditions (B8) and (B9) are automatically satisfied.

(ii) $\eta_{g} = 1$. Notice that under a global gauge transformation $W(x,y,s) \equiv W_{\xi} \in SU(2)$ the PSG elements transform as

$$
\begin{align*}
    g_{x}(u) & \rightarrow W_{u}g_{x}(u)W_{u}, \\
    g_{x}(w) & \rightarrow W_{w}g_{x}(w)W_{w}, \\
    g_{C}(u) & \rightarrow W_{u}g_{C}(u)W_{u}, \\
    g_{C}(v) & \rightarrow W_{v}g_{C}(v)W_{v}, \\
    g_{C}(w) & \rightarrow W_{w}g_{C}(w)W_{w}.
\end{align*}
$$

Thus from (B2) and (B4) we can always have $g_{x}(s) = t^{0}$ and $g_{x}(u) = t^{0}$, $g_{C}(v) = \eta_{C}t^{0}$ by choosing a proper gauge.

(A) $\eta_{C} = 1$. From (B3) we have $g_{C}(w) = t^{0}$.

(B) $\eta_{C} = \eta_{C} = -1$. From (B3) we have $g_{C}(w) = t^{3}$ by gauge fixing.

To summarize, there are $2 \times (2 + 3) = 10$ different algebraic PSGs with $\eta_{T} = 1$ and $g_{T}(s) = t^{0}$.

(ii) $\eta_{T} = 1$. According to (B2) and (B8), by choosing a proper gauge we can have $g_{s}(u) = t^{0}$ and $g_{s}(w) = t^{3}$ by gauge fixing. Also from (B4) we can choose a gauge so that $g_{C}(u) = -t^{0}$ and $g_{C}(v) = -i\eta_{C}t^{3}$.

(A) $\eta_{C} = 1$. In this case (B4) requires $g_{C}(w) = t^{0}$ and thus $\eta_{C} = -1$ according to (B3).

(B) $\eta_{C} = 1$.

(a) $\eta_{C} = -1$. Now from (B4) and (B3) we must have $g_{C}(w) = t^{0}$.

(b) $\eta_{C} = 1$. By (B4) and (B3) we have $g_{C}(w) = t^{3}$.

A. Symmetry conditions on the mean-field $\textit{Ansatz}$

Let us denote the mean-field bonds connecting sites $(0,0,u)$ and $(x,y,s)$ as $[x,y,s] \equiv [x,y,s]_{(0,0,u)}$. Using (8) we can generate any other mean-field bonds through symmetry operations (such as translations $G_{x}, G_{y}, G_{z}$ and mirror reflection $G_{x}\sigma$) from $[x,y,s]$. However, these mean-field bonds cannot be chosen arbitrarily since they possess symmetry relation (8): $G_{T}(j)G_{T}^{\dagger}(j) = \eta_{T}(j)$.

Where $U$ is any element in the symmetry group. Notice that for time reversal $T$ we have

$$
G_{T}(j)G_{T}^{\dagger}(j) = \eta_{T}(j).
$$

We summarize these symmetry conditions on the mean-field bonds here:

(i) For $s = u$,

$$
\begin{align*}
T_{1}^{t} & \equiv g_{T}[x,y,u]g_{T}^{\dagger} = [-x,-y,u], \\
T_{2}^{t} & \equiv \sigma_{x}[x,-y,u] \rightarrow [x,-x,u], \\
T_{3}^{t} & \equiv \sigma_{x}[x,u] \rightarrow [x,x,u].
\end{align*}
$$

(ii) For $s = v$,

$$
\begin{align*}
T_{1}^{v} & \equiv g_{T}[x,y,v]g_{T}^{\dagger} = [x,y,v], \\
T_{2}^{v} & \equiv \sigma_{x}[0,y,v] \rightarrow [0,y,v], \\
T_{3}^{v} & \equiv \sigma_{x}[0,v] \rightarrow [0,v,v].
\end{align*}
$$

(iii) For $s = w$,

$$
\begin{align*}
T_{1}^{t} & \equiv g_{T}[x,y,w]g_{T}^{\dagger} = [x,y,w], \\
T_{2}^{t} & \equiv \sigma_{x}[x,1-2x,w] \rightarrow [x,1-2x,w], \\
T_{3}^{t} & \equiv \sigma_{x}[0,w] \rightarrow [0,0,w].
\end{align*}
$$
Now let us consider several very simple examples. First, the on-site chemical potential terms \( \Lambda(x,y,s) = \Lambda_s \) satisfy the following consistent conditions:

\[
\begin{align*}
\tau^1 \Lambda_s \tau^1 &= -\Lambda_s, \\
g_\sigma(u)\Lambda_s g_\sigma^\dagger(u) &= \Lambda_u, \\
g_\sigma(v)\Lambda_s g_\sigma^\dagger(v) &= \Lambda_v, \\
g_\sigma(w)\Lambda_s g_\sigma^\dagger(w) &= \Lambda_w, \\
g_\sigma(u)\Lambda_w g_\sigma^\dagger(u) &= \Lambda_y, \\
g_\sigma(v)\Lambda_w g_\sigma^\dagger(v) &= \Lambda_v, \\
g_\sigma(w)\Lambda_w g_\sigma^\dagger(w) &= \Lambda_w.
\end{align*}
\]

In fact in all \( 20 \mathbb{Z}_2 \) spins on a kagome lattice we have \( \Lambda_u = \Lambda_v = \Lambda_w = \Lambda_w \equiv \Lambda_s \) with a proper gauge choice.

All the first-NN mean-field bonds can be generated from \( u_{\alpha} \equiv [0,0,v] \). For a generic \( \mathbb{Z}_2 \) spin liquid with PSG elements \( G_T(x,y,s) = i \tau^1 \) and (B1), (B5), and (B7), the bond \( u_{\alpha} = [0,0,v] \) satisfies the following consistent conditions:

\[
\begin{align*}
\tau^1 u_{\alpha} \tau^1 &= -u_{\alpha}, \\
g_\sigma(u)g_\sigma^\dagger(v)u_{\alpha}g_\sigma^\dagger(v)g_\sigma^\dagger(w) &= u_{\alpha}^\dagger.
\end{align*}
\]

(B13)

It follows immediately that for six \( \mathbb{Z}_2 \) spin liquids, i.e., nos. 7–12 in Table II, all NN mean-field bonds must vanish since \( u_{\alpha} = 0 \) as required by (B13). Therefore it is unlikely that the \( \mathbb{Z}_2 \) spin liquid realized in the kagome Hubbard model will be one of these six states. In the following we study the remaining 14 \( \mathbb{Z}_2 \) spin liquids on the kagome lattice.

All second-NN mean-field bonds can be generated from \( u_{\gamma} \equiv [0,1,w] \), which satisfies the following symmetry conditions:

\[
\begin{align*}
\tau^1 u_{\gamma} \tau^1 &= -u_{\gamma}, \\
g_\sigma(u)g_\sigma^\dagger(v)u_{\gamma}g_\sigma^\dagger(v)g_\sigma^\dagger(w) &= u_{\gamma}^\dagger.
\end{align*}
\]

(B14)

There are two kinds of third-NN mean-field bond. The first kind can all be generated by \( u_{\rho} \equiv [1,0,u] \), which satisfies

\[
\begin{align*}
\tau^1 u_{\rho} \tau^1 &= -u_{\rho}, \\
g_\sigma(u)g_\sigma^\dagger(v)u_{\rho}g_\sigma^\dagger(v)g_\sigma^\dagger(w) &= u_{\rho}^\dagger.
\end{align*}
\]

(B15)

The second kind can all be generated by \( u_{\tau} \equiv [-1,0,u] \), which satisfies

\[
\begin{align*}
\tau^1 u_{\tau} \tau^1 &= -u_{\tau}, \\
g_\sigma(u)g_\sigma^\dagger(v)u_{\tau}g_\sigma^\dagger(v)g_\sigma^\dagger(w) &= u_{\tau}^\dagger.
\end{align*}
\]

(B16)

The projected symmetry group corresponding to the above mean-field Ansätze (C4) is

\[
G_{\sigma}(x,y,w) = G_{\sigma}(x,y,w) = (-1)^{x+y} G_{\sigma}(x,y,w),
\]

so that the mean-field Ansätze satisfy (8).

2. Classification of \( \mathbb{Z}_2 \) spin liquids around the \( \text{U}(1) \) SL \([0,\pi] \) state

Plugging (C5) into the algebraic consistency conditions (A4)–(A15), we obtain four algebraic solutions of \( \mathbb{Z}_2 \) PSGs around the \( \text{U}(1) \) SL \([0,\pi] \) state. With choice of a proper gauge they all satisfy

\[
\begin{align*}
g_{\tau} &= i \tau_1, \\
g_{\tau} &= g_{\tau} = \tau^0, \\
\eta_{\tau} &= \eta_{\tau} = \eta_{\tau} = -1, \\
\eta_{\tau_2} &= \eta_{\tau_2} = \eta_{\tau_2} = 1.
\end{align*}
\]

(C6)

The four \( \mathbb{Z}_2 \) PSGs near the \( \text{U}(1) \) SL \([0,\pi] \) state have the following features:

(No. 6) \( Z_2[0,\pi] \alpha \): \( g_{\sigma} = g_{\tau} = \tau^0 \), \( \eta_{\sigma} = \eta_{\tau} = 1 \),

(No. 2) \( Z_2[0,\pi] \beta \): \( g_{\sigma} = \tau^0 \), \( g_{\tau} = i \tau^3 \),

\[
\begin{align*}
\begin{align*}
(x+y,u|x,y,w) &= (x,y+1,u|x,y,u) \\
&\quad = -i(x,y,u|x,y,w) \\
&\quad = \chi r^3.
\end{align*}
\end{align*}
\]
Z$_2$ SPIN LIQUIDS IN THE S = $\frac{1}{2}$ . . .
\(\eta_\sigma = \eta_\sigma T = 1,\) \(\eta_\sigma C_x = \eta_\sigma C_T = -\eta_\sigma C_x = 1;\)
\(\text{(No. 14) } Z_2[0,\pi] y' \text{: } g_u = i\tau^3, \quad g_{C_x} = \tau^0,\)
\(\eta_\sigma = \eta_\sigma T = -1,\) \(\eta_\sigma C_x = \eta_\sigma C_T = -\eta_\sigma C_x = 1;\)
\(\text{(No. 16) } Z_2[0,\pi] b \text{: } g_u = g_{C_x} = i\tau^3,\)
\(\eta_\sigma = \eta_\sigma T = -1,\) \(\eta_\sigma C_x = \eta_\sigma C_T = -\eta_\sigma C_x = 1.\)

(8)

(9)

(10)

Of course they belong to the 20 Z$_2$ spin liquids summarized in Table II.

3. Four possible Z$_2$ spin liquids around the U(1) SL [0,\pi] state:

\textit{Mean-field Ansätze}

\textit{a. Consistent conditions on mean-field bonds}

Implementing the generic conditions mentioned earlier on several near-neighbor mean-field bonds with PSG (C6)–(C10), we obtain the following consistent conditions:

(i) For on-site chemical potential terms \(\Lambda_i(x,y,s) = \bar{\lambda}(x,y,s) \cdot \tau,\) the translation operations \(g_{\overrightarrow{B}_{12}}, 1, \pi\) in the PSG guarantee that \(\Lambda_i(x,y,s) = \tilde{\lambda}_i(0,0,0) \equiv \Lambda_i, s = u,v,w.\) They satisfy
\[
gr T \Lambda_i g_T^\dagger = -\Lambda_i,
\]
\[
g_r \Lambda_u g_r^\dagger = -\Lambda_u, \quad g_r \Lambda_v g_r^\dagger = -\Lambda_v, \quad g_r \Lambda_w g_r^\dagger = -\Lambda_v;
\]
\[
g_{\overrightarrow{C}_x} \Lambda_u g_{\overrightarrow{C}_x}^\dagger = \Lambda_v, \quad g_{\overrightarrow{C}_x} \Lambda_v g_{\overrightarrow{C}_x}^\dagger = -\Lambda_u, \quad g_{\overrightarrow{C}_x} \Lambda_w g_{\overrightarrow{C}_x}^\dagger = -\Lambda_v.
\]

(ii) For the first-neighbor mean-field bond \(u_a \equiv [0,0,v]\) (there is only one independent mean-field bond, meaning all other first-neighbor bonds can be generated from \([0,0,v]\) through symmetry operations),
\[
gr T u_a g_T^\dagger = u_a,
\]
\[
(g_\sigma g_{\overrightarrow{C}_x}^2 \tau^3) u_a (g_{\overrightarrow{C}_x}^2 \tau^3) = -u_a.
\]

(iii) For the second-neighbor mean-field bond \(u_b \equiv [0,1,w]\) we have
\[
gr T u_b g_T^\dagger = u_b,
\]
\[
(g_\sigma g_{\overrightarrow{C}_x} u_b (g_{\overrightarrow{C}_x}^2 \tau^3) = -u_b.
\]

(iv) For third-neighbor mean-field bonds \(u_{c1} \equiv [1,0,u]\) and \(u_{c2} \equiv [1,1,0]\) we have
\[
gr T u_{c1} g_T^\dagger = -u_{c1},
\]
\[
(g_{\overrightarrow{C}_x}^3 \tau^3) u_{c1} (g_{\overrightarrow{C}_x}^3 \tau^3) = u_{c1}^\dagger.
\]
and
\[
gr T u_{c2} g_T^\dagger = -u_{c2},
\]
\[
(g_{\overrightarrow{C}_x}^3 \tau^3) u_{c2} (g_{\overrightarrow{C}_x}^3 \tau^3) = u_{c2}^\dagger.
\]

\[b. \text{Mean-field Ansätze of the four } Z_2 \text{ spin liquids near the U(1) SL [0,\pi] state}\]

For the \(Z_2[0,\pi] a\) state with \(g_u = g_{C_x} = \tau^0\) the mean-field Ansätze are (up to third-neighbor mean-field bonds)
\[
u_a = i\alpha_0 \tau^0 + \alpha_1 \tau^1, \quad u_b = i\beta_0 \tau^0 + b_1 \tau^1,
\]
\[
u_{c1} = c_1 \tau^3, \quad u_{c2} = c_2 \tau^3,
\]
\[
\Lambda_u = \lambda_3 \tau^3, \quad s = u,v,w.
\]

Since we are considering a phase perturbed from the U(1) SL [0,\pi] state, we shall always assume \(a_0 \neq 0\) (first-neighbor hopping terms) in the following discussion. A \(Z_2[0,\pi] b\) spin liquid can be realized by first-neighbor mean-field singlet pairing terms with \(a_1 \neq 0.\)

For the \(Z_2[0,\pi] b\) state with \(g_u = \tau^0, g_{C_x} = i\tau^3\) the mean-field Ansätze are (up to third-neighbor mean-field bonds)
\[
u_a = i\alpha_0 \tau^0 + \alpha_1 \tau^1, \quad u_b = i\beta_0 \tau^0 + b_1 \tau^1,
\]
\[
u_{c1} = c_1 \tau^3, \quad u_{c2} = c_2 \tau^3,
\]
\[
\Lambda_u = \lambda_2 \tau^2 + \lambda_3 \tau^3, \quad s = u,v,w.
\]

A \(Z_2[0,\pi] c\) spin liquid can be realized by second-neighbor pairing terms with \(a_0 b_1 - a_1 b_0 \neq 0.\)

For the \(Z_2[0,\pi] c\) state with \(g_u = i\tau^3, g_{C_x} = \tau^0\) the mean-field Ansätze are (up to third-neighbor mean-field bonds)
\[
u_a = i\alpha_0 \tau^0, \quad u_b = i\beta_0 \tau^0 + b_1 \tau^1,
\]
\[
u_{c1} = c_1 \tau^3, \quad u_{c2} = 0,
\]
\[
\Lambda_u = \lambda_3 \tau^3, \quad s = u,v,w.
\]

A \(Z_2[0,\pi] d\) spin liquid can be realized by third-neighbor pairing terms with \(c_2 \neq 0.\)

4. Low-energy effective theory

The reciprocal unit vectors (corresponding to unit vectors \(\hat{a}_{1,2}\) on a kagome lattice) are \(\overrightarrow{B}_1 = \frac{1}{\sqrt{3}}(\hat{x} - \frac{2}{\sqrt{3}}\hat{y})\) and \(\overrightarrow{B}_2 = \frac{1}{\sqrt{3}}(\hat{x} + \frac{2}{\sqrt{3}}\hat{y})\), satisfying \(\overrightarrow{\hat{a}} \cdot \overrightarrow{B}_i = \delta_i,j.\) In the mean-field Ansatz (C4) of U(1) SL [0,\pi] the unit cell is doubled; the translation unit vectors are \(\overrightarrow{A}_1 = 2\overrightarrow{\hat{a}_1}\) and \(\overrightarrow{A}_2 = \overrightarrow{\hat{a}_2}\). Accordingly, the first BZ for such a mean-field Ansatz is only half of the original first BZ with the new reciprocal unit vectors being \(\overrightarrow{B}_1 = \overrightarrow{B}_1/2\) and \(\overrightarrow{B}_2 = \overrightarrow{B}_2/2\). Denoting the momentum as \(k \equiv (k_x, k_y)/a = k_1 \overrightarrow{B}_1 + k_2 \overrightarrow{B}_2\) with \(|k_{1,2}| \leq \pi\), we have
\[
k_1 = 2k_x, \quad k_2 = (k_x + \sqrt{3}k_y)/2.
\]

The two Dirac cones in the spectra of the U(1) SL [0,\pi] state (C4) are located at \(\pm \overrightarrow{Q}\) with
\[
\overrightarrow{Q} = \left(0, \frac{\pi}{\sqrt{3}}\right) = \frac{\pi}{2} \overrightarrow{B}_2
\]
with the proper chemical potential \( \Lambda(i) = (i|i) = \chi(\sqrt{3} - 1)\tau \) added to the mean-field Ansatz (C4).

For convenience we choose the following basis for the Dirac-like Hamiltonian obtained from expansion around \( \pm \mathbf{Q} \):

\[
\phi_{\pm, \tau, A} = \frac{1}{\sqrt{6}} e^{-ix\frac{\pi}{3}} \times \begin{pmatrix}
0, e^{i\frac{\pi}{3}} \tau, 0, e^{-i\frac{\pi}{3}} \tau, 0, e^{i\frac{\pi}{3}} \tau, 0, \sqrt{2} \phi
\end{pmatrix}^T,
\]

\[
\phi_{\pm, \tau, B} = \frac{1}{\sqrt{6}} e^{-ix\frac{\pi}{3}} \times \begin{pmatrix}
1, 0, e^{-i\frac{\pi}{3}} \tau, 0, \sqrt{2} e^{-i\frac{\pi}{3}} \tau, 0, 1, -e^{i\frac{\pi}{3}} \tau, 0, 0, 0
\end{pmatrix}^T,
\]

where \( \pm \) are the valley indices for two Dirac cones at \( \pm \mathbf{Q} \) with Pauli matrices \( \mu \) and \( b = A, B \) are band indices with Pauli matrices \( v \). The spin indices \( \Sigma = \uparrow, \downarrow \) are as usual, with Pauli matrices \( \sigma \). The corresponding creation operators for these modes are \( \Psi_{\pm, \Sigma, \tau, b} = \psi_{0, \phi_{\pm, \Sigma, \tau, b}} \) in the order of \( (0,0,\upsilon),(0,0,\upsilon),(0,0,\upsilon),(0,0,\upsilon),(0,0,\upsilon) \) for the six sites per doubled new unit cell.

Here

\[
R_T \equiv I_{2 \times 2} \otimes \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \otimes \begin{pmatrix}
\tau
\end{pmatrix}
\]

and

\[
R_{T_i}(k) = \begin{pmatrix}
0 & -e^{ik_2} \\
1 & 0
\end{pmatrix} \otimes \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \otimes \begin{pmatrix}
\tau
\end{pmatrix}
\]

are transformation matrices on 12-component eigenvectors for time reversal \( T \) and translation \( T_{1,2} \) operations. By definition of the PSG the eigenvectors \( \phi_{\pm} \) with momentum \( k = k_1 \mathbf{B}_1 + k_2 \mathbf{B}_2 \equiv (k_1, k_2) \) and energy \( E \) have the following symmetric properties:

\[
T : \phi_{k_1, k_2}(E) = R_T(\phi_{-k_1, -k_2}(E)),
\]

\[
T_1 : \phi_{k_1, k_2}(E) = R_{T_1}(k_1, k_2) \phi_{-k_1, -k_2}(E),
\]

\[
T_2 : \phi_{k_1, k_2}(E) = R_{T_2}(k_1, k_2) \phi_{-k_1, -k_2}(E).
\]

The transformation rules for mirror reflection \( \sigma \) and \( \pi/3 \) rotation \( C_6 \) depend on the choice of \( \sigma \cdot g_{C_6} \) in the PSG. In general we have

\[
\sigma: \Psi^{\dagger}_k \rightarrow \Psi^{\dagger}_{-\mu k} M_{\sigma}(g_{C_6}),
\]

\[
C_6: \Psi^{\dagger}_k \rightarrow \Psi^{\dagger}_{\mu k} \sigma M_{C_6}(g_{C_6}).
\]

Using the basis (C22) the 8 \times 8 matrices \( M_{\sigma, C_6} \) can be expressed in terms of Pauli matrices \( \mu \otimes \sigma \otimes v \). For the four \( \mathbb{Z}_2 \) spin liquids we have

\[
M_{\sigma}(g_{\sigma} = \tau^0) = \mu^3 \otimes \sigma^0 \otimes \begin{pmatrix}
0 & e^{-i\frac{\pi}{3}} \\
e^{-i\frac{\pi}{3}} & 0
\end{pmatrix},
\]

\[
M_{\sigma}(g_{\sigma} = i\tau^3) = \mu^3 \otimes \sigma^3 \otimes \begin{pmatrix}
0 & e^{i\frac{\pi}{3}} \\
e^{i\frac{\pi}{3}} & 0
\end{pmatrix},
\]

\[
M_{C_6}(g_{C_6} = \tau^0) = \begin{pmatrix}
1 & 0 \\
0 & i
\end{pmatrix} \otimes \sigma^0 \otimes \sigma^0, \quad \sigma^3 \otimes \sigma^0, \quad e^{i\frac{\pi}{3}} \upsilon^0,
\]

\[
M_{C_6}(g_{C_6} = i\tau^3) = \begin{pmatrix}
i & 0 \\
0 & -i
\end{pmatrix} \otimes \sigma^0 \otimes \sigma^0, \quad \sigma^3 \otimes \sigma^0, \quad e^{i\frac{\pi}{3}} \upsilon^0.
\]

It turns out that in the \( \mathbb{Z}_2 [0, \pi] \beta \) state, only the first mass term \( m_1 = \mu^0 \sigma^0 \upsilon^0 \) is invariant under \( \sigma \) and \( C_6 \) operations. In the other three states neither of the mass terms \( m_{1,2} \) is symmetry allowed. As a result we only have one gapped \( \mathbb{Z}_2 \) spin liquid, i.e., the \( \mathbb{Z}_2 [0, \pi] \beta \) state in the neighborhood of the U(1) Dirac SL \( [0, \pi] \) state.

Let us consider mean-field bonds up to second neighbor for the Ansatz \( \mathbb{Z}_2 [0, \pi] \beta \). Perturbations to the two Dirac cones of U(1) SL \( [0, \pi] \) with \( \lambda_3 = (\sqrt{3} - 1) \upsilon_0 \) in general have the following form:

\[
\delta H_0 = [\chi - (\sqrt{3} - 1) \upsilon_0 - (\sqrt{3} + 1) b_0] \mu^0 \sigma^0 \upsilon^0 + [(\sqrt{3} + 1) b_1 - \lambda_2 - (\sqrt{3} - 1) a_1] \mu^0 \sigma^1 \upsilon^0. \]

This means that we need either a first-neighbor \( (a_1) \) or a second-neighbor \( (b_1) \) pairing term to open up a gap in the spectrum. Meanwhile these pairing terms break the original U(1) symmetry down to \( \mathbb{Z}_2 \) symmetry.
Z$_2$ SPIN LIQUIDS IN THE S = $\frac{1}{2}$ ... 

APPENDIX D: Z$_2$ SPIN LIQUIDS IN THE NEIGHBORHOOD OF THE UNIFORM RVB STATE

The mean-field Ansatz of the uniform RVB state is simple:
\[ H_{MF} = \chi \sum_{(ij):\sigma} f_{i,\sigma} f_{j,\sigma}, \tag{D1} \]
where $\chi$ is a real parameter and $(ij)$ represents sites $i,j$ that are nearest neighbors of each other. It is straightforward to show that the PSG elements of such a mean-field Ansatz are
\[ G_U(x,y,s) = g_U, \quad U = T_{1,2}, T, \sigma, C_6 \tag{D2} \]
and the SU(2) matrices $g_U$ satisfy
\[ g_T \tau^1 g_T^{-1} = -\tau^3, \]
\[ g_U \tau^3 g_U^{-1} = \tau^3, \quad U = T_{1,2}, \sigma, C_6. \tag{D3} \]
It turns out that there are only four gauge-inequivalent Z$_2$ PSGs as solutions to (A4)–(A16) with the form (D2). In other words, there are only four different Z$_2$ in the neighborhood of a uniform RVB state. With choice of a proper gauge they all satisfy $g_T = i\tau^1, g_{T,2} = \tau^0$, and $\eta_{T,2} = \eta_{T,1} = \eta_{C,2} = \eta_{T,1} = 1, \eta_T = -1$. These four states are characterized by

\[ \begin{align*}
\text{(No. 1)} & \quad Z_2[0,0]A: g_\sigma = g_{C_6} = \tau^0, \\
\text{(No. 5)} & \quad Z_2[0,0]B: g_\sigma = \tau^0, \quad g_{C_6} = i\tau^3, \\
\text{(No. 15)} & \quad Z_2[0,0]C: g_\sigma = i\tau^3, \quad g_{C_6} = \tau^0, \\
\text{(No. 13)} & \quad Z_2[0,0]D: g_\sigma = g_{C_6} = i\tau^3, \\
\end{align*} \tag{D4} \]

It turns out that these four Z$_2$ SLs around the uniform RVB state are all gapped as shown in Table II.