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Stable Control of Distributed Hysteretic Systems Using Cellular Broadcast Stochastic Feedback

Levi B. Wood and H. Harry Asada, Fellow, ASME

Abstract—This work develops a provably stable broadcast feedback approach to regulating the aggregate output of a collection of stochastically behaving cellular units with pronounced hysteresis. Similar to skeletal muscle, comprised of a number of muscle fibers, the cellular units are independent and each contributes to an aggregate system output. The control architecture consists of a central controller that only observes the aggregate system output and cell-level controllers that manage cell state on the hysteresis loop. The central controller stochastically recruits cells to reduce expected squared error. Stability analysis based on supermartingale theory guarantees that the control framework is stable.

I. INTRODUCTION

There exist many systems in nature where a multitude of small units work together in a collective effort to create an aggregate output, as illustrated in Fig. 1. A skeletal muscle, for example, consists of a number of muscle fibers, which are activated by motor neurons. The aggregate output is the resultant force, or displacement, which is accommodated by regulating the number of muscle fibers recruited to be active. In general, biological systems generate meaningful function from collective behaviors of the cells that constitute a tissue or an organ.

These cellular systems exhibit superb robustness and sustainability, increased flexibility and versatility, and unique functionality that single units cannot create. With an effective control mechanism, a cellular system can maintain required functionality even if a significant fraction of the units are not functional [1]. The multitude of cellular units can replace each other and coordinate their behaviors to exhibit the type of functionality that a single bulk process cannot produce.

The objective of this paper is to explore the potential of the cellular control architecture and elaborate a novel control method for a particular class of nonlinear distributed systems. Artificial muscle actuators will be used as an example for formulating the control problem. Artificial muscles are made of active materials, such as shape memory alloy (SMA) and conducting polymers, which have pronounced hysteresis and long latency time for activation. The cellular architecture will be an effective solution to coping with the distributed nonlinearity of the material. In this cellular architecture, a bulk of actuator material is segmented into many independent units, and their aggregate output is regulated by coordinating the multitude of units. It will be shown that, although the actuator material is hysteretic and slow in response, the aggregate output response can be improved significantly by coordinating the individual units.

As illustrated in Fig. 1, the control system consists of a central controller and local controllers. We aim to develop a control system that meets the following requirements:

- A central controller monitors only the collective behavior of the cells, \( Y \).
- Each cellular unit has a local controller regulating its own state based on a simple command from the central controller.
- The central controller broadcasts the same command to all the cellular units ubiquitously.

In the following, we will describe properties of these cellular systems and their control architecture, followed by mathematical modeling and control problem formulation. We will derive a control law by minimizing future mean squared error and prove system stability.

II. ARCHITECTURE AND ALGORITHM

A. Hysteresis and Refraction Control

Consider a material having a pronounced hysteresis loop, as illustrated in Fig. 2. The process is bi-stable, taking a binary output value, on or off, when the input is higher or lower than a threshold value: \( u_{as} \) for the ascending branch and \( u_{sr} \) for the descending branch as shown in the figure. Shape memory alloys and other smart materials exhibit this type of nonlinearity distributed across the material body.

Fig. 1. Broadcast control with aggregate output feedback.

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Feedback control of this type of nonlinear distributed system is a challenging problem. In the past, many advanced control methods have been studied. Song et. al. in [2], for example, has an extensive literature survey of SMA and hysteresis modeling and control during the last three decades. Recently, Segmented Binary Control (SBC) has been found to be an effective approach to overcoming bistable, hysteric, distributed nonlinearities [3]. SBC is a technique wherein a single actuator material is segmented into many discrete cellular units, each of which can be fully contracted or fully relaxed. Thus, the aggregate output is proportional to the number of the cellular units that are turned on, if each unit produces the same magnitude of displacement. In SBC, a local controller responsible for each cellular unit regulates the binary state of its own segment. It merely drives the state of the unit towards on or off, rather than stopping it at a specific point between the two.

This segmented binary control architecture provides a means of overcoming not only distributed nonlinearity, but also time delay in the system response, when combined with another technique described next. Suppose that a unit is being turned off and moving from $S_i$ to $S_j$ in Fig. 2. In order to move in the opposite direction, the unit must move horizontally across the hysteresis band. No output change is observed until it arrives at the ascending branch between $S_i$ and $S_j$. This latency time can be eliminated by using a technique known as Hysteresis Loop Control (HLC). In HLC, each unit, after making an on-off transition, is pushed toward a hold position, $S_i$ or $S_j$. Note that a unit at these states is ready for making an immediate transition, having no latency time for moving across the hysteresis band. Prohibiting the units that are still in a transition phase and calling upon only the ready-state units ($S_i$ and $S_j$), the latency time can be eliminated. An output change can be observed immediately after the ready-state units have reacted to the central controller’s command.

This kind of coordination is impossible for a single bulk process, but is feasible for segmented cellular systems. Furthermore, this coordination can be implemented easily and effectively, as addressed in the following section.

**B. Anonymous Control and Stochastic Recruitment**

One drawback of the coordinated HLC for reducing latency time is that the central controller needs to know the status of every cell in the collective so that it knows which cells are ready to make a transition immediately. To know the status of each cell and make coordinated control decisions, each cell must report to the central controller whether it is ready at each time step. As the number of cellular units grows to yield high resolution in the actuator output, this local-to-central communication may become complex and impractical.

Though the status of each cell is needed for coordination, the output of any particular cell does not need to be regulated specifically and only the collective system output is important. Any cell contributes to the collective output in the same way and a particular collective output can be caused by any of the cells being on as long as the correct number is on, i.e., the cells contribute to the collective equivalently and thus anonymously.

Anonymous contribution means that a central controller does not really need to command each cell specifically, but can instead issue a global command so that a certain fraction of cells may change output. The challenge is to use global commands that are brief, requiring little bandwidth, and that require very little interpretation/computation at the cells. An approach that the authors’ group has previously proposed is to design the cells to behave stochastically according to a signal that is broadcasted to all cells [1],[4]-[6]. Each cell makes a stochastic decision to change its output with probability $p$, from off to on, and probability $q$, from on to off. In broadcast control, these state transition probabilities are modulated by the central controller. If it is desired that more cells turn on, the central controller broadcasts a higher value of probability $p$, in order to recruit more cells to turn on, and vice versa. This stochastic recruitment control is totally anonymous and broadcast in nature, thus meeting our requirements. However, previous results were derived only for the case of simple on-off units having no dynamics and hysteresis.

This work combines the refraction rule and the stochastic broadcast control so that a multitude of hysteric cellular units can be coordinated and the aggregate system output can be stably controlled to its desired value.

**III. SYSTEM REPRESENTATION**

**A. Collective Behavior**

We are interested in regulating the collective behavior of a population of $N$ stochastic on-off cellular units to a reference. Our objective it to use only the collective output in the feedback loop, and a single control that is broadcast to all cells. Depending on the application, a collective output may be defined in different ways. For clarity, the following development will be made in the context of muscle actuators consisting of serially connected on-off cellular units so that the ensemble output is the summation of the outputs of each individual cell. However, this development is generalizable to any system where the collective output can be written in terms of a sum of the outputs of the individual cells.
In the present development, the \( i \)th on-off unit can exhibit an output that is either displaced or not at time \( t \):
\[
y_i(t) = \begin{cases} \eta, & \text{on} \\ 0, & \text{off} \end{cases}
\]
where \( \eta \) is the difference in length between a cell when it is displaced and when it is not. The ensemble output is
\[
y(t) = \sum_{i=1}^{N} y_i(t)
\]
and the error between the collective output and the desired output, \( r \), is
\[
e(t) = r - Y(t)
\]
which will be used for broadcast feedback control.

**B. Local Behaviors of Hysteresis Loop Controller**

Each cellular unit has a local controller for regulating its state within the hysteresis loop. Figure 3 summarizes the behavior of individual cellular units with a state transition network. From the states \( F_x \) and \( B_x \), a cell is ready for an output transition and responds to the output transition probabilities \( p \) and \( q \) as specified by the central controller. After an output transition, a cell must traverse the intermediate states \( F_i, \ldots, F_x \), or \( B_i, \ldots, B_x \), before it will respond to the central controller once again. This behavior, where a cell becomes unresponsive, is termed a “refractory rule.”

![Figure 3. Markov chain representation of single cell local control system.](image)

**C. Aggregate State, Internal Dynamics, and Observability**

We are interested in controlling the collective output of a system of cells. The states and outputs of the individual cells are inconsequential. Since we can only measure the global output, \( Y(t) \), it cannot be expected that we are able to infer in which node of the Markov chain in Fig. 3 each individual cell exists. Since each cell contributes anonymously and in the same way as any other cell would, the most we can tell is the number of cells that are on and off. Since the number of on and off cells determines the aggregate output, the number is all that we wish to regulate and we have no interest in the outputs of particular cells.

Given a control, \( p \) and \( q \), the probability distribution over the aggregate number of on and off cells during the next time step can be completely described in terms of the numbers of forward and backward units ready for changing outputs, \( N_{Fx} \) and \( N_{Bx} \), respectively. Likewise, the number of cells in the ready node \( F_x \) during the next time step, \( N_{F_{i+1}} \), can be described by the number of cells in the \( F_x \) node, \( N_{Fx} \), which is then determined by the number of cells in the \( F_{i-1} \) node, \( N_{F_{i-1}} \), and so on. It is clear then that all of the information required to describe the future behavior of the collective output is given by the number of cells in each node. Thus, the aggregate system state can be defined as
\[
X = \begin{bmatrix} N_{F_1} & \ldots & N_{F_x} & N_{F_{x+1}} & \ldots & N_{F_{N-1}} & N_{F_N} & N_{B_{x+1}} & \ldots & N_{B_{N-1}} & N_{B_N} \end{bmatrix}
\]
Note that the elements in vector \( X \) sum to \( N \).

Counting the number of cells in the same node, we can obtain the distribution of the \( N \) cells over the \((n_p + n_s + 2)\) nodes. Defining a state space as a collection of all possible values of the aggregate state vector in (4),
\[
S = \{ X | 0 \leq N_{F1} \leq N, 0 \leq N_{B1} \leq N, 0 \leq N_{F2} \leq N, 0 \leq N_{B2} \leq N, \ldots, 0 \leq N_{FN} \leq N, 0 \leq N_{BN} \leq N \}
\]
we can treat the aggregate system as a vector Markov process in \( S \):
\[
\Pr(X_{i+1} | X_i) = \Pr(X_{i+1} | X_i)
\]
Note that the \( N \) cellular units are independent, identical Markov processes. Therefore, this vector Markov process is a type of Markov population model with time-varying transition probabilities [7].

While Eq. (4) contains all of the information required for describing the future behavior of the system, it is not possible to infer the number of cells in each node based on output observation, \( Y(t) \), alone and the system state is unobservable. To see this, consider the case where a number of cells transition from \( F_x \) to \( B_x \) and simultaneously the same number of cells transition back from \( B_x \) to \( F_x \). There is no change in the output but the system state clearly changed.

Though the output is not enough to determine aggregate system state if transitions are allowed in both directions at the same time, it is possible to make the system deterministically observable if we limit allowable transitions. If a unilateral control strategy [1] is used such that
\[
p = \begin{cases} 1, & \epsilon(t) \leq 0 \\ p(\epsilon), & \epsilon(t) > 0 \end{cases} ; q = \begin{cases} q(\epsilon), & \epsilon(t) < 0 \\ 0, & \epsilon(t) \geq 0 \end{cases}
\]
then the system becomes observable. That is to say that the change in output \((Y(t) - Y(t-1))/\eta\) completely describes the number of cells that transitioned from \( F_x \) to \( B_x \) or from \( B_x \) to \( F_x \). Namely, for \( \epsilon(t) > 0 \) we can write
\[
N_{F_i} = 0 \\
N_{F_{i+1}} = N_{F_{i+1}} \quad N_{F_{i+1}} = N_{F_{i+1}} \\
N_{F_{i+1}} = N_{F_{i+1}} \\
N_{B_{i+1}} = N_{B_{i+1}} \\
N_{F_{i+1}} = N_{F_{i+1}} \quad N_{F_{i+1}} = N_{F_{i+1}} \\
N_{B_{i+1}} = N_{B_{i+1}} \quad N_{B_{i+1}} = N_{B_{i+1}} \\
N_{B_{i+1}} = N_{B_{i+1}} \quad N_{B_{i+1}} = N_{B_{i+1}} \\
N_{B_{i+1}} = N_{B_{i+1}} \quad N_{B_{i+1}} = N_{B_{i+1}}
\]
A similar set of expressions can be written for \( \epsilon(t) < 0 \).

Therefore, all the components of the aggregate state vector \( X \), can be constructed from the previous \( \delta = \max(n_p, n_s) \).
time steps of collective output observations:

\[ Z_t = \left[ Y_t, \ldots, Y_{t-\delta} \right]^T \]  \hspace{1cm} (9)

Note that if initial conditions are unknown, \( X_t \) can be fully determined after \( t \geq \delta \). This implies that the aggregate system is observable under the unilateral transition constraint.

The unilateral control assumption can be well justified for engineering applications like SMA actuators because it is wasteful to command any cells opposite to the direction of the desired output.

IV. STABLE CONTROL

A. Stochastic Control Law

The objective of this section is to find a stochastic control law in terms of transition probabilities, \( p \) and \( q \), to be broadcasted by the central controller, that will bring the error \( e_t = r - Y_t \) to zero in a stochastic sense, given a constant reference input \( r \). Various types of stability and convergence criteria can be applied to this problem [8]. As long as a control law allows us to hit the target with a probability that is lower bounded by a uniform limit:

\[ \Pr(e_{t+1} = 0 | e_t, p, q) \geq \epsilon > 0, \forall e_t, \forall t \],

then

\[ \Pr(e_t = 0, \forall t \leq t_f) \leq (1 - \epsilon)^{t_f-t_t} \rightarrow 0 \]. Many control laws can in fact satisfy this stability condition [1],[6]. Depending on specific control laws, however, the convergence process will be different.

In this paper we consider the mean squared error \( E[e_{t+1}^2 | X_t, p, q] \) as a metric of the probability distribution of error during the next time step, and derive a feedback control law based on a stability criterion using this metric [8]. In the case of general hysteretic systems, the behavior is complex, and it is difficult to guarantee that the mean squared error monotonically reduces at every time step. Exploiting the preloading scheme and the refinement rule, however, we can guarantee smooth and effective convergence in a straightforward manner. The specific control law will be derived below.

Given the current state \( X_t \), which is derived from the observation \( Z_t \), the expected next-step squared error can be written as

\[ J(p, q, X_t) \triangleq E[e_{t+1}^2 | X_t] \]

\[ = Var[Y_{t+1} | X_t] + \{E[Y_{t+1} | X_t] - r\}^2 \]  \hspace{1cm} (10)

where broadcast probabilities \( p \) and \( q \) are used at all of the cellular units at time \( t \). A control law can be obtained by finding \( p \) and \( q \) that reduce the next-step squared error as much as possible. Such a control law is merely a local, incremental control law. Nonetheless, useful global properties, e.g. supermartingale convergence and decaying rate, can be derived from the incremental expected squared error, as will be discussed later in this section.

The mean squared error is broken down to the output variance term and the output offset term, as shown in the second line of (10). Both terms can be computed explicitly as functions of \( p \) and \( q \). Without loss of generality we can assume \( e_t > 0 \) and set \( q = 0 \) by (7), which yields

\[ E[Y_{t+1} | X_t] = E\left[Y_r + \eta N_r^{\delta} \right] = Y_r + \eta N_r^{\delta} \]

\[ where the second line follows from the fact that only cells in \( F_\delta \) can transition. Likewise, the variance becomes

\[ Var[Y_{t+1} | X_t] = \eta^2 (p - p^2) N_r^{\delta} \]

where the second line follows from the binomial distribution of the cellular units to transition. A similar set of expressions can be written for \( e_t < 0 \).

Substituting the mean and variance of the next-step output \( Y_{t+1} \) given by (11) and (12) into (10) yields

\[ J(p, 0; X_t) = \eta^2 N_r^{\delta} (p - 1) + \eta N_r^{\delta} (\eta - 2\epsilon) p + \epsilon \] \hspace{1cm} (13)

Our objective is to minimize \( J \) subject to the range of meaningful probabilities \( 0 \leq p \leq 1 \). Equation (13) is a convex parabola if \( N_r^{\delta} > 1 \) and a line for \( N_r^{\delta} = 0.1 \). The minimum of the parabola can be found by setting \( dJ(p, 0; X_t) / dp = 0 \) which yields

\[ p_{\text{min}} = \frac{2\epsilon - \eta}{2\eta (N_r^{\delta} - 1)} \]

Let \( p_{\text{min}} \) be the value of \( 0 \leq p \leq 1 \) that minimizes \( J(p, 0; X_t) \). Given that \( N_r^{\delta} > 1 \) and Eq. (14), when \( N_r^{\delta} > e_t / \eta \) and \( e_t / \eta > 1/2 \), \( p_{\text{min}} = p_{\text{opt}} \). For \( 1 \leq N_r^{\delta} \leq e_t / \eta \), all available units should be turned on so \( p_{\text{min}} = 1 \). If \( N_r^{\delta} = 0 \), then the broadcast does not matter and no units can activate. Thus, the complete control law for \( e_t > 0 \) is

\[ p_{\text{opt}} = \begin{cases} \frac{2\epsilon - \eta}{2\eta (N_r^{\delta} - 1)}, & N_r^{\delta} > 1 \text{ and } \frac{1}{2} \leq e_t / \eta < N_r^{\delta} \\ 1, & 1 \leq N_r^{\delta} \leq e_t / \eta \text{ and } e_t / \eta \geq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases} \] \hspace{1cm} (15)

A similar law for \( e_t < 0 \) immediately follows.

A few useful insights can be obtained from the above control law. First, the solution is a trade-off between the output variance \( Var[Y_{t+1} | X_t] \) and the squared output offset \( \{E[Y_{t+1} | X_t] - r\}^2 \) expected for the next step. Choosing \( p \) that brings the mean offset to zero does not necessarily reduce the expected squared error most effectively in the current step. It is clear from the variance \( Var[Y_{t+1} | X_t] = \eta^2 (p - p^2) N_r^{\delta} \) that, as \( p \) gets closer to either 0 or 1, the next step output becomes more predictable. For example, if the number of ready units, \( N_r^{\delta} \), is the same as the error \( e_t / \eta \), the system can reach the reference point deterministically in one step. The above control law tends to
pick a value for $p$ that produces more predictable, low variance output for the next step, which is better in reducing the expected squared error.

B. Convergence

The objective of this section is to guarantee that the error $\epsilon_t$ converges with probability 1 despite the intricate behavior due to hysteresis. Further, we want to analyze the decay rate of convergence. This can be addressed based on the expected squared error used in the above control law derivation.

Consider:

$$E\left[\epsilon_t^2 \mid X_t\right] \leq a\epsilon_t^2$$

If there exists a parameter $a$ in $0 < a < 1$ that satisfies (16) for all $t \geq 1$, the process is a non-negative supermartingale, and $a$ provides an exponential expected decay rate [6]. Unfortunately, in the case of hysteric systems, there are periods when no cells are ready to transition, $N^t_{\sigma^y} = 0$ or $N^t_{\sigma^0} = 0$, during which no decay is expected: $E[\epsilon_t^2 \mid X_t] = \epsilon_t^2$, or $a = 1$. This invariant situation does not continue forever, since preloading is completed within $\delta = \max(n_y, n_x)$ time steps.

**Lemma** If $\epsilon > 0$, then expected squared error decreases, $E[\epsilon_t^2 \mid X_t] < \epsilon_t^2$, at least once in every $\delta = \max(n_y, n_x)$ time steps and, at that time, there is a uniform bound $\bar{a}$ such that, if $\epsilon \neq 0$,

$$\bar{a} \leq \frac{E[\epsilon_t^2 \mid X_t]}{\epsilon_t^2} < 1$$

where $\bar{a}$ does not depend on $t$.

**Proof** See Appendix A.

Instead of evaluating the decay rate in every time step, using the decay rate of $\delta$ time steps yields the following convergence theorem.

**Theorem** The stochastic broadcast control system consisting of $N$ hysteric cellular units given by the Markov chain in Fig. 3 and a central controller that monitors aggregate output $Y$, and that broadcasts transition probabilities, $p_{mn}$ and $q_{mn}$, based on the control law of (18), converges to a constant reference point $r \in [0, \eta N]$ with probability 1:

$$e_t \to 0 \text{ w.p.1}$$

with the following exponential convergence properties:

$$E[\epsilon_t^2 \mid X_t] \leq \bar{a}e^t$$

$$\Pr\left(\sup_{s \leq t} \epsilon_s^2 \geq \mu\right) \leq \frac{\bar{a}e^t}{\mu}$$

where $0 \leq \bar{a} < 1$, $X_t = x$ is an arbitrary initial state, $c$ is the initial squared error $c = \epsilon_0^2$, and $m$ is the floor $m = \lfloor t/\delta \rfloor$ with $\delta = \max(n_y, n_x)$.

**Proof** The control law (18) does not increase the mean squared error, i.e., $E[\epsilon_t^2 \mid X_t] \leq \epsilon_t^2$ for each $\epsilon_t^2$ in the sample space and in every time step. Taking expectation conditioned on $X_t$ yields $E[\epsilon_t^2 \mid X_t] \leq E[\epsilon_t^2 \mid X_t]$. For the first $\delta$ time steps, we can write $c \geq E[\epsilon_t^2 \mid X_t] \geq \cdots \geq E[\epsilon_t^2 \mid X_t]$. From the Lemma it follows that during this period there exists at least one time step $t' \in (0, \delta]$ such that $E[\epsilon_t^2 \mid X_t] \leq \bar{a}e^t$ with $\bar{a}'$ less than 1. Therefore, we can bound $E[\epsilon_t^2 \mid X_t] \leq \bar{a}c$ with $0 \leq \bar{a} < 1$ for all $t \in [\delta, 2\delta]$. For $(m-1)\delta \leq t < m\delta$, we can repeat this process $m$ times since the fractional bound $\bar{a}$ is uniform in time. This yields (19).

From the above argument, $E[\epsilon_t^2 \mid X_t] \leq \bar{a}c$. Rewriting this yields $c - E[\epsilon_t^2 \mid X_t] \geq (1 - \bar{a})c$. Similarly for the next $\delta$ time steps, $E[\epsilon_t^2 \mid X_t] - E[\epsilon_t^2 \mid X_t] \geq (1 - \bar{a})E[\epsilon_t^2 \mid X_t]$. Repeating this process recursively for the following $(m-2)$ periods we can obtain a total of $(m-1)$ inequalities. Adding the left-hand sides of these $(m-1)$ inequalities yields $c - E[\epsilon_{\text{out}}^2 \mid X_t]$, which is larger or equal to the sum of the right-hand sides:

$$(1 - \bar{a}) \sum_{i=0}^{m-1} E[\epsilon_{i+1}^2 \mid X_t] \leq c - E[\epsilon_{\text{out}}^2 \mid X_t] \leq c$$

Since $1 - \bar{a} > 0$,

$$\lim_{m \to \infty} \sum_{i=0}^{m-1} E[\epsilon_{i+1}^2 \mid X_t] = \frac{c}{(1 - \bar{a})} < \infty$$

Namely, $\sum_{i=0}^{m} E[\epsilon_{i+1}^2 \mid X_t]$ is bounded. Applying the Borel-Cantelli Lemma [9] to this yields,

$$E[\epsilon_{\text{out}}^2 \mid X_t] \to 0$$

Since $E[\epsilon_{\text{out}}^2 \mid X_t] \geq E[\epsilon_t^2 \mid X_t]$ for $\forall t \in [m\delta, (m+1)\delta]$,

$$E[\epsilon_t^2 \mid X_t] \to 0$$

This implies (18); as $t$ tends to infinity, $e_t$ converges to 0 with probability 1. Furthermore, from the supermartingale inequality and the Chebyshev inequality [8] it follows that

$$\Pr\left(\sup_{s \leq t} \epsilon_s^2 \geq \mu\right) \leq \frac{E[\epsilon_t^2 \mid X_t]}{\mu} \leq \frac{\bar{a}e^t}{\mu}$$

Thus, the theorem holds and the convergence properties are guaranteed. Q.E.D.

C. Performance Characterization

While the Theorem is useful for proving convergence, it gives a very conservative bound on convergence rate, since in many cases, the fractional convergence will be much smaller than $\bar{a}$, and will be so much more frequently than once every $\delta$ time steps. Therefore, it is much more insightful to compute the ensemble expected squared error at each time step.

We illustrate performance through simulation of $N = 100$ cells, to be controlled to a constant reference, $r = 50$, and repeated 10,000 times for each of the following cases: (a) $n_y = 0$, $n_x = 0$, and $N^{FR} = N$; (b) $n_y = 4$, $n_x = 4$, and $N^{FR} = N$; (c) $n_y = 20$, $n_x = 4$, and $N^{FR} = N$; (d) $n_y = 4$, $n_x = 4$, and random initial conditions. Case (c) is an extreme asymmetric...
system with an extremely long backward preloading time. In case of (d), each cell has a 1/2 probability of starting on or off with initial state distributed uniformly across the intermediate and ready states of each on or off side. The ensemble means for each of the cases can be computed by averaging across the 10,000 trials.

Figure 4 shows $E[e_r^2]$ estimated from 10,000 trials for each case. Case (a) does not have any intermediate nodes and thus converges very quickly. Case (d) starts much closer to the desired reference on average, but requires slightly more time than (a) to converge, on average, because of intermediate nodes. Cases (b) and (c) both have many intermediate nodes, and these cases do not initially have any on units. Thus, if the target is overshot, then, it is not possible for any units to respond until the units which just turned on traverse all of the intermediate nodes. The system converges even for these extreme cases, as guaranteed by the Theorem.

![Figure 4. RMS error versus time for cases (a), (b), (c), and (d).](image)

This paper has developed and proved stability of a broadcast feedback method for controlling the collective behavior of an ensemble of stochastically behaving, hysteretic cellular units.

The presented control methodology has the potential to be a useful approach to many application areas, which include both natural biological systems and engineered systems. The latter includes not only shape memory alloy actuators that we used as an exemplary case study for developing the control architecture and theory, but also any physical system exhibiting distributed hysteretic dynamics whose aggregate output must be controlled stably and robustly.

Such cellular architecture naturally resides in real biological systems, too. The authors are currently working on the control of biological cell populations, where collective behaviors of a multitude of cells must be controlled. The stochastic broadcast feedback approach is a promising methodology for those real biological cells, whose behaviors can be modeled as stochastic agents with prominent hysteresis, time delay, and refractory periods.

APPENDIX A: PROOF OF THE LEMMA

Without loss of generality we can assume $e_i > 0$. First we can show that there is at least one cell at the states on the forward branch of the hysteresis loop. Otherwise, all the cells are on and $Y = N$. Then $e_i = r - Y$, cannot be positive. Therefore, at least one cell must exist on the forward states, $F_{i_1}, \ldots, F_{i_n}$ and $F_e$. Although no ready cell is available at time $t$, $N_i = 0$, other cells at states $F_{i_1}, \ldots, F_{i_n}$ can reach the ready state $F_e$ in at most $n_i$ time steps.

This implies that at least once in $n_i \leq \delta$ time steps, ready cells appear, $N_i^{FR} \geq 1$, for which the following computations apply. If $1 \leq N_i^{FR} < \epsilon_i / \eta$, then $p_{on} = 1$ from (15), and the quotient of $E[e_i^2 | X] / e_i^2$ is given by

$$\alpha_a = E[e_i^2 | X] / e_i^2 = \left(1 - \eta \frac{N_i^{FR}}{\eta_i} \right)^2 \leq \left(1 - \frac{1}{N} \right)^2 < 1 \quad (26)$$

If $\frac{1}{2} \leq e_i / \eta \leq N_i^{FR}$, $p_{on} = p_o$ and

$$\alpha_b = E[e_i^2 | X] / e_i^2 = 1 - \eta \frac{N_i^{FR}}{\eta_i} \leq 1 - \frac{1}{4} \left( \frac{N}{N - 1} \right) < 1 \quad (27)$$

Both cases yield $E[e_i^2 | X] < e_i^2$. Let $\tilde{a} = \max(\alpha_a, \alpha_b) < 1$, then $E[e_i^2 | X] \leq \tilde{a} e_i^2$ at least once in $n_i \leq \delta$ time steps. For $e_i < 0$, at least one cell can reach the backward ready state $B_e$ in at most $n_i \leq \delta$ time steps, and at that time $E[e_i^2 | X] < e_i^2$. Therefore, the expected squared error decreases at least once in $\delta = \max(n_i, n_o)$ time steps and the expected squared error is bounded with the same $\tilde{a} < 1$ as in the case of $e_i > 0$.

$$E[e_i^2 | X] \leq \tilde{a} e_i^2. \text{ Q.E.D.}$$

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