On the invertibility of the XOR rotations of a binary word
On the invertibility of the XOR of rotations of a binary word

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Abstract

We prove the following result regarding operations on a binary word whose length is a power of two: computing the exclusive-or of a number of rotated versions of the word is an invertible (one-to-one) operation if and only if the number of versions combined is odd.

(This result is not new; there is at least one earlier proof, due to Thomsen in his PhD thesis [12]. Our proof may be new.)

Keywords: invertibility, exclusive-or, rotation, binary words, circulant matrix.

1 Introduction and proof of main result

This short note considers some simple operations on binary words.

We only consider binary words whose length is a power of two, as this is typically the case for actual computer operations (e.g., with 32-bit or 64-bit words).

We focus on operations based on rotations and exclusive-ors, as these are typically standard built-in operations.

Simple invertible operations such as these are used in many applications, such as pseudo-random number generation [7, 9], encryption [4], and cryptographic hash function design [10].

We state and prove the main result, and then provide some related discussion afterwards.

Theorem 1 If \( n \) is a power of two, \( v \) is an \( n \)-bit word, and \( r_1, r_2, \ldots, r_k \) are distinct fixed integers modulo \( n \), then the function

\[
R(v) = R(v; r_1, r_2, \ldots, r_k) = (v \ll r_1) \oplus (v \ll r_2) \oplus \cdots \oplus (v \ll r_k)
\]

(1)

is invertible if and only if \( k \) is odd, where \( (v \ll r) \) denotes the \( n \)-bit word \( v \) rotated left by \( r \) positions, and where \( \oplus \) denotes the bit-wise “exclusive-or” of \( n \)-bit words.

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Proof: Let \( V = \{0, 1\} \), and let \( V^n \) denote the set of all \( n \)-bit words. We identify \( V^n \) with \( GF(2)^n \), the set of \( n \)-element vectors over the finite field \( GF(2) \).

With this identification, \( R \) is a linear operation over \( V^n \); \( R(v) \) may be obtained by multiplying \( v \) by an \( n \times n \) circulant matrix over \( GF(2) \) having \( k \) ones per row and per column. (An equivalent statement of our theorem is that when \( n \) is a power of two, an \( n \times n \) circulant matrix over \( GF(2) \) is invertible if and only if the number \( k \) of ones in each row is odd.)

We define the Hamming weight (or weight) of an \( n \)-bit word \( v \) to be the number of ones in \( v \).

Our proof identifies words in \( V^n \) with polynomials in \( GF(2)[x] \) of degree less than \( n \).

For each \( n \)-bit word \( v \) we define an associated polynomial \( v(x) \) in \( GF(2)[x] \) in the natural way: if

\[
    v = (v_{n-1}, v_{n-2}, \ldots, v_1, v_0)
\]

then the associated polynomial \( v(x) \) is

\[
    v(x) = \sum_{i=0}^{n-1} v_i x^i.
\]

For example, the unit-weight word \( u_i \) having a one in position \( i \) is associated with the polynomial \( u_i(x) = x^i \). This association between words and polynomials is one-to-one.

Let \( f_n(x) = x^n + 1 \), a polynomial in \( GF(2)[x] \). We now work with polynomials modulo \( f_n(x) \), so that rotation can be effected by polynomial multiplication modulo \( f_n(x) \), as is typically done when working with cyclic error-correcting codes (see [6, Section 9.2]) or circulant matrices (see [1]).

Now the word

\[
    (v \ll r)
\]

is associated with the polynomial

\[
    v(x) * u_r(x) \pmod{f_n(x)}
\]

reducing modulo \( f_n \) captures the effects of the rotation. In other words, multiplying by \( u_r(x) \) modulo \( f_n(x) \) represents a left-rotation by \( r \) positions.

Computing \( R(v) \) combines the effect of several rotations, so the word \( R(v) \) is associated with the polynomial

\[
    v(x) * r(x) \pmod{f_n(x)}
\]

where

\[
    r(x) = x^{r_1} + x^{r_2} + \cdots + x^{r_k}.
\]

Note that \( R \) is an invertible operation if and only if \( r(x) \) is relatively prime to \( f_n(x) \); (This result is due to Guan et al. [5] Theorem 2.4; see also Bini et al. [1] Theorem 2.2.) If \( \gcd(r(x), f_n(x)) = 1 \), then an inverse to \( r(x) \) modulo \( f_n(x) \) can be found by the extended version of Euclid’s algorithm, otherwise no inverse exists. These propositions hold whether or not \( n \) is a power of two.
If \( n \) is a power of two, then
\[
R_n(x) = x^n + 1 = (x + 1)^n,
\]
since we are working in \( GF(2) \) (see [6, Thm. 1.46]). In this case, \( r(x) \)
is relatively prime to \( f_n(x) \) if and only if \( r(x) \) is relatively prime to the
polynomial \( x + 1 \).

Polynomials that are not relatively prime to \( x + 1 \) must be multiples of
\( x + 1 \), since \( x + 1 \) is irreducible. A polynomial in \( GF(2)[x] \) is a multiple of
\( x + 1 \) if and only if its value at \( x = 1 \) is 0. But \( r(1) = 0 \) if and only if \( r(x) \)
has an even number of non-zero coefficients. Therefore \( r(x) \) is relatively
prime to \( f_n(x) \) if and only if \( k \) is odd.

Thus, when \( n \) is a power of two, \( R \) is an invertible operation on \( GF(2)^n \)
if and only if \( k \) is odd.

2 Discussion

The inverse operation to \( R \) can be found using Euclid’s extended algorithm
on input polynomials \( r(x) \) and \( f_n(x) \), to find polynomials \( s(x) \) and \( t(x) \) such that
\[
s(x) \cdot r(x) + t(x) \cdot f_n(x) = 1.
\]
The inverse operation \( S \) to \( R \) corresponds to the polynomial \( s(x) \), rep-
resenting another function of the same form as \( R \) (that is, an xor of rotations).
In matrix terms, the inverse of a circulant matrix is another
circulant matrix.

In terms of computational complexity, \( R(v) \) is easy to compute when
\( k \) is small, requiring not more than \( k \) rotations and \( k - 1 \) xors. Although
the inverse \( S \) has the same form as \( R \), it may require considerably more
work to compute. For example, if \( r(x) \) has degree \( d \), then \( s(x) \) must
have degree at least \( n/d \) and at least \( n/d \) terms, so that evaluating \( S(v) \)
requires at least \( \log_2(n/d) \) additions, since each addition in a computation
chain can at most double the number of terms. Here multiplication by \( x^r \)
(rotations) are “free” and we are only counting exclusive-ors. The exact
complexity, in terms of rotations and xors, of evaluating \( R(v) \) or \( S(v) \) may
be non-trivial to determine precisely, and we leave these questions as open
problems. Thus, when \( k \) and \( d \) are small \( R \) may be considered to be in
some sense “very modestly one-way”—easier to compute in one direction
than another. Stephen Boyack [3] has interesting related results on the
complexity of matrix operations over \( GF(2) \) and their inverses.

Efficient invertible operations are useful in many applications. A linear
operation somewhat similar to the one studied here is the “xorshift”
operation:
\[
v = v \oplus (v \ll r)
\]
where “\( \ll \)” is the “left-shift” operator; xorshift has been used in pseudo-
random number generation [7, 9] and hash-function design [10]. Schnorr
and Vaudenay [11, Lemma 5] study the related operation
\[
(v \land d) \oplus (v \ll \ll r)
\]
where “∧” denote bitwise “and” and where $d$ is a constant $n$-bit word; they show that this operation is invertible if and only if the iterates $(d \llll (r \cdot i))$ take for each bit position the value 0 for some $i$.

The result of this paper may be useful to those working on similar applications. For example, we began our study of $R$ when thinking about possible improvements to the MD6 hash function [10]. We also note that the $k = 3$ version of the operation discussed here is used in the C2 cipher [2] (although not in manner that required its invertibility (it is part of the feedback function in a Feistel block-cipher)), and in the SHA hash function standard message expansion computation [8] (as the $\Sigma$ function; invertibility of $\Sigma$ is not claimed or proven).

When $n$ is not a power of 2, we don’t know of any comparably simple characterization of when $R(v)$ is invertible, other than the requirement that \( \gcd(f_n(x), r(x)) = 1 \); perhaps simpler characterizations can be found for some cases, such as when $n = 3 \cdot 2^k$.

### 3 Related Work

Lars Knudsen points out that a different proof for the same result is available in the Ph.D. thesis [12, Theorem 3.3, pages 86–87] of Søren Thomsen. Thomsen’s cute proof considers powers $R^{2^i}$ of the original operation, notes that

$$R^{2^i}(v; r_1, r_2, \ldots, r_k) = R(v; 2r_1, 2r_2, \ldots, 2r_k)$$

from which it follows that $R$ is invertible since $R^n$ will be the identity function (if and only if $k$ is odd).

### 4 Conclusions

This note provides an alternate proof of a characterization as to when an easily computed operation, based on the exclusive-or of rotated versions of a word, is invertible.

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### References


