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<td>As Published</td>
<td><a href="http://dx.doi.org/10.1103/PhysRevD.84.061701">http://dx.doi.org/10.1103/PhysRevD.84.061701</a></td>
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<tr>
<td>Publisher</td>
<td>American Physical Society (APS)</td>
</tr>
<tr>
<td>Version</td>
<td>Final published version</td>
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<td>Citable Link</td>
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Implications of the Babinet Principle for Casimir interactions

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(Received 30 March 2011; published 1 September 2011)

We formulate the Babinet Principle (BP) as a relation between scattering amplitudes and combine it with multiple scattering techniques to derive new properties of electromagnetic Casimir forces. We show that the Casimir force exerted by a planar conductor or dielectric on a self-complementary perforated planar mirror is approximately half that on a uniform mirror independent of the distance between them. Also, the BP suggests that Casimir edge effects are generically anomalously small. Furthermore, the BP can be used to relate any planar object to its complementary geometry, a relation we use to estimate Casimir forces between two screens with apertures.

In this case, we use the BP to show that the electromagnetic Casimir force between such a screen and a perfect mirror is approximately half the force between two perfect mirrors. Subject to a proviso about the locality of Casimir forces, we suggest that Casimir edge effects are anomalously small in electromagnetism. Finally, we show how the BP can be used to compute the Casimir force between perforated screens.

The scattering of waves from a screen is described in terms of a scattering amplitude, $\tilde{F}$, related to the unitary $S$-matrix by $\tilde{F} = \frac{1}{2}(S - I)$. In the absence of the screen, $\tilde{F}$ vanishes. It is convenient to distinguish the contribution to $\tilde{F}$ corresponding to transmission of waves across a screen, denoted by $\tilde{T}$, from the contribution corresponding to waves that are reflected, denoted by $\tilde{R}$. Thus, a plane wave $|k\rangle$ incident from the left on a screen $\Sigma$ gives rise to a reflected wave, $\tilde{R}|k\rangle$, on the left and a transmitted wave, $\tilde{T}|k\rangle$, on the right.

For the sake of simplicity and clarity, we discuss the case of a scalar field and then quote the generalization to electromagnetism, where the predictive power of the BP is considerably greater. Without loss of generality, we assume a planar incident wave $|k\rangle$ impinging on the screen from the left-hand side, $k_z > 0$. The scattering off the screen $\Sigma$ is defined by the ansatz

$$|\phi\rangle = \begin{cases} |k\rangle + \sum_{k'} \tilde{R}_{k,k'} |k'\rangle & \text{on the left,} \\ |k\rangle + \sum_{k'} \tilde{T}_{k,k'} |k'\rangle & \text{on the right.} \end{cases}$$

Note that the trivial forward scattering is separated out from the diffraction described by $\tilde{R}$ and $\tilde{T}$. If the same incident wave, $|k\rangle$, shines on the complementary screen, $\tilde{\Sigma}$, the scattering can be described by a similar ansatz with the corresponding scattering amplitudes denoted by...
The scattering is determined by the BC that the scalar field obeys on the screen, which we take to be either Dirichlet or Neumann. Applied to scalar fields, the BP relates the diffraction of a field that obeys a Dirichlet boundary condition on $\Sigma$, to the diffraction experienced by a field that obeys a Neumann boundary condition on $\tilde{\Sigma}$, and vice versa.

The proof of the BP is discussed in many textbooks, where it is usually assumed that one screen has compact support, while its complement extends to infinity. We have extended the BP for both scalar and electromagnetic fields to the case where neither screen has compact support, e.g., each is a half-plane. The proof is inspired by an argument in Ref. [15], and is based on the fact that a general solution to the wave equation can be cast as a sum of terms even or odd in $z$. In the Dirichlet (Neumann) case, the odd (even) solution is trivial. It is not hard to show that an odd solution in the Neumann case for screen $\Sigma$ can be constructed from the even solution for $\tilde{\Sigma}$ in the Dirichlet case and vice versa. Uniqueness of solutions to the Helmholtz equation fixes the correspondence of solutions to be one-to-one. A linear combination of even and odd terms sets up the scattering problem. The result is a relation between transmission and reflection matrices for the screen and its complement. For example, in the former case, $T_{k'}^N = \frac{1}{C_6} \frac{k}{D=N} I k$, where $I k$ is the null set, then $T_{k'=k}$, as expected. For the other channel, the BP dictates $T_{k'^N}^{D/N} + T_{k'}^{N/D} = -I_{k'}$, once again, this can be checked in the limiting case where $\Sigma$ is either a full screen or the null set. For example, in the former case, $T_{k'=k}^{D/N} = I_{k'}$, which cancels the incident wave on the right side of the screen.

Similar equations hold for electromagnetism. However, a slight complication arises since the scattering from a screen generally mixes electric (E) and magnetic (M) polarizations. The BP must then be cast in a matrix form,

$$
\begin{pmatrix}
E_{k'k}^{MM} - E_{k'k}^{EE} \\
E_{k'k}^{EM} + E_{k'k}^{ME}
\end{pmatrix} =
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix},
$$

(2)

and

$$
\begin{pmatrix}
T_{k'k}^{MM} - T_{k'k}^{EE} \\
T_{k'k}^{EM} + T_{k'k}^{ME}
\end{pmatrix} =
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix},
$$

(3)

where the labels $\{k, k'\}$ have been suppressed. To summarize, the scalar version of the BP relates Dirichlet to Neumann BC and vice versa, while the electromagnetic version relates scattering with conducting BC back to itself, merely interchangeing polarizations.
this configuration. For an infinite plate, \( R_{pp} \) is diagonal in polarization, hence, \( R_{pp} = R_{pp} \delta_{pp} \). So are the translation matrices \( U_{pp} = U_{pp} \delta_{pp} \). The Casimir energy in the first reflection then becomes

\[
\mathcal{E}_1 = -\frac{\hbar c}{2\pi} \sum_{p-E,M} \int_0^\infty dk \text{tr}(R_{pp}^{p}\mathcal{U}_{pp}^p R_{sc}^{pp}\mathcal{U}_{pp}^{pp1}),
\]

where \( R_{sc} \) is the reflection amplitude for the self-complementary geometry. Equation (2) relates the reflection matrices of opposite polarizations, \( R_{MM} - R_{EE} = -I \) and \( R_{EE} - R_{MM} = I \). Using this, one can show that the sum \( \mathcal{E}_1 + \mathcal{E}_2 \) is equal to the energy of two infinite plates in the first reflection. However, for a self-complementary geometry, \( R_{sc} = R_{sc} \) up to a trivial translation or rotation matrix. \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are therefore equal and both equal to half the interaction of two infinite plates in the first reflection approximation. For a perfect conductor, this reads

\[
\mathcal{E}_1 = \frac{1}{2} \mathcal{E}_1(\text{parallel-plates}) = -\frac{\hbar c A}{16\pi^2 d^3},
\]

independent of any internal length scale that characterizes the self-complementary geometry. Thus, for example, the size of the strips, the squares, or the wedges in Fig. 1 does not enter the expression for the energy in the first reflection.

The higher-reflection corrections to Eq. (5) are no worse than the case of two parallel plates, where they are less than 8%. This is because the higher-reflection terms involve higher powers of the \( R_{sc} \). The absolute value of the eigenvalues of \( R_{sc} \) are all less than (or equal to) unity (whereas those of an infinite reflecting plate, are all unity), so higher reflections’ contributions are further suppressed.

If the separation between \( \Sigma \) and the reflecting plate is much smaller than the length scale of the structure on the screen (\( d \ll a \) in Fig. 1), then Eq. (5) follows from the (first reflection approximation to the) proximity force approximation (PFA) and is expected [20]. This approximation treats the objects locally as parallel plates and is exact in the limit of close proximity. When the separation is comparable to or larger than the length scales of \( \Sigma \), then our result is by no means obvious. In fact, one might have thought that a mesh like the checkerboard completely reflects electromagnetic waves with wavelength \( \sim d \), when \( d \gg a \) leading to \( \mathcal{E}_1 \rightarrow \mathcal{E}_1(\text{parallel plates}) \) as is the case, for example, for corrugated or pitted conductor when \( d \) is large compared to the length scale of the corrugations or pitting [21].

An interesting special case is a half-plate. On dimensional grounds, the energy can be separated into two contributions, \( \mathcal{E} = \mathcal{E}_{\text{area}} + \mathcal{E}_{\text{edge}} \), the first proportional to the area and the second proportional to the edge length. The former is given by \( \mathcal{E}_{\text{area}} = -\pi^2 \hbar c A/720d^3 \). Equation (5) rules out the edge term in the first reflection for electromagnetism, \( \mathcal{E}_{\text{edge}} = 0 \), since the energy differs from parallel plates only by a trivial factor of one-half—which accounts for the area of the half-plate. Likewise, the scalar Dirichlet and Neumann edge terms are equal in magnitude and opposite in sign in the first reflection approximation. The absence of the edge term in the first reflection for a half-plate geometry was first observed in Ref. [18] as a limiting case of a wedge with a zero opening angle.

We believe this argument can be generalized further. Consider a screen \( \Sigma \) with a structure characterized by a minimum length scale, \( a \). Then, as the separation from a reflecting plate, \( d \), goes to zero, the leading contribution to the Casimir interaction energy, proportional to \( A/d^3 \), is captured, as always, by the PFA. If the next term in the expansion in \( 1/d \) is also determined locally, then it must be proportional to the perimeter of the screen \( \Sigma \),

\[
\mathcal{E} \sim_{d \to 0} -\hbar c (\alpha_A A/d^3 + \alpha_p P/d^2 + \cdots).
\]

The coefficient of the first term is set by the PFA, \( \alpha_A = \pi^2/720 \). Locally as \( d \to 0 \), any edge looks like a half-plate, therefore, we can conclude that \( \alpha_p \) is zero in first reflection, so the edge effects are very small. More precisely, if edge effects are local, for electromagnetism \( |\alpha_p| < \alpha_A \), while for Dirichlet and Neumann BC (for which, the edge term is not absent in the first reflection), we have \( |\alpha_p| \sim \alpha_A \) [18,22]. The absence of an edge term in the electromagnetic Casimir energy for a piston geometry [23] supports this conjecture.

Finally, we consider a screen with small apertures. The BP relates diffraction by an aperture to diffraction by its complement. If the aperture is small, then its complement is a small object whose Casimir interactions can be computed in the Casimir-Polder limit [24]. The exact relation involves interchanging polarizations [see Eqs. (2) and (3)]. For a screen with a small hole opposite an infinite reflecting plate, it is easy to show that in the first reflection approximation, the energy is given by the interaction of two infinite plates minus the interaction of the complementary object with the infinite plate. This gives a small correction to the force between parallel plates.

An interesting case is the lateral force between two screens with perforations. The energy can be written as the sum of the interaction between parallel plates (without the holes), the hole-plate and plate-hole interactions, and the hole-hole interaction, plus higher reflections, \( \mathcal{E} = \mathcal{E}_{\text{plates}} + \mathcal{E}_{\text{plate-hole}} + \mathcal{E}_{\text{hole-plate}} + \mathcal{E}_{\text{hole-hole}} + \cdots \). Only the interaction between the holes can give rise to a lateral force between the two plates and \( \mathcal{E}_{\text{hole-hole}} \) is given by the electromagnetic interaction between two objects of the same size and position as the holes. If the holes are small compared to the interscreen separation, the higher reflections are negligible.

As an application, consider two parallel perforated plates separated by a distance \( d \), each with a square array of small circular holes of radius \( R \) and center-to-center separation \( \Delta \). According to the BP and in leading order in...
FIG. 2 (color online). The lateral force per area as a function of the displacement \( \delta \). For this graph, we chose \( d = 2\Delta \). The point \( \delta = 0 \) is an stable equilibrium of this configuration.

\[
R/d \text{ and } R/\Delta, \text{ the lateral force between the two plates is identical to the lateral force between two arrays of discs with the same size and spacing as the holes. The Casimir interaction between two objects with electric polarizability matrices } \alpha_1 \text{ and } \alpha_2 \text{ is given by } [26,27]
\]

\[
\mathcal{E} = -\frac{\hbar c}{8\pi^2} \left[ 13\text{tr}(\alpha_1\alpha_2) - 56\text{tr}(\alpha_1\alpha_2\Omega) + 63\text{tr}(\Omega\alpha_1\Omega\alpha_2) \right],
\]

where \( \Omega = \hat{n}n^T \) with \( \hat{n} \) being the unit vector connecting the two dipoles separated by a distance \( r \). The force between the two arrays of dipoles can be computed easily. For a perfect conductor, \( \alpha_1 = \alpha_2 = \text{diag}(\alpha, \alpha, 0) \), where \( \alpha = 4R^3/3\pi \); only the polarizability components parallel to the disk are nonzero. Figure 2 shows the lateral force as a function of the lateral displacement \( \delta \).

Finally, we comment on the physical conditions that must be satisfied for the BP to apply to a screen made of a good conductor like gold. On the one hand, the screen thickness, \( t \), should be large compared to the skin depth, \( \delta \). On the other hand, \( t \) should be negligible compared to the separation \( d \). So the thickness \( t \) should satisfy \( \delta \ll t \ll d \). \( \delta \) is determined from the frequency by \( \delta = \sqrt{2/\mu_0\omega\sigma} \), where \( \omega \sim 2\pi c/d \), and the inequalities become \( \sqrt{d/\pi c\mu_0\sigma} \ll t \ll d \). For gold, the skin depth is \( \delta \approx 5 \text{ nm} \) at \( d = 1 \text{ \mu m} \) and decreases proportional to \( \sqrt{d} \) as \( d \) decreases. For measurements in the \( d \sim 0.5–1.0 \text{ \mu m} \) range, a thickness of 100 nm should suffice.

We thank T. Emig, N. Graham, M. Kardar, M. Krüger, and U. Mohideen for helpful conversations. This work was supported in part by the U. S. Department of Energy under cooperative research Agreement No. DE-FG02-05ER41360 (R. L. J. and M. F. M.) and the NSF Grant No. DMR-08-03315, DARPA Contract No. S-000354 (M. F. M.).

16. The BP was used to relate the Casimir energy of an isolated screen to its complement in N. Graham and K. D. Olum, Phys. Rev. D 72, 025013 (2005).
25. For simplicity, we ignore magnetic response.