C[script 3], semi-clifford and generalized semi-clifford operations

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\section{Introduction}

The theory of fault-tolerant quantum computation is one of the main parts of the theory of quantum computation. In this theory we are introducing a quantum code, a universal set of gates, and then a method to apply these gates fault-tolerantly \cite{1}. The most important quantum codes are quantum stabilizer codes, and teleportation is an idea to apply a universal set of quantum gates on these codes \cite{2}.

It is well-known that all Pauli gates as well as Clifford operators can be applied fault-tolerantly using teleportation. However, these are not the only gates with such a property. Indeed, a Clifford operator can be applied fault-tolerantly via teleportation because by conjugation it sends the Pauli group to itself. Generalizing this idea, we can define the so called $C_k$ hierarchy operators.

\begin{definition}
Let $\mathcal{P} = C_1$ denote the Pauli group. For $k \geq 1$ define
\begin{equation}
C_{k+1} = \{ U : UPU^\dagger \subseteq C_k \}.
\end{equation}
\end{definition}

Gottesman and Chuang in \cite{2} introduce these sets and show that all $C_k$ gates can be applied fault-tolerantly via teleportation. However, our knowledge of these operators is poor.

By definition, $C_2$ is the Clifford group, and there is a rich theory for characterizing and representing these operators \cite{3}. Also, by definition $C_k \subseteq C_{k+1}$. But for $k \geq 3$, $C_k$ is no longer the Clifford group.
longer a group \[ \mathcal{C}_k \], and because of that, the problem of studying \( \mathcal{C}_k \) hierarchy turns out to be a hard one. However, if we restrict ourselves to diagonal gates in \( \mathcal{C}_k \), it is a group \[ \mathcal{C}_k \].

Let us think of the diagonal gates in another direction. If \( U \in \mathcal{C}_k \) is diagonal, then \( U \) commutes with all \( \sigma_z \) operators, and then \( UPU^\dagger \) contains a maximal abelian subgroup of \( \mathcal{P} \). Also, for such a \( U \), it is not hard to see that \( QUQ_2 \) is in \( \mathcal{C}_k \) and satisfies the same property for all Clifford operators \( Q_1 \) and \( Q_2 \). This observation leads us to the following definition.

**Definition 1.2** A unitary operator \( U \) acting on \( n \) qubits is called semi-Clifford if \( UPU^\dagger \) contains a maximal abelian subgroup of \( \mathcal{P} \). In other words, \( U \), under conjugation, sends a maximal abelian subgroup of \( \mathcal{P} \) to another maximal abelian subgroup of \( \mathcal{P} \).

Semi-Clifford operators are first defined in \[ 5 \] for one qubit operators, and are generalized for \( n \) qubits in \[ 4 \]. Also in \[ 4 \], Zeng et al. raise the question of the relation between semi-Clifford operators and \( \mathcal{C}_k \) hierarchy. They show that all \( \mathcal{C}_k \) hierarchy gates are semi-Clifford if \( n = 1 \), \( 2 \) (\( n \) is the number of qubits). For \( n = 3 \) they prove the same property if \( k = 3 \), and propose the following conjecture for larger \( n \).

**Conjecture 1.1** \[ 4 \] All gates in \( \mathcal{C}_3 \) are semi-Clifford operations.

Moreover, for \( k \geq 4 \) by giving an example they show that there are gates in \( \mathcal{C}_k \) which are not semi-Clifford. But they realize that those gates are generalized semi-Clifford.

**Definition 1.3** A generalized semi-Clifford operator on \( n \) qubits is defined to send, by conjugation, the linear span of at least one maximal abelian subgroup of \( \mathcal{P} \) to the linear span of another maximal abelian subgroup of \( \mathcal{P} \).

Clearly, any semi-Clifford operator is generalized semi-Clifford. But we can think of an abelian group of \( 2^n \) diagonal matrices which are all linearly independent and are different from \( \sigma_z \) gates; then, the span of this group is the same as the span of all \( \sigma_z \) operators in \( \mathcal{P} \). Thus, semi-Clifford and generalized semi-Clifford operators are not the same.

Here is the second conjecture made in \[ 4 \].

**Conjecture 1.2** \[ 4 \] All gates in \( \mathcal{C}_k \) are generalized semi-Clifford operations.

The main result of this paper is that Conjecture 1.2 holds for \( k = 3 \).

**Theorem 1.1** Every gate in \( \mathcal{C}_3 \) is generalized semi-Clifford.

### 1.1 Related works

Gottesman and Mochon (personal communication) have disproved Conjecture 1.1, i.e. they have found a gate in \( \mathcal{C}_3 \) which is not semi-Clifford. Here we briefly discuss their counterexample.

Consider seven qubits and call them \( A_1, A_2, A_3, B_1, B_2, B_3, \) and \( R \). Let \( U \) be the multiplication of the three controlled-swap gates which act on \( (R, A_i, B_i) \), \( i = 1, 2, 3 \) (it swaps \( A_i \) and \( B_i \) if \( R \) is \( |1\rangle \)). Also, let \( V \) be the multiplication of four controlled-\( \sigma_z \) gates which act on \( (A_1, A_2, A_3), (A_1, B_2, B_3), (B_1, A_2, B_3), \) and \( (B_1, B_2, A_3) \). By computing the action of \( UV \) on Pauli matrices it can be seen that \( UV \in \mathcal{C}_3 \); however, \( VU \) is not in \( \mathcal{C}_3 \) since \( (VU)\sigma_z(VU)^\dagger \), where \( \sigma_z \) acts on qubit \( R \), does not belong to \( \mathcal{C}_2 \).

Now, we claim that \( UV \) is a \( \mathcal{C}_3 \) gate which is not semi-Clifford. Suppose \( UV \) is semi-Clifford; then, by Proposition 1 of \[ 4 \] there are Clifford operations \( Q_1, Q_2 \) such that \( D = \)
$Q_1 U V Q_2$ is diagonal. On the other hand, since $UV$ is in $C_3$, $D \in C_3$, which means that $D P D^T \subseteq C_2$. Note that, for any $\sigma \in P$ and $Q \in C_2$, $\bar{\sigma}$ and $\bar{Q}$ (the entry-wise complex conjugate of $\sigma$ and $Q$) also belong to $P$ and $C_2$, respectively. Hence, $D P D^T \subseteq C_2$ implies $D \bar{P} D^T \subseteq C_2$, or equivalently $D = D^T = Q_1^T U V Q_1^T \in C_3$. Therefore, $VU$ is in $C_3$, which is a contradiction.

1.2 Structure of the paper

In Section 2, we fix some notations on Pauli operators and then review the characterization of Clifford operations from [3], which represents each Clifford gate by a $C$-matrix (a symplectic matrix) and an $h$-vector over the binary field.

In Section 3, we formulate our main idea for proving Theorem 1.1, which is to express the whole assumptions in terms of some relations on $C$-matrices and $h$-vectors.

In Section 4, we reduce the problem to a special case where $C$-matrices contain a block of zeros, which is easier to handle.

In Section 5, we are trying to obtain more information from the representation of Clifford operations based on $C$-matrices and $h$-vectors. Indeed, $C$ and $h$ characterize Clifford operators up to an overall phase. Thus, having $C$ and $h$, we should be able to determine entries of a Clifford operator as a matrix, up to an overall phase. In Section 5, we explicitly find the matrix representation of Clifford operations in the special case introduced in Section 4. The generalization of these results are studies in Appendix A, in which each Clifford operator is expressed as a linear combination of Pauli matrices, and the coefficients of this expansion are computed.

In Section 6, we find a formula that given two Clifford operators $Q$ and $Q'$ represented by $(C, h)$ and $(C', h')$, respectively, expresses whether $QQ' = Q'Q$ or $QQ' = -Q'Q$. Notice that this is a valid question since the representation of Clifford operations by $C$-matrices and $h$-vectors, is independent of an overall phase, and cannot distinguish $Q'Q$ and $-Q'Q$.

In Section 7, we put all pieces together and finish the proof of Theorem 1.1.

2 Preliminaries

2.1 Pauli operators

First of all let us fix some notations for Pauli matrices.

$$
\sigma_{00} = \tau_{00} = \sigma_0 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \quad \sigma_{01} = \tau_{01} = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma_{10} = \tau_{10} = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad i \sigma_{11} = \tau_{11} = i \sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
$$

Pauli operators over $n$ qubits are denoted by $\sigma_a$ and $\tau_a$, where $a = \begin{pmatrix} v \\ w \end{pmatrix}$ is in $\mathbb{Z}_2^{2n}$ and

$$
\sigma_a = \sigma_{v_1 w_1} \otimes \cdots \otimes \sigma_{v_n w_n}, \\
\tau_a = \tau_{v_1 w_1} \otimes \cdots \otimes \tau_{v_n w_n}.
$$

Also, we represent the phases $\pm 1, \pm i$ by $i^\delta (-1)^\epsilon$, where $\delta, \epsilon \in \mathbb{Z}_2$. Then, it is easy to see that the multiplication of two Pauli operators $(i^{\delta_1} (-1)^{\epsilon_1} \tau_{a_1})(i^{\delta_2} (-1)^{\epsilon_2} \tau_{a_2})$ is equal to $i^{\delta} (-1)^{\epsilon} \tau_{a}$, where

$$
\delta = \delta_1 + \delta_2, \\
\epsilon = \epsilon_1 + \epsilon_2 + \delta_1 \delta_2 + a_2^T J a_1,
$$

$$
a = a_1 + a_2,
$$

where $J$ is the symplectic form.
in which \( T \) denotes the transposed matrix,

\[
J = \begin{pmatrix}
0 & I_n \\
0 & 0
\end{pmatrix},
\]

and \( I_n \) is the identity matrix of size \( n \). As a result,

\[
\tau_a \tau_b = (-1)^{b^\mathsf{T} P a \tau_b \tau_a},
\]

where

\[
P = J + J^\mathsf{T} = \begin{pmatrix}
0 & I_n \\
I_n & 0
\end{pmatrix}.
\]

### 2.2 Clifford operators

Assume that \( Q \) is a Clifford operator. By definition for each Pauli matrix \( \tau_a \), \( Q \tau_a Q^\dagger \) is also a Pauli operator. According to Eq. (8) and Eqs. (10) and (14), the map

\[
Q_{\tau_a \tau_a} Q^\dagger = (Q_{\tau_a} Q^\dagger)(Q_{\tau_a} Q^\dagger)
\]

to compute the image of Pauli operators under conjugation by \( Q \), it is sufficient to know \( Q \tau_j Q^\dagger \), \( j = 1, \ldots, 2n \), where \( \{ e_1, \ldots, e_{2n} \} \) is the standard basis of \( \mathbb{Z}_2^{2n} \) (all coordinates of \( e_j \) are 0 except the \( j \)-th which is 1). Hence, assume that \( Q \tau_j Q^\dagger = i^{\delta_j}(-1)^{\epsilon_j} \tau_j \), where \( d, h \in \mathbb{Z}_2^n \), and \( c_j \in \mathbb{Z}_2^n \). Notice that \( \tau_j \) and then \( Q \tau_j Q^\dagger \) are hermitian, so \( d_j \) can be determined in terms of \( e_j \); \( d_j = e_j^T J e_j \). Thus, if we define a \( 2n \times 2n \) matrix \( C \) whose \( j \)-th column is equal to \( e_j \), then

\[
d = \text{diag}(C^T J C),
\]

where \( \text{diag}(M) \) denotes a vector whose \( j \)-th coordinate is the \( j \)-th entry on the diagonal of \( M \).

Now, using the matrix \( C \) and Eq. (12) we can compute \( Q(i^{\delta_1}(-1)^{\epsilon_1} \tau_{a_1}) Q^\dagger \) in terms of \( C \) and \( h \). In fact,

\[
Q(i^{\delta_1}(-1)^{\epsilon_1} \tau_{a_1}) Q^\dagger = i^{\delta_2}(-1)^{\epsilon_2} \tau_{a_2}
\]

in which \( \text{low}(M) \) denotes the strictly lower triangular part of matrix \( M \).

In order to determine the conditions that \( C \) and \( h \) should satisfy, note that if \( \tau_a \) and \( \tau_b \) commute, then their image under conjugation by \( Q \), commute as well. Therefore, by Eqs. (10) and (14), the map \( a \rightarrow Ca \) should preserve the symplectic inner product, i.e.

\[
a^T C^T P C b = a^T P b,
\]

or equivalently

\[
C^T P C = P.
\]

Matrices \( C \) that satisfy Eq. (17) are called symplectic.

Dehaene and De Moor prove the following theorem in [3].

**Theorem 2.1** [3] For any symplectic matrix \( C \) (satisfying Eq. (17)) and any vector \( h \) over \( \mathbb{Z}_2 \), there is a unique (up to a phase) Clifford operator \( Q \) such that \( Q(i^{\delta_1}(-1)^{\epsilon_1} \tau_{a_1}) Q^\dagger = i^{\delta_2}(-1)^{\epsilon_2} \tau_{a_2} \) where \( a_2, \delta_2 \) and \( \epsilon_2 \) are given by Eqs. (14), (15), (16).

According to this theorem any Clifford operator \( Q \) can be represented by a pair \((C, h)\), where \( C \) is a symplectic matrix. Then, to get a full representation of Clifford operators as a group, it is sufficient to compute the inverse and product of these operators based on \( C \)-matrices and \( h \)-vectors.
Theorem 2.2 \[3\]

(a) Given \((C_1, h_1)\) and \((C_2, h_2)\) defining two Clifford operators \(Q_1\) and \(Q_2\), respectively, the product \(Q_{12} = Q_2Q_1\) is represented by \((C_{12}, h_{12})\) such that

\[
\begin{align*}
C_{12} &= C_2C_1, \\
h_{12} &= h_1 + C_1^T h_2 + \text{diag}(C_1^T h_2 + d_2 d_2^T)C_1 + d_1 d_2^T C_1,
\end{align*}
\]

(18) where \(d_1\) and \(d_2\) are defined in Eq. \[13\].

(b) Given \((C, h)\) defining a Clifford operator \(Q\), the inverse \(Q' = Q^{-1}\) is represented by \((C', h')\) such that

\[
\begin{align*}
C' &= C^{-1}, \\
h' &= C^{-T} h + \text{diag}(C^{-T} h C + dd^T)C^{-1} + d'd^T C^{-1},
\end{align*}
\]

(19) where \(M^{-T} = (M^{-1})^T\), and \(d' = \text{diag}(C^{-T} J C^{-1})\).

Theorems 2.1 and 2.2 give a full representation of Clifford group. However, this representation is up to a phase, i.e. two Clifford operations that differ only on a global phase have the same \(C\)-matrix and \(h\)-vector. As the last remark on this representation notice that each Pauli operator is a Clifford gate as well. So, we can represent it by a \(C\)-matrix and an \(h\)-vector. Also, by Eq. \[11\] every two Pauli matrices either commute or anti-commute. Therefore, the \(C\)-matrix for all Pauli operators is identity. More explicitly, the Pauli operator \(\tau_a\) corresponds to \((C = I_{2n}, h = Pa)\).

3 Main ideas

In this section, we formulate our main idea for proving Theorem 1.1. Let \(U\) be a \(C_3\) gate on \(n\) qubits. Then, by definition \(U P U^\dagger\) is a subset of Clifford group, so if we define \(Q_i = U \sigma_e U^\dagger\), \(i = 1, \ldots, 2n\), then \(Q_i\) is a Clifford gate and

\[
Q_i^2 = I,
\]

(22)

\[
Q_i Q_j = (-1)^{\delta_{i+n,j}} Q_j Q_i,
\]

(23)

where \(\delta_{i+n,j}\) is the Kronecker delta function, and \(i \leq j\).

Conversely, let \(Q_1, \ldots, Q_{2n}\) be Clifford operators that satisfy Eqs. \[22\] and \[23\]. Then, \(Q_1, \ldots, Q_{2n}\) commute, and they have a common eigenvector \(|\alpha\rangle\) with eigenvalues +1 or -1. Let \(Q_i|\alpha\rangle = (-1)^{\lambda_i}|\alpha\rangle\), for \(i = 1, \ldots, n\), where \(\lambda_i \in \mathbb{Z}_2\). Define the linear operator \(U\) by

\[
U|x_1 \ldots x_n\rangle = Q_{n+1}^{\xi_1} \ldots Q_{2n}^{\xi_n} |\alpha\rangle,
\]

(24)

for every standard basis vector \(|x_1 \ldots x_n\rangle\), where \(x_1, \ldots, x_n \in \mathbb{Z}_2\).

We claim that \(U\) is unitary and \(Q_i = U \sigma_e U^\dagger\) for every \(i\). Since all \(Q_i\)'s are unitary, the vectors \(U|x_1 \ldots x_n\rangle\) are normal. Also, if \((x_1, \ldots, x_n)\) and \((y_1, \ldots, y_n)\) are different, say at the first coordinate \((x_1 + y_1 = 1)\), then

\[
\begin{align*}
\langle x_1 \ldots x_n | U^\dagger U | y_1 \ldots y_n \rangle &= \langle \alpha | Q_{n+1}^{x_1+y_1} \ldots Q_{2n}^{x_n+y_n} | \alpha \rangle \\
&= (-1)^{\lambda_1} \langle \alpha | Q_{n+1}^{x_1+y_1} \ldots Q_{2n}^{x_n+y_n} Q_1 | \alpha \rangle \\
&= -(-1)^{\lambda_1} \langle \alpha | Q_{n+1}^{x_1+y_1} \ldots Q_{2n}^{x_n+y_n} Q_1 | \alpha \rangle \\
&= -\langle \alpha | Q_{n+1}^{x_1+y_1} \ldots Q_{2n}^{x_n+y_n} | \alpha \rangle \\
&= -\langle x_1 \ldots x_n | U^\dagger U | y_1 \ldots y_n \rangle.
\end{align*}
\]

(25)
Thus, $\langle x_1 \ldots x_n | U^\dagger U | y_1 \ldots y_n \rangle = 0$ and $U$ is unitary. $Q_i = U \sigma_i U^\dagger$ can be proved by showing that the action of $U^\dagger Q_i U$ on the basis vectors is equal to the action of $\sigma_i$.

This observation shows that any $C_3$ gate corresponds to a subgroup of the Clifford group which is isomorphism to the Pauli group, and vice versa. Also, it suggests to study subgroups of Clifford group, isomorphic to Pauli group, instead of $C_3$ gates directly.

**Theorem 3.1** Theorem [1.7] is equivalent to the following:

Let $G$ be a subgroup of the Clifford group which is isomorphic to the Pauli group. Then there exist maximal abelian subgroups $H \subset G$ and $H' \subset P$ such that the linear span of $H$ is equal to the linear span of $H'$. □

To proceed through this idea, let the group $G$ be generated by Clifford operations $Q_1, \ldots, Q_{2n}$ that satisfy Eqs. (22) and (23). As a Clifford operator, let $Q_i$ be represented by the pair $(C_i, h_i)$, $i = 1, \ldots, 2n$. Then according to Theorem 2.1, Eq. (22) implies

$$C_i^2 = I,$$

$$h_i + C_i^T h_i + \text{diag}(C_i^T J C_i + d_i d_i^T) C_i + d_i d_i^T C_i) = 0,$$

and Eq. (23) implies

$$C_i C_j = C_j C_i,$$

$$h_i + C_i^T h_j + \text{diag}(C_i^T J C_j + d_j d_j^T) C_i + d_j d_j^T C_i)$$

$$= h_j + C_j^T h_i + \text{diag}(C_j^T J C_i + d_i d_i^T) C_j + d_j d_j^T C_j),$$

for every $i, j$.

Notice that representing a Clifford operator by a $C$-matrix and an $h$-vector is independent of a global phase. This is why here we do not see the sign of Eq. (23). Indeed, we need another equation, which we call the sign formula, to compute this sign and get a full representation of Eqs. (22) and (23).

Unlike Eqs. (26) and (28), Eqs. (27) and (29) are not easy to handle, so we need to somehow reduce these equations to simpler ones.

Let $Q$ be an arbitrary Clifford operator represented by the pair $(C, h)$. Define $Q'_i = QQ_i Q^\dagger$; then, $Q'_1, \ldots, Q'_{2n}$ satisfy Eqs. (22) and (23) as well, so they generate a subgroup of the Clifford group isomorphic to the Pauli group. Also, since $Q$ sends Pauli operators to Pauli operators under conjugation, proving the claim in Theorem 3.1 for the group generated by $\{Q'_i : i = 1, \ldots, 2n\}$, implies the theorem for the group generated by $\{Q_i : i = 1, \ldots, 2n\}$.

This argument shows that in Eqs. (26) and (29), we can replace the symplectic matrices $C_1, \ldots, C_n$ by $CC_1 C^{-1}, \ldots, CC_{2n} C^{-1}$ for every symplectic matrix $C$. Here we use the fact that every symplectic matrix corresponds to the $C$-matrix of some Clifford operation $Q$ (Theorem 2.1), and the formulas for the $C$-matrix of the inverse and multiplication of Clifford operators (Theorem 2.2).

### 4 Symplectic involutions

In the previous section we see that each of the symplectic matrices $C_i$ is an involution ($C_i^2 = I$). Therefore, if $C_i$ was a matrix in a field of characteristic different from 2, $C_i$ was diagonalizable with $+1, -1$ on the diagonal. In fact, a stronger property holds; it is proved in [6] that for any commutative set of symplectic involutions over a field of characteristic $\neq 2$, there exists a symplectic matrix $M$ such that $MCM^{-1}$ is diagonal with $+1, -1$ diagonal entries, for every matrix $C$ in the set.

Here, all the symplectic involutions are over $\mathbb{Z}_2$, and the above proposition does not hold anymore. However, following almost the same steps as in [6], an analogous result can be proved.
Theorem 4.1  For every symplectic involution \( C \) of size \( 2n \) over \( \mathbb{Z}_2 \) there exists a symplectic matrix \( M \) such that
\[
MCM^{-1} = \begin{pmatrix} I & E \\ 0 & I \end{pmatrix}, \tag{30}
\]
where \( E \) is a symmetric matrix \((E^T = E)\).

Proof: Let us consider the block form of \( C \)
\[
C = \begin{pmatrix} A & E \\ F & B \end{pmatrix}, \tag{31}
\]
where \( A, B, E \) and \( F \) are \( n \times n \) matrices. \( C^2 = I \) and \( C^T PC = P \). Then \( PC = C^T P \), or equivalently \( PC \) is symmetric. In other words, \( B = A^T \), and \( E \) and \( F \) are symmetric. Therefore,
\[
C = \begin{pmatrix} A & E \\ F & A^T \end{pmatrix}. \tag{32}
\]
Also, \( C^2 = I \) gives \( AE \) and \( FA \) are symmetric and \( A^2 + EF = I \).

Assume that \( \text{rank}(E) = r \). Then there is an invertible matrix \( R \) such that
\[
RER^T = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}, \tag{33}
\]
where \( e \) is a full-rank matrix of size \( r \times r \). Since \( E \) is symmetric, \( e \) is symmetric as well.

Now notice that
\[
M_1 = \begin{pmatrix} R & 0 \\ 0 & R^{-T} \end{pmatrix} \tag{34}
\]
is symplectic and the upper-right block of \( M_1CM_1^{-1} \) is equal to \( RER^T \). So, we may assume that
\[
E = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}. \tag{35}
\]

Let
\[
A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \tag{36}
\]
where the matrices \( a_1, a_2, a_3 \) and \( a_4 \) are of sizes \( r \times r, r \times (n-r), (n-r) \times r \) and \( (n-r) \times (n-r) \), respectively. \( AE \) is symmetric and \( e \) is invertible; then \( a_3 = 0 \) and \( a_1e \) is symmetric.

Now define
\[
S = \begin{pmatrix} e^{-1}a_1 & e^{-1}a_2 \\ (e^{-1}a_2)^T & 0 \end{pmatrix}, \tag{37}
\]
and
\[
M_2 = \begin{pmatrix} I & 0 \\ S & I \end{pmatrix}. \tag{38}
\]
Since \( S \) is symmetric, \( M_2 \) is symplectic. Also, the upper-left block of \( M_2CM_2^{-1} \) is equal to
\[
\begin{pmatrix} 0 & 0 \\ 0 & a_4 \end{pmatrix}. \tag{39}
\]
Hence, we can assume that \( a_1 \) and \( a_2 \) are zero as well. (Note that in \( M_2CM_2^{-1} \), \( E \) remains unchanged.)

Let
\[
F = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}. \tag{40}
\]
using $A^2 + EF = I$ we get $a_4^2 = I$, $f_1 = e^{-1}$, $f_2 = 0$, and $f_3 = 0$ since $F$ is symmetric. Then

$$C = \begin{pmatrix} 0 & 0 & e & 0 \\ 0 & a_4 & 0 & 0 \\ e^{-1} & 0 & 0 & 0 \\ 0 & f_4 & 0 & a_4^T \end{pmatrix}. \quad (41)$$

Now, observe that the map

$$\Phi(X, Y) = \begin{pmatrix} u_1 & 0 & u_2 & 0 \\ 0 & v_1 & 0 & v_2 \\ u_3 & 0 & u_4 & 0 \\ 0 & v_3 & 0 & v_4 \end{pmatrix}, \quad (42)$$

where

$$X = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}, \quad Y = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}, \quad (43)$$

preserves multiplication: $\Phi(X_1X_2, Y_1Y_2) = \Phi(X_1, Y_1)\Phi(X_2, Y_2)$. Hence, according to the above block form of $C$, it is sufficient to prove the theorem for the special cases

$$C = \begin{pmatrix} 0 & e \\ e^{-1} & 0 \end{pmatrix}, \quad C = \begin{pmatrix} a & 0 \\ f & a^T \end{pmatrix}. \quad (44)$$

In the first case if we let

$$M = \begin{pmatrix} I & 0 \\ e^{-1} & I \end{pmatrix}, \quad (45)$$

then

$$MCM^{-1} = \begin{pmatrix} I & e \\ 0 & I \end{pmatrix}, \quad (46)$$

and the theorem holds.

In the second case, notice that

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} C \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} a^T & f \\ 0 & a \end{pmatrix}. \quad (47)$$

Then if $f \neq 0$, by the same steps as before, we can reduce $C$ to smaller matrices and use induction. Thus, we may assume that $f = 0$, and

$$C = \begin{pmatrix} a^T & 0 \\ 0 & a \end{pmatrix}. \quad (48)$$

Since $C^2 = I$, $a^2 = I$. Hence, there exists an invertible matrix $R$ on $\mathbb{Z}_2$ such that $RaR^{-1}$ is in the Jordan normal form, and it is a block diagonal matrix, each of its blocks is either $I_1 = (1)$ or

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (49)$$

On the other hand,

$$M = \begin{pmatrix} R^{-T} & 0 \\ 0 & R \end{pmatrix} \quad (50)$$

is symplectic and

$$MCM^{-1} = \begin{pmatrix} R^{-T}aTR^{-T} & 0 \\ 0 & RaR^{-1} \end{pmatrix}. \quad (51)$$
Therefore, if we prove the theorem for the two cases \( a = (1) \) and

\[
a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\]

we are done.

In the first case there is nothing to prove, and in the second case we have

\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Recall that if the characteristic of the field is not 2 we can replace the matrix in Eq. (52) by a diagonal one. Also, a commutative set of symplectic involutions over such a field can be simultaneously transformed to a set of diagonal matrices under a symplectic change of basis [6]. Comparing to this result and based on Theorem 4.1, one may expect that on a field of characteristic 2, we can transform a commutative set of symplectic involutions to matrices of the form of Eq. (30). However, this is not the case; for a counterexample consider the following matrices:

\[
C_1 = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad C_2 = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1
\end{pmatrix}.
\]

Both \( C_1 \) and \( C_2 \) are symplectic, \( C_1^2 = C_2^2 = I \), and \( C_1C_2 = C_2C_1 \). If \( C_1 \) and \( C_2 \) could be transformed to the form of Eq. (30), then \( (I + C_1)(I + C_2) = 0 \), which does not hold.

**Theorem 4.2** Let \( \mathcal{M} \) be a commutative set of symplectic involutions of size \( 2n \times 2n \) over \( \mathbb{Z}_2 \). Then there exists a symplectic matrix \( M \) such that for every \( C \in \mathcal{M} \), \( MCM^{-1} \) is of the form

\[
MCM^{-1} = \begin{pmatrix} A & E \\ 0 & A^T \end{pmatrix},
\]

where \( A \) and \( E \) are \( n \times n \) matrices.

**Proof:** We prove the theorem by induction on \( n \). If \( \mathcal{M} \) contains only the identity matrix, then there is nothing to prove. So, let \( C \in \mathcal{M} \) be unequal to identity. According to Theorem 4.1 we may assume

\[
C = \begin{pmatrix} I & E \\ 0 & I \end{pmatrix}.
\]

Also, as in the proof of Theorem 4.1 we may assume

\[
E = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix},
\]

where \( e \) is a full-rank symmetric matrix of size \( r \times r \).

Let \( C' \in \mathcal{M} \) be different from \( C \). If we consider the block form of \( C' \)

\[
C' = \begin{pmatrix} A' & E' \\ F' & B' \end{pmatrix},
\]

\end{align*}
then $B'^T = A'$, and $E', F'$ are symmetric (because $C'$ is symplectic and $C'^2 = I$.) Thus, we may assume

$$C' = \begin{pmatrix} a_1 & a_2 & e_1 & e_2 \\ a_3 & a_4 & e_2^T & e_4^T \\ f_1 & f_2 & e_1 & a_3 \\ f_2^T & f_4 & a_2 & a_4^T \end{pmatrix}. \quad (59)$$

Now, writing the constraint $CC' = C'C$, and using the fact that $e$ is full-rank, we conclude that $a_3 = 0$, $f_1 = 0$ and $f_2 = 0$. Hence, every matrix in $\mathcal{M}$ is of the form

$$C' = \begin{pmatrix} a_1 & a_2 & e_1 & e_2 \\ 0 & a_4 & e_2^T & e_4 \\ 0 & 0 & a_1^T & 0 \\ 0 & f_4 & a_2^T & a_4^T \end{pmatrix}. \quad (60)$$

Therefore, if $r = n$, which covers the base case $n = 1$, we are done. So, assume that $n > r \geq 1$.

Suppose we map such a matrix $C'$ to its sub-matrix

$$D' = \begin{pmatrix} a_4 \\ f_4 \\ a_4^T \end{pmatrix}. \quad (61)$$

This map over matrices in the form of Eq. (60) preserves addition and multiplication. Therefore, all matrices $D'$ are symplectic involutions and commute. Thus, by induction we may assume $f_4 = 0$, and we are done. □

The following corollary is a conclusion of the above theorem and the argument at the end of Section 3.

**Corollary 4.1** To prove the claim of Theorem 3.1, it is sufficient to consider the case that the group $G$ is generated by Clifford operators $Q_1, \ldots, Q_{2n}$ which are represented by pairs $(C_1, h_1), \ldots, (C_{2n}, h_{2n})$ such that

$$C_i = \begin{pmatrix} A_i & E_i \\ 0 & A_i^T \end{pmatrix}, \quad (62)$$

where $A_i^2 = I$, and $E_i$ and $A_iE_i$ are symmetric. □
5 Clifford operators as linear transformations

In Section 2, we present the characterization of Clifford operators based on how they transform Pauli matrices under conjugation. In this section, we describe these operators as linear transformations.

In Appendix A, a Clifford operator $Q$, given by a $C$-matrix and an $h$-vector, is represented as a linear combination of Pauli matrices, and the coefficients of this expansion are extracted. These results are very general; however, according to Corollary 4.1 for the purpose of proving our main theorem, we may assume that the matrix $C$ is in the form of Eq. (62). In the following, we show that in this special case, $Q$ is a permutation times a diagonal matrix. This result is used in Section 6 in order to find a formula which indicates whether two Clifford operators commute or anti-commute ($QQ' = Q'Q$ or $QQ' = -Q'Q$).

Assume that $Q$ is a Clifford gate, represented by the pair $(C, h)$, where $Q^2 = I$ and

$$C = \begin{pmatrix} A & E \\ 0 & A^T \end{pmatrix}, \quad h = \begin{pmatrix} f \\ g \end{pmatrix}. \quad (63)$$

In this special case we can find a simpler expression for Theorem 2.1. In fact,

$$C^TJC = \begin{pmatrix} 0 & 0 \\ 0 & AE \end{pmatrix}, \quad (64)$$

and

$$d = diag(C^TJC) = \begin{pmatrix} 0 \\ d_0 \end{pmatrix}, \quad (65)$$

where $d_0 = diag(AE)$. Thus,

$$Q \tau_a Q = (i)^d_0 a_2 (-1)^{h^T a + a_2^T \text{tr}(AE + d_0 a_2^T)\alpha_2} \tau_{Ca}, \quad (66)$$

where

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \quad (67)$$

Also, since $Q^2 = I$ by Eq. (27),

$$A^T f = f. \quad (68)$$

Let $\{ |x\rangle : x \in \mathbb{Z}^2 \}$ be the standard basis for the Hilbert space of $n$ qubits. Then we have

$$\tau_a |x\rangle = (-1)^{a_1^T (x + a_2)} |x + a_2\rangle. \quad (69)$$

Let $a_2 = 0$. By Eq. (66)

$$\tau_a Q |x\rangle = Q (Q \tau_a Q) |x\rangle = Q (-1)^{f^T a_1} \tau_{Ca} |x\rangle,$$

and by Eq. (69)

$$\tau_a Q |x\rangle = (-1)^{f^T a_1} (-1)^{a_1^T A^T x} Q |x\rangle = (-1)^{a_1^T (f + A^T x)} Q |x\rangle. \quad (70)$$

As a result, $Q |x\rangle$ is the simultaneous eigenvector of all $\tau_a$, where $a_2 = 0$, with eigenvalue $(-1)^{a_1^T (f + A^T x)}$. Therefore, there exists $\lambda_x$ such that

$$Q |x\rangle = \lambda_x |f + A^T x\rangle. \quad (71)$$

This equation shows that the action of $Q$ on standard basis vectors is the same as a permutation with some phases $\lambda_x$. So, if we could compute these phases, then we had a complete characterization of $Q$ as a linear operator.
Let $b$ be such that $b_1 = 0$. By Eq. 71,
\[ \tau_b Q \ket{0} = \lambda_0 \tau_b \ket{f} = \lambda_0 \ket{f + b_2}. \]
Hence,
\[ \lambda_0 Q \ket{f + b_2} = Q \tau_b Q \ket{0}. \]
Equivalently,
\[
\lambda_0 \lambda_f + b_2 \ket{f + A^T (f + b_2)} = (i) d_{b_2}' b_2 (-1)^\gamma + b_2 T \text{ lows}(AE + d_{b_2}) b_2 \tau_{C'b} \ket{0} = (i) d_{b_2}' b_2 (-1)^\gamma + b_2 T \text{ lows}(AE + d_{b_2}) b_2 \ket{A^T b_2}
\]
where in the last equation we use $b_2^T A E b_2 = d_{b_2}' b_2$. Therefore, by $A^T f = f$ we obtain
\[
\lambda_0 \lambda_f + y = (i) d_{b_2}' y (-1) d_{b_2}' y + g^T y + g^T \text{ lows}(AE + d_{b_2}) y,
\]
which determines entries of $Q$ up to an overall sign.

**Theorem 5.1** Let $Q$ be a Clifford operator, represented by $C, h$ which are given by Eq. 71, and let $Q^2 = I$. Then, the action of $Q$ on standard basis vectors is described by Eq. 73, where $\lambda_z$ are phases that can be computed by Eq. 73. □

### 6 Sign formula

Recall that in Eqs. 26-29 we express equations $Q_i^2 = Q_j^2 = I$ and $Q_i Q_j = \pm Q_j Q_i$ for two Clifford operations $Q_i$ and $Q_j$, in terms of their $C$-matrices and $h$-vectors. However, these formulas are independent of the plus or minus sign, so we need another formula, which we call the sign formula, to recognize two cases $Q_i Q_j = Q_j Q_i$ and $Q_i Q_j = -Q_j Q_i$.

Suppose $Q = Q_i$ and $Q' = Q_j$ are represented by $(C = C_i, h = h_i)$ and $(C' = C_j, h' = h_j)$, respectively, and satisfy Eqs. 26-29. Following Corollary 4.1 let us assume
\[
C = \begin{pmatrix} A & E \\ 0 & A^T \end{pmatrix}, \quad C' = \begin{pmatrix} A' & E' \\ 0 & A'^T \end{pmatrix},
\]
where $A^2 = A'^2 = I$. Also, let
\[
h = \begin{pmatrix} f \\ g \end{pmatrix}, \quad h' = \begin{pmatrix} f' \\ g' \end{pmatrix}.
\]
Then by Eq. 29 we have $f + A^T f' = f' + A^T f$, or equivalently
\[ (I + A^T) f' = (I + A^T) f. \]

According to Eq. 71
\[ Q' Q \ket{0} = Q' \lambda_0 \ket{f} = \lambda_0 \lambda_f' \ket{f' + A^T f}. \]
Similarly, $Q Q' \ket{0} = \lambda_0 \lambda_f \ket{f + A^T f'}$. Therefore, we conclude that $Q Q' = -Q Q'$ if and only if $\lambda_0 \lambda_f' = -\lambda_0 \lambda_f$, or equivalently
\[ (\lambda_0^2)(\lambda_0 \lambda_f') = -(\lambda_0^2)(\lambda_0 \lambda_f'). \]
Now note that using Eq. 73 we can explicitly compute all the terms in this equation. As a result, the sign in $Q Q' = \pm QQ'$ can be determined in terms of $C, h, C'$, and $h'$. We do not express this formula here because of its complexity; however, it is clear from Eq. 73 that if $f = f' = 0$, then $Q$ and $Q'$ commute.

**Lemma 6.1** Suppose $Q$ and $Q'$ are represented by Eqs. 76 and 77, and satisfy $Q^2 = Q'^2 = I$ and $QQ' = \pm QQ'$. Then, if $f = f' = 0$, $QQ' = QQ'$. □
7 Proof of Theorem [1.1]

Using Theorem[3] let \( \mathcal{G} \) be a subgroup of the Clifford group, isomorphic to the Pauli group. Suppose that \( \mathcal{G} \) is generated by \( Q_1, \ldots, Q_{2n} \) which are represented by pairs

\[
(C_1, h_1), \ldots, (C_{2n}, h_{2n}),
\]

respectively, and satisfy Eqs. (22) and (23), and then (20)-(24). By Corollary [1] we may assume that

\[
C_i = \begin{pmatrix} A_i & E_i \\ 0 & A_i^T \end{pmatrix}, \quad h_i = \begin{pmatrix} f_i \\ g_i \end{pmatrix}.
\]

We prove that there exists a maximal abelian subgroup \( \mathcal{H} \) of \( \mathcal{G} \) such that all of matrices in \( \mathcal{H} \) are diagonal. In that case, the linear span of \( \mathcal{H} \) would be equal to the linear span of the group generated by all \( \sigma_z \) operators (which is a maximal abelian subgroup of the Pauli group), and we are done.

Define the map \( T : \mathbb{Z}_2^{2n} \to \mathbb{Z}_2^n \) that sends \( x = (x_1, \ldots, x_{2n}) \) to the \( f \)-vector of the Clifford operator \( Q_{1}^{x_1} \cdots Q_{2n}^{x_{2n}} \). (By \( f \)-vector we mean the upper part of its \( h \)-vector.) \( T \) is not linear but almost linear.

**Lemma 7.1**

1. \( \ker T = T^{-1}(0) \) is a linear subspace of \( \mathbb{Z}_2^{2n} \).
2. For any \( x \) and \( x' \), where \( T(x) = T(x') \), \( x + x' \in \ker T \).
3. If \( y \in \ker T \), then \( T(x + y) = T(x) \), for any \( x \).
4. For every \( y \in \mathbb{Z}_2^n \), \( T^{-1}(y) \) is either empty or equal to \( x + \ker T \), for some \( x \in \mathbb{Z}_2^{2n} \).
5. \( |\ker T|, |\text{Im } T| = 2^{2n} \)

**Proof:** Define \( Q_x = Q_{1}^{x_1} \cdots Q_{2n}^{x_{2n}} \) and \( A_x = A_{1}^{x_1} \cdots A_{2n}^{x_{2n}} \).

1. We should show that if the \( f \)-vectors of \( Q_x \) and \( Q_{x'} \) are both zero, then the \( f \)-vector of \( Q_{x+x'} = \pm Q_x Q_{x'} \) is also zero. By Theorem [22] the \( f \)-vector of \( Q_x Q_{x'} \) is equal to \( T(x) + A_x^T y(x') = 0 \).
2. Let \( T(x) = T(x') \). Then, again by Theorem [22] and Eq. (82)

\[
T(x + x') = T(x) + A_x^T T(x') = T(x) + A_x^T T(x) = 0.
\]

Thus, \( x + x' \) is in \( \ker T \).

1. \( T(x + y) = T(x) + A_x^T T(y) = T(x) \).
2. (iv) and (v) are clear from (ii) and (iii). \( \square \)

**Lemma 7.2** \( \dim \ker T = n \), and \( T \) is surjective.

**Proof:** By the previous lemma, \( |\ker T|, |\text{Im } T| = 2^{2n} \), so if we prove \( \dim \ker T = n \), then \( T \) would be surjective automatically.

\( |\text{Im } T| \leq 2^n \), then \( |\ker T| \geq 2^n \), or equivalently \( \dim \ker T = r \geq n \).

Let \( x, x' \in \ker T \), so by definition, the \( f \)-vector of \( Q_x \) and \( Q_{x'} \) are both zero. Thus, by Lemma [5.1] \( Q_x \) and \( Q_{x'} \) commute. Therefore, since by Lemma [7.1] \( \ker T \) is a linear subspace, \( \mathcal{H} = \{ \pm Q_x, \pm i Q_x : x \in \ker T \} \) is an abelian subgroup of \( \mathcal{G} \). On the other hand, every maximal abelian subgroup of the Pauli group is of size \( 4 \times 2^n \). (The factor 4 is due to the phases \( \pm 1 \) and \( \pm i \).) Hence, \( 4 \times 2^n = |\{ \pm Q_x, \pm i Q_x : x \in \ker T \}| \leq 4 \times 2^n \). As a result, \( \dim \ker T = r = n \). \( \square \)

This lemma and its proof show that \( \mathcal{H} \) is a maximal abelian subgroup of \( \mathcal{G} \).
Lemma 7.3 \[ A_x = I, \text{ for any } x \in \text{Ker} T. \]

Proof: By Eq. (78), if \( x \in \text{Ker} T \), we have

\[
(I + A_x^T)T(y) = (I + A_x^T)T(x) = 0, \tag{84}
\]

for every \( y \). On the other hand, by Lemma 7.2 \( T \) is surjective; thus, \( (I + A_x^T)y = 0 \), for every vector \( y \). Equivalently, \( A_x = I \). \( \square \)

Now, we are ready to finish the proof. \( \mathcal{H} \) is a maximal abelian group of \( \mathcal{G} \) which is a group isomorphic to the Pauli group. Therefore, if we show that all elements of \( \mathcal{H} \) are diagonal, the linear span of \( \mathcal{H} \) is equal to the linear span of the maximal abelian subgroup of the Pauli group generated by \( \sigma_z \) gates, and then we are done.

Let \( x \in \text{Ker} T \); we prove that \( Q_x \) is diagonal. Using Eq. (71), since the \( f \)-vector of \( Q_x \) is zero, and \( A_x = I \), \( Q_x|y\rangle = \lambda_y|y\rangle \), for some \( \lambda_y \). This equality means that \( Q_x \) is diagonal in the standard basis. We are done.

8 Conclusion

In this paper we develop some techniques to characterize \( C_3 \) gates based on the subgroups of the Clifford group isomorphic to the Pauli group. We prove that any such group, after conjugation by a Clifford operation, contains a maximal abelian subgroup, all of whose elements are diagonal. This result proves the conjecture that any \( C_3 \) gate is a generalized semi-Clifford gate. Using Proposition 2 of [4], we conclude that any \( C_3 \) gate is of the form \( Q\Pi\Lambda Q' \), where \( \Pi \) is a permutation, \( \Lambda \) is diagonal, and \( Q, Q' \) are Clifford operations. To obtain a deeper understanding of \( C_3 \) gates we should characterize all of these groups (subgroups of Clifford group isomorphic to Pauli group). Such a characterization leads us to a better understanding of \( C_3 \), and then \( C_k, k \geq 4 \).

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Appendix A

This appendix contains some results regarding the coefficients of the linear expansion of a Clifford operator in terms of Pauli matrices.

Let \( Q \) be an arbitrary Clifford operation represented by the pair \((C, h)\). Since Pauli operators consist a basis for the linear space of matrices, there are complex numbers \( r_a \), \( a \in \mathbb{Z}_2^n \), such that

\[
Q = \sum_a r_a (i a^T J a \tau_a), \tag{85}
\]

Using Eqs. (14)-(16), for every \( b \in \mathbb{Z}_2^n \) we have

\[
Q \tau_b Q^\dagger = (i)^d b (-1)^{h^T b + b^T J a = \text{Ker} T} J C + dd^T) b \tau_{C b}, \tag{86}
\]

where \( d \) is defined by Eq. (13), and then

\[
Q \tau_b = (i)^d b (-1)^{h^T b + b^T J a = \text{Ker} T} J C + dd^T) b \tau_{C b} Q. \tag{87}
\]
Therefore, replacing $Q$ by Eq. (85), we get

$$\sum_a r_a(i)^a T J a (-1)^{b T J a} r_a + b$$

$$= (i)^{d T b} (-1)^{b T b + b T J \text{lows} (C T J C + d d T ) b} \sum_a r_a(i)^a T J a' (-1)^{a' T J C b} r_{a'\text{T}C b}. \quad (88)$$

Equivalently, if $a' = a + b + C b$, then

$$r_a(i)^a T J a (-1)^{b T J a} = r_a'(i)^{d T b} (-1)^{b T b + b T J \text{lows} (C T J C + d d T ) b} (-1)^{a' T J C b}. \quad (89)$$

Suppose $b$ is an eigenvector of $C$ with eigenvalue one ($C b = b$). Then $a' = a$, and by the above equation if $r_a \neq 0$

$$(-1)^{b T J a} = (i)^{d T b} (-1)^{b T b + b T J \text{lows} (C T J C + d d T ) b} (-1)^{a' T J C b}. \quad (90)$$

Thus, $d T b$ must be zero.

To see this fact more explicitly, consider the block form of $C$

$$C = \begin{pmatrix} A & E \\ F & B \end{pmatrix}. \quad (91)$$

Since $C$ is symplectic, $B T A + E T F = I$ and $A T F, E T B$ are symmetric. Also, $C b = b$ is equivalent to

$$A x + E y = x, \quad F x + B y = y. \quad (92)$$

where

$$b = \begin{pmatrix} x \\ y \end{pmatrix}. \quad (93)$$

By the definition of $d$,

$$d = \text{diag}(C T J C) = \text{diag} \left( \begin{array}{cc} A T F & A T B \\ E T F & E T B \end{array} \right)$$

$$= \text{diag} \left( \begin{array}{cc} F T A & F T E \\ E T F & E T B \end{array} \right)$$

$$= \text{diag} \left( \begin{array}{cc} F T & 0 \\ 0 & E T \end{array} \right) C. \quad (94)$$

Since the last matrix is symmetric, and the operations are on a field of characteristic 2, we have

$$d T b = b T \left( \begin{array}{cc} F T & 0 \\ 0 & E T \end{array} \right) C b = b T \left( \begin{array}{cc} F T & 0 \\ 0 & E T \end{array} \right) b = x T F T x + y T E T y. \quad (95)$$

Then we should show that $x T F T x + y T E T y = 0$. In fact, we can prove a stronger equality:

$$b' T \left( \begin{array}{cc} F T & 0 \\ 0 & E T \end{array} \right) b = x' T F T x + y' T E T y = 0,$$

where

$$b' = \begin{pmatrix} x' \\ y' \end{pmatrix} \quad (96)$$

is another eigenvector of $C$ with eigenvalue one.
Using $B^TA + E^TF = I$, we have $y^TB^Ax + y^TE^Fx = y^Tx$, and by Eq. (102) we conclude that

\[
0 = y^TB^Ax + y^TE^Fx + y^T x = y^TB^T(x + Ey) + y^TE^T(y + By) + y^T x
= y^TB^T x + y^TE^Ty + y^T x = y^TE^Ty + (y^T + x^TF^T)x + y^T x
= y^TE^Ty + x^TF^T x,
\]

where in the third line we use $B^TE = E^TB$.

**Lemma 1** The map $S(b) = b^T\text{lows}(C^TJC + dd^T)b$ that is defined on the set of eigenvectors of $C$ with eigenvalue one is a linear map. As a result, there exists a vector $\alpha$ such that $S(\alpha) = \alpha^T b$.

**Proof:** It is sufficient to show that $b^T\text{lows}(C^TJC + dd^T)b + b^T\text{lows}(C^TJC + dd^T)b = 0$ for two eigenvectors $b$ and $b'$ with eigenvalue one.

\[
b^T\text{lows}(C^TJC + dd^T)b + b^T\text{lows}(C^TJC + dd^T)b
= b^T\text{lows}(C^TJC + dd^T)b + b^T\text{lows}(C^TJC + dd^T)b'
= b^T\left[\text{lows}(C^TJC + dd^T) + \text{lows}(C^TJC + dd^T)\right] b'
= b^T\left(\begin{pmatrix} F^T & 0 \\ 0 & E^T \end{pmatrix}C + dd^T \right) b'
= b^T\left(\begin{pmatrix} F^T & 0 \\ 0 & E^T \end{pmatrix}b' + b^T dd^T b' \right) = 0,
\]

where in the fourth line we used the same idea as in Eq. (104), and the fact that $d = \text{diag}(dd^T)$ which gives

\[
\text{diag}\left(\begin{pmatrix} F^T & 0 \\ 0 & E^T \end{pmatrix}C + dd^T \right) = 0.
\]

$\square$

Now, let us return to Eq. (99). If $r_a \neq 0$, then for any $b$ such that $Cb = b$ we have

\[
(-1)^a^T J a = (-1)^a^T b + b^T\text{lows}(C^TJC + dd^T)b(-1)^a^T J b,
\]

or equivalently

\[
a^T P b + h^T b + b^T\text{lows}(C^TJC + dd^T)b = 0.
\]

Using Lemma 1, we can write this equality as

\[
(Pa + h + \alpha)^T b = 0.
\]

Suppose $\text{dim Ker}(I + C) = s$, i.e., there are $s$ independent eigenvectors of $C$ with eigenvalue one. Thus, by Eq. (102), there are $s$ independent linear constraints on the vectors $a$ for which $r_a \neq 0$. Therefore, there are at most $2^{2n-s}$ vectors $a$ such that $r_a \neq 0$. On the other hand, if $r_a$ is non-zero, then by Eq. (99), $r_{a+e} \neq 0$, for every vector $e$ in the image of $I + C$. Moreover, $\text{dim Im}(I + C) = 2n - s$, and then the number of such vectors $e$ is equal to $2^{2n-s}$. This means that all of vectors $a'$ for which $r_{a'}$ is non-zero, are of the form $a' = a + e$ for some $e \in \text{Im}(I + C)$.

As a summary, we have the following theorem.
Theorem 1. Let $Q$ be a Clifford operation represented by the pair $(C, h)$, and let $r_a$, $a \in Z_2^{2n}$, be the coefficients of $Q$ as a linear combination of Pauli matrices as in Eq. (85). Also, let $\alpha$ be the vector defined in Lemma 1. Then, $r_a$ is non-zero for $a = P(h + \alpha)$. Moreover, every $a'$ where $r_{a'} \neq 0$, is of the form $a' = P(h + \alpha) + b + Cb$ for some $b$. In this case, $r_a$ and $r_{a'}$ are related by Eq. (89). In particular, $|r_a| = |r_{a'}|$. ☐

References


