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Analytical results on Casimir forces for conductors with edges and tips

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The formalism recently implemented in refs. 18 and 19 enables a systematic exploration of Casimir interactions between any arrangement of such shapes. Edges are explored in the section devoted to wedge geometries, where for perfect conductors the EM field can be parameterized in terms of two scalar fields. The corresponding scattering amplitudes in this case were previously known; their application to compute Casimir forces is developed here. In particular, we obtain simple analytical expressions for the force between a knife edge and a plate from the first two terms of the multiple-scattering series, and verify (by numerical computation of higher order terms) that these expressions provide highly accurate estimates of the force. The limitations of common approximation schemes can be illustrated by simple examples in this geometry. Analogous computations relevant to a sharp knife edge (23–27) and a conducting plate (and among themselves) is then computed systematically by a multiple-scattering series. For the wedge, we obtain analytical expressions for the interaction with a plate, as functions of opening angle and tilt, which should provide a particularly useful tool for the design of microelectromechanical devices. Our result for the Casimir interactions between conducting cones and plates applies directly to the force on the tip of a scanning tunneling probe. We find an unexpectedly large temperature dependence of the force in the cone tip which is of immediate relevance to experiments.

The inherent appeal of the Casimir force as a macroscopic manifestation of quantum “zero-point” fluctuations has inspired many studies over the decades that followed its discovery (1). Casimir’s original result (2) for the force between perfectly reflecting mirrors separated by vacuum was quickly extended to include slabs of material with specified (frequency-dependent) dielectric response (3). Quantitative experimental confirmation, however, had to await the advent of high-precision scanning probes in the 1990s (4–7). Recent studies have aimed to reduce or reverse the attractive Casimir force for practical applications in micron-sized mechanical machines, where Casimir forces may cause components to stick and the machine to fail. In the presence of an intervening fluid, experiments have indeed observed repulsion due to quantum (8) or critical thermal (9) fluctuations. Metamaterials, fabricated designs of microcircuitry, have also been proposed as candidates for Casimir repulsion across vacuum (10).

Although there have been many studies of the role of materials (dielectrics, conductors, etc.), the treatment of shapes and geometry has remained comparatively less investigated. Interactions between nonplanar shapes are typically calculated via the proximity force approximation (PFA), which sums over infinitesimal segments treated as locally parallel plates (11). This approximation represents a serious limitation because the majority of experiments measure the force between a sphere and a plate with precision that is now sufficient to probe deviations from PFA in this and other geometries (12, 13). Practical applications are likely to explore geometries further removed from parallel plates. Several numerical schemes (14–16), and even an analog computer (17), have recently been developed for computing Casimir forces in general geometries. However, analytical formulae for quick and reliable estimates remain highly desirable.

The formalism recently implemented in refs. 18 and 19 enables systematic computations of electromagnetic (EM) Casimir forces in terms of a multipole expansion. Using these methods, we have been able to compute forces between various combinations of planes, spheres, and circular and parabolic cylinders (19–24) (see also refs. 25–27). Both perfect conductors and dielectrics have been studied. However, with the notable exception of the knife edge (23–28), which is a limit of the parabolic cylinder geometry, systems with sharp edges have not yet been studied analytically.1

In what follows, we provide a summary of the approach and highlights of our results. The first section introduces the scattering formalism, whose main ingredients are scattering amplitudes from individual objects. A multiple-scattering series then enables systematic exploration of Casimir interactions between any arrangements of such shapes. Edges are explored in the section devoted to wedge geometries, where for perfect conductors the EM field can be parameterized in terms of two scalar fields. The corresponding scattering amplitudes in this case were previously known; their application to compute Casimir forces is developed here. In particular, we obtain simple analytical expressions for the force between a knife edge and a plate from the first two terms of the multiple-scattering series, and verify (by numerical computation of higher order terms) that these expressions provide highly accurate estimates of the force. The limitations of common approximation schemes can be illustrated by simple examples in this geometry. Analogous computations relevant to a sharp

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tip are carried out in the section on the cone. Scattering amplitudes for the cone are obtained via a representation of the EM Green’s function and sketched briefly in Methods. The expressions for the interaction of a cone and a plate (both perfectly reflecting) are rather complex in general, but reduce to simple forms in the limit of small opening angle (approaching a needle). For the interaction between parallel metal plates, thermal corrections at room temperature are known to be practically negligible (at micron separations). Although these corrections are small for the wedge, we find that the room temperature result for the cone is considerably larger than at zero temperature (by roughly a factor of 2 at micron separations). This difference should prove quite important to experiments and designs involving sharp tips. In particular, the prospects for experimental detection of the force on the tip of an atomic force microscope are discussed in Conclusion and Outlook. Detailed derivations, supporting formulae, and graphs for each of these sections are provided in the SI Appendix.

**Casimir Forces via the Scattering Formalism**

The conceptual foundations of the scattering approach can be traced back to earlier multiple-scattering formalisms (7, 34–36), but these were not sufficiently efficient to enable more complicated calculations. The ingredients in our method are depicted in Fig. 1. The Casimir energy associated with a specific geometry depends on the way that the objects constrain the EM waves that can bounce back and forth between them. The dependence on the properties of the objects is completely encoded in the scattering amplitude or the properties of the objects is completely encoded in the scattering waves. This, in turn, can be traced back to earlier multiple-scattering formalisms (7, 34–36).

For the interaction between parallel metal plates, thermal corrections for the interaction of a cone and a plate (both perfectly reflecting) are rather complex in general, but reduce to simple forms in the limit of small opening angle (approaching a needle). For a perfectly reflecting wedge, translation symmetry makes it possible to decompose the EM field into two scalar components: a polarization that vanishes on its surface (Dirichlet boundary condition) and an M-polarization field that has vanishing normal derivative (Neumann). In the cylindrical coordinate system $(r, \phi, z)$, a wedge has surfaces of constant opening angle $\theta_0$. Whereas for describing scattering from cylinders, a natural basis is $e^{\pm \text{i} m \phi} H_T^{(1)}(i \sqrt{k^2 + \mu^2})$, with Bessel-$H_T^{(1)}$ functions indexed by $m = 0, 1, 2, \ldots$, for a wedge we must choose $e^{\pm \text{i} m \mu} H_T^{(1)}(i \sqrt{k^2 + \mu^2})$ with real $\mu \geq 0$, corresponding to imaginary angular momenta that are no longer quantized (see SI Appendix, Section 1). The $T$ matrices, diagonal in $\mu$, take the simple forms indicated in Fig. 1. Dimensional analysis indicates that the interaction energy of a wedge of edge length $L$ at a separation $d$ from a plate is $\mathcal{E} = -(hcL/d^2 f(\theta_0)$, where $f(\theta_0, \phi_0)$ is a dimensionless function of the opening angle $\theta_0$ and inclination $\phi_0$ to the plate. This geometry, and the corresponding function $f(\theta_0, \phi_0)$, are plotted in Fig. 2, Middle.

The limit $\theta_0 \to 0$ corresponds to a knife edge which was previously studied as a limiting form of a parabolic cylinder (23, 28). The matrix $N'$ for the wedge simplifies in this limit, enabling exact calculation of the first few terms in the expansion of $\text{tr} \ln[1 - N']$ in Eq. 1.

\[
\mathcal{E} = \frac{hc}{2\pi} \int_0^\infty dx \text{tr} \ln[1 - N'] = -\frac{hc}{2\pi} \int_0^\infty dx \left[ \text{tr} N' + \frac{1}{2} \text{tr} N'^2 + \cdots \right],
\]

1

![Fig. 1. Ingredients in the scattering theory approach to EM Casimir forces. See the text for further discussion.](image-url)
The Casimir interaction energy of a wedge at a distance \( d \) above a plane, as a function of its semiopening angle \( \theta_0 \) and tilt \( \phi_0 \). The rescaled energy as a function of \( \theta_0 \) and \( \phi_0 \) is shown in the middle panel. The symmetric case, \( \phi_0 = 0 \), is displayed in the top panel and the interesting case where the back side of the wedge is hidden from the plane is shown in the bottom panel. See the text for further discussion.

\[
\mathcal{E} = \frac{\hbar c}{\pi d^2} \left[ \sec \phi_0 \frac{1}{16\pi^2} + \frac{1}{192\pi^3} \csc^3 \phi_0 \sec \phi_0 \left( 2\phi_0 - \sin 2\phi_0 \right) \right] + \ldots
\]

[2]

The first square brackets corresponds to \(\mathcal{N} \) (depicted by an orange line in Fig. 2) and the second to \(\mathcal{N}/2\); their sum (depicted by a red line) is in remarkable agreement with the full result (blue surface). As \( \phi_0 \to \pi/2 \), the knife edge becomes parallel to the plate and the interaction energy diverges as it becomes proportional to the area rather than \( L \). For parallel plates, we know that the terms in the multiple-scattering series \( \mathcal{N}/n \) are proportional to \( 1/n^4 \) (1). Numerically, we find that the convergence is more rapid for \( \phi_0 < \pi/2 \), and that the first two terms in Eq. 2 are accurate to within 1%. Including more than three terms in the series will not modify the curve at the level of accuracy for this figure. Casimir’s calculation for parallel plates gives an exact result at \( \phi_0 = \pi/2 \), marked with an \( \times \) in Fig. 2, Middle.

The panels in Fig. 2 display some of the more interesting aspects of the wedge-plate geometry. In the middle panel, the energy, rescaled by an overall factor of \( \hbar cL/d^2 \) and multiplied by \( \cos(\theta_0 + \phi_0) \), is plotted versus \( \theta_0 \) and \( \phi_0 \). The factor of \( \cos(\theta_0 + \phi_0) \) is introduced to remove the divergence as one face of the wedge becomes parallel to the plate and the energy becomes proportional to the area rather than just the length of the wedge. The blue surface is obtained by numerical evaluation of the first three terms in the multiple-scattering series of Eq. 1, with \( \mathcal{N} \) constructed in terms of the plate and wedge \( T \) matrices given in Fig. 1. The front curve (\( \theta_0 = 0 \)) corresponds to a knife edge as previously mentioned. The top panel depicts the “butterfly” configuration where the wedge is aligned symmetrically with respect to the normal to the plate. As the wings open up to a full plate at \( \theta_0 = \pi/2 \), the energy again approaches the classic parallel plate result, also marked by an \( \times \). Finally, and perhaps most interestingly from a qualitative point of view, the bottom panel depicts the case where one wing is fixed at \( \pi/4 \), and the other opens up by \( \psi = 2\theta_0 \). This case displays the sensitivity of the Casimir energy to the back side of the wedge, which is “hidden” from the plate. In the proximity force approximation, the energy is independent of the orientation of the back side of the wedge, and thus the PFA result (solid line) is constant until the back surface becomes visible to the plate. The correct result (dotted line) varies continuously with the opening angle and differs from the proximity force estimate by nearly a factor of 2, showing that the effects responsible for the Casimir energy are more subtle than can be captured by the PFA.

**Tips via the Cone**

Computations for a cone—the surface of constant \( \theta \) in spherical coordinates \((r, \theta, \phi)\)—require a similar passage from spherical waves labeled by \((\ell, m)\) to complex angular momentum, \( \ell \to i\lambda - 1/2 \). In this case \( \lambda \ge 0 \) is real, whereas \( m \) remains quantized to integer values. However, unlike the wedge case, the EM field can no longer be separated into two scalar parts; the more complicated representation that we report in the SI Appendix, Section II involves an additional field, similar to the ghost fields that appear elsewhere in quantum field theory. This representation of EM scattering can also be of use for describing reflection of ordinary EM waves from cones. Dimensional analysis indicates that, for a cone poised vertically at a distance \( d \) from a plate, the interaction energy scales as \( (\hbar c/d) \) times a function of its opening angle \( \theta_0 \). This arrangement and the resulting interaction energy are depicted in Fig. 3, Left, with the energy scaled by \( \cos^2 \theta_0 \) to remove the divergence as the cone opens up to a plate for \( \theta_0 \to \pi/2 \). The PFA (11) (depicted by the dashed line) becomes exact in this limit, but it is progressively worse as \( \theta_0 \) decreases from \( \pi/2 \). In particular, this approximation predicts that the energy vanishes linearly as \( \theta_0 \to 0 \), although in fact it vanishes as

\[
\mathcal{E} \sim \frac{\hbar c \ln 4 - 1}{16\pi^3} \frac{1}{d} \left| \ln \frac{4}{\pi} \right|
\]

[3]

where the logarithmic divergence is characteristic of the remnant energy in this limit (33). The EM results shown as the bold blue curve in Fig. 3 are obtained by including two terms in the series of Eq. 1, with \( \mathcal{N} \) constructed from the plate and cone \( T \) matrices in Fig. 1, including the additional ghost field. We have also included the corresponding curves for scalar fields subject to Dirichlet and Neumann boundary conditions which are depicted as fine red curves where the top (bottom) one corresponds to the Dirichlet (Neumann) boundary condition. The limit of \( \theta_0 \to 0 \) is shown in the left panel of Fig. 3 as an orange dashed line. As the sharp tip is tilted by an angle \( \beta \), the prefactor \( (\ln 4 - 1)/16\pi \) is replaced by \( g(\beta)/\cos \beta \), where \( g(\beta) \) can be computed from integrals of trigonometric functions (see SI Appendix, Section II). We plot this quantity in the right panel of Fig. 3.

**Thermal Corrections**

For practical applications, the above results have to be corrected for finite temperature. These are easily incorporated by replacing the integral in Eq. 1 with a sum over Matsubara frequencies \( \kappa_n = (2\pi k_B T/\hbar c)n \) (37). For the knife edge, the first (single-scattering) term in the force resulting from Eq. 2 is modified to
\[ F_{\text{EM}} = \frac{\hbar c L}{4\pi^2} \cos \theta_0 \frac{1}{d^3} \left[ 1 + \frac{1}{45} \left( \frac{d}{\lambda_T} \right)^4 + \cdots \right]. \tag{4} \]

where \( \lambda_T = \frac{\hbar}{\sqrt{2kT}} \) is the thermal wavelength. At room temperature \( \lambda_T \approx 7 \mu m \), indicating that thermal corrections are significant at separations \( d \approx 1 \mu m \). (A similar correction appears in the first scattering term for parallel plates.)

By contrast, the thermal corrections for a cone are quite significant, and the force from Eq. 3 on a sharp cone is modified at low temperatures to

\[ F_{\text{needle}} = \frac{-\hbar c}{16\pi} \ln \frac{4 - 1}{d^2} - \frac{2\ln 2d}{3\lambda_T} + \frac{0.810}{\lambda_T} + \cdots. \tag{5} \]

More generally, for a sharp cone tilted by an angle \( \beta \), and at a temperature \( T \), the result of Eq. 3 is multiplied by a (semi)analytic function \( g(\beta d/\lambda_T)/\cos \beta \) as reported in SI Appendix, Section III. The function \( g(\beta d/\lambda_T) \) is an increasing function of both \( \beta \) and the temperature \( T \). We plot this function for \( d = 1 \mu m \) in Fig. 3. Interestingly, the room temperature force is more than 100% higher than the force for \( T = 0 \). In contrast, the corresponding increase for parallel plates at \( d = 1 \mu m \) is only about 0.1%. This enhanced role of thermal corrections for specific geometries has been noted before (38, 39) and appears essential to the design of microelectromechanical (MEM) devices.

**Conclusion and Outlook**

Construction of electromechanical devices at the micrometer scale requires mastery of Casimir forces, which depend nontrivially on geometry and shape. The few analytic solutions available previously apply generically to smooth objects such as cylinders and spheres. Mechanical devices, however, can optimally manipulate forces by adjusting the alignment and proximity of sharp components such as tips and edges. In this paper, we have presented a synthesis of EM scattering results from wedges and cones to compute Casimir forces for perfect conductors with sharp edges and tips. The interaction of these objects with a metal plate can then be computed systematically by a multiple-scattering analysis, which provides an analytical expressions for the interaction of a plate and a knife edge with arbitrary opening and tilt angles, and the singular behavior of the force on the tip of a sharp scanning needle.

In practice, the ideal Casimir results have to be corrected due to imperfect conductivity and finite temperatures. We find that thermal fluctuations at room temperature are quite significant for a cone and have to be properly accounted for. Fortuitously, for a (semiinfinite) needle, a shape-induced, drastic reduction of magnetic against electric polarizability limits the influence of imperfect conductivity, which for low frequencies can be easily understood from the polarizability of a needle (40). In particular, there is ongoing debate in the literature (41–44) regarding the use of Drude or Plasma models of conductivity—the former does not shield low-frequency magnetic modes leading to decreased Casimir force at high temperatures (24). The magnetic modes actually penetrate the needle at all temperatures and do not contribute to the Casimir force plotted in Fig. 3. Imperfect conductivity can also modify the electric modes. However, although we do not compute forces for a general frequency-dependent dielectric response \( \epsilon(\omega) \), we argue in the SI Appendix, Section IV that for typical metals (e.g., Au or Al), even at zero temperature and for needles (the cases where these corrections are largest), the corrections due to imperfect conductivity are at most around 5% for \( d = 0.3 \mu m \).

Thus for separations \( 0.3 \mu m \lesssim d \lesssim 10 \mu m \), relevant to MEM devices and experiments, the perfect conductor results, with the important finite temperature corrections, should suffice. For example, let us consider the tip of an atomic force microscope: At separations, \( d \approx 0.3 \mu m \), where the tip may be well approximated as a metal cone, our results predict a force that is a fraction of a piconewton. Such forces are at the limit of current sensitivities (45) and will likely become accessible with future improvements. Current experiments are performed on spheres of relatively large radius \( R \), where the force is greater by a factor of \( (R/d) \) (a typical \( R = 100 \mu m \)). A rounded wedge with radius of curvature \( R \) falls in an intermediate range, with forces larger (23) by \( \sqrt{R/d} \) than the sharp case. The force can also be enhanced by using arrays of cones or wedges, at the cost of the difficulty of maintaining their alignment.

**Methods**

The primary tool in finding the scattering matrix of an object is the (free) Green’s function given in a coordinate system appropriate to the object’s shape. The scalar Green’s function for the wedge and cone geometries were previously known (46, 47). Although the translation symmetry of a perfect conducting wedge makes it possible to treat the EM field as a combination of two scalar problems, the vector nature of the field is unavailable in the case of the cone.

We obtain the EM Green’s function for the cone by analytic continuation of the standard representation in terms of spherical partial waves. Electromagnetism, unlike the scalar field, does not allow monopole fluctuations, and this constraint introduces an additional subtlety. The full Green’s function becomes
where, in addition to the usual magnetic (transverse electric) modes $M$ and electric (transverse magnetic) modes $N$, we also find a set of discrete modes $R$, which have no analog in other representations of the Green's function. The total angular momentum of the latter modes is zero, which might appear in contradiction with the absence of the $\epsilon=0$ channel in electromagnetism. Here, however, these modes cancel the unphysical contributions of the other two modes to this channel. For this reason, we designate them as ghost fields. This Green's function is the primary tool for the result in Tips via the Cone.

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