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Zeeman splitting of photonic angular momentum states in a gyromagnetic cylinder

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We show that, under the presence of a static magnetic field, the photon eigenfrequencies of a circular gyromagnetic cylinder experience a splitting that is proportional to the angular momentum density of light at the cylinder surface. Such a splitting of the photonic states is similar to the Zeeman splitting of electronic states in atoms. This leads to some unusual decoupling properties of these nondegenerate photonic angular momentum states, which are demonstrated through numerical simulations.

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I. INTRODUCTION

Recently, the effect of static magnetic field on photonic states has attracted a lot of attention due to the discovery of protected one-way photonic edge states in gyromagnetic photonic crystals,1–3 which are analog to the topologically protected edge states in electronic systems.4 Although there have been many designs of one-way waveguide devices,5 analytical studies on simple photonic states of related systems are rare when compared to the extensive studies of electronic states. One of the most fundamental examples for electronic states is the famous Zeeman effect.6 In analogy with the Zeeman splitting of electronic states in atom, here we study a similar basic effect on photonic states in a gyromagnetic cylinder. We also introduce some unusual wave decoupling phenomena, which are consequences of such a splitting.

Since Zeeman splitting of electronic states is associated with the broken degeneracies of electron states with different angular momenta, we expect to see a relation between the Zeeman-like splitting in our photonic system and the angular momentum of light. Angular momentum of light,7 which could be used for storage of quantum information,8 has drawn serious interest in recent years.9 Therefore, it is our motivation in this paper to calculate the frequency splitting of photonic angular momentum states and derive a formula that relates the angular momentum of light to the frequency splitting.

The paper is organized as follows. Section II describes the physical system. Section III A shows the Zeeman splitting of photonic angular momentum states with exact numerical solution, followed by its relation to angular momentum of light in Sec. III B. Finally, some decoupling effects due to the Zeeman splitting are discussed in Sec. III C.

II. PHYSICAL SYSTEM

We consider the splitting of photonic states in a gyromagnetic cylinder as shown in Fig. 1 with its radius \( r = a \) and permittivity \( \varepsilon_d \). When we apply a static magnetic field along the z direction, the magnetic permeability tensor of the gyromagnetic cylinder can be written as

\[
\tilde{\mu}_c = \begin{pmatrix}
\mu_r & -i\mu_k & 0 \\
 i\mu_k & \mu_r & 0 \\
 0 & 0 & 1
\end{pmatrix}.
\]

where \( \mu_r = 1 + \omega_m\omega_0/(\omega_0^2 - \omega^2) \), \( \mu_k = \omega_m\omega_0/(\omega_0^2 - \omega^2) \),\(^10\) \( \omega_0 = \gamma H_0 \) is the precession frequency, \( \gamma \) is the gyromagnetic ratio, \( \omega_m = 4\pi\gamma M_z \), and \( 4\pi M_z \) is the saturation magnetization. For the sake of simplicity, we consider the transverse electric (TE) polarization (i.e., electric field along the z direction). Due to the cylindrical geometry, the fields of every eigenstate have the \( e^{i\omega t} \) dependence with \( \phi \) being the azimuthal angle. It should be noted that the \( e^{-i\omega t} \) time-dependent convention for harmonic field is used in this work.

III. RESULTS

A. Splitting of photonic states

Here, we first discuss the properties of frequency splitting of photonic states using the exact solutions. We will explain an interesting interpretation by comparing the exact solutions to the perturbation solutions in Sec. III B. Solving the Maxwell’s equations in the cylindrical coordinates with the boundary conditions of \( E_z \) and \( H_z \) continuous at \( r = a \), we obtain the Mie resonance condition,\(^11\)

\[
\sqrt{\frac{\varepsilon_d}{\mu_r}} J'_m(ka)H_m^{(1)}(k_0a) - J_m(ka)H_m^{(1)}(k_0a) - \frac{m\mu_k}{(\mu_r^2 - \mu_k^2)k_0\varepsilon_d}J_m(ka)H_m^{(1)}(k_0a) = 0, \tag{2}
\]

where \( m \) is the azimuthal quantum number, \( J_m \) is the \( m \)-th order Bessel function, \( H_m^{(1)} \) is the \( m \)-th order Hankel function of the first kind, \( k' = k_0\sqrt{\mu_r\varepsilon_d} \), \( k_0 = \omega/c \) is free-space wave number, and \( \mu'_z = (\mu_r^2 - \mu_k^2)/\mu_r \). The frequencies satisfying Eq. (2) are the frequencies of the photonic states. It should be noted that Eq. (2) does not have real root frequencies because of radiation loss, but we will only consider the states with relatively low radiation loss (i.e., frequencies with small imaginary parts).

In comparison, when the applied magnetic field is absent, the first two terms in Eq. (2) are the same as in the Mie condition for a cylinder with isotropic permeability \( \mu_r \),

\[
\sqrt{\frac{\varepsilon_d}{\mu_r}} J'_m(ka)H_m^{(1)}(k_0a) - J_m(ka)H_m^{(1)}(k_0a) = 0, \tag{3}
\]

where \( k = k_0\sqrt{\mu_r\varepsilon_d} \). The main differences between Eq. (2) and Eq. (3) are the replacement of \( \mu_r \) with \( \mu'_z \) and the last linear \( m \) term in Eq. (2), which causes the broken symmetry between negative-\( m \) and positive-\( m \) states. Therefore, we can
\[
\begin{align*}
\hat{\mu} &= \left( \begin{array}{ccc}
\mu_x & -j\mu_y & 0 \\
-j\mu_y & \mu_x & 0 \\
0 & 0 & 1 \\
\end{array} \right) \\
\end{align*}
\]

FIG. 1. (Color online) Zeeman splitting of photonic states when a static magnetic field is applied along the \( z \) direction. The top figure shows a gyromagnetic cylinder surrounded by air. Formula in the dashed box displays the relation between frequency shift and angular momentum density of light \( (j_z) \). The first column (black lines) shows the original degenerate states in the gyromagnetic cylinder in the absence of magnetic field \( H_0 = 0 \) (i.e., with isotropic permeability \( \mu_r \)). The second column (green lines) shows the shifts due to the change of effective permeability from \( \mu_r \) to \( \mu'_r \) (indicated as "index shift") when a static magnetic field of \( H_0 = 800 \text{ Oe} \) is considered. The third column (red and blue lines) shows the final Zeeman splitting with the adjacent signs indicating the sign of \( m \). Only the states with \( n = 1.2 \) and \( |m| = 4.5 \) are shown.

interpret the effect of the static magnetic field as two steps: (i) a shift associated with an index change (from permeability \( \mu_r \) to \( \mu'_r \)) and (ii) a splitting of frequencies.

For a numerical demonstration, we consider yttrium-iron-garnet, which is a type of commercially available gyromagnetic material, as the material of the cylinder supporting photonic states at microwave frequencies. Using parameters provided in a previous experimental study \( (\epsilon_g = 15.26, H_0 = 800 \text{ Oe}, \text{ and } 4\pi M_s = 18844G) \), we plot the frequency splitting diagram in Fig. 1 for a cylinder of radius \( a = 1 \text{ cm} \) by finding the frequency roots of Eq. (2). Here, in addition to the azimuthal quantum number \( m \), we denote \( n \) as the quantum number in the radial direction and \( (n, m) \) as a specific photon state. To have a clear picture, we first focus on \( |m| = 4.5 \) in the lowest \( (n = 1) \) and the second lowest \( (n = 2) \) radially quantized levels. In the absence of the static magnetic field (first column of Fig. 1), the positive-\( m \) and negative-\( m \) states are degenerate. However, when the static magnetic field is present, the original degenerate states shift up to higher frequencies (second column of Fig. 1) and split into two counter-rotating states \((n, m) \) and \((n, -m) \), as indicated respectively by the red lines and blue lines in the third column of Fig. 1. Physically, the effect of the static magnetic field can be understood as a broken time reversal symmetry (and reciprocity) so that photonic states are no longer degenerate. We will show in a later part of this paper that such a splitting is proportional to the angular momentum of light.

To verify the splitting, we use a finite element solver (COMSOL Multiphysics) to calculate the electric field profile of the nondegenerate rotational states after splitting. In Fig. 2, we plot the wave profiles excited by an out-of-plane oscillating line current source lying on the surface of the gyromagnetic cylinder for the resonant frequencies of \( n = 1.2 \) and \( m = \pm 4 \) states. When the \((1, -4) \) state [Fig. 2(a)] and the \((2, -4) \) state [Fig. 2(b)] are excited, we see clockwise rotating fields with four complete oscillations along the azimuthal directions. On the contrary, the corresponding \( m = +4 \) states have electric fields rotating in the counterclockwise direction [see Figs. 2(c) and 2(d)]. These results confirm the Zeeman splitting diagram in Fig. 1. It should be noted that for the case of \( n = 2 \), we have two maxima across the radial direction. One may expect that the frequency splitting should depend also on the field oscillations within the gyromagnetic cylinder. However, we will see that the splitting depends only on the fields at the surface of the cylinder.

For a complete picture of the splitting of photonic states, we plot in Fig. 3 the exact solutions [roots of Eq. (2)] for the resonant frequency as a function of the azimuthal momentum number \( m \). It is found that the resonant frequencies show pronounced differences between positive and negative \( m \) states, stemmed from the effect of the static magnetic field. For \( n = 1 \), the most obvious frequency splitting can be observed and the splitting gap width decreases gradually as \( m \) increases. We also observe that the splitting is relatively weak for larger \( n \) and it is almost independent of \( m \) for large \( m \) when \( n \) is fixed.

B. Relation to angular momentum of light

To understand the frequency splitting and its relation to the angular momentum of light, we employ a Hamiltonian
Loeff-quality states with eigenvalue approach for electromagnetic waves. In this Zeeman splitting of photonic angular momentum is subjected to a scattering boundary condition. Here, we take the approximation that the dispersive material \( \frac{\Delta \mu}{\mu_1} / \Omega_1 \).

\( \vec{r} > \) and \( \vec{r} < \) and \( \omega h \) modes, respectively.

Equation (9) can be rewritten to display the relation to the angular momentum of light:

\[
\Delta \omega_{nm} = \frac{\mu_k}{\mu_k^2 - \mu_r^2} \langle \psi_{0,n,m} | S \Delta \Omega | \psi_{0,n,m} \rangle,
\]

where \( \psi_{0,n,m} \) is the unperturbed state ket, and

\[
\langle \psi_{0,n,m} | S \Delta \Omega | \psi_{0,n,m} \rangle = \frac{\int [\epsilon_0 \epsilon(r)|E|^2 + \mu_0 \mu(r)|H|^2] dA}{\int [\epsilon_0 \epsilon(r)|E|^2 + \mu_0 \mu(r)|H|^2] dA}.
\]

Here, Eqs. (9) and (10) are derived by transforming the original eigenvalue problem Eq. (6) without the perturbation term to a Hermitian eigenvalue problem that guarantees the orthogonality among eigenkets. Details are given in the Appendix. Equation (9) can be rewritten to display the relation to the angular momentum of light:

\[
\Delta \omega_{nm} = 2 \pi \epsilon_0^2 \frac{j\mu \omega_{wa}}{U} \frac{\mu_k}{\mu_k^2 - \mu_r^2} \langle \psi_{0,n,m} | S \Delta \Omega | \psi_{0,n,m} \rangle,
\]

where

\[
j \mu \omega_{wa} = \frac{\epsilon_0 m}{2 \omega_h} |E_z(k'_n a)|^2
\]

is the “angular momentum density” evaluated at the surface of the cylinder, \( \omega_h \) is the frequency of the unperturbed state, \( k'_n = \omega_h / c \), and

\[
U = \frac{1}{2} \int [\epsilon_0 \epsilon(r)|E|^2 + \mu_0 \mu(r)|H|^2] dA.
\]

Here, the angular momentum density at the boundary is defined as

\[
j \mu = \epsilon_0 \epsilon T \times [E^* \times (\nabla \times E)]
\]

for TE polarization, \( U \) can be considered as the total electromagnetic energy integrated on the \( xy \) plane. To avoid the problem of normalization for an infinite spatial domain, we consider an approximate problem of finite cylindrical spatial domain of a radius several times larger than that of the gyromagnetic cylinder. For the sake of simplicity, we can approximate \( U \) for this finite domain as

\[
U \approx 2 \pi \epsilon_0 \epsilon T \int_0^a E_x^2 r dr
\]

by using the fact that energy is mostly concentrated in the volume of the gyromagnetic cylinder. Since we consider frequencies far above \( \omega_h \), the third factor in Eq. (11), \( \mu / \mu_k^2 - \mu_r^2 \), is positive and the sign of frequency shift is, therefore, the same as the sign of \( m \). We thus conclude that the frequency shift is proportional to the angular momentum density at the cylinder surface per photon.

In comparison, the Zeeman effect on electronic states in atoms has a similar formula for the energy shift: \( \Delta E = \mu_B g J \cdot B / \hbar \), where \( J \) is the total projected angular momentum.
in the direction of the static magnetic field \((B)\), \(\mu_B\) is the Bohr magneton, \(\hbar = h/2\pi\) is the reduced Plank’s constant, and \(g\) is a dimensionless constant that depends on the type of atoms. Such an energy shift is proportional to the magnetic moments associated with the angular momentum of electrons. Here in Eq. (11), we also have the proportionality between frequency shift and angular momentum of light. With such proportionality, we thus call our work the Zeeman splitting of photonic angular momentum states in gyromagnetic cylinder, which is the most important result in this paper. It should be noted that Eq. (11) is not limited to microwaves. One could easily extend this new theory to terahertz or optical frequencies in other systems, such as plasma systems, where the roles of electric and magnetic fields are switched.

To verify our analytical formula [Eq. (11)], we evaluate and compare it with the exact results in Fig. 3. Using the fact that the radiation field outside the gyromagnetic cylinder should not contribute significantly to the frequency shift, we get a closed-form expression for Eq. (11):

\[
\Delta \omega_{\text{nm}} \approx -\frac{\mu_k c^2}{\mu_k^2 - \mu_r^2} m J_2^2(k_n^*a) \int_0^1 |J_0(k_r^*r)|^2 2r dr
\]

\[
= -\frac{\mu_k m \omega_{\text{in}}}{\mu_r k_n^*a^2} (J_2^2(k_n^*a) - J_{m-1}(k_n^*a)J_{m+1}(k_n^*a)).
\]

(14)

As shown in Fig. 3, we have a very good agreement between the frequencies evaluated from the closed-form solution [Eq. (14)] and the exact results given by Eq. (2). Discrepancies exist only in the first few low-order resonances, which may be due to the first-order perturbation, the neglected radiation field, and the dispersive property of gyromagnetic materials. This proves the validity of Eq. (11).

C. Decoupling effects of nondegenerate states

With a nondegenerate angular momentum state that allows only a single-direction rotating field, the Zeeman-like effect can lead to some useful mode-decoupling properties for photonics application. For example, we consider the setup shown in Fig. 4, where a resonant cylinder is placed between two waveguides. The wave in one waveguide can efficiently couple to the other through a resonant coupling. For the case without the static magnetic field, the resonant coupling does not depend on which port the wave comes from because the resonator supports both degenerate modes [see Fig. 4(a) for left-incident wave and Fig. 4(b) for right-incident waves launched at the resonant frequency, 7.70 GHz, of the \((1, \pm 4)\) state]. However, when the states are nondegenerate, not both incoming waves can couple to the other waveguide. At the resonant frequency, 8.72 GHz, of the \((1, -4)\) state, we find that the left-incident wave can couple to the other waveguide [Fig. 4(c)], but the right-incident wave fails to interact with the cylinder and pass straightly through the top waveguide [Fig. 4(d)]. The one-way transport properties can be understood by the mode coupling [Fig. 4(c)] and mode decoupling [Fig. 4(d)] between the incident guided wave and the single-mode state. The unusual properties shown in Fig. 4 can be used for experimental demonstration of the Zeeman-like splitting.

Recently, similar nonreciprocal transport properties of light in the presence of a static magnetic field have been widely studied.\(^{1-3,12,15-18}\) The phenomena here may also be
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APPENDIX: EVALUATION OF THE PERTURBATION TERM

To evaluate the correct first-order frequency shift, we transform the original eigenvalue problem for the unperturbed states,

\[ \Omega_0 |\psi_{0,n,m}⟩ = \omega_{nm} |\psi_{0,n,m}⟩, \]

into a Hermitian eigenvalue problem:\(^{13}\)

\[ \sqrt{S} \Omega_0 \sqrt{S}^{-1} |\tilde{\psi}_{0,n,m}⟩ = \omega_{nm} |\tilde{\psi}_{0,n,m}⟩, \]

(A2)

where \( |\tilde{\psi}_{0,n,m}⟩ = \sqrt{S} |\psi_{0,n,m}⟩ \), and

\[ S \doteq \left( \begin{array}{ccc} \epsilon_0 \epsilon(r) & 0 & 0 \\ 0 & \mu_0 \mu(r) & 0 \\ 0 & 0 & \mu_0 \mu(r) \end{array} \right). \]

(A3)

It can be proven that \( S \Omega_0 \) is Hermitian, which implies that \( \sqrt{S} \Omega_0 \sqrt{S}^{-1} = \sqrt{S}^{-1} (S \Omega_0) \sqrt{S}^{-1} \) is also Hermitian. The Hermitian operator \( \sqrt{S} \Omega_0 \sqrt{S}^{-1} \) guarantees the orthogonality among its eigenkets:

\[ \langle \tilde{\psi}_{0,n,m} | \tilde{\psi}_{0',n',m'}⟩ = \delta_{nn'} \delta_{mm'}. \]

(A4)

The frequency shift in first-order perturbation theory is then given by

\[ \langle \tilde{\psi}_{0,n,m} | \sqrt{S} \Delta \Omega \sqrt{S}^{-1} | \tilde{\psi}_{0,n,m}⟩ \]

\[ = \langle \psi_{0,n,m} | \sqrt{S} \sqrt{S} \Delta \Omega | \psi_{0,n,m}⟩ \]

\[ = \langle \psi_{0,n,m} | \Delta \Omega | \psi_{0,n,m}⟩ \]

\[ = \frac{\int \mu_0^{-1} \theta(a-r) \left[ \mu_0 \mu(r) H^* \frac{\partial E_z}{\partial r} + \mu_0 \mu(r) H^* \frac{\partial E_z}{\partial \phi} + \frac{\partial E_z}{\partial r} \right] dA}{\int [\epsilon_0 |E|^2 + \mu_0 \mu(r) |H|^2] dA} \]

\[ = \frac{2\pi \frac{\mu_0 a}{\epsilon_0} \int_0^a \left( E_z^* \frac{\partial E_z}{\partial r} + \frac{\partial E_z}{\partial \phi} \right) dr}{\mu_0 \int [\epsilon_0 |E|^2 + \mu_0 \mu(r) |H|^2] dA} \]

\[ = \frac{2\pi \frac{\mu_0 a}{\epsilon_0} \int_0^a \left( E_z^* \frac{\partial E_z}{\partial r} + \frac{\partial E_z}{\partial \phi} \right) dr}{\mu_0 \int [\epsilon_0 |E|^2 + \mu_0 \mu(r) |H|^2] dA} \]

\[ = 2\pi c^2 \frac{\int_{r=a}^b |E|^2}{U}. \]

(A5)

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\(^{5}\)M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).


\(^{10}\)J. P. Torres and L. Torner, Twisted Photons: Applications of Light with Orbital Angular Momentum (Wiley-VCH, Bristol, 2011).


