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Contributions of the wall boundary layer to the formation of the counter-rotating vortex pair in transverse jets

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Using high-resolution 3-D vortex simulations, this study seeks a mechanistic understanding of vorticity dynamics in transverse jets at a finite Reynolds number. A full no-slip boundary condition, rigorously formulated in terms of vorticity generation along the channel wall, captures unsteady interactions between the wall boundary layer and the jet – in particular, the separation of the wall boundary layer and its transport into the interior. For comparison, we also implement a reduced boundary condition that suppresses the separation of the wall boundary layer away from the jet nozzle. By contrasting results obtained with these two boundary conditions, we characterize near-field vortical structures formed as the wall boundary layer separates on the backside of the jet. Using various Eulerian and Lagrangian diagnostics, it is demonstrated that several near-wall vortical structures are formed as the wall boundary layer separates. The counter-rotating vortex pair, manifested by the presence of vortices aligned with the jet trajectory, is initiated closer to the jet exit. Moreover tornado-like wall-normal vortices originate from the separation of spanwise vorticity in the wall boundary layer at the side of the jet and from the entrainment of streamwise wall vortices in the recirculation zone on the lee side. These tornado-like vortices are absent in the case where separation is suppressed. Tornado-like vortices merge with counter-rotating vorticity originating in the jet shear layer, significantly increasing wall-normal circulation and causing deeper jet penetration into the crossflow stream.

Key words: jets, vortex flows

1. Introduction

The mixing properties of a jet issuing normally into a uniform crossflow, or transverse jet, are important in many engineering applications. Transverse jets are canonical examples of a flow exhibiting a complex set of coherent vortical structures. These jets are used as fuel sources in industrial furnaces and as diluent jets for blade or exhaust gas cooling in industrial or airborne gas turbines. In many of these examples,
complex flow dynamics may offer an opportunity for the control of the mixing rate between the jet and the crossflow, and hence improvements in performance. For instance, gas turbines may benefit from the enhancement of the mixing rate between fuel and air, such as achieving smaller size, higher efficiency and lower noise over a wider range of operating conditions.

The structure of a transverse jet flow field is governed by three major dimensionless parameters: the Reynolds number, $Re_\infty \equiv U_\infty d/\nu$, the jet-to-crossflow momentum ratio, $r \equiv (\rho_j V_j^2/\rho_\infty U_\infty^2)^{1/2}$, and the relative thickness of the wall boundary layer to the jet diameter, $\delta/d$. Here, $\rho_j$ and $V_j$ are the density and mean velocity of the jet, while $\rho_\infty$ and $U_\infty$ are the density and velocity of the crossflow. The jet diameter is denoted by $d$, $\nu$ is the kinematic viscosity and $\delta$ is the wall boundary layer thickness of the incoming crossflow near the jet nozzle. The Reynolds number is based on the jet velocity and is given by $Re_j \equiv V_j d/\nu = r Re_\infty$.

Experimental observations by Kelso et al. (1996) and Fric & Roshko (1994) identify the key vortical structures of the transverse jet, shown schematically in figure 1. The jet shear layer is the result of the advected in-pipe boundary layer. At sufficiently high $Re$, the jet shear layer, especially on its windward side, develops a Kelvin–Helmholtz instability and generates distinct bands of vortices around the jet column. The roll-up phenomena leading to the formation of these structures show some dependence on the Reynolds number. Kelso et al. (1996) report that the shear layer roll-up is limited to the windward side of the jet at small Reynolds numbers, while large-scale roll-up occurs along the entire perimeter for $Re_\infty > 1000$.

At certain flow conditions, upright vortices are observed downstream of the jet nozzle. Although these upright wake vortices show apparent similarity to the vortices shed from a solid cylinder, the origin of the upright wake vortices is thought to be fundamentally different. Instead of being formed from the vortex sheet on the boundary of the solid cylinder, Fric & Roshko (1994) have shown that they result
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Figure 2. Behaviour of the jet shear layer for \( r = 4.0 \) at \( Re_\infty = 1600 \) and \( \delta/d = 0.61 \). Reproduced from Kelso et al. (1996). Blue dye is injected from a circumferential slot into the jet-pipe boundary layer 1.6\( d \) upstream of the exit, while red dye is released from a dye injection port immediately upstream of the exit.

from separation events in the wall boundary layer. As a result, \( Re_\infty \) and \( r \) strongly control their dynamics. Wake vortices are not expected when \( Re_\infty \ll 500 \) (Kelso et al. 1996). The most orderly wake vortices are reported around \( r = 4 \) (Fric & Roshko 1994; Kelso et al. 1996).

Horseshoe vortices, which develop close to the wall upstream of the jet column, are believed to result from the presence of an adverse pressure gradient and the associated separation of the wall boundary layer upstream of the jet nozzle (Fric & Roshko 1994). These vortices are initially formed within the wall boundary layer, whose circulation per unit length is \( 1/r \) of that of the jet shear layer. As a result, the horseshoe vortex is a weak structure. Kelso et al. (1996) assert that ‘the horseshoe vortex system seems to play only a minor role in the overall structure’.

From a practical perspective, the counter-rotating vortex pair (CVP) is the most important vortical structure observed in a transverse jet. It is a robust feature of the flow over a wide parameter range and has been a focus of the numerous studies (Kelso et al. 1996; Cortelezzi & Karagozian 2001; Lim, New & Luo 2001). According to Broadwell & Breidenthal (1984), the impulse of the jet normal to the crossflow results in a streamwise CVP. This view is quite effective in explaining the far-field jet trajectory, but provides only a limited explanation of how vorticity is transformed mechanistically to create counter-rotating vorticity in the near field. Moreover, the CVP feature significant unsteadiness that cannot be effectively characterized by global momentum-balance arguments.

The origin of the CVP has been one of the most subtle issues in transverse jet flows. Differing accounts of the mechanism by which the counter-rotating vortices form still persist. Experimental observations (Kelso et al. 1996; Lim et al. 2001) suggest that the CVP is initiated just above the jet exit as jet shear layer vorticity folds onto itself and Kelvin–Helmholtz instability leads to a simultaneous roll-up. A water-tunnel dye visualization of the folding shear layer is shown in figure 2. The resulting flow pattern can be interpreted as the tilting and folding of vortex rings as they are ejected from the nozzle, where the downstream side of each vortex ring is approximately aligned
with the jet trajectory. A slightly different mechanism reported by Yuan, Street & Ferziger (1999) points to quasi-steady ‘hanging vortices’ formed in the skewed mixing layers on the lateral edges of the jet; the authors suggest that an adverse pressure gradient causes these vortices to break down into a weak CVP.

While these mechanisms account primarily for the influence of the jet shear layer on the CVP formation, some researchers have also suggested that the formation of counter-rotating vortices is not due to vorticity from the jet shear layer alone. Water-tunnel dye visualizations by Kelso et al. (1996) suggest that the CVP contains vorticity generated in the channel wall boundary layer. Since the circulation per unit length of the wall boundary layer is relatively small – that is, 1/r of that of the jet shear layer – the evolution of vorticity from the wall boundary layer was neglected in a recent investigation (Cortelezzi & Karagozian 2001), while our previous work (Marzouk & Ghoniem 2007) partially included this effect using a simplified model. In the present work, we will quantify the influence of the wall boundary layer on the formation of the CVP by constructing more complete vorticity-flux boundary conditions.

Our previous work provided a detailed, mechanistic description of inviscid vorticity dynamics in transverse jets at a high Reynolds number (Marzouk & Ghoniem 2007), while ignoring the possible separation of the wall boundary layer. In that work, the vorticity generation mechanisms accounted for the interactions between the channel wall vortex sheet and the jet flow immediately around the nozzle edge. It was demonstrated that the vorticity introduced by the jet shear layer must contain not only an azimuthal component but also a wall-normal component and an additional azimuthal component, both resulting from the jet–crossflow interaction. It was shown that the vorticity introduced by the jet into the flow evolves to form several coherent vortical structures. By the axial lifting of the lee side of the shear layer, the vortical structure, primarily dominated by its azimuthal component, forms two arms, resulting in a CVP aligned with the jet trajectory. The shear layer also shows periodic roll-ups, which create alternate vortex arcs on the lee and windward sides. Eventually, these vortex arcs approach each other on the windward boundary of the jet by the action of the already formed CVP, creating an interlocking structure that persists for several diameters above the jet exit.

This paper extends and generalizes the previous investigation of transverse jets by Marzouk & Ghoniem (2007). Instead of focusing on inviscid dynamics, we introduce viscous diffusion of momentum and consider transverse jets at a moderate Reynolds number. In order to capture the fundamental processes responsible for entrainment and mixing of fluid from the crossflow into the jet, we seek a mechanistic understanding of vorticity structure and evolution by analysing results of a high-resolution simulation. Lagrangian vortex methods are used to perform the numerical simulations. A viscous vortex method enables direct simulation of the evolution of the wall boundary layer as well as that of the jet shear layer, and facilitates simulation of the transverse jet flow field over long times. Our simulations reveal a rich set of near-wall vortical structures resulting from boundary layer separation, many of which interact with the jet and affect the shear layer dynamics, the initiation of counter-rotating vorticity, and the overall jet trajectory. The impact of vorticity from the wall boundary layer on the CVP is clearly identified.

The remainder of the paper is structured as follows. Section 2 describes the vorticity formulation of transverse jets. The governing equations and the boundary conditions are introduced and subsequently discretized for numerical implementation. Two distinct vorticity-flux boundary conditions are constructed: the full no-slip boundary condition and a reduced vorticity influx model. These two boundary conditions
are used separately in two numerical simulations to investigate the impact of wall boundary layer separation on the near-field structure of the jet in §3, where two sets of simulation results are presented for comparison. We discuss the evolution of the CVP under the effect of separation. Finally, a brief summary of the results is provided in §4.

2. Vorticity formulation of transverse jets

2.1. Governing equations

We focus on incompressible flow, assume constant density \( \rho_j = \rho_\infty \), and for relevance to mixing in engineered systems, we consider \( r \gg 1 \). The equations of motion for viscous, incompressible flow are written in the following velocity–vorticity formulation, where \( \omega = \nabla \times u \):

\[
\frac{D\omega}{Dt} = \omega \cdot \nabla u + \nu \Delta \omega, \quad \nabla \cdot u = 0.
\] (2.1)

Using the Helmholtz decomposition, we write

\[
u = u_\omega + u_p,
\] (2.2)

where \( u_\omega \) is recovered by the Biot–Savart law

\[
u_\omega(x, t) = -\frac{1}{4\pi} \int_{\Omega} \frac{(x - x') \times \omega(x', t)}{|x - x'|^3} \, dx' = K \ast \omega,
\] (2.3)

where the following notation is used:

\[
u(F \ast G)(x) \equiv \int_{\Omega} F(x, y) \times G(y) \, dy,
\] (2.4)

and

\[
u(x, y) = -\frac{1}{4\pi} \frac{x - y}{|x - y|^3}.
\] (2.5)

\( u_p \) is a divergence-free potential velocity field \( (u_p = -\nabla \phi) \). It satisfies a prescribed normal velocity on the boundary of the given domain \( \Omega \):

\[
\Delta \phi = 0 \quad \text{in} \ \Omega, \quad n \cdot u_\omega + n \cdot u_p = n \cdot u \quad \text{on} \ \partial \Omega.
\] (2.6)

Vorticity is generated at the boundary and is introduced into the flow by diffusion normal to the wall or by advection at corners. Together, these equations completely describe the motion of an incompressible, viscous flow.

2.2. Three-dimensional vortex methods

A three-dimensional vortex element method (Leonard 1985; Cottet & Koumoutsakos 2000) is used to simulate an unsteady, incompressible transverse jet at a finite Reynolds number. We employ a viscous splitting algorithm: the evolution of the flow field is considered in discrete fractional steps, where the vorticity field is first convected and then diffused (Cottet & Koumoutsakos 2000; Majda & Bertozzi 2002). The algorithm consists of substeps where the convective and the diffusive effects are considered separately.

During the convection substep, we solve the equations of motion for an inviscid incompressible flow in a vorticity transport form:

\[
\frac{D\omega}{Dt} = \omega \cdot \nabla u, \quad \nabla \cdot u = 0.
\] (2.7)
Numerical solution of this equation proceeds by discretization of the vorticity field onto overlapping vector elements, each centred at \( \chi_i \) with volume \( dV_i \) and vorticity \( \omega_i \):

\[
\omega(x, t) \approx \sum_{i}^{N} [\omega_i dV_i](t) f_\sigma (x - \chi_i(t)).
\]

(2.8)

The vorticity associated with each element is localized by a radially symmetric core function \( f_\sigma \) of radius \( \sigma \), where \( f_\sigma(x) = f(|x|/\sigma)/\sigma^3 \). Here we employ the Rosenhead–Moore kernel as the core function (Rosenhead 1931; Moore 1972), which yields

\[
u_\omega(x, t) \approx \sum_{i}^{N} f_\delta(x - \chi_i(t)).
\]

(2.9)

from the Biot–Savart law (2.3).

The vorticity field is discretized onto Lagrangian computational elements, or particles, with weights \( \alpha_i(t) \equiv (\omega_d V_i)(t) \) and trajectories \( \chi_i(t) \):

\[
\omega(x, t) \approx \sum_{i}^{N} \alpha_i(t) f_\delta (x - \chi_i(t)).
\]

(2.10)

\( f_\delta(r) \) is a radially symmetric core function of radius \( \delta \). We are solving the equations of motion for both trajectories and weights

\[
\frac{d\chi_i}{dt} = u(\chi_i)
\]

(2.11)

\[
\frac{d\alpha_i}{dt} = \alpha_i \cdot \nabla u(\chi_i, t).
\]

(2.12)

A second-order predictor/corrector scheme with adaptive time-step control is used for time integration of the ordinary differential equations (2.11), (2.12), where the velocity and its gradients at each node \( u(\chi_i) \) and \( \nabla u(\chi_i) \) are evaluated by an adaptive tree-code (Lindsay & Krasny 2001; Schlegel, Wee & Ghoniem 2008). The velocity is used to advect the particles, while the velocity gradient is computed to account for the stretching and tilting of particles. The parallel implementation of the adaptive tree-code is achieved by a domain decomposition using the \( k \)-means clustering technique (Marzouk & Ghoniem 2005).

During the diffusion substep, a modified interpolation kernel is used (Wee & Ghoniem 2006). At each step, the solution of the diffusion equation is approximated by a new set of particles, which lie on a uniform grid over a region encompassing the support of the particle distribution from the previous step. These new sets of particles are obtained by interpolating each particle from the previous step into its neighbouring grid points, i.e. for each grid point, whose index is given as \( j \):

\[
[\omega_j dV_j]^{n+1} = \sum_{i}^{N} f_{ij} [\omega_i dV_i]^n,
\]

(2.13)

where \( f_{ij} \) is the redistribution fraction from the \( i \)th particle to the grid point. \( f_{ij} \) is obtained by using the interpolation kernel \( A_3 \) (Wee & Ghoniem 2006):

\[
f_{ij} = A_3 \left( \frac{x_j - x_i}{\Delta x} \right) A_3 \left( \frac{y_j - y_i}{\Delta x} \right) A_3 \left( \frac{z_j - z_i}{\Delta x} \right)
\]

(2.14)
where
\[
\Lambda_3(\xi; c) = \begin{cases} 
1 - 2c^2 + |\xi| (3c^2 - \frac{1}{2}) - \xi^2 + \frac{|\xi|^3}{2} & : |\xi| < 1 \\
(2 - |\xi|) \left( \frac{1}{6} (3 - |\xi|)(1 - |\xi|) + c^2 \right) & : 1 \leq |\xi| < 2, \\
0 & : 2 \leq |\xi|. 
\end{cases}
\tag{2.15}
\]

Here \( c = \sqrt{\nu \Delta t_d / \Delta x} \), which represents the ratio between the diffusion length scale and the grid size \( \Delta x \). \( \Delta t_d \) is the time-step size for the diffusion substep, which can be different from the time-step size for the convection substep \( \Delta t_c \). Often, to have enough resolution in convection calculation, \( \Delta t_c \) is chosen to be a fraction of \( \Delta t_d \), and in that case, a few convection substeps are performed for each diffusion substep.

After interpolation, we eliminate particles with \( |\omega_jdV_j| < |\omega_dV|_{del} \) to control the problem size. The value of \( |\omega_dV|_{del} \) is chosen to be at least two orders of magnitude smaller than the strength of elements representing the vortical structures we are interested in. After elimination, each remaining particle on the grid is converted back into an element, having its centre \( \chi_j \) at the grid point on which it lies. This conversion completes the entire time step including convection and diffusion.

2.3. Computational domain and boundary conditions

The coordinate system used in computation is shown in figure 1. The centre of the circular jet orifice is the origin. All variables are made dimensionless by \( d \), the jet diameter, and \( U_{\infty} \), the speed of the crossflow. The jet velocity, \( V_j \), is specified by the jet-to-crossflow velocity ratio \( r \), i.e. \( r = V_j / U_{\infty} \). The crossflow is directed in the positive \( x \)-direction, and the jet centreline is aligned with the \( y \)-axis. The plane of \( y = 0 \) is considered as a solid wall, except at the jet orifice. We assume symmetry across \( z = 0 \). This assumption has been verified by full three-dimensional simulations under similar conditions, for shorter runs.

The boundary conditions discussed in this section only concern the evolution of the vorticity already in the computational domain. Introduction of vorticity is separately dealt in § 2.4. To enforce the normal-velocity boundary condition at \( y = 0 \) during each convection substep, an image vorticity distribution is placed in \( y < 0 \): \( \omega_{img,conv}(x, y, z) = -\omega_x(x, -y, z)\hat{e}_x + \omega_y(x, -y, z)\hat{e}_y - \omega_z(x, -y, z)\hat{e}_z. \) (2.16)

The jet outflow is represented by a semi-infinite cylindrical vortex sheet of radius 1/2 extending from \( y = 0 \) to \( y = -\infty \), with strength \( y = -2r\hat{e}_\theta \). For numerical implementation, this vortex cylinder is terminated at \( y = -5 \), which is long enough for the domain we are interested in. The crossflow velocity is modelled by a potential \( \phi_{\infty} = -x \).

During each diffusion substep, a wall-tangential vorticity flux is imposed using the homogeneous Neumann boundary condition, i.e. \( \partial \omega_x / \partial y = \partial \omega_z / \partial y = 0 \), at \( y = 0 \). On the other hand, the wall-normal vorticity flux is determined by satisfying \( \nabla \cdot \omega = 0 \) at \( y = 0 \). Just as in the velocity evaluation, this apparently complicated set of boundary conditions is implemented by placing an image in \( y < 0 \): \( \omega_{img,diff}(x, y, z) = -\omega_{img,conv}(x, y, z). \) (2.17)

Finally, to control the number of vortex elements, we gradually increase our deletion cutoff \( |\omega_dV|_{del} \) from \( x = 7 \), and we delete all the elements with \( x \geq 10 \) as the exit boundary condition. We have not seen a significant impact on the near-field jet evolution from this exit boundary treatment.
2.4. Boundary generation of vorticity

The scheme presented so far describes the numerical implementation of vorticity transport in the computational domain, i.e. \( y > 0 \). Here, we describe how vorticity is introduced into the domain.

In the case of transverse jets, vorticity is introduced into the domain from two sources.

(i) Vorticity generated in the jet-pipe boundary layer for \( y < 0 \) enters the domain at the nozzle edge. The introduction of vorticity is purely convective. We refer to it as \( \gamma_j \).

(iii) A vortex sheet is formed on the wall \(( y = 0)\) as previously described. We refer to this vortex sheet as \( \gamma_w \). Vorticity formed at the wall by the action of no-slip diffuses normal to the wall \( y_\perp \). This vorticity is introduced into the computational domain \(( y > 0)\) via two mechanisms. On the one hand, it may be introduced slowly via convection and viscous diffusion as the wall boundary layer grows. On the other hand, it may be introduced more abruptly by being lifted near the jet exit. In this case, the solenoidal continuation of \( \gamma_w \), which is referred to as \( \gamma_c \), must be carefully considered.

Figure 3 schematically shows each of these sources of vorticity. Major coordinate variables are also shown in the figure. In the following, we describe each source in detail.

Vorticity generated in the jet boundary layer at \( y < 0 \) is represented by a single sheet of azimuthal vorticity (in this study we only consider the case of a very thin jet boundary layer). Introducing this vorticity into the flow as a cylindrical vortex sheet, we have

\[
\gamma_j = -r\hat{e}_\theta \quad \text{for } y \ll 1
\]  

(2.18)

in the jet column.
As shown in Marzouk & Ghoniem (2007), the azimuthal vorticity in (2.18) does not provide a complete picture of the jet near field, however. For \( \rho > 1/2 \), a vortex sheet is formed on \( y = 0 \):

\[
\mathbf{\gamma}_w = \hat{e}_y \times \mathbf{u}_{\text{slip}},
\]

where \( \mathbf{u}_{\text{slip}} \) is the slip velocity on the surface of the wall. The wall vortex sheet, \( \mathbf{\gamma}_w \), is by construction solenoidal for \( \rho > 1/2 \), since

\[
\int_0^{0^+} (\nabla \cdot \mathbf{\gamma}_w) \, dy = \int_0^{0^+} \left( \frac{\partial \omega_x}{\partial x} + \frac{\partial \omega_z}{\partial z} \right) \, dy + \omega_{y,y=0^+} - \omega_{y,y=0^-} = 0.
\]

We have used the following relations:

\[
\omega_x = u_z \delta(y),
\]

\[
\omega_z = -u_x \delta(y),
\]

and

\[
\int_0^{0^+} \left( \frac{\partial \omega_x}{\partial x} + \frac{\partial \omega_z}{\partial z} \right) \, dy \left|_{y=0}^{y=0^+} \right. = -\omega_{y,y=0^+}.
\]

This solenoidality is, however, violated at \( \rho = 1/2 \), unless each vortex filament is continued from the wall (\( \rho > 1/2 \) and \( y = 0 \)) to the jet column (\( \rho = 1/2 \) and \( y > 0 \)) in an appropriate way.

To make this appropriate connection, we separate \( \mathbf{\gamma}_w \) into azimuthal and radial components, and examine how each component behaves at the jet nozzle boundary. The azimuthal component, \( \mathbf{\gamma}_{w,\theta} = \mathbf{\gamma}_w \cdot \hat{e}_\theta \), is simply advected by the local velocity, which is taken to be \( V_j \hat{e}_y/2 = r \hat{e}_y/2 \), without experiencing any tilting or stretching. Thus, writing

\[
\mathbf{\gamma}_c = \gamma_{c,y} \hat{e}_y + \gamma_{c,\theta} \hat{e}_\theta,
\]

we have

\[
\gamma_{c,\theta} \big|_{\rho=1/2, y=0} = \gamma_{w,\theta} \big|_{\rho=1/2, y=0}.
\]

In the following, we denote \( f(\theta) \equiv \gamma_{w,\theta} \big|_{\rho=1/2, y=0} \). The radial component, \( \gamma_{w,\rho} = \mathbf{\gamma}_w \cdot \hat{e}_\rho \), on the other hand, does experience tilting towards the direction of the jet. At the nozzle boundary, solenoidality requires that

\[
|\mathbf{\gamma}_w| = |\mathbf{\gamma}_c|.
\]

This is obtained by applying the divergence theorem to the vortex filament, which essentially states that the circulation should remain constant along each vortex filament. Additionally considering the sense of rotation that \( \omega_{\rho} \) experiences across the jet shear layer, we find

\[
\gamma_{c,y} \big|_{\rho=1/2, y=0} = -\gamma_{w,\rho} \big|_{\rho=1/2, y=0}.
\]

In the following, we denote \( g(\theta) \equiv -\gamma_{w,\rho} \big|_{\rho=1/2, y=0} \).

Now, we extend (2.25) and (2.27) to \( y > 0 \) by forcing \( \mathbf{\gamma}_c \) to be solenoidal. For \( y \ll 1 \), we assume that the jet column is a cylinder pointing straight upwards. We apply \( \nabla \cdot \mathbf{\omega} = 0 \) on this cylindrical surface, which yields

\[
\frac{\partial \gamma_{c,y}}{\partial y} + 2 \frac{\partial \gamma_{c,\theta}}{\partial \theta} = 0.
\]

Integrating (2.28) from \( y = 0 \), we obtain

\[
\mathbf{\gamma}_c = f(\theta) \hat{e}_\theta + (g(\theta) - 2yf'(\theta)) \hat{e}_y.
\]
By summing (2.18) and (2.29), the complete jet column vortex sheet for $y \ll 1$ is now given as follows:

$$\gamma_j + \gamma_c = (\alpha + f(\theta))\hat{e}_\theta + (g(\theta) - 2yf'(\theta))\hat{e}_y.$$  \hspace{1cm} (2.30)

Equations (2.19) and (2.30) completely describe the vortex sheet on the wall and that on the jet column, respectively. The only prescription is that $u = V_j\hat{e}_y/2$ along the jet nozzle boundary. Note that this condition is actually the one we impose around the jet nozzle boundary – i.e. a boundary condition, rather than an assumption.

In figure 3, the geometry of each vortex sheet is illustrated. These vortex sheets represent vorticity that is newly generated at the boundary at each time step, but their dynamics are distinct. The vortex sheet generated along the jet column (2.30) is located at $y > 0$ and experiences the influence of both convection and diffusion. On the other hand, the vortex sheet on the wall (2.19) leaves the wall first by diffusion, then by the combined action of diffusion and convection. Before leaving the wall, it does not affect the flow in $y > 0$, since its effect is exactly cancelled by its image (2.16). This is a particularly important observation, since an unseparated thin wall vortex sheet, whose boundary layer thickness is much smaller than other major jet structures, can effectively be modelled by preventing the wall vortex sheet from diffusing into the domain of $y > 0$. That is, by suppressing diffusion of $\gamma_w$, we can study the impact of an unseparated wall vortex sheet on the jet. With this statement in mind, in the rest of this section, we describe two numerical approaches to discretize (2.19) and (2.30). The first method of discretization allows the wall vortex sheet to diffuse outwards, while the second method does not. Comparing the results, we can evaluate how the separation of the wall vortex sheet affects the behaviour of the jet.

In the second approach, the full no-slip boundary condition is implemented along the solid wall, and both the wall vortex sheet and the jet vortices evolve as described by the governing equations. To account for the wall vortex sheet, the surface of the wall is divided into triangular and rectangular elements. Each surface element has its area, $dA_i$, and a collocation point at its centre, $x_{col,i}$. The slip velocity is computed at each collocation point. Once the slip velocity, $u_{slip,i} = u(x_{col,i})$, is obtained, a vortex element with its strength $[\omega_i dV_i] = (\hat{e}_y \times u_{slip,i}) dA_i$ is generated at the collocation point. When $u_{slip}$ is evaluated on the plane of $y = 0$, a vortex element close to the collocation point, i.e. within a distance comparable to $\sigma$, is interpreted as having its core size smaller than $\sigma$, proportional to its distance from the plane of $y = 0$. The reason for this special treatment is to avoid the cancellation of vorticity due to the existence of the image of the vortex element, which may otherwise lead to excessive vorticity introduction to the computational domain. This vorticity generation mechanism is incorporated in the second-order predictor/corrector scheme as follows: (i) the vortex particles are advected and stretched during the predictor step, (ii) new vortex elements are generated at each collocation point to cancel the slip velocity. These elements are immediately diffused, so that their contribution to the velocity field is not cancelled by the use of the image method. The vortex element added in this predictor step will be used only to compute the velocity and its gradients during the corrector step, and then discarded. (iii) Vortex particles are advected and stretched during the corrector step, (iv) a new set of vortex elements is generated at each collocation point to cancel the slip velocity. These new particles are added to the current set of particles and (v) these particle are diffused and the time step is now complete.

In order to reduce the computational domain size, vortex particles are introduced, at each time step, on the $x = x_{inflow}$ plane. Their corresponding velocity field satisfies the Blasius solution.
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To account for the jet column vortex sheet, we introduce new elements near the nozzle boundary every $\Delta t_{\text{noz}}$ time units. Each new set of vortex elements introduced near the jet nozzle boundary is the discretized version of (2.30) for $0 < y < r/\Delta t_{\text{noz}}/2$, which corresponds to the vorticity introduced into the flow during the period of $\Delta t_{\text{noz}}$. We thus introduce elements such that their centres lie at $y = r/\Delta t_{\text{noz}}/4$. In the azimuthal direction, we divide it among $n_\theta$ vortex elements. Taking the centre of each element as the collocation point, we obtain the following expression for the total strength of the vortex elements:

$$[\omega_i \, dV_i] = \left( -\frac{r^2}{4} + \frac{r}{4} f(\theta) \right) \hat{e}_\theta \Delta t_{\text{noz}} \Delta \theta + \left( \frac{r}{4} g(\theta) - \frac{r^2 \Delta t_{\text{noz}}}{8} f'(\theta) \right) \hat{e}_y \Delta t_{\text{noz}} \Delta \theta. \quad (2.31)$$

To avoid numerical error, we only retain terms at the same order in $\Delta t$ to $\gamma_j$. This eliminates the contributions from the terms containing $f(\theta)$ and $g(\theta)$, which are at $O(\Delta t^2)$, once the wall boundary layer is well developed. Note that this approach completely accounts for all the interactions between the wall vortex sheet and the jet column.

In the first approach, which was used previously by Marzouk & Ghoniem (2007), the wall vortex sheet exists because of the slip induced by the crossflow, but is assumed to stay at the wall without being diffused into the computational domain of $y > 0$. Neglecting the feedback from the vorticity in the domain, we approximately express the wall vortex sheet as $\gamma_w = -\hat{e}_z$. As mentioned earlier, since $\gamma_w$ never leaves the wall due to the assumed lack of diffusion here, we do not have to explicitly generate vortex elements for $\gamma_w$. The existence of this vortex sheet only affects the jet at the nozzle boundary, where non-trivial values of $\gamma_c$ are generated by the solenoidal connection of vortex filaments. Since $f(\theta) = -\cos \theta$ and $g(\theta) = \sin \theta$ in this case, substitution into (2.31) gives new vortex elements near the jet nozzle boundary every $\Delta t_{\text{noz}}$ time units:

$$[\omega_i \, dV_i] = \left( -\frac{r^2}{4} - \frac{r}{4} \cos(\theta) \right) \hat{e}_\theta \Delta t_{\text{noz}} \Delta \theta + \left( \frac{r}{4} \sin(\theta) - \frac{r^2 \Delta t_{\text{noz}}}{8} \sin(\theta) \right) \hat{e}_y \Delta t_{\text{noz}} \Delta \theta. \quad (2.32)$$

Expression (2.32) is identical to the expression given by Marzouk & Ghoniem (2007). Note that (2.31) is more rigorously derived here by considering the solenoidality of the vorticity field. Equation (2.32) is a special case with the assumption that $\gamma_w = -\hat{e}_z$.

In this article, we call the first model the reduced vorticity influx model and the second model the full no-slip boundary condition. By comparing the results of these two vorticity introduction mechanisms, we identify the effect of wall boundary layer separation on the dynamics of the transverse jet. Specifically, many near-field vortical structures, observed on the lee side of the jet (Kelso et al. 1996), are believed to result from separation of the wall boundary layer, and hence are excluded a priori in the reduced model, where the effect of the wall boundary layer is only implicitly included by its solenoidal continuation, i.e. $\gamma_c$.

3. Results: vorticity dynamics of transverse jets

In the following, we first investigate a jet at $r = 5$ with the reduced vorticity influx model; we refer to this simulation as case I. We then present a simulation with the full no-slip boundary condition, which is referred to as case II. Case II has the identical conditions as those of case I. The same resolution is used in both cases, and contrasting the two cases reveals differences in jet dynamics, particularly in the near
Table 1. Numerical and physical parameters of each case, normalized by $U_\infty$ and $d$. $\Delta t_c$ is initially taken to be the same as $\Delta t_{noz}$ and is adapted automatically during the calculation. $\delta$ is the wall boundary layer thickness at the jet nozzle exit.

<table>
<thead>
<tr>
<th>Case</th>
<th>$Re_\infty$</th>
<th>$Re_j$</th>
<th>$r$</th>
<th>$\Delta t_{noz}$</th>
<th>$\Delta t_d$</th>
<th>$\sigma$</th>
<th>$\Delta x$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
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<td>1225</td>
<td>5</td>
<td>0.02</td>
<td>0.02</td>
<td>0.1</td>
<td>0.035</td>
<td>0.0</td>
</tr>
<tr>
<td>II</td>
<td>245</td>
<td>1225</td>
<td>5</td>
<td>0.02</td>
<td>0.02</td>
<td>0.1</td>
<td>0.035</td>
<td>0.5</td>
</tr>
</tbody>
</table>

3.1. A transverse jet with the reduced vorticity influx model

We employ the reduced vorticity influx model in simulations of a jet at $r = 5$ (case I). The Reynolds number based on the crossflow velocity, $Re_\infty$, is set to 245. Accordingly, the Reynolds number based on the jet flow velocity, $Re_j$, is 1225. The core size of the vortex elements, $\sigma$, is chosen to be 0.1, which was the value used in Marzouk & Ghoniem (2007). The grid size for interpolation, $\Delta x$, is 0.035, which yields an overlap ratio $\sigma/\Delta x$ of approximately 3, ensuring the smoothness of the velocity field. Simulations are performed with $\Delta t_d = 0.02$. $\Delta t_{noz}$ is also set to 0.02, ensuring overlap between vortex elements introduced around the jet nozzle boundary. $\Delta t_c$ is automatically adjusted.

3.1.1. Overall flow features

Instantaneous snapshots of vorticity isosurfaces in case I are shown in figure 4. Though the simulation is performed only in half of the domain ($z > 0$), we also plot the mirror image across the $z = 0$ plane to provide a more intuitive representation of the flow. Two important features are readily identified: the roll-up of the jet shear layer at the windward side of the jet, as seen in figure 4(a), resulting from a Kelvin–Helmholtz-like instability; and the formation of the CVP at the lee side. The latter feature is particularly visible in figure 4(b). While both structures have been reported in our previous inviscid study (Marzouk & Ghoniem 2007), the organized structures in figure 4 persist further downstream, in part because the present results are allowed to evolve in time further beyond the startup transients. Other differences from the previous inviscid results are apparent. For instance, the roll-up of the shear layer is delayed from what was reported in Marzouk & Ghoniem (2007). This delay is ascribed to the effect of viscosity, which attenuates the growth of the instabilities leading to the observed dynamics.

The flow field of the transverse jet exhibits significant unsteady dynamics. The periodic roll-up of the jet shear layer, i.e. the Kelvin–Helmholtz instability, is one of several mechanisms leading to unsteadiness. The frequency of roll-up is $St = f d/U_\infty \approx 2.5$, where $f$ is the frequency; the Strouhal number based on the jet velocity, $St_j = St/r$, is thus around 0.5, which is lower than that observed by Marzouk & Ghoniem (2007), which was 0.8 for a velocity ratio $r = 7$. For a better comparison, Marzouk’s simulations have been reproduced at $r = 5$, and the Strouhal number is found to be around 0.9.

Organized counter-rotation is evident in figure 5, showing the time-averaged vorticity-magnitude isosurfaces, $|\omega| = 10$, for $t \in [10, 11.5]$. Mechanisms leading to the formation of these counter-rotating vortices are discussed next.
Figure 4. Vorticity-magnitude isosurfaces, $|\omega| = 15$, coloured by $\omega_z$, at $t = 12.0$ from two perspectives (case I). (a) Windward side and (b) lee side.
3.1.2. Shear layer roll-up and counter-rotating vorticity formation

Since vorticity dynamics are more easily understood in a Lagrangian reference frame rather than in an Eulerian perspective, we use material element tracking to provide a mechanistic description of vorticity evolution in the transverse jet. Vortex lines do not remain attached to the material lines in viscous flows. However, given the current values of $Re$, vortex lines and material lines are reasonably matched in our simulations. The characteristic length scale of momentum diffusion is $\sqrt{\nu T}$, where $T$ is the duration of tracking. Since each material line is typically tracked for about unit computational time, $T \sim 1$. The normalized kinematic viscosity is given by $\nu = Re^{-1}$, which results in $\sqrt{\nu T} \sim 0.06$. This value is even smaller than the core size $\sigma$, which defines the spatial resolution of the simulation. Therefore, these material lines deviate from actual vortex lines only at very small length scales. We thus use material lines as approximate surrogates for vortex lines in order to trace the development of large-scale vortical structures.

Two systems of coherent vortical structures were identified in the previous section: shear layer roll-ups at the windward side of the jet and the CVP. To investigate the vorticity dynamics leading to the formation of these structures, we introduce 10 planar material rings at the nozzle exit at equal intervals during $t \in [10.0, 10.18]$ in case I. Snapshots of the evolution of these material rings are given in figure 6.

On the lee side, we observe a complex out-of-plane distortion of the material lines, ultimately leading to the formation of the CVP. Vorticity introduced at the nozzle is primarily oriented in the azimuthal direction, and thus aligned with the initial
configuration of each material ring. The rings shown at $t = 10.2$ gradually distort out-of-plane on the lee side of the jet as they evolve into the domain. Lift-up on the lee side forms a tongue-like structure ($t = 10.7$) including two arms that are essentially aligned with the jet trajectory and that carry counter-rotating vorticity. This deformation constitutes the initiation of the CVP. In contrast with previous inviscid results (Marzouk & Ghoniem 2007), however, lift-up of the material line elements on the lee side precedes the roll-up of the shear layer. Roll-up is delayed by viscosity, but this sequence then implies that roll-up of the shear layer – i.e. the formation of large, coherent vortex rings – is not a necessary condition for the initiation of counter-rotating vorticity. A detailed description of the CVP formation mechanism can be found in Marzouk & Ghoniem (2007). Results from the present viscous simulations suggest that this mechanism persists in the presence of viscosity and, moreover, is independent of large-scale roll-up.

In summary, vorticity dynamics in the transverse jet at a finite Reynolds number – simulated in case I with the reduced vorticity influx model – exhibit many broad similarities to previous inviscid results. Essential features include roll-up of the jet shear layer, the initiation of counter-rotating vorticity via lee-side deformations of the jet’s azimuthal vorticity and the eventual breakdown of large vortical structures into smaller scales. The results do reveal some important differences, however. In the present viscous simulations, roll-up of the jet shear layer on its windward side is delayed relative to the inviscid case. Large-scale vortical structures persist further
downstream before breaking up. The mechanism by which counter-rotating vorticity is initiated appears similar to that reported previously (Cortelezzi & Karagozian 2001; Marzouk & Ghoniem 2007), and the present simulation shows that the formation of counter-rotating vorticity does not require shear layer roll-up as a prerequisite.

3.2. A transverse jet with the full no-slip boundary condition

The reduced model whose results have just been analysed is only capable of providing a partial portrait of the flow dynamics. Unlike its idealized counterpart, a real transverse jet may experience the influence of a separated wall boundary layer. To investigate the evolution of vorticity generated on the wall, in this section, we present simulations from case II, in which we fully account for the interaction of the wall boundary layer with the transverse jet. The conditions are identical to those of case I, where \( Re_j = 1225 \) and \( r = 5 \). Identical spatial discretization parameters are used. Numerical parameters for time discretization are \( \Delta t_{\text{noz}} = \Delta t_d = 0.02 \). Vorticity is generated at the wall over the domain \(-3 \leq x < 7\) and \(-5 \leq z < 5\). Near the nozzle exit, the surface of the wall is discretized with triangular elements. The typical area of these elements is chosen to be smaller than \( \Delta x^2 \). The remaining part of the wall is discretized into square elements of side length 0.025, which is smaller than \( \Delta x \).

To maintain the solenoidality of the wall vortices, we put mirror images of vortices across the planes of \( z = -5 \) and \( z = 5 \).

By the end of the simulation, i.e. \( t = 12 \), the trajectory in case II is stationary, and the number of vortex elements has reached over 4 million.

3.2.1. Overall flow features

The overall flow features are comparable to those observed with the reduced model in §3.1 but exhibit greater spatial complexity and higher vorticity magnitudes. Figure 7 shows time-averaged vorticity-magnitude isosurfaces in the transverse jet for \( t \in [10.0, 11.5] \), with colours indicating wall-normal vorticity \( \omega_y \). Results of case I and case II are presented side by side for comparison.

The lee-side structures of case II show striking differences from those of case I. The CVP in case II is initiated very close to the wall and has a stronger wall-normal velocity. In the near field, the counter-rotating vorticity in case II does not correspond to a single vortex pair. Instead, we observe two distinct strands of wall-normal vorticity on each side of the centreplane. Both strands start very near the wall at \( x \approx 0 \), one of them being similar to that observed in case I. The new vorticity strand initiated in the recirculation zone behind the jet is advected within the jet column by the second strand, i.e. the CVP already observed in case I. These two strands remain separate and do not merge for around three diameters above the nozzle exit.

By comparing figures 4 and 8, we note that the shear layer on the windward boundary of the jet rolls up to form ring-like structures in both cases I and II. The roll-up frequency in case II is around 0.5, which is similar to that in case I. However, there are some notable differences. The roll-up in case II, for instance, occurs closer to the jet nozzle than in case I. Isosurfaces of \( |\omega_z| \) in figure 8(a) show more explicitly the roll-up of the jet shear layer in case II.

Finally, we show the jet trajectory of case II in figure 9 (solid line). The trajectory is represented by the jet-centre streamline, obtained from the mean velocity field for \( t \in [10.0, 11.5] \). For comparison, we also plot the jet-centre streamline of the mean velocity field from case I (dashed line). The trajectory of case II is more upright, penetrating further into the crossflow than that of case I. Experimental observations of the jet trajectory are also reproduced in figure 9. While there is some scatter among
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Figure 7. Time-averaged vorticity-magnitude isosurfaces, $|\omega| = 17.5$, coloured by $\omega_y$, from two perspectives. Cases I and II are presented in left and right, respectively. (a) Windward side and (b) lee side.

these measurements, all plotted in $rd$-scaled coordinates, the majority fall closer to the jet-centre streamline of case II. Indeed, case II more closely matches the correlation given by Margason (1968):

$$\frac{y}{rd} = 4^{1/3} \left( \frac{x}{rd} \right)^{1/3}. \quad (3.1)$$

Deeper penetration in case II seems to be the result of stronger counter-rotating vorticity in the near field, a feature that we discuss in detail below.

3.2.2. Near-wall flow structures

By continually satisfying the no-slip boundary condition on the wall, the wall boundary layer, which is initiated as a vortex sheet, grows into a layer of finite
Figure 8. Vorticity-magnitude isosurfaces, $|\omega| = 17.5$ at $t = 12$, coloured by $\omega_y$ from two perspectives (case II). (a) Windward side and (b) lee side.
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Figure 9. Computed trajectories verses experimental observations. The solid and dashed curves represent jet-centre streamlines from the time-averaged velocity fields $t \in [10.0, 11.5]$ in case II and $t \in [10.0, 11.5]$ in case I, respectively. The upright crosses, squares, triangles and slanted crosses represent the experimental data with $r = 4$, $r = 6$, $r = 8$ and $r = 10$ obtained by Keffer & Baines (1962), respectively. The circles show the data with $r = 7.72$ obtained by Kamotani & Greber (1972). The dots show the data with $r = 10$ obtained by Smith & Mungal (1998). The dash-dotted line represents an experimental correlation (3.1) for $r = 7$ from Margason (1968).

thickness. Interactions between the wall boundary layer and the jet then yield a rich set of near-wall flow structures in the vicinity of the jet exit. For instance, figure 10(b) shows a side-by-side comparison of the time-averaged velocity field on the $y/d = 0.4$ plane between case I and case II. Case II contains a strong recirculation zone evident behind the jet column. This recirculation zone is not captured in case I or in previous inviscid simulations (Cortelezzi & Karagozian 2001; Marzouk & Ghoniem 2007) where the wall boundary layer was not included in the inviscid simulations. This suggests that a recirculation zone indeed results from separation of the wall boundary layer behind the jet, a feature we will revisit below. Also, the crossflow shows little or no penetration across the jet shear layer, consistent with what has been observed in experiments (Fric & Roshko 1994).

The recirculation zone in figure 10 is part of a larger, interconnected system of near-wall flow structures. Indeed, simulation results of case II largely confirm the qualitative streamline patterns originally suggested by flow visualization and hot-wire measurements in Kelso et al. (1996). Here we note key topological features of the flow; in the next section, we connect these features to the generation and evolution of vorticity.
Figure 10. Time-averaged velocity field on two planes parallel to the wall. Contours on the $z < 0$ side and the $z > 0$ side correspond respectively to the velocity field of cases I and II. The half-circle indicates the location of the nozzle boundary. A, B and S denote three critical points, A and B being two ‘tornado-like’ critical points and S being a saddle node. (a) $y/d = 0.2$ and (b) $y/d = 0.4$. 
Figures 10 and 11 show time-averaged streamlines on four different planes near the nozzle: two planes parallel to the wall ($y = 0.2$ and $y = 0.4$), the centreplane ($z = 0$) and the transverse cross-section just downstream of the jet ($x/d = 1.25$).

The centreplane ($z = 0$) contains an unstable spiral node downstream of the jet, marked with an N in figure 11(a); this node was inferred from topological arguments and confirmed by dye visualization in Kelso et al. (1996). Three critical points, two
distinct foci and a saddle node, marked by A and B and S respectively in figure 10, are observed on the y = 0.2 and y = 0.4 planes parallel to the wall. Special attention should be given to the spiralling foci A and B of figure 10. These foci were referred to as ‘tornado-like critical points’ by Kelso et al. (1996) and were conjectured to be the location at which wall boundary layer vorticity is lifted away from the wall and pulled into the jet shear layer. The sense of rotation of the tornado-like structure emanating from the focus B, and corresponding to the recirculation zone behind the jet, is the same as that of the counter-rotating vortices. Finally, the node found in figure 11(a) has also been reported by Kelso et al. (1996) and Wu, Vakili & Yu (1988).

Figure 11(b) shows the streamwise ‘wall vortices’ noted by Kelso et al. (1996) and Wu et al. (1988) on the x = 1.25 plane, just behind the jet. These wall vortices can also be seen in figures 12 and 13 as the continuation of the new vorticity strand behind the jet. In figure 12, we show a superposition of the time-averaged vorticity isosurface corresponding to |ω| = 4.5 and selected streamlines. The plot combines the two velocity fields seen in figure 11 and clearly shows how the flow is entrained into the jet wake, wrapping around the wall vortices and being lifted upwards in the vorticity column associated in figures 7 and 8 with the critical point B. In figure 13 the two upright vorticity strands associated with points A and B, as well as one wall vortex, are shown. A similar structure was reported in a previous LES simulation (Yuan et al. 1999), suggesting that they are robust features that persist over a wide range of flow conditions. While Yuan et al. (1999) concluded that these wall vortices
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Figure 13. Vorticity-magnitude isosurfaces showing the wall vortices on the lee side of the jet at $t = 12.0$, coloured by $\omega_x$ (case II). A and B indicate the ‘tornado-like’ critical points.

originated from the legs of the horseshoe vortex, which forms upstream of the jet column and deflects laterally around the jet, evidence from our simulations suggest that they originate from a different source. Our results show that the wall boundary layer vorticity, initially in the spanwise direction before reaching the jet column, is subject to a complex motion. Wall boundary layer vortex lines reaching the jet are lifted upwards into the jet column, around $z = 0$ and end at infinity on both sides, in the negative and positive $z$-directions. In between infinity and the part lifted upwards, these vortex lines are stretched and deflected around the jet column. Once these vortex lines meet with their symmetrical part on the lee side of the jet, they are now oriented in the crossflow direction, and have opposite vorticity signs. Due to the opposite sign, the two vortical structures cannot merge and simply roll up on themselves, creating two wall vortices. These vorticity lines follow the same path as the recirculation vorticity column formed around critical point B, behind the jet, explaining thus why these two vortical structures merge. Dye visualization of the crossflow (e.g. figure 22 in Kelso et al. 1996) suggests that the upstream portion of these wall vortices, having been lifted away from the wall, merge with the CVP. Figure 13 clearly shows that this lifting and merging actually occurs.

3.2.3. Impact of near-wall structures on jet evolution

Previous inviscid simulations (Cortelezzi & Karagozian 2001; Marzouk & Ghoniem 2007) attribute the formation of counter-rotating vortices to the evolution of vorticity in the jet shear layer as it separates from the nozzle. Our simulations in case I confirm this mechanism, even in the presence of viscosity. Allowing detailed interactions between the wall boundary layer and the jet shear layer, however, changes the picture significantly.

To examine the mechanism of counter-rotating vortex formation under the impact of the wall boundary layer, we again track material lines. Unlike case I, however, near-wall vortex lines do not coincide with rings introduced around the jet nozzle. Tracking planar material rings will therefore not capture the essential vortex dynamics. Instead,
we will first identify the vortex lines near the jet exit plane; then we will track these vorticity lines to identify possible mechanisms of vorticity evolution.

Vortex lines are found by numerical integration of the instantaneous vorticity field at $t = 10.0$. The starting points for integration are chosen slightly above the wall ($y = 0$), either in the centreplane ($z = 0$) or far from the jet nozzle ($z > 3$). From the centreplane starting points, integration is performed both forward and backward. Representative near-wall vortex lines can be classified into two distinct groups. One group, represented by a slightly deformed ring around the jet nozzle, is analogous to the vortex lines produced by the reduced model in case I. The ring is tilted upwards into the jet flow on its lee side, similar to the closed vortex filaments in Marzouk & Ghoniem (2007) and Coelho & Hunt (1989). Because the ring is confined to the nascent jet shear layer and largely azimuthal in orientation, its vorticity must originate primarily in the jet-pipe boundary layer.

The other group of vortex lines has its origin in the vorticity of the wall boundary layer. Most of the vortex lines in this group are unterminated (i.e. infinitely long). The closed vortex loop located downstream of the jet is an exception, and it corresponds to both the streamwise wall vortices shown in figure 11(b) and the recirculation zone just downstream of the jet column. Vorticity in this loop points in the $+z$-direction on its lee (large-$x$) side, consistent with the sense of rotation of the wall vortices.

The contribution of the two groups to the formation of counter-rotating vorticity is assessed by tracking these initial vortex lines as material lines. As described in §3.1.2, using material lines as surrogates for the vortex lines is a fairly good approximation at the present values of $Re$. The evolution of the first group of vortex lines is very similar to the first group in case I and is thus not replotted, see figure 14. As the rings are convected upwards by the jet, they tilt further into the jet flow on its lee side. The lee-side lift-up generates two segments of counter-rotating vorticity, very similar to the dynamics described in §3.1.2 (Cortelezzi & Karagozian 2001; Marzouk & Ghoniem 2007). However, significant lift-up only happens several diameters away from the jet nozzle exit. Counter-rotating vorticity in the immediate vicinity of the jet nozzle – i.e. as identified by vorticity isosurfaces for $y/d < 2$ in figures 7(b), 15(a) and 15(b) – does not correspond to this later-stage lift-up. Near-wall counter-rotating vorticity therefore cannot result from deformation of the jet shear layer vorticity, as captured by this first group of vortex lines.

Figure 14 shows the evolution of the vortex lines in the second group, including both the vortex loop in the wake of the jet and the vortex lines stretching to infinity on both sides, originating in the wall boundary layer. Again, we begin with representative vortex lines at $t = 10.00$ – distinguished by colours in figure 14(a) – and trace their evolution until $t = 10.5$. All four vortex lines are stretched upwards into the jet shear layer. As they are pulled upwards, all four lines carry vorticity oriented in the positive wall-normal direction for $z > 0$ and vice versa for $z < 0$; in other words, they provide near-wall counter-rotating vorticity. Upwards-pointing vorticity on the lee side of the jet, in particular, corresponds to the focus A shown in figure 10(a). Streamwise vorticity just above the wall on the lee side of the jet – i.e. the wall vortices, coinciding with the purple vortex loop in figure 14(a) – is pulled upwards in the vicinity of this focus. At the same time, the bottom of the purple vortex loop is pulled against the crossflow, towards the lee side of the jet nozzle; this action is consistent with the recirculation behind the jet column shown in figure 10(b). Overall, these results demonstrate that near-wall counter-rotating vorticity is primarily formed from the vorticity entrained from the wall boundary layer.
Figure 14. Evolution of vortex lines originating in the wall boundary layer at (a) $t = 10.0$ and (b) $t = 10.5$ (case II), observed from the lee side of the jet.

Figure 14 thus indicates that the reduced vorticity influx model only partially explains CVP formation in transverse jets. The reduced model captures the initiation of counter-rotating vorticity several diameters above the nozzle by lee-side deformation.
of the shear layer. Closer to the jet (e.g. for \( y/d < 2.0 \)), the contribution of the wall boundary layer vorticity is actually more important.

A more quantitative assessment is performed to substantiate the importance of the vorticity in the near-wall flow structures. To better identify the new vorticity strands corresponding to the recirculation zone on the lee side of the jet, from the CVP that was observed in case I, figure 15 show time-averaged contours from cases I and II. These contours are shown side by side for comparison, with case I in the \( z < 0 \) half-plane. In this manner, we can observe the distribution and magnitude of wall-normal vorticity in the earliest stages of jet evolution, with and without the effects of wall boundary layer separation. Contour lines from cases I and II exhibit radical differences. For instance, very near the wall, e.g. at \( y/d = 0.4 \), case I shows a single, crescent-shaped region of wall-normal vorticity at the spanwise edge of the nozzle. This vorticity corresponds to the wall-normal component of \( \gamma_c \) (figure 3). On the other hand, case II shows two clusters of wall-normal vorticity on the \( y/d = 0.4 \) plane, one curved along the nozzle edge (\( x/d \in [-0.5, 0.5] \)) and the other located downstream of the jet. These clusters match the two separate strands of counter-rotating vorticity identified in vorticity isosurface plots above. The lee-side cluster of vorticity in case II corresponds to the separation of the wall boundary layer and upward entrainment of the wall vortices; in other words, this vorticity originates from \( \gamma_w \), not from \( \gamma_c \) or \( \gamma_j \).

While cases I and II both contain crescent-shaped regions of wall-normal vorticity at the spanwise edges of the nozzle, vorticity in the latter region is much stronger. Circulation in the \(-0.5 \leq x/d \leq 0.5\) region of case II is approximately twice that of the corresponding region in case I.

Moving down to figure 15(b), we find that the two clusters of wall-normal vorticity in case II, A and B, gain in strength yet remain somewhat separate on the \( y/d = 1.0 \) plane. In figure 15(c), in case I, the initiation of significant 'coherent' counter-rotating vorticity is finally visible on the \( y/d = 1.6 \) plane, consistent with the 3-D isosurface plot of figure 5. In case II, at the \( y/d = 1.6 \) plane, the vorticity strand B has now been entrained inside the jet column by the outer vorticity strand A, i.e. the one corresponding to the CVPs in case I.

We can quantify the evolution of wall-normal vorticity in the near field of the jet by integrating it across successive half-planes of constant \( y \). In particular, the circulation contained in the counter-rotating vortices is \( \Gamma(y) = \int \int_{z > 0} \omega_y \, dz \, dx \). Figure 16 shows the evolution of \( \Gamma(y) \) over the range \( 0 < y/d < 2.5 \) for cases I and II. The jet is essentially upright in \( 0 < y/d < 2.5 \) for both cases, so wall-normal and jet-streamwise vorticity are almost equivalent in those cases. Rates of growth are qualitatively different in the two cases. Case I shows a gradual increase in circulation for \( y/d > 1.0 \), consistent with the gradual deformation of the jet shear layer and the evolution of the vortex rings. Case II, however, shows a sharp rise of circulation within a relatively thin region near the wall. This circulation profile starts from a value of zero, since case II satisfies the no-slip boundary condition everywhere. The thin layer, wherein the sharp increase in circulation is observed, is therefore indicative of a sudden transition, i.e. separation and re-orientation of the wall boundary layer in the wake of the jet. In contrast, the circulation in case I starts at a value near 1.0 at \( y/d = 0 \), since the initial wall-normal vorticity is due entirely to \( \gamma_c \).

The existence of the strong near-field counter-rotating vortices in case II leads to jet evolution features which are different from what is observed in case I. For example, strong near-field counter-rotating vortices, in the early near-field stage of the jet, cause it to penetrate deeper into the flow as shown in figure 9. This is consistent with previous observations that the in-pipe jet profile and the crossflow boundary layer
Formation of the counter-rotating vortex pair in transverse jets

Figure 15. Contours of $\omega_y$ on planes of constant $y$. Contours on the $z < 0$ side and the $z > 0$ side correspond respectively to the time-averaged vorticity field of cases I and II. The dashed lines represent negative values. A and B indicate the ‘tornado-like’ critical points.

(a) $y/d = 0.4$; (b) $y/d = 1$ and (c) $y/d = 1.6$. 
thickness affect the penetration of the jet near the nozzle exit, and impact the overall trajectories (Muppidi & Mahesh 2005). The observation indicates that separation of the wall boundary layer is critically important to analysis of the jet in the near field.

4. Summary and conclusions

In this article, high-resolution vortex simulations are used to gain a mechanistic understanding of the structure and evolution of vorticity in viscous transverse jets. A previous investigation of transverse jets by Marzouk & Ghoniem (2007) has been extended by including viscous diffusion of momentum while allowing the wall boundary layer to grow and separate according to the dynamics imposed by the flow. We have developed a rigorous vorticity-flux boundary condition, which includes full interaction between the wall boundary layer and the jet, by generalizing the vorticity-flux boundary condition proposed by Marzouk & Ghoniem (2007). We have also shown that the previous vorticity-flux boundary condition is actually a reduced model of the generalized vorticity-flux boundary condition, in which separation of the wall boundary layer is suppressed. Two separate simulations have been performed using these two different boundary conditions. By comparing the results, we characterize the impact of wall boundary layer separation on the evolution of the jet.

In the absence of wall boundary layer separation, the results of our simulation show minor mechanistic differences from those observed by Marzouk & Ghoniem (2007). Vortex lines ejected from the jet nozzle evolve into two nearly horizontal arcs, connected by two initially vertical segments that contain counter-rotating vorticity. These segments tilt into the streamwise direction, and the ‘train’ of successive segments constitutes the counter-rotating vortices. Shear layer roll-up is delayed by the action of viscosity and is not required for the initiation of counter-rotating vorticity.

When the wall boundary layer is allowed to separate, however, the evolution of the jet vorticity is altered significantly. Many additional near-wall vortical structures,
whose existence was qualitatively predicted on the basis of experimental measurements (Kelso et al. 1996), emerge. These structures leave topological signatures that can be identified in the instantaneous streamlines. Among them, tornado-like wall-normal structures on the lee side of the jet, close to the nozzle, contribute significantly to the counter-rotating vortex pair. Indeed, counter-rotating vorticity in the first few diameters above the jet exit results primarily from the entrainment of wall boundary layer vorticity via these structures. These vortices are absent in the case where separation is suppressed. The circulation of the counter-rotating vortex pair increases significantly, and deeper jet penetration into the crossflow is observed. Any mechanistic description lacking the influence of the wall boundary layer will miss many important physical phenomena in transverse jets.

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REFERENCES


Margason, R. J. 1968 The path of a jet directed at large angles to a subsonic free stream. NASA Technical Note D-4919.


