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Superstring theory in ten dimensions is arguably the most promising candidate for a unified quantum mechanical description of gravity and other interactions. This theory, however, takes different guises. For instance, there are two different string theories with maximal supersymmetry, the type-IIA and the type-IIB theory. The ten-dimensional superstring theories, together with a global symmetry, the type-IIA and the type-IIB theory. The scalar dilaton $\phi$, where $i, j, \ldots = 1, \ldots, D$ are space-time indices. The DFT is formulated in terms of a dilaton density $d$, which is related to $\phi$ via the field redefinition $e^{-2d} = \sqrt{g} e^{-2\phi}$, and the “generalized metric”

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -g^{ik} b_{kj} \\ b_{ik} g^{kj} & g_{ij} - b_{ik} b_{kj} \end{pmatrix},$$

which combines $g$ and $b$ into an $O(D, D)$ covariant tensor with indices $M, N, \ldots = 1, \ldots, 2D$. All fields depend on the doubled coordinates $X^M = (\xi, x)$, and view (1) as just a particular parametrization. The action can be written as

$$S = \int dx \bar{\xi} e^{-2d} \mathcal{R}(\mathcal{H}, d),$$

where $\mathcal{R}(\mathcal{H}, d)$ is an $O(D, D)$ invariant scalar, cf. Eq. (4.24) in the second reference of [4], and we use the shorthand notation $dx = d^D x$. The action is invariant under the gauge transformations

$$\delta_\xi \mathcal{H}_{MN} = \xi^P \partial_P \mathcal{H}_{MN} + 2(\partial_M \xi^P - \partial_P \xi^M) \mathcal{H}_{NP},$$

$$\delta_\xi d = \xi^M \partial_M d - \frac{1}{2} \partial_M \xi^M,$$

with the derivatives $\partial_M = (\partial_\xi, \partial_\xi)$. Here, $O(D, D)$ indices $M, N$ are raised and lowered with the invariant metric

$$\eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and (anti)symmetrizations are accompanied by the combinatorial factor $\frac{1}{2}$. The consistency of the above theory requires the constraint

$$\partial^M \partial_M A = \eta^{MN} \partial_M \partial_N A = 0, \quad \partial^M A \partial_M B = 0,$$

for all fields and parameters $A$ and $B$. This constraint implies that locally the fields depend only on half of the

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**Unification of Type-II Strings and $T$ Duality**

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We present a unified description of the low-energy limits of type-II string theories. This is achieved by a formulation that doubles the space-time coordinates in order to realize the $T$-duality group $O(10, 10)$ geometrically. The Ramond-Ramond fields are described by a spinor of $O(10, 10)$, which couples to the gravitational fields via the Spin(10, 10) representation of the so-called generalized metric. This theory, which is supplemented by a $T$-duality covariant self-duality constraint, unifies the type-II theories in that each of them is obtained for a particular subspace of the doubled space.

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coordinates, and one can always find an $O(D, D)$ transformation into a frame in which the fields depend only on the $x^i$. If one drops the dependence on the "dual coordinates" $\tilde{x}_i$ in (2) or, equivalently, sets $\tilde{\delta}^i = 0$, the action reduces to the conventional low-energy effective action

$$S = \int d^Dx \sqrt{-g} e^{-2\Phi} \left[ R + 4(\partial \Phi)^2 - \frac{1}{12} H^{ijk} H_{ijk} \right].$$

(6)

where $H_{ijk} = 3\tilde{\delta}_{ij} b_{kj}$ is the field strength of the 2-form. Moreover, for $\tilde{\delta}^i = 0$ the gauge transformations (3) with parameter $\xi^M = (\tilde{\xi}_i, \tilde{\xi}^i)$ reduce to the conventional general coordinate transformations $x^i \rightarrow x^i - \xi^i(x)$ and to the gauge transformations of the 2-form, $\delta b_{ij} = 2\partial_i \tilde{\xi}^j$.

Let us now turn to the extension by the RR sector. In this we make significant use of the work of Fukuma, Oota, and Tanaka [10]. (See also [11,12].) The RR sector consists of forms of degrees 1 and 3 for type IIA and of degree 2 and 4 for type IIB, where the 5-form field strength of the 4-form is subject to a self-duality constraint. Here, we will use a democratic formulation that simultaneously uses dual forms, such that type IIA contains all odd forms and type IIB contains all even forms, both being supplemented by duality relations [10]. The set of all forms naturally combines into a Majorana spinor of $O(10,10)$. Imposing an additional Weyl condition yields a spinor containing either all even or all odd forms, and we will show that the DFT extension of the RR sector can be formulated in terms of such a spinor.

We start by fixing our conventions for the spinor representation, setting $D = 10$ from now on. More precisely, these are representations of the double covering groups $\text{Pin}(10,10)$ of $O(10,10)$ and $\text{Spin}(10,10)$ of $SO(10,10)$. The gamma matrices satisfy the Clifford algebra

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}1.$$  

(7)

A convenient representation can be constructed using fermionic oscillators $\psi^i$ and $\psi_i$, satisfying

$\{\psi_i, \psi^j\} = \delta^i_j$,  $\{\psi_i, \psi_j\} = 0$,  $\{\psi^i, \psi^j\} = 0$,  

(8)

where $(\psi_i)^\dagger = \psi^i$. With (4) we infer that they realize the algebra (7) via

$$\Gamma_i = \sqrt{2}\psi_i,  \quad \Gamma^i = \sqrt{2}\psi^i.$$  

(9)

Introducing a "Clifford vacuum" $|0\rangle$ with $\psi_i|0\rangle = 0$ for all $i$, and the normalization $\langle 0|0\rangle = 1$, we can construct the representation by successive application of the raising operators $\psi^i$. A general spinor state then reads

$$\chi = \sum_{p>0} \frac{1}{p!} C_{i_1\ldots i_p} \psi^{i_1} \ldots \psi^{i_p} |0\rangle,$$

(10)

whose coefficients $C_{i_1\ldots i_p}$ can be identified with $p$-forms $C^{(p)}$. Any element $S$ of the Pin group projects, via a group homomorphism $\rho$: $\text{Pin}(10,10) \rightarrow O(10,10)$, to an element

$$\rho(S)$$

for all $h \in O(10,10)$,

$$\Gamma_M \rho(S)^{-1} = S_{\rho(h)}.$$

(11)

where $h \eta h^T = \eta$. Conversely, for any $h \in O(10,10)$, there is a $S \in \text{Pin}(10,10)$ such that both $\pm S$ project to $h$. A spinor can be projected to a spinor of fixed chirality, i.e., to eigenstates $\chi_{\pm}$ of $(-1)^{N_F}$ with eigenvalues $\pm 1$, where $N_F = \sum_i \psi^i \psi_i$ is the “fermion number operator.” The spinor $\chi_+$ of positive chirality then contains only even forms, and the spinor $\chi_-$ of negative chirality contains only odd forms. Imposing a chirality constraint reduces the symmetry from $\text{Pin}(10,10)$ to $\text{Spin}(10,10)$ since only the latter leaves this constraint invariant. Finally, we need the charge conjugation matrix satisfying

$$C \Gamma^M C^{-1} = (\Gamma^M)^\dagger.$$

(12)

A particular realization is given by

$$C = (\psi^1 - \psi_1)(\psi^2 - \psi_2) \cdots (\psi^{10} - \psi_{10}),$$

which satisfies $C \psi_i C^{-1} = \psi^i$ and thereby (12).

Given a spinor (10) we can act with the Dirac operator

$$\gamma = \frac{1}{\sqrt{2}} \Gamma^M \partial_M = \psi^i \partial_i + \psi^i \tilde{\delta}^i,$$

(14)

which can be viewed as the $O(10,10)$ invariant extension of the exterior derivative. In fact, for $\tilde{\delta} = 0$, it differentiates with respect to $x^i$ and increases the form degree by one, thus acting like $d$. Moreover, it squares to zero,

$$\gamma^2 = \frac{1}{2} \Gamma^M \Gamma^N \partial_M \partial_N = \frac{1}{2} \eta^{MN} \partial_M \partial_N = 0,$$

(15)

using (7) and the constraint (5).

In order to write an action that couples the NS-NS fields represented by the generalized metric $\mathcal{H}$ in (1) to the RR fields represented by a spinor $\chi$, we note that the matrix $\mathcal{H}$ is an $SO(10,10)$ group element and thus has a representative in $\text{Spin}(10,10)$, as has been used in dimensionally reduced theories [10]. In our case, however, a subtlety arises because (1) contains the full space-time metric, which we assume to be of Lorentzian signature. $SO(10,10)$ has two connected components, $SO^+(10,10)$, which contains the identity, and $SO^-(10,10)$. Because of the Lorentzian signature of $g$, $\mathcal{H}$ is actually an element of $SO^-(10,10)$. It turns out that a spin representative $\mathcal{S}_{\mathcal{H}} \in \text{Spin}(10,10)$ of $\mathcal{H}$ cannot be constructed consistently over the space of all $\mathcal{H}$. For instance, one may find a closed loop $\mathcal{H}(i)$, $i \in [0,1]$, $\mathcal{H}(0) = \mathcal{H}(1)$, in $SO^-(10,10)$, with the initial and final elements related by a timelike $T$ duality, for which a continuously defined spin representative yields $\mathcal{S}_{\mathcal{H}(1)} = -\mathcal{S}_{\mathcal{H}(0)}$. As a result, timelike $T$ dualities cannot be realized as transformations of the conventional fields $g$ and $b$. Nevertheless, a fully $T$-duality invariant action can be written if we treat the spin representative itself as the dynamical field. We thus introduce a field $\mathcal{S}$, satisfying

$$S \Gamma_M S^{-1} = \Gamma_N h^N_M,$$

(11)
\[ S = S^\dagger, \quad S \in \text{Spin}^{-}(10, 10). \] (16)

The generalized metric is then defined by the group homomorphism, \( \rho(S) = H \). By (16) and the general properties of the group homomorphism [7], \( H^T = \rho(S^\dagger) = H \) and so, as required, \( H \) is symmetric.

We are now ready to define the DFT formulation of type-II theories, whose independent fields are \( S, d, \) and \( \chi \). The action reads

\[ S = \int dxd\bar{\chi}(e^{-2d}\mathcal{R}(H, d) + \frac{1}{4}(\partial \chi)^\dagger \partial \chi). \] (17)

and is supplemented by the self-duality constraint

\[ \partial \chi = -\mathcal{K} \partial \chi, \quad \mathcal{K} = C^{-1}S. \] (18)

For the special case of type IIA, a similar duality relation has also been proposed in [8].

The field equation of \( \chi \) reads

\[ \delta(\mathcal{K}\partial \chi) = 0, \] (19)

which also follows as an integrability condition from the duality relation (18), upon acting with \( \partial \) and using (15). The field equation of \( S \) reads

\[ \mathcal{R}_{MN} + \mathcal{E}_{MN} = 0, \] (20)

where \( \mathcal{R}_{MN} \) is the DFT extension of the Ricci tensor [4], and the “energy-momentum” tensor reads, using (18),

\[ \mathcal{E}^{MN} = -\frac{1}{16} H^{(M} \rho(\chi) \Gamma^{N)P} \partial \chi. \] (21)

Let us now discuss the symmetries of this theory. First, it is invariant under a global action by \( S \in \text{Spin}^{+}(10, 10), \)

\[ \chi \to S \chi, \quad S \to S' = (S^{-1})^\dagger S S^{-1}, \] (22)

implying \( \partial \chi \to S \partial \chi \). Specifically, \( \chi \) is assumed to have a fixed chirality, which breaks the invariance group of the action from \( \text{Pin}(10, 10) \) to \( \text{Spin}(10, 10) \), while the duality relations break the invariance group to \( \text{Spin}^{+}(10, 10) \). The gauge symmetries of this theory are given by

\[ \delta_\lambda \chi = \delta \lambda, \] (23)

with spinorial parameter \( \lambda \), leaving (17) and (18) manifestly invariant by (15), and the gauge symmetry (3) parametrized by \( \xi^M \). On the new fields \( S \) and \( \chi \) it reads

\[ \delta_\xi \chi = \xi^M \partial_M \chi + \frac{1}{2} \partial_M \xi_N \Gamma^M \Gamma^N \chi, \]

\[ \delta_\xi \mathcal{K} = \xi^M \partial_M \mathcal{K} + \frac{1}{2} [\Gamma^P \mathcal{K}] \partial_P \xi^Q, \] (24)

where we have written the gauge variation of \( S \) in terms of \( \mathcal{K} \) defined in (18). It can be checked that this gauge transformation gives rise to the required variation (3) of \( H \) upon application of \( \rho \).

We will now evaluate the DFT defined by (17) and (18) in particular \( T \)-duality frames, starting with \( \delta^T = 0 \). To this end, we have to choose a particular parametrization of \( S \).

Writing

\[ H = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} = h_b^T h_g^{-1} h_b, \] (25)

we have to find spin representatives of the group elements \( h_b \) and \( h_g \). The subtlety here is that, with \( g \) Lorentzian, \( h_g \) takes values in \( SO^-(10, 10) \) and thus is not in the component connected to the identity. It is then convenient to write \( g \) in terms of vielbeins,

\[ g = e \bar{e}^T, \quad h_g = h_e h_h h_h^T, \] (26)

where \( e \) has positive determinant, i.e., \( e \in GL^+(10) \), and \( k \) is the flat Minkowski metric \( \text{diag}(-1, 1, \ldots, 1) \). The group elements \( h_e \) and \( h_h \) are in the component connected to the identity and so their spin representatives can be written as simple exponentials,

\[ S_b = e^{-(1/2)b_{ij} \psi^i \psi^j}, \quad S_e = \frac{1}{\sqrt{\det e}} e^{\psi^1 \psi^j}, \] (27)

with \( e = \exp(E) \), as can be verified with (11). A spin representative for the matrix \( k \) can be chosen to be [13]

\[ S_k = \psi^1 \psi_1 - \psi_1 \psi^1, \] (28)

where \( 1 \) labels the timelike coordinate. This can also be verified with (11). A spin representative \( S_{\mathcal{H}} \) of \( \mathcal{H} \) can then locally be defined as

\[ S_{\mathcal{H}} = S_b^{-1} S_g^{-1} S_b, \quad S_g = S_e S_k S_e^\dagger. \] (29)

We now set \( S = S_{\mathcal{H}} \), but we stress that this is just a particular parametrization in much the same way that (1) is just a particular parametrization of \( H \).

It is now straightforward to evaluate the action (17) for \( \mathcal{H} = 0 \). First, as noted above, \( \delta \chi \) reduces to the exterior derivatives of the \( C(p) \), \( F(p+1) \equiv dC(p) \). The action of \( S_k \) in \( S_{\mathcal{H}} \) then modifies this, using (27), to

\[ \hat{F} = e^{-b} \wedge F = e^{-b} \wedge dC. \] (30)

Second, (29) implies for the action of \( S_g^{-1} \)

\[ S_g^{-1} \psi^{i_1} \ldots \psi^{i_p} |0\rangle = -\sqrt{g} g^{i_1 j_1} \ldots g^{i_p j_p} \psi^{j_1} \ldots \psi^{j_p} |0\rangle. \] (31)

The Lagrangian corresponding to the RR part of (17) then reduces to kinetic terms for all forms,

\[ L_{\text{RR}} = -\frac{1}{4} \sqrt{g} \sum_{p=1}^{D} \frac{1}{p!} g^{i_1 j_1} \cdots g^{i_p j_p} \hat{F}_{i_1 \ldots i_p} \hat{F}_{j_1 \ldots j_p}, \] (32)

where we recall that the sum extends over all even or all odd forms, depending on the chirality of \( \chi \). Similarly, using (13), the self-duality constraint (18) reduces to the conventional duality relations (with the Hodge star *),

\[ F^{(p)} = (-1)^{(D-p)(D-p-1)/2} \ast \hat{F}^{(D-p)}. \] (33)

We have thus obtained the democratic formulation of type-II theories, whose field equations are equivalent to
the conventional field equations of type IIA for odd forms and of type IIB for even forms [10].

Let us briefly comment on the gauge symmetries for \( \tilde{\delta} = 0 \). The transformations (24) for \( \chi \), parametrized by \( \xi^M = (\tilde{\xi}_i, \tilde{\xi}^i) \), reduce to the conventional general coordinate transformations \( x^i \to x^i - \tilde{\xi}_i(x) \) of the \( p \)-forms \( C^{(p)} \), and also to nontrivial transformations under the \( b \)-field gauge parameter \( \tilde{\xi}_i, \tilde{\delta}_i C = d\tilde{\xi} \wedge C \).

We turn now to the discussion of other \( T \)-duality frames, starting with \( \tilde{\delta}_i = 0, \tilde{\delta}^i \neq 0 \). For the analysis of this case it is convenient to perform a field redefinition according to the \( T \)-duality transformation \( J \) that exchanges \( x^i \) and \( \tilde{x}_i \), and which, as a matrix, coincides with \( \eta \) defined in (4),

\[
\mathcal{H}^I = J \mathcal{H} J = \mathcal{H}^{-1}.
\]

(34)

It has been shown in [4] that the NS-NS part of the DFT reduces for \( \tilde{\delta}_i = 0 \) to the same action (6), but written in terms of the primed (\( T \)-dual) variables. Next, we define a corresponding field redefinition for the RR fields, using a spin representative \( S^I \) of \( J \),

\[
\chi' = S_J \chi, \quad \delta^I = \psi^I \tilde{\delta}^i + \psi_i \delta_i, \quad \delta^I \chi' = S_J \delta \chi.
\]

(35)

For the RR action we then find

\[
L_{RR} = \frac{1}{4} (\partial^I \chi')^s S_{J^s} \partial^I \chi = \frac{1}{4} (\tilde{\delta}^I \chi')^t (S_{J^t})^s S_{J^s} \tilde{\delta}^I \chi' = \frac{1}{4} (\tilde{\delta}^I \chi')^s S_{J^s} \tilde{\delta}^I \chi',
\]

(36)

where we used that \( J \) contains a timelike \( T \) duality such that, as mentioned above, this leads to a sign factor in the transformation of \( S_{J^I} \). Thus, in the new variables the action takes the same form as in the original variables, up to a sign. The transformed Dirac operator in (35) implies that setting \( \delta_i = 0 \) in the first form in (36) is equivalent to setting \( \delta^I = \psi^I \tilde{\delta}^i \) in the final form in (36). This way to evaluate the action is, however, equivalent to our computation above of setting \( \tilde{\delta} = 0 \) in the original action, just with fields and derivatives replaced by primed fields and derivatives. Thus, we conclude that the DFT action reduces for \( \delta_i = 0 \) to a type-II theory with the overall sign of the RR action reversed. These are known as type-II* theories and have been introduced by Hull in the context of timelike \( T \) duality [14]. They are defined such that the timelike circle reductions of type IIA (IIB) and type IIB* (IIA*) are equivalent. This result also implies that the overall sign of \( \mathcal{S} \) has no physical significance in that it merely determines for which coordinates \( (x, \tilde{x}) \) we obtain the type-II or type-II* theory.

More generally, one finds that evaluating the DFT in a \( T \)-duality frame that is obtained by an odd (even) number of \( T \)-duality inversions from a frame in which the theory reduces, say, to type IIA, it reduces to the \( T \)-dual theory, i.e., type IIB (IIA) for spacelike transformations and type IIB* (IIA*) for timelike transformations. Summarizing, the DFT defined by (17) and (18) combines all type-II theories in a single universal formulation. We hope that this theory may provide insights into the still elusive formulation of string theory as, e.g., for a yet to be constructed type-II string field theory.

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