Interacting One-Dimensional Fermionic Symmetry-Protected Topological Phases

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In free fermion systems with given symmetry and dimension, the possible topological phases are labeled by elements of only three types of Abelian groups, 0, $\mathbb{Z}_2$, or $\mathbb{Z}$. For example, noninteracting one-dimensional fermionic superconducting phases with S_z spin rotation and time-reversal symmetries are classified by $\mathbb{Z}$. We show that with weak interactions, this classification reduces to $\mathbb{Z}_4$. Using group cohomology, one can additionally show that there are only four distinct phases for such one-dimensional superconductors even with strong interactions. Comparing their projective representations, we find that all these four symmetry-protected topological phases can be realized with free fermions. Further, we show that one-dimensional fermionic superconducting phases with Z_n discrete S_z spin rotation and time-reversal symmetries are classified by $\mathbb{Z}_4$ when n is even and $\mathbb{Z}_2$ when n is odd; again, all these strongly interacting topological phases can be realized by noninteracting fermions. Our approach can be applied to systems with other symmetries to see which one-dimensional topological phases can be realized with free fermions.

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Symmetry-protected topological (SPT) phases [1,2] are short-range entangled states with symmetry-protected gapless edge excitations [3–8]. The Haldane phase on a spin-1 chain [9,10] and two- or three-dimensional topological insulators [11–16] are examples of SPT states. Using K theory or topological terms, all free fermion SPT phases can be classified [17,18] for all 10 Altland-Zirnbauer symmetry classes [19] of single-body Hamiltonians. It turns out that different free fermion SPT phases are described by only three types of Abelian groups, 0, $\mathbb{Z}_2$, or $\mathbb{Z}$.

With interactions the classification is more varied; however, we must first describe the symmetry differently. Instead of specifying the symmetry of single-body Hamiltonians, we treat the free fermion systems as many-body systems and specify the many-body symmetry of their many-body Hamiltonians. Only in this case can we accurately add interaction terms to the many-body Hamiltonians that preserve the many-body symmetry and study their effect on the SPT phases of free fermions. A classification of various free fermion gapped phases, given their many-body symmetry, can be found in Ref. [20].

Fidkowski and Kitaev (also Turner, Pollmann, and Berg) studied the interaction effects in one case: In their one-dimensional time-reversal (TR) invariant topological superconductors [21–23], the $\mathbb{Z}$ classification in the free case breaks down to $\mathbb{Z}_8$ with interactions that preserve TR symmetry. Here, we present another model beginning with a lattice Hamiltonian for a one-dimensional superconductor with both TR and S_z spin-rotation symmetries, described by the $\mathbb{Z}$ classification in the free case. With the addition of weak interactions that preserve these symmetries, the classification reduces to $\mathbb{Z}_4$ (see Table I). Our interaction results are obtained by assuming that edge degeneracy fully distinguishes each gapped phase; e.g., all states without edge degeneracy belong to the same trivial phase.

We compare these four fermionic phases to the four phases predicted separately from group cohomology [4,5,24] (a method valid for strong interactions). We find that each fermionic phase has a distinct projective representation [3,4], and since the group cohomology also gives rise to four, and only four distinct phases [7], we conclude that free fermions can realize all strongly interacting SPT phases in this case. We further study interaction effects on a one-dimensional superconductor with Z_n discrete S_z spin rotations and TR symmetries. For this symmetry group, we find that the SPT phases are classified by $\mathbb{Z}_4$ when n is even and $\mathbb{Z}_2$ when n is odd. Again, these results are separately obtained both from perturbing our fermionic lattice Hamiltonian and from the group cohomology classification for strong interactions—again showing that all strongly interacting topological phases can be realized by noninteracting fermions.

Free fermion lattice model.—We write a one-dimensional Hamiltonian with one trivial and two nontrivial phases,

$$H = -t\sum_{\langle ij\rangle\sigma} c^\dagger_{i\sigma}c_{j\sigma} - 2\Delta_s\sum_j c^\dagger_{j\uparrow}c^\dagger_{j\downarrow} + \text{H.c.} \pm i\Delta_p/2\sum_j c^\dagger_{j+1\uparrow}c^\dagger_{j\downarrow} + c_{j+1\downarrow}c_{j\uparrow} + \text{H.c.} \tag{1}$$

where the first term is typical nearest-neighbor hopping, the second term Δ_s represents on-site pairing, and the last term with Δ_p pairs electrons on adjacent sites.

This Hamiltonian satisfies time-reversal T and S_z spin-rotation symmetries specified on $c^T_{i\sigma} = \{\hat{c}_{i\uparrow}, \hat{c}_{i\downarrow}\}$ as

TABLE I. Symmetry groups described by the one-dimensional Hamiltonian in Eq. (1) (where Z_2^T is time-reversal), showing their free fermion classification and how they reduce with interactions. The latter remains true with strong interactions, so all such phases can be realized with free fermions.

Symmetry	Free classification	With interactions
$U(1) \times Z_2^T$	$\mathbb{Z}$	$\mathbb{Z}_4$
$Z_n \times Z_2^T$ (n even)	$\mathbb{Z}$	$\mathbb{Z}_4$
$Z_n \times Z_2^T$ (n odd and $n > 1$)	$\mathbb{Z}$	$\mathbb{Z}_2$

$$\hat{T} c_{i\sigma} \hat{T}^{-1} = i\sigma_y c_{i\sigma};$$
$$e^{i\theta\hat{S}_z} c_{i\sigma} e^{-i\theta\hat{S}_z} = \begin{pmatrix} e^{-i\theta/2} & \\ & e^{i\theta/2} \end{pmatrix} c_{i\sigma}$$

so that $\hat{T}H\hat{T}^{-1} = H$ and $e^{i\theta\hat{S}_z} H e^{-i\theta\hat{S}_z} = H$. As the band gap closes to leave just the hopping component when $\Delta_s = \pm\Delta_p$, we obtain the phase diagram in Fig. 1.

We start by identifying the trivial phase $N = 0$. When $|\Delta_s| > |\Delta_p|$, we can arbitrarily increase the strength of Δ_s without closing the gap. In the limit of $\delta_s \gg t, \delta_p$, we can neglect the hopping and p-wave pairing terms so the Hamiltonian reduces to on-site pairing—any cut separates the system into two parts leaving no boundary states (see Fig. 1); this is the trivial $N = 0$ phase. Next, we look for ground-state degeneracy at the interface between this phase and its neighbors. This is most conveniently done in a low-energy continuum model, where the effective Hamiltonian becomes

$$H = -i \int dx \tilde{\Psi}^\dagger \left[(\sigma_z \otimes \mathbb{I})\partial_x + \begin{pmatrix} & m \\ -m^T & \end{pmatrix} \right] \tilde{\Psi}$$

in a basis of right and left-moving fermion operators $\tilde{\Psi}^T = (\psi_{R\uparrow}, i\psi^\dagger_{R\downarrow}, i\psi^\dagger_{L\downarrow}, \psi_{L\uparrow})$ close to the Fermi surface. Here, $m = \Delta_p \mathbb{I} - \Delta_s \sigma_z$.

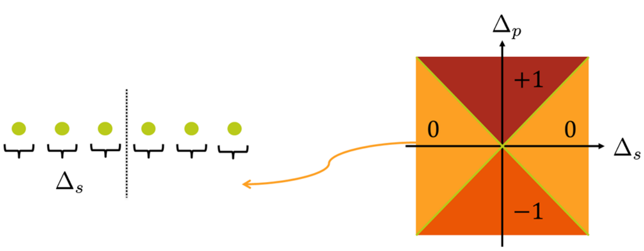

FIG. 1 (color online). Phase diagram when varying the parameters Δ_s and Δ_p: The phase boundaries are $\Delta_s = \pm\Delta_p$, which separate three phases denoted by $N = +1$, 0, and -1. We can make Δ_s arbitrarily large without closing the gap: this limit describes on-site pairing where any cut cleanly separates the system into two without leaving edge states—allowing identification of this trivial $N = 0$ phase.

Smoothly varying our mass term $m(x)$ across an interface, we set $\Delta_p(x) = \frac{1}{2}(1 + \tanh x)$ and $\Delta_s(x) = \frac{1}{2}(1 - \tanh x)$. This has the zero-energy solution

$$\hat{\psi}_{0+} = \int dx \operatorname{sech}(x)(\hat{\psi}_{R\uparrow} + i\hat{\psi}^\dagger_{L\downarrow}) \tag{2}$$

This complex fermion operator ($\hat{\psi}_{0+} \neq \hat{\psi}^\dagger_{0+}$) with energy $E = 0$ contains a double degeneracy (empty or filled) that allows labelling of $\Delta_p > |\Delta_s|$ as the nontrivial $N = 1$ phase. This mode transforms under symmetry as

$$\hat{T}\begin{pmatrix} \hat{\psi}_{0+} \\ \hat{\psi}^\dagger_{0+} \end{pmatrix}\hat{T}^{-1} = -\sigma_y \begin{pmatrix} \hat{\psi}_{0+} \\ \hat{\psi}^\dagger_{0+} \end{pmatrix},$$
$$e^{i\theta\hat{S}_z}\hat{\psi}_{0+}e^{-i\theta\hat{S}_z} = e^{-i\theta/2}\hat{\psi}_{0+} \tag{3}$$

Since the two degenerate states differ by $S_z = 1/2$ and are related by time-reversal, they carry the quantum numbers $S_z = \pm 1/4$.

Using the symmetry relations in Eq. (3), we check if any perturbations in the Hamiltonian can shift the energy of this mode. We find that the density terms $\delta H = c\hat{\psi}^\dagger_{0+}\hat{\psi}_{0+}$ are forbidden by TR; hence, our ground state degeneracy is protected by system symmetries—this $N = 1$ phase is stable against perturbations.

To find the $N = -1$ phase, we change $\Delta_p(x) \to -\Delta_p(x)$, and upon repeating our procedure, we find a different zero mode solution, which we label as

$$\hat{\psi}_{0-} = \int dx \operatorname{sech}(x)(i\hat{\psi}^\dagger_{R\downarrow} - \hat{\psi}_{L\uparrow}) \tag{4}$$

which transforms as

$$\hat{T}\begin{pmatrix} \hat{\psi}_{0-} \\ \hat{\psi}^\dagger_{0-} \end{pmatrix}\hat{T}^{-1} = \sigma_y \begin{pmatrix} \hat{\psi}_{0-} \\ \hat{\psi}^\dagger_{0-} \end{pmatrix},$$
$$e^{i\theta\hat{S}_z}\hat{\psi}_{0-}e^{-i\theta\hat{S}_z} = e^{-i\theta/2}\hat{\psi}_{0-} \tag{5}$$

This state has stable ground state degeneracy, as $\delta H = c\hat{\psi}^\dagger_{0-}\hat{\psi}_{0-}$ is also forbidden by TR, indicating $\Delta_p < |\Delta_s|$ is a nontrivial phase as well.

Let us see if it is meaningful to label this second nontrivial phase $N = -1$. We examine what happens upon stacking the two chains with nontrivial phases, the first with $\Delta_p > |\Delta_s|$ and the second with $\Delta_p < |\Delta_s|$. (The first chain would have the zero mode $\hat{\psi}_{0+}$ and the second $\hat{\psi}_{0-}$.) We find that the coupling $\delta H = c\hat{\psi}^\dagger_{0+}\hat{\psi}_{0-} + \mathrm{H.c.}$ is allowed within system symmetries, making the ground state nondegenerate. So two chains with two distinct zero modes (labelled $+$ and $-$) combine to become trivial, indicating that the two phases should be labelled with opposite indices. Naturally, the phase with $\hat{\psi}_{0-}$ would then be the $N = -1$ phase, so this model indeed gives three symmetry-protected phases $N = -1$, 0, and $+1$.

While two chains containing zero modes with opposite indices become trivial, we further consider the stability of

FIG. 2 (color online). We stack two chains in the same nontrivial phase with positive (or negative) index to see if their edge states are stable. With the first chain $a = 1$ and the second $b = 2$, M_{ab} is any coupling between them. We find that all possible couplings are forbidden by our system symmetries, and so the two similar modes are stable and form the $N = 2$ phase.

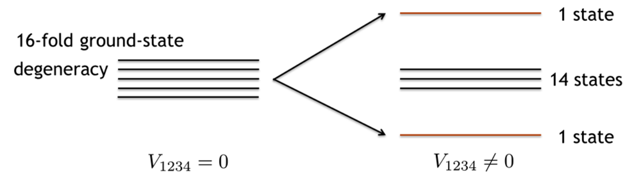

FIG. 3 (color online). Without interactions $V_{1234} = 0$, the ground state degeneracy for four zero modes is $2^4 = 16$. With interactions $V_{1234} \neq 0$, the two states are split by $\delta E = \pm|V_{1234}|$, making the ground state nondegenerate and this $N = 4$ phase trivial. As this $N = 4$ phase is now smoothly connected to the trivial $N = 0$ phase, our classification reduces from $\mathbb{Z}$ to $\mathbb{Z}_4$ with interactions.

two chains containing zero modes with the same positive (or negative) index. This may generalize to larger integers in the $\mathbb{Z}$ group, so now we examine the stacking of two chains with a similar index more systematically.

A generic coupling term (see Fig. 2) is $\delta H = c\hat{\psi}^\dagger_{+a} M_{ab} \hat{\psi}_{+b}$ + H.c. Here, a and b are indices running over the chain number $1, 2\ldots$; e.g., $\hat{\psi}_{+1}$ denotes a zero mode from the $N = 1$ phase in the first chain, and M_{ab} is any generic coupling between these two operators. We examine the simplest case of $a = 1$ and $b = 2$. Terms of the form $\delta H = c\hat{\psi}^\dagger_{+1} M_{12} \hat{\psi}_{+2}$ + H.c. are forbidden by TR symmetry, as specified in Eq. (3), while fermion pairing terms such as $\delta H = c\hat{\psi}^\dagger_{+1} M_{12} \hat{\psi}^\dagger_{+2}$ + H.c. violate S_z spin rotation symmetry. As there are no other quadratic fermion terms, the stacking of two chains is stable against perturbations and combine to give an $N = 2$ phase.

Thus, adding a number of one-dimensional chains with a positive index gives a positive integer in the $\mathbb{Z}$ group. Negative numbers are obtained simply by stacking chains with $\hat{\psi}_{0-}$. As we showed earlier, a pair of $\hat{\psi}_{0+}$ and $\hat{\psi}_{0-}$ coupled together become trivial, so the integer N in our $\mathbb{Z}$ group is the difference between all positive and negative zero modes. Then each phase labelled by N has $2^{|N|}$ degenerate ground states.

Interaction effects.—Now let us allow couplings with an arbitrary number of fermion operators. We look at terms with four and two operators that take the general form

$$\delta H = V_{abcd} \hat{\psi}^\dagger_{+a} \hat{\psi}_{+b} \hat{\psi}^\dagger_{+c} \hat{\psi}_{+d} + W_{ab} \hat{\psi}^\dagger_{+a} \hat{\psi}_{+b} + \text{H.c.} \quad (6)$$

δH is compatible with both TR and S_z spin rotation symmetry when V_{abcd} and W_{ab} satisfy certain conditions.

A possible term couples four separate chains through an interaction with only $V_{1234} \neq 0$. This δH is invariant under both TR and S_z spin rotation symmetry and couples two states $|0101\rangle$ and $|1010\rangle$ in our four-mode basis (0 and 1 denote unoccupied and occupied, respectively, for each of the four chains). Without interactions, we have a ground state degeneracy of $2^4 = 16$; with interactions, two of these 16 states split in energy by $\delta E = \pm|V_{1234}|$; see Fig. 3. This makes the ground state nondegenerate and the phase $N = 4$ trivial.

Since four chains with all positive (or negative) indices are equivalent to the trivial phase, we can smoothly connect the $N = 3$ phase to the $N = -1$ phase by adding four chains with all negative indices. So, with only three distinct nontrivial phases, the $\mathbb{Z}$ integer classification for free fermions reduces to $\mathbb{Z}_4$ in the presence of interactions.

Four-fermion interaction terms also reduce the ground state degeneracy in the $N = 2$ phase from $2^2 = 4$ to a twofold degeneracy. The term

$$\delta H = V_{1122}(\hat{\psi}^\dagger_{+1}\hat{\psi}_{+1} - \hat{\psi}_{+1}\hat{\psi}^\dagger_{+1})(\hat{\psi}^\dagger_{+2}\hat{\psi}_{+2} - \hat{\psi}_{+2}\hat{\psi}^\dagger_{+2})$$

causes two states $|00\rangle$ and $|11\rangle$ to shift in energy by V_{1122}, while two other states $|01\rangle$ and $|10\rangle$ shift by $-V_{1122}$. As we still have doubled ground state degeneracy, the state $N = 2$ remains nontrivial. To summarize, interaction effects reduce our degeneracy, leaving three nontrivial phases each with a twofold ground state degeneracy.

Four distinct projective representations.—Our results demonstrate the stability of free fermion phases with weak interactions. This method may not capture all possible interacting phases because strongly interacting topological phases may not adiabatically connect to free fermion phases. Alternatively, different phases from weak interactions may become the same phase with strong interactions. To address these issues, we illustrate a distinct projective representation [25] for each phase corresponding to a different one-dimensional SPT phase.

Using the symmetry operations defined in Eqs. (3) and (5), we write their matrix representations on the degenerate subspace. In the $N = 1$ phase with basis $|0_+\rangle$ and $|1_+\rangle$

$$U_\theta \to M(U_\theta) = \begin{pmatrix} 1 & \\ & e^{-i\theta/2} \end{pmatrix}, \qquad \tilde{T} \to M(\tilde{T})K = \sigma_x K$$

where $\tilde{T} = U_{-\pi}T$, a rotated TR operator we can introduce since U_θ and T commute, and K is the antiunitary operator corresponding to complex conjugation. Then

$$M(\tilde{T})KM(U_\theta) = e^{i\theta/2}M(U_\theta)M(\tilde{T})K \quad (7)$$

and $M(\tilde{T})KM(\tilde{T})K = 1$. This is a projective representation, as the phase in Eq. (7) cannot be removed by adding any phase factor to $M(U_\theta)$.

Moving to the $N = -1$ phase with ground states $|0_-\rangle$ and $|1_-\rangle$, we have the same representation for $M(U_\theta)$, while $M(\tilde{T})K = e^{-i\sigma_z\pi/2}\sigma_x K$ in this case. Equation (7) remains true, but now $M(\tilde{T})KM(\tilde{T})K = -1$, so we have a different projective representation.

In the $N = 2$ phase, our ground states are fourfold: $|0_+0_+\rangle$, $|0_+1_+\rangle$, $|1_+0_+\rangle$, and $|1_+1_+\rangle$. Here

$$M(U_\theta) = \begin{pmatrix} e^{i\theta/4} & \\ & e^{-i\theta/4} \end{pmatrix} \otimes \begin{pmatrix} e^{i\theta/4} & \\ & e^{-i\theta/4} \end{pmatrix}, \quad M(\tilde{T})K = \sigma_x \otimes \sigma_y K. \tag{8}$$

While $M(\tilde{T})K$ and $M(U_\theta)$ commute this time, $M(\tilde{T})KM(\tilde{T})K = -1$, again making a third nontrivial projective representation.

As each nontrivial phase has a distinct nontrivial projective representation, they remain distinct phases even when interactions are strong. We can compare our results to the unperturbative bosonic classification in one dimension obtained by group cohomology [4,7,24], as we can bosonize our fermionic model. The resulting bosonic model would have the same symmetry $U(1) \times Z_2^T$, with phases classified by $\mathbb{Z}_2 \times \mathbb{Z}_2$; i.e., the four distinct projective representations there correspond to the four different strongly interacting phases. Our fermionic results similarly contain all the four phases with these distinct projective representations, so this model realizes all possible nontrivial phases with strong interactions.

Modifying symmetry from $U(1)$ to Z_n spin rotation.—As our fermion model respects S_z spin rotation and TR symmetry, it naturally contains Z_n discrete spin rotations as well. We can replace $U(1)$ spin rotation by Z_n spin rotation; i.e., rotation by an arbitrary angle is now constrained to values of $\theta = 2\pi/n$ and our new symmetry group has generators time-reversal T and discrete S_z rotation $R = e^{iS_z 2\pi/n}$, satisfying

$$T^2 = (-)^{N_F}, \qquad R^n = (-)^{N_F}, \qquad \mathrm{TR} = RT. \tag{9}$$

Here $(-)^{N_F}$ is the fermion number parity operator. When n is even, this group $G(T, Z_n)$ is generated by R and $\tilde{T} = R^{n/2}T$; so $G(T, Z_n) = Z_{2n} \times Z_2^{\tilde{T}}$. When n is odd, we find that $\tilde{R} = RT$ alone generates this group $G(T, Z_n) = Z_{4n}^T$ (see Supplemental Material [26]).

For $n \geq 2$, no new fermion bilinear terms are allowed, so the free fermion classification does not change from $\mathbb{Z}$. In the case of $n = 1$, new quadratic terms of the form $\delta H = c\hat{\psi}^\dagger_{+1}\hat{\psi}^\dagger_{+2} + \mathrm{H.c.}$ are permitted. This term couples two chains forming the $N = 2$ phase to make the ground state nondegenerate. The $N = 2$ phase becomes trivial, and the classification for $n = 1$ reduces to $\mathbb{Z}_2$.

Similarly, for higher n, we can always add interacting terms with $2n$ $\hat{\psi}_+$ operators similar to the term in the $n = 1$ case above. For $n = 2$, for instance, this term is $\delta H = c\hat{\psi}^\dagger_{+1}\hat{\psi}^\dagger_{+2}\hat{\psi}^\dagger_{+3}\hat{\psi}^\dagger_{+4} + \mathrm{H.c.}$. Such interactions couple $2n$ zero modes each in the $N = 1$ phase to render the ground state nondegenerate. In effect, Z_n spin rotation symmetry allows interactions that reduce the classification to $\mathbb{Z}_{2n}$.

We have established that under $U(1)$ spin rotation symmetry, interactions reduce the classification to $\mathbb{Z}_4$. Including more interactions as allowed by Z_n spin rotation further reduces the classification to $\mathbb{Z}_{2n}$. Taken together, we find that there is no effect on even n, which remains $\mathbb{Z}_4$, since $2n$ is a multiple of 4. Odd n, however, reduces to a $\mathbb{Z}_2$ classification (as 2 becomes the largest common denominator between $2n$ and 4).

The number of nontrivial phases can be compared to and matched with the group cohomology prediction $\mathbb{Z}_2 \times \mathbb{Z}_2$ for even n and $\mathbb{Z}_2$ for odd n (see Supplemental Material [26]). We find that different symmetry groups with the same free fermion classification reduce to various results (here $\mathbb{Z}_4$ or $\mathbb{Z}_2$ are examples) in the presence of interactions (see the summary in Table I). As verified by comparison of these phases with group cohomology, all the possible strongly interacting phases can be realized by free fermions in this model.

Finally, we note that our classification is protected only by system symmetries of spin rotation and TR. As shown earlier, without such symmetry a term $\delta H = c\hat{\psi}^\dagger_{0+}\hat{\psi}_{0+}$ would be permitted, which would render the ground state nondegenerate and the classification trivial ($\mathbb{Z}_1$).

Discussion.—We study the SPT phases of one-dimensional fermionic superconductors with TR and S_z spin rotation symmetries. If fermions do not interact, their classification is given by the $\mathbb{Z}$ group; with weak interactions, this reduces to a $\mathbb{Z}_4$ classification. As each of our four fermion phases have distinct projective representations, they correspond to four distinct phases by comparison with group cohomology, which predicts four, and only four, different gapped phases, even with strong interactions.

Hence, all distinct symmetric gapped phases with strong interactions are realized by noninteracting fermions in this case. Fermion parity is part of our $U(1)$ symmetry, which cannot be spontaneously broken. Therefore, this model does not have the fermion parity symmetry broken phases corresponding to Majorana topological modes [24]. The edge states in our one-dimensional superconductors are described by complex fermions, and it is not surprising that our interacting classification comprises half of the results from Kitaev and Fidkowski's Majorana model [21–23].

We further studied the SPT phases of one-dimensional superconductors with TR and Z_n discrete S_z spin rotation symmetries, to find they are classified by $\mathbb{Z}_4$ when n is even and $\mathbb{Z}_2$ when n is odd. Again, as the phases in our fermionic model match with the group cohomology prediction, all gapped phases of these one-dimensional fermionic superconductors are also realized by noninteracting fermions.

Interactions on different symmetry groups with the same free fermion classification give rise to varied results (Table I). Here, perturbing from a free fermion model gives all strongly interacting phases; however, in other cases such phases may not be realizable with free fermions. Finally, the effectiveness of this method remains open, especially in higher dimensions, where additional tools may be needed. Further study of different symmetry groups or in higher dimensions would be worthwhile.

Toward the completion of this Letter, we noted the study of A. Rosch (arXiv:1203.5541), which shows "a topological insulator made of four chains of superconducting spinless fermions characterized by four Majorana edge states can adiabatically be deformed into a trivial band insulator" via "interactions to spinful fermions," which has some relation to our $\mathbb{Z}_4$ classification of one-dimensional fermionic superconducting phases with TR and S_z spin rotation symmetries.

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[25] More accurate math terminology is "equivalence class of projective representations" where we merely write "projective representation" so as to not distract the readers from the physics. This is also true in some subsequent instances; e.g., "different projective representation."
[26] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.109.096403 for details on our free fermion model and the group cohomology classification when spin-rotation is discrete.