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Unification of Type-II Strings and T Duality

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We present a unified description of the low-energy limits of type-II string theories. This is achieved by a formulation that doubles the space-time coordinates in order to realize the T-duality group $O(10,10)$ geometrically. The Ramond-Ramond fields are described by a spinor of $O(10,10)$, which couples to the gravitational fields via the Spin$(10,10)$ representative of the so-called generalized metric. This theory, which is supplemented by a T-duality covariant self-duality constraint, unifies the type-II theories in that each of them is obtained for a particular subspace of the doubled space.

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Superstring theory in ten dimensions is arguably the most promising candidate for a unified quantum mechanical description of gravity and other interactions. This theory, however, takes different guises. For instance, there are two different string theories with maximal supersymmetry, the type-IIA and the type-IIB theory. The ten-dimensional superstring theories, together with 11-dimensional supergravity, are different limits of a single underlying theory and are related through a web of dualities (see, e.g., [1]). The simplest of these dualities is $T$-duality that, for instance, relates type-IIA string theory on the circular background $\mathbb{R}^{8,1} \times S^1$ of radius $R$ to type-IIB string theory on the same background, but with radius $1/R$.

In its low-energy limit string theory is described by Einstein’s theory of general relativity, coupled to particular matter fields. In this description, $T$ duality results in the appearance of the hidden symmetry group $O(d,d)$ upon dimensional reduction on a torus $T^d$. Moreover, the low-energy limits of type IIA and type IIB give rise to the same theory, consistent with their equivalence under $T$ duality [2].

The general coordinate invariance of Einstein gravity naturally explains the presence of the $GL(d,d)$ subgroup, but the emergence of the full $O(d,d)$ upon dimensional reduction requires the precise matter couplings predicted by string theory, hinting at a novel geometrical structure. Recently, a “double field theory” (DFT) has been found which realizes a $T$-duality group prior to dimensional reduction [3,4] (see also [5,6]). By doubling the space-time coordinates, the low-energy effective action of bosonic string theory or, equivalently, of the Neveu-Schwarz–Neveu-Schwarz (NS-NS) sector of superstring theory, can be extended to an action that has $O(D,D)$ as a global symmetry, where $D$ is the space-time dimension.

In this Letter we introduce the extension to the Ramond-Ramond (RR) sector of type-II strings, which will lead to a theory that contains all type-II theories simultaneously in different $T$-duality “frames.” Here we will not present explicit derivations, but a more detailed exposition will appear elsewhere [7]. Related work has appeared in [8,9].

We start by reviewing the NS-NS subsector. It consists of the metric $g_{ij}$, the Kalb-Ramond 2-form $b_{ij}$, and the scalar dilaton $\phi$, where $i,j,\ldots=1,\ldots,D$ are space-time indices. The DFT is formulated in terms of a dilaton density $e^{-2\phi} = \sqrt{\det g}$, and the “generalized metric”

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} & \end{pmatrix},$$

which combines $g$ and $b$ into an $O(D,D)$ covariant tensor with indices $M,N,\ldots=1,\ldots,2D$. All fields depend on the doubled coordinates $X^M = (\xi, x)$.

where $\mathcal{R}(\mathcal{H},d)$ is an $O(D,D)$ invariant scalar, cf. Eq. (4.24) in the second reference of [4], and we use the shorthand notation $dx = d^Dx$. The action is invariant under the gauge transformations

$$\delta_\xi \mathcal{H}_{MN} = \xi^p \partial_p \mathcal{H}_{MN} + 2(\partial_{(M} \xi^p - \partial^p \xi_{(M}) \mathcal{H}_{N)P}),$$

$$\delta_\xi d = \xi^M \partial_M d - \frac{1}{2} \partial_M \xi^M,$$

with the derivatives $\partial_M = (\partial^i, \partial_i)$. Here, $O(D,D)$ indices $M,N$ are raised and lowered with the invariant metric

$$\eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and (anti)symmetrizations are accompanied by the combinatorial factor $\frac{1}{2}$. The consistency of the above theory requires the constraint

$$\partial^M \partial_M A = \eta^{MN} \partial_M \partial_N A = 0, \quad \partial^M A \partial_M B = 0,$$

for all fields and parameters $A$ and $B$. This constraint implies that locally the fields depend only on half of the
The gamma matrices satisfy the Clifford algebra

\[ \{\Gamma^M, \Gamma^N\} = 2\eta^{MN}\mathbf{1}. \]  

A convenient representation can be constructed using fermionic oscillators \( \psi^i \) and \( \psi_i \), satisfying

\[ \{\psi_i, \psi^j\} = \delta^j_i, \quad \{\psi_i, \psi_i\} = 0, \quad \{\psi^i, \psi^j\} = 0, \]

where \( (\psi_i)^\dagger = \psi^i \). With (4) we infer that they realize the algebra (7) via

\[ \Gamma_i = \sqrt{2}\psi_i, \quad \Gamma^i = \sqrt{2}\psi^i. \]

Introducing a “Clifford vacuum” \( |0\rangle \) with \( \psi_i|0\rangle = 0 \) for all \( i \), and the normalization \( \langle 0|0 \rangle = 1 \), we can construct the representation by successive application of the raising operators \( \psi^i \). A general spinor state then reads

\[ \chi = \sum_{p=0}^D \frac{1}{p!} C_{i_1...i_p} \psi^{i_1}...\psi^{i_p}|0\rangle, \]

whose coefficients \( C_{i_1...i_p} \) can be identified with \( p \)-forms \( C^{(p)} \). Any element \( S \) of the Pin group projects, via a group homomorphism \( \rho: \text{Pin}(10, 10) \to O(10, 10) \), to an element \( h \in O(10, 10) \),

\[ S\Gamma^M S^{-1} = \Gamma_N h^N_M, \quad h = \rho(S), \]

where \( h\eta h^T = \eta \). Conversely, for any \( h \in O(10, 10) \), there is an \( S \in \text{Pin}(10, 10) \) such that both \( \pm S \) project to \( h \). A spinor can be projected to a spinor of fixed chirality, i.e., to eigenstates \( \chi_{\pm} \) of \( (-1)^{N_F} \) with eigenvalues \( \pm 1 \), where \( N_F = \sum_i \psi^i \psi_i \) is the “fermion number operator.” The spinor \( \chi_+ \) of positive chirality then contains only even forms, and the spinor \( \chi_- \) of negative chirality contains only odd forms. Imposing a chirality constraint reduces the symmetry from \( \text{Pin}(10, 10) \) to \( \text{Spin}(10, 10) \) since only the latter leaves this constraint invariant. Finally, we need the charge conjugation matrix satisfying

\[ C\Gamma^M C^{-1} = (\Gamma^M)^\dagger. \]

A particular realization is given by

\[ C = (\psi^1 - \psi_1)(\psi^2 - \psi_2)\cdots(\psi^{10} - \psi_{10}), \]

which satisfies \( C\psi_i C^{-1} = \psi^i \) and thereby (12).

Given a spinor (10) we can act with the Dirac operator

\[ \tilde{\delta} = \frac{1}{\sqrt{2}} \Gamma^M \partial_M = \psi^i \partial_i + \psi_i \tilde{\delta}^i, \]

which can be viewed as the \( O(10, 10) \) invariant extension of the exterior derivative \( d \). In fact, for \( \tilde{\delta} = 0 \), it differentiates with respect to \( x^i \) and increases the form degree by one, thus acting like \( d \). Moreover, it squares to zero,

\[ \tilde{\delta}^2 = \frac{1}{2} \Gamma^M \Gamma^N \partial_M \partial_N = \frac{1}{2} \eta^{MN} \partial_M \partial_N = 0, \]

using (7) and the constraint (5).

In order to write an action that couples the NS-NS fields represented by the generalized metric \( \mathcal{H} \) in (1) to the RR fields represented by a spinor \( \chi \), we note that the matrix \( \mathcal{H} \) is an \( SO(10, 10) \) group element and thus has a representative in \( \text{Spin}(10, 10) \), as has been used in dimensionally reduced theories [10]. In our case, however, a subtlety arises because (1) contains the full space-time metric, which we assume to be of Lorentzian signature. \( SO(10, 10) \) has two connected components, \( SO^+(10, 10) \), which contains the identity, and \( SO^-(10, 10) \). Because of the Lorentzian signature of \( g \), \( \mathcal{H} \) is actually an element of \( SO^-(10, 10) \). It turns out that a spin representative \( S_{\mathcal{H}} \in \text{Spin}(10, 10) \) of \( \mathcal{H} \) cannot be constructed consistently over the space of all \( \mathcal{H} \). For instance, one may find a closed loop \( \mathcal{H}(t), t \in [0, 1], \mathcal{H}(0) = \mathcal{H}(1) \), in \( SO^-(10, 10) \), with the initial and final elements related by a timelike \( T \) duality, for which a continuously defined spin representative yields \( S_{\mathcal{H}(1)} = -S_{\mathcal{H}(0)} \). As a result, timelike \( T \) dualities cannot be realized as transformations of the conventional fields \( g \) and \( b \). Nevertheless, a fully \( T \)-duality invariant action can be written if we treat the spin representative itself as the dynamical field. We thus introduce a field \( \mathbb{S} \), satisfying
\[ S = S^\dagger, \quad S \in \text{Spin}^-(10, 10). \] (16)

The generalized metric is then defined by the group homomorphism, \( \rho(S) = \mathcal{H} \). By (16) and the general properties of the group homomorphism \([7]\), \( \mathcal{H}^T = \rho(S^\dagger) = \mathcal{H} \) and so, as required, \( \mathcal{H} \) is symmetric.

We are now ready to define the DFT formulation of type-II theories, whose independent fields are \( S, d, \) and \( \chi \). The action reads

\[
S = \int dx^d \left( e^{-2d} \mathcal{R}(\mathcal{H}, d) + \frac{1}{4} (\partial \chi)^\dagger \mathcal{S} \partial \chi \right),
\] (17)

and is supplemented by the self-duality constraint

\[
\partial \chi = -\mathcal{K} \partial \chi, \quad \mathcal{K} = C^{-1} S.
\] (18)

For the special case of type IIa, a similar duality relation has also been proposed in \([8]\).

The field equation of \( \chi \) reads

\[
\delta (\mathcal{K} \delta \chi) = 0,
\] (19)

which also follows as an integrability condition from the duality relation (18), upon acting with \( \partial \) and using (15). The field equation of \( S \) reads

\[
\mathcal{R}_{MN} + \mathcal{E}^M_N = 0,
\] (20)

where \( \mathcal{R}_{MN} \) is the DFT extension of the Ricci tensor \([4]\), and the “energy-momentum” tensor reads, using (18),

\[
\mathcal{E}^{MN} = -\frac{1}{16} \mathcal{H}^{(M} \rho \partial M N \Gamma \rho N \partial \chi).
\] (21)

Let us now discuss the symmetries of this theory. First, it is invariant under a global action by \( S \in \text{Spin}^+(10, 10), \)

\[ \chi \to S \chi, \quad \mathcal{S} \to \mathcal{S}' = (S^{-1})^\dagger \mathcal{S} S^{-1}, \] (22)

implying \( \delta \chi \to S \delta \chi \). Specifically, \( \chi \) is assumed to have a fixed chirality, which breaks the invariance group of the action from \( \text{Pin}(10, 10) \) to \( \text{Spin}(10, 10) \), while the duality relations break the invariance group to \( \text{Spin}^+(10, 10) \). The gauge symmetries of this theory are given by

\[
\delta_{\lambda} \chi = \lambda \chi,
\] (23)

with spinorial parameter \( \lambda \), leaving (17) and (18) manifestly invariant by (15), and the gauge symmetry (3) parametrized by \( \xi^M \). On the new fields \( \mathcal{S} \) and \( \chi \) it reads

\[
\delta_{\xi} \mathcal{S} = \xi^M \partial_M S + \frac{1}{2} \partial_M \xi_N \Gamma^M \Gamma^N S,
\] (24)

\[
\delta_{\xi} \mathcal{K} = \xi^M \partial_M \mathcal{K} + \frac{1}{2} [\Gamma^\rho Q, \mathcal{K}] \partial_\rho \xi_Q,
\]

where we have written the gauge variation of \( \mathcal{S} \) in terms of \( \mathcal{K} \) defined in (18). It can be checked that this gauge transformation gives rise to the required variation (3) of \( \mathcal{H} \) upon application of \( \rho \).

We will now evaluate the DFT defined by (17) and (18) in particular T-duality frames, starting with \( \delta^i = 0 \). To this end, we have to choose a particular parametrization of \( \mathcal{S} \). Writing

\[
\mathcal{H} = \begin{pmatrix} 1 & 0 & g^{-1} & 0 \\ b & 1 & 0 & g \\ 0 & 0 & 1 & -b \\ h_b^T h_g^{-1} h_b \end{pmatrix},
\] (25)

we have to find spin representatives of the group elements \( h_b \) and \( h_g \). The subtlety here is that, with \( g \) Lorentzian, \( h_g \) takes values in \( SO^-(10, 10) \) and thus is not in the component connected to the identity. It is then convenient to write \( g \) in terms of vielbeins,

\[
g = e k e^T, \quad h_g = h_{1g} h_{Tg},
\] (26)

where \( e \) has positive determinant, i.e., \( e \in GL^+(10) \), and \( k \) is the flat Minkowski metric \( diag(-1, 1, \ldots, 1) \). The group elements \( h_1 \) and \( h_T \) are in the component connected to the identity and so their spin representatives can be written as simple exponentials,

\[
S_b = e^{-(1/2) b_{ij} \phi^i \phi^j}, \quad S_e = \frac{1}{\sqrt{\text{det} e}} e^{\phi^i \phi^i},
\] (27)

with \( e = \exp(E) \), as can be verified with (11). A spin representative for the matrix \( k \) can be chosen to be \([13]\)

\[
S_k = \psi^{\dagger} \psi - \psi \psi^\dagger,
\] (28)

where \( 1 \) labels the timelike coordinate. This can also be verified with (11). A spin representative \( S_{3s} \) of \( \mathcal{H} \) can then locally be defined as

\[
S_{3s} = S_b S_g^{-1} S_b, \quad S_g = S_c S_k S_e^\dagger.
\] (29)

We now set \( \mathcal{S} = S_{3s} \), but we stress that this is just a particular parametrization in much the same way that (1) is just a particular parametrization of \( \mathcal{H} \).

It is now straightforward to evaluate the action (17) for \( \delta = 0 \). First, as noted above, \( \delta \chi \) reduces to the exterior derivatives of the \( C^{(p)} \), \( F^{(p+1)} \equiv d C^{(p)} \). The action of \( S_b \) in \( S_{3s} \) then modifies this, using (27), to

\[
\hat{F} = e^{-b_{(2)}} \wedge F = e^{-b_{(2)}} \wedge d C.
\] (30)

Second, (29) implies for the action of \( S_g^{-1} \)

\[
S_g^{-1} \psi^{ij} \ldots \psi^{ij} |0\rangle = -\sqrt{sg} g_{ij}^{(i} \ldots g_{ij}^{j) \psi^{ij} \ldots \psi^{ij} |0\rangle}.
\] (31)

The Lagrangian corresponding to the RR part of (17) then reduces to kinetic terms for all forms,

\[
L_{RR} = -\frac{1}{4} \sqrt{g} \sum_{p = 1}^{D-1} g^{ij \ldots} g^{ij \ldots} \hat{F}_{i \ldots j^p} \hat{F}_{j \ldots i^p} \hat{F}_{j \ldots i^p} \hat{F}_{j \ldots i^p},
\] (32)

where we recall that the sum extends over all even or odd forms, depending on the chirality of \( \chi \). Similarly, using (13), the self-duality constraint (18) reduces to the conventional duality relations (with the Hodge star \( * \)),

\[
\hat{F}^{(p)} = (-1)^{(D-p)(D-p-1)/2} * \hat{F}^{(D-p)}.
\] (33)

We have thus obtained the democratic formulation of type-II theories, whose field equations are equivalent to
the conventional field equations of type IIA for odd forms and of type IIB for even forms [10].

Let us briefly comment on the gauge symmetries for \( \delta i = 0 \). The transformations (24) for \( \chi \), parametrized by \( \xi^M = (\xi_i, \xi^i) \), reduce to the conventional general coordinate transformations \( x^i \rightarrow x^i - \xi^i(x) \) of the \( p \)-forms \( C(p) \), and also to nontrivial transformations under the \( b \)-field gauge parameter \( \delta_i C = d\xi \wedge C \).

We turn now to the discussion of other \( T \)-duality frames, starting with \( \delta i = 0, \delta' i \neq 0 \). For the analysis of this case it is convenient to perform a field redefinition according to the \( T \)-duality transformation \( J \) that exchanges \( x^i \) and \( \tilde{x}_i \) and which, as a matrix, coincides with \( \eta \) defined in (4),

\[
\mathcal{H}' = J \mathcal{H} = \mathcal{H}^{-1}.
\]

It has been shown in [4] that the NS-NS part of the DFT reduces for \( \delta_i = 0 \) to the same action (6), but written in terms of the primed (\( T \)-dual) variables. Next, we define a corresponding field redefinition for the RR fields, using a spin representative \( S_j \) of \( J \),

\[
\chi' = S_j \chi, \quad \delta' = \psi_i \delta^i + \psi_i \delta_i, \quad \delta' \chi' = S_j \delta \chi. \quad (35)
\]

For the RR action we then find

\[
L_{RR} = \frac{1}{4} (\partial \chi')^\dagger S_{\delta \ell} \partial \chi = \frac{1}{4} (\partial' \chi')^\dagger (S_j^{-1})^\dagger S_{\delta \ell} S_j^{-1} \partial' \chi'
\]

\[
= -\frac{1}{4} (\partial' \chi')^\dagger S_{\delta \ell} \partial' \chi', \quad (36)
\]

where we used that \( J \) contains a timelike \( T \) duality such that, as mentioned above, this leads to a sign factor in the transformation of \( S_{\delta \ell} \). Thus, in the new variables the action takes the same form as in the original variables, up to a sign. The transformed Dirac operator in (35) implies that setting \( \delta_i = 0 \) in the first form in (36) is equivalent to setting \( \delta' = \psi_i \delta_i \) in the final form in (36). This way to evaluate the action is, however, equivalent to our computation above of setting \( \delta = 0 \) in the original action, just with fields and derivatives replaced by primed fields and derivatives. Thus, we conclude that the DFT action reduces for \( \delta i = 0 \) to a type-II theory with the overall sign of the RR action reversed. These are known as type-II* theories and have been introduced by Hull in the context of timelike \( T \) duality [14]. They are defined such that the timelike circle reductions of type IIA (IIb) and type IIB* (IIA*) are equivalent. This result also implies that the overall sign of \( \mathbb{S} \) has no physical significance in that it merely determines for which coordinates (\( x \) or \( \tilde{x} \)) we obtain the type-II or type-II* theory.

More generally, one finds that evaluating the DFT in a \( T \)-duality frame that is obtained by an odd (even) number of \( T \)-duality inversions from a frame in which the theory reduces, say, to type IIA, it reduces to the \( T \)-dual theory, i.e., type IIB (IIA) for spacelike transformations and type IIB* (IIA*) for timelike transformations. Summarizing, the DFT defined by (17) and (18) combines all type-II theories in a single universal formulation. We hope that this theory may provide insights into the still elusive formulation of string theory as, e.g., for a yet to be constructed type-II string field theory.

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