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Noise correlations in a flux qubit with tunable tunnel coupling

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We have measured flux-noise correlations in a tunable superconducting flux qubit. The device consists of two loops that independently control the qubit’s energy splitting and tunnel coupling. Low-frequency flux noise in the loops causes fluctuations of the qubit frequency and leads to dephasing. Since the noises in the two loops couple to different terms of the qubit Hamiltonian, a measurement of the dephasing rate at different bias points provides a way to extract both the amplitude and the sign of the noise correlations. We find that the flux fluctuations in the two loops are anticorrelated, consistent with a model where the flux noise is generated by randomly oriented unpaired spins on the metal surface.

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I. INTRODUCTION

The performance of qubits based on superconducting circuits is affected by the ubiquitous low-frequency flux noise present in such devices. Early measurements on superconducting quantum interference devices (SQUIDs) showed the existence of flux noise with a 1/f-like power spectrum.1 The noise was found to be universal in the sense that it depended only weakly on materials and sample dimensions.2 In more recent years, it has become clear that the same excess flux noise limits the phase coherence of superconducting qubits. A slowly varying flux leads to fluctuations of the qubit energy levels, which has been shown to be the dominant dephasing mechanism in both flux3–5 and phase qubits.6 Finding the origin and a possible remedy for the excess flux noise is therefore of importance to facilitate further improvements of intrinsic coherence times in these devices.

Theories have been developed to explain the microscopic origins of the noise, involving the existence of real or effective unpaired spins in the vicinity of the superconducting structures.7–10 Various experiments have shown the presence of a large number of localized spins on the surface of thin films of normal11 and superconducting metals,12 as well as in Si/SiO2 interfaces.13 Sendelbach et al. measured correlations between flux and inductance noise in SQUIDs, possibly suggesting the formation of spin clusters.14 The results suggest that the flux noise is related to the nonequilibrium dynamics of the spin system, possibly described by spin glass models15 or fractal spin clusters.16 Recent measurements on qubits with different geometries indicate that the flux noise scales as (l/w), in agreement with the surface spin model17 (l is the length and w is the width of the superconducting wires).

Flux noise correlations have been studied in a system of coupled qubits sharing parts of their loops.18 It was found that the flux fluctuations originating from the shared branch lead to correlations in the noise of the two qubits. In this work, we use a single, two-loop qubit to investigate flux noise correlations between different parts within a single qubit [Fig. 1(b)]. The fluxes in the two loops couple to the longitudinal and transverse components of the qubit’s Hamiltonian, respectively. The ability to control both parameters of the Hamiltonian allows us to investigate qubit coherence properties at different frequencies while remaining at the optimal bias point.19 By measuring the qubit dephasing rate as a function of the flux bias, we can extract the correlations between the flux fluctuations in the two loops. Knowing that the two fluxes couple differently to the qubit energy allows us to determine the sign of the correlations. We find the flux noise to be strongly anticorrelated, in agreement with a model where the noise is generated by spins on the superconductor surface.

II. SETUP

The standard flux qubit consists of a superconducting loop with three or more Josephson junctions [Fig. 1(a)]. The diabatic states correspond to clockwise and counterclockwise circulating currents, respectively.20,21 When the flux in the loop is close to half a flux quantum, Φ = 0.5Φ0, the Hamiltonian is

\[ H = -\left( \phi_1 + \phi_2 \right) / 2 \] within a two-level approximation. Here, \( \phi = 2I_P / (\Phi - 0.5\Phi_0) \), where \( I_P \) is the persistent current and \( \Phi_0 = h / 2e \). The tunnel coupling \( \Delta \) is fixed by fabrication and determined by the size of the smallest Josephson junction. By replacing the smallest junction with a second loop, the tunnel coupling \( \Delta \) can be tuned in situ by the flux \( \Phi_2 = f_2 \Phi_0 \) in the second loop [Fig. 1(b)]. Due to the device geometry, the detuning \( \epsilon \) is controlled by the combined flux \( (f_1 + f_2 / 2) \Phi_0 \), where \( f_1 = \Phi_1 / \Phi_0 \) is the normalized flux in loop 1.3 To clarify the relation between the qubit parameters and the two fluxes, we introduce the effective fluxes

\[ f_x = (f_1 + f_2 / 2) - 1 / 2, \quad f_y = f_2. \] (1)

The energy separation of the qubit is then given by

\[ E_{01} / h = \sqrt{\epsilon^2 + \Delta^2}, \quad \text{with} \quad \epsilon = \epsilon(f_x), \quad \Delta = \Delta(f_x). \] (2)

The qubit is embedded in a dc SQUID for reading out the qubit state. The readout is implemented by applying a short current pulse to the SQUID; due to the inductive coupling between the SQUID and the qubit, the SQUID switching probability will vary depending on the qubit state.23
FIG. 1. (Color online) (a) A standard flux qubit. The two diabatic states correspond to clockwise and counterclockwise circulating currents. (b) A tunable flux qubit. The smallest junction has been replaced by an additional loop. (c) The actual sample design. The qubit is embedded in a SQUID used for readout of the qubit state. The line widths are not drawn to scale; a detailed drawing can be found in Appendix A.

The design of the actual device is shown in Fig. 1(c). The structure is made of aluminum and is fabricated using e-beam lithography and standard two-angle shadow evaporation techniques (see Appendix A for a detailed drawing). There are two local current-bias lines, which, together with the global external field, allow us to control the fluxes in the two qubit loops and in the readout SQUID independently. Due to the close proximity of the loops and the bias lines, there is substantial cross-coupling between the different elements. The measured inductive couplings are given in Appendix B. During the experiments, we applied appropriate compensation currents to the bias lines to compensate the unwanted couplings. The residual unwanted coupling is less than 1%.

III. RESULTS

Figure 2(a) shows two qubit spectra measured for two values of \( f_x \). The measurement was done by applying a long microwave pulse (3 \( \mu \)s) to saturate the qubit before reading out its state. The spectra have the form expected from Eq. (2), but with different values of \( \Delta \). The horizontal feature around 3.2 GHz is due to the plasma mode of the readout SQUID. In Fig. 2(b) we plot the measured \( \Delta \) as a function of \( f_x \). The tunnel coupling \( \Delta \) is symmetric around \( f_x = 0 \) and can be tuned experimentally from 1.5 to 15 GHz. The dashed line shows the results of a numerical simulation of the qubit energy levels for realistic fabrication parameters. To quantify the relation between \( \Delta \) and \( f_x \), we plot the numerical derivative of the data for a region around \( f_x = 0.39 \) [Fig. 2(c)]. We have marked the positions of the two operating points with \( \Delta = 2.5 \) GHz and \( \Delta = 3.5 \) GHz, where we perform coherence measurements. The sensitivity \( \partial \Delta / \partial f_x \) is different for those two points.

The results in Fig. 2 show that our device behaves as expected and that we can tune the qubit parameters \( \Delta \) and \( \varepsilon \) independently by applying fluxes in the two loops. To characterize the qubit coherence as a function of those parameters, we perform energy-relaxation \( (T_1) \), free-induction decay (FID), and Hahn-echo measurements. \( T_1 \) is measured by applying a \( \pi \) pulse to the qubit and delaying the readout. Figure 3(a) shows the extracted \( T_1 \) decay times as a function of detuning \( f_z \), measured for the two values of \( \Delta \) stated in Fig. 2(c). \( T_1 \) is fairly independent of \( f_x \). The difference in \( T_1 \) for the two values of \( \Delta \) can be attributed to differences in detuning from the SQUID plasma mode at \( f_{\text{plasma}} = 3.2 \) GHz.

The FID sequence consists of two \( \pi/2 \) pulses separated by a time \( t \). The FID is sensitive to low-frequency noise...
that causes adiabatic fluctuations in the qubit energy splitting $E_{01}$. The Hahn echo contains an extra $\pi$ pulse in the middle of the sequence to refocus low-frequency fluctuations. For Gaussian-distributed noise with a $1/f$-type spectrum, the decay of both the FID and the Hahn echo has the form

$$p(t) = e^{-t^2/2\Gamma_{\phi F}} e^{-(t/T_{\phi F})^2}. $$

By first measuring $T_1$, we can extract the pure dephasing times $T_{\phi F}$ and $T_{\phi E}$ for the FID and the Hahn-echo sequences, respectively.\(^3\)

In Fig. 3(b) we plot $T_{\phi F}$ and $T_{\phi E}$ as a function of $f_z$ for the two values of $\Delta$. We make a few observations: (i) Time $T_{\phi E}$ is 4–5 times longer than $T_{\phi F}$, consistent with the $1/f$-type spectrum.\(^3\)(ii) Both $T_{\phi F}$ and $T_{\phi E}$ fall off as $|f_z|$ is increased; this is because $\partial E_{01}/\partial f_z$ is 0 only at $f_z = 0$ and increases approximately linearly with $|f_z|$ over this range of $f_z$. However, contrary to Refs. 3 and 25, the echo dephasing times are considerably shorter than the limit set by energy relaxation ($2T_1$) even close to $f_z = 0$. This is a result of our device having an extra loop that controls $\Delta$: fluctuations in $f_z$ will couple to $\Delta$, which will couple to $E_{01}$ even when $f_z = 0$. (iii) Close to $f_z = 0$, the dephasing times are longer for $\Delta = 2.5$ GHz than for $\Delta = 3.5$ GHz: this is because the sensitivity $\partial \Delta/\partial f_z$ to noise in $f_z$ is stronger at $\Delta = 3.5$ GHz [see Fig. 2(c)]. (iv) The longest dephasing times do not occur at $f_z = 0$ but are slightly shifted to positive $f_z$. The effect is visible for both values of $\Delta$, but the shift is larger for $\Delta = 3.5$ GHz.

We investigate the shift in more detail by measuring the FID at $\Delta = 3.5$ GHz for positive and negative values of $f_z$. The results are shown in Fig. 3(c): at $f_z = -0.39$, the longest dephasing times are shifted toward negative $f_z$. Even though $\Delta$ is the same at both $f_z = 0.39$ and $f_z = -0.39$, the sensitivity $\partial \Delta/\partial f_z$ is negative for $f_z < 0$ and positive for $f_z > 0$. The fact that the shift depends on the sign of $\partial \Delta/\partial f_z$ leads us to suspect that there are correlations between $f_z$ and $\Delta$. We have checked that the shift does not depend on the SQUID bias current for small excursions from the operating point.\(^26\) Residual unwanted coupling between the bias lines and the loops may induce a small flux in $f_z$ when sweeping $f_z$, which will cause uncertainty when determining the point of $f_z = 0$. However, having residual unwanted couplings of less than 1% gives an upper bound of $0.01 \times \Delta \times (\partial \Delta/\partial f_z)/(\partial \Delta/\partial f_z)^2 \approx 3 \times 10^{-6}$ for the uncertainty in $f_z$, which is considerably smaller than the shifts shown in Fig. 3(c).

IV. CORRELATIONS

The phase decay rate $\Gamma_{\phi F} = 1/T_{\phi F}$ is due to a combination of noise in both $f_z$ and $f_c$. To calculate $\Gamma_{\phi F}$ in the presence of correlations, we assume that the fluctuations $\delta f_z$ and $\delta f_c$ are described by noise spectra of the form $S_{f_z}(\omega) = (A_{x} \Phi_{0}^{-2})/|\omega|$ and $S_{f_c}(\omega) = (A_{x} \Phi_{0}^{-2})/|\omega|$. In addition, we introduce the correlation spectrum $S_{f_z f_c}(\omega) = (A_{x} A_{z} \Phi_{0}^{-2})/|\omega|^{1/2}$ and the correlation coefficient $c_{f_z} = \langle \delta f_z \delta f_c \rangle/\sqrt{\langle \delta f_z^2 \rangle \langle \delta f_c^2 \rangle} = A_{z}/\sqrt{A_{x} A_{z}}$. The correlation spectrum is thus assumed to have the same form as the individual noise spectra $S_{f_z}$ and $S_{f_c}$. The total decay rate is\(^25\)

$$\Gamma_{\phi F} = \frac{\ln(1/\omega_{\text{hom}})}{\hbar \Phi_{0}} \left[ A_{x} \left( \frac{\partial E_{01}}{\partial f_z} \right)^2 + A_{z} \left( \frac{\partial E_{01}}{\partial f_c} \right)^2 \right]$$

$$+ 2 c_{f_z} \sqrt{A_{x} A_{z}} \left( \frac{\partial E_{01}}{\partial f_z} \left( \frac{\partial E_{01}}{\partial f_c} \right)^{1/2} \right). \quad (3)$$

The low-frequency cutoff $\omega_{\text{hom}}/2\pi = 1$ Hz is imposed by the measurement protocol. Figure 4(a) shows the result of Eq. (3), plotted for $A_x = A_z = (3 \mu \Phi_0^2)$, $A_{x} = (3 \mu \Phi_0^2)$, and $c = -0.25 \pm 0.05$. We stress that the same fitting parameters are used for both values of $\Delta$. The reason why the data for $\Delta = 2.5$ GHz show a less dramatic shift in $f_z$ is because $\partial E_{01}/\partial f_z$ is smaller at this value of $\Delta$. The values for $\partial E_{01}/\partial f_z$ and $\partial E_{01}/\partial f_c$ used in the fits were extracted from Fig. 2, giving $\partial \delta/\partial f_c = 1.37$ THz, $\partial \delta/\partial f_z = 110$ GHz for $\Delta = 2.5$ GHz.

![FIG. 4. (Color online) (a) Phase decay rate $\Gamma_{\phi F}$ expected from the sensitivities shown in (b), with $A_x = A_z = (3 \mu \Phi_{0}^2)$. (b) Sensitivity of the qubit energy to flux fluctuations. For perfectly correlated noise ($c_{f_z} = \pm 1$), the fluctuations cancel out when $\partial E_{01}/\partial f_z = \mp \partial E_{01}/\partial f_c$. (c) Measured decay rate $\Gamma_{\phi F}$. The fit gives a correlation factor of $c_{f_z} = -0.25$. (d) Schematic picture of the surface-spin model. Spins located on the shared line labeled $d_{c}$ induce fields pointing in opposite directions in the two loops.](image)
and \(\partial \varepsilon /\partial f_c = 1.28 \text{ THz}, \ \partial \Delta /\partial f_c = 142 \text{ GHz}\) for \(\Delta = 3.5 \text{ GHz}\).

It is important to point out that the correlations discussed so far have been between fluctuations \(\delta f_1, \delta f_2\) in the effective fluxes defined by Eq. (1). The fluctuations and the correlations in the geometric fluxes \(f_1, f_2\) can be found by inverting Eq. (1). Assuming that the fluctuations \(\delta f_1, \delta f_2\) are described by spectra \(S_f(\omega) = (A_1 \Phi_0^2)/|\omega|\) and \(S_f(\omega) = (A_2 \Phi_0^2)/|\omega|\), we find (see Appendix C for a derivation)

\[
A_1 = A_x + A_x/4 - c_{xx} \sqrt{A_x A_x} = (4.5 \mu \Phi_0)^2, \\
A_2 = A_x = (3.1 \mu \Phi_0)^2, \\
c_{12} = c_{xx} \sqrt{A_x/A_x} - 1/2 \sqrt{A_x/A_x} = -0.55. 
\]

Fabrication imperfections may cause deviations in the sizes of the two junctions in loop 2, which will affect Eq. (4). From the results in Fig. 2(b), we know that the junction asymmetry is less than \(\pm 10\%\), which will lead to an uncertainty of the correlation coefficient \(c_{12}\) of about \(\pm 0.1\). Note that the measurement is sensitive to correlations in the frequency range \(1 \text{ Hz to } \sim 1 \text{ MHz}\), as set by the FID weighting function.\(^{24,25}\)

To exclude the influence from room-temperature electronics, we note that the noise from the current source has an \(A_1/f\)-type noise spectrum with \(A_1 \sim (1 \text{ nA})^2\).\(^{28}\) This will create \(A_1 \sim (0.3 \mu \Phi_0)^2\) of flux noise in loop 1, which is an order of magnitude lower than the extracted noise amplitudes.

The results of Eq. (4) show that the flux fluctuations in the two loops of the sample are anticorrelated. This is clear evidence that the noise is not due to a global fluctuating magnetic field, since such fluctuations would give positive correlations. Instead, we consider a model where the noise is due to a large number of randomly oriented spins distributed over the metal surface [see Fig. 4(d)]. The spins will generate local magnetic fields that are picked up by the loops, and fluctuations of the spin orientations will give rise to flux noise.\(^{7}\)

In our sample, spins that are located on the line that is shared by the loops [labeled \(d_1\) in Fig. 4(d)] will induce fields of opposite directions in loop 1 and loop 2. This will give rise to negative correlations between the fluxes in the loops.

To get an estimate of the expected correlations, we assume (i) that the field of a single spin only couples to a loop if it is sitting on that loop’s circumference, (ii) that the width of the sample wires is constant, and (iii) that the amount of flux that a loop picks up from a single spin is independent of the spin’s position along the width of the wire. The assumptions seem reasonable based on numerical simulations of the coupling between a SQUID and the magnetic moment of a single spin.\(^{7}\)

Under these conditions the flux fluctuations generated by an ensemble of randomly oriented spins will scale as \(\delta f \propto \sqrt{\rho d}\), where \(\rho\) is the spin density and \(d\) the length of the loop. The correlation coefficient becomes

\[
c_{12} = \frac{\langle \delta f_1 \delta f_2 \rangle}{\sqrt{\langle \delta f_1^2 \rangle \langle \delta f_2^2 \rangle}} = \frac{-d_c}{\sqrt{(d_1 + d_2)(d_2 + d_2)}}. 
\]

Here \(d_c\) is the length of the line segment shared by both loops, and \(d_1, d_2\) are the lengths of the remaining sections of the two loops [Fig. 4(d)]. With values relevant for our sample \((d_1 = 10.8 \mu \text{m}, d_2 = 6.0 \mu \text{m}, d_3 = 4.8 \mu \text{m})\), we find \(c_{12} = -0.37\).

This is smaller than the value extracted experimentally, but the agreement is reasonable considering the simplicity of the model. Further work is needed to develop a more realistic model of the device (see Appendix A) and to describe the correlations when spin interactions are taken into account.\(^{15,16}\)

We also note that the noise ratio \(A_1/A_2\) roughly agrees with the ratio \((d_1 + d_2)/(d_2 + d_3)\) of the circumferences of the two loops, as expected from the surface spin model.

To conclude, we have investigated flux noise correlations by measuring dephasing rates in a tunable flux qubit. We find the flux fluctuations in neighboring loops to be anticorrelated, in agreement with predictions from models where the flux noise is generated by randomly oriented spins on the metal surface. The ability to extract noise correlations provides important information about the microscopic origin of the flux noise. The method presented here is general and can be extended to samples with different geometries. Combining this method with recent improvements in multipulse noise spectroscopy\(^{25}\) opens up the possibility of measuring correlations at different frequencies and thereby probing the dynamics of the surface spin system.

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APPENDIX A: SAMPLE DETAILS

Figure 5(a) shows a drawing of the qubit, indicating the displacement of the two superconducting layers and the overlapping regions where Josephson junctions are formed. In Fig. 5(b), we show an SEM picture of a device with the same geometry as the one used in the experiments. The qubit and the readout SQUID are shown at the upper right in the figure. The two lines next to the qubit are the local flux bias lines, while the thicker line at the lower left is the microwave drive line.
APPENDIX B: INDUCTANCE MATRIX

The fluxes and currents are related by the inductance matrix

\[
\begin{bmatrix}
 f_{SQ} \\
 f_z \\
 f_x
\end{bmatrix} =
\begin{bmatrix}
 M_{11} & M_{12} & M_{13} \\
 M_{21} & M_{22} & M_{23} \\
 M_{31} & M_{32} & M_{33}
\end{bmatrix}
\begin{bmatrix}
 I_{coil} \\
 I_z \\
 I_x
\end{bmatrix}.
\]

The matrix elements were determined by measuring \( \varepsilon, \Delta \), and the SQUID switching current as a function of the three bias currents, giving

\[
M = \begin{bmatrix}
\frac{1}{0.0355 \text{mA}} (1.0247 & 0.248) \\
\frac{1}{0.0355 \text{mA}} (0.303 & 0.232) \\
\frac{1}{0.03817 \text{mA}} (1 & 0.244 & 0.285)
\end{bmatrix}.
\]

APPENDIX C: DERIVATION OF FLUX CORRELATIONS

We start by defining the normalized fluxes \( f_1 = \Phi_1/\Phi_0 \) and \( f_2 = \Phi_2/\Phi_0 \) in the two qubit loops. Correlations between fluctuations \( \delta f_1 \) and \( \delta f_2 \) in these parameters are characterized by the correlation coefficient \( c_{12} \), defined as

\[
c_{12} = \frac{\langle \delta f_1 \delta f_2 \rangle}{\sqrt{\langle \delta f_1^2 \rangle \langle \delta f_2^2 \rangle}}.
\]

The qubit parameters \( \varepsilon = \varepsilon(f_z) \) and \( \Delta = \Delta(f_x) \) are controlled by the effective fluxes

\[
f_z = (f_1 + f_2)/2 - 1/2,
\]

\[
f_x = f_2.
\]

The coefficient for describing correlations between fluctuations \( \delta f_z \) and \( \delta f_x \) in the effective fluxes is given by

\[
c_{zx} = \frac{\langle \delta f_z \delta f_x \rangle}{\sqrt{\langle \delta f_z^2 \rangle \langle \delta f_x^2 \rangle}} = \frac{\langle \delta f_z + \delta f_x \rangle}{\sqrt{\langle \delta f_z^2 \rangle \langle \delta f_x^2 \rangle}}.
\]

We want to be able to extract \( c_{12} \) from the measured correlations \( c_{zx} \) between the effective fluxes. We start by inserting the expression for \( \langle \delta f_1 \delta f_2 \rangle \) from Eq. (C1) into Eq. (C3). This gives

\[
c_{zx} = \frac{c_{12} \sqrt{\langle \delta f_z^2 \rangle \langle \delta f_x^2 \rangle}}{\sqrt{\langle \delta f_z^2 \rangle \langle \delta f_x^2 \rangle}} + \frac{2}{\sqrt{\langle \delta f_x^2 \rangle \langle \delta f_z^2 \rangle}}
\]

\[
= c_{12} \sqrt{\langle \delta f_z^2 \rangle \langle \delta f_x^2 \rangle} + \frac{2}{\sqrt{\langle \delta f_z^2 \rangle \langle \delta f_x^2 \rangle}} \frac{1}{\sqrt{\langle \delta f_z^2 \rangle \langle \delta f_x^2 \rangle}}.
\]

In addition, we need to express the fluctuations \( \langle \delta f_2^2 \rangle \) in terms of the effective fluctuations \( c_{zx}, \langle \delta f_z^2 \rangle, \) and \( \langle \delta f_x^2 \rangle \). From the definition in Eq. (C2), we have

\[
\langle \delta f_z^2 \rangle = \langle \delta f_1^2 + \delta f_2^2/2 \rangle = \langle \delta f_1^2 \rangle + \langle \delta f_2^2 \rangle/2.
\]

Using the expression for \( \langle \delta f_1 \delta f_2 \rangle \) from Eq. (C1), we get

\[
\langle \delta f_z^2 \rangle = \langle \delta f_1^2 \rangle + c_{12} \sqrt{\langle \delta f_z^2 \rangle \langle \delta f_x^2 \rangle} + \langle \delta f_x^2 \rangle/4.
\]

Inserting the expression for \( c_{12} \) from Eq. (C5) gives

\[
\langle \delta f_z^2 \rangle = \langle \delta f_1^2 \rangle + c_{12} \sqrt{\langle \delta f_z^2 \rangle \langle \delta f_x^2 \rangle} - \frac{1}{2} \sqrt{\langle \delta f_z^2 \rangle \langle \delta f_x^2 \rangle} + \langle \delta f_x^2 \rangle/4.
\]

Note that \( \langle \delta f_z^2 \rangle = \langle \delta f_2^2 \rangle \), we have

\[
\langle \delta f_z^2 \rangle = \langle \delta f_1^2 \rangle + c_{12} \sqrt{\langle \delta f_z^2 \rangle \langle \delta f_x^2 \rangle} - \frac{1}{2} \sqrt{\langle \delta f_z^2 \rangle \langle \delta f_x^2 \rangle} + \langle \delta f_x^2 \rangle/4.
\]

Finally, solving for \( \langle \delta f_z^2 \rangle \) gives

\[
\langle \delta f_z^2 \rangle = \langle \delta f_1^2 \rangle + \langle \delta f_x^2 \rangle/4 - c_{12} \sqrt{\langle \delta f_z^2 \rangle \langle \delta f_x^2 \rangle}.
\]

We assume that the fluctuations \( \delta f_1, \delta f_2, \delta f_z, \) and \( \delta f_x \) are all described by noise spectra of the form \( S_f(\omega) = A_f/|\omega| \), \( S_\epsilon(\omega) = A_\epsilon/|\omega| \), and \( S_e(\omega) = A_e/|\omega| \). Using this and summarizing the results of Eq. (C5) and Eq. (C10), we get

\[
A_1 = A_z + A_x/4 - c_{12} \sqrt{A_z A_x},
\]

\[
A_2 = A_x,
\]

\[
c_{12} = c_{12} \sqrt{A_z/A_1} = \frac{1}{2} \sqrt{A_z/A_1}.
\]

This is Eq. (4) above.