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Projective non-Abelian statistics of dislocation defects in a $Z_N$ rotor model

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Non-Abelian statistics is a phenomenon of topologically protected non-Abelian geometric phases as we exchange quasiparticle excitations. Here we construct a $Z_N$ rotor model that realizes a self-dual $Z_N$ Abelian gauge theory. We find that lattice dislocation defects in the model produce topologically protected degeneracy. Even though dislocations are not quasiparticle excitations, they resemble non-Abelian anyons with quantum dimension $\sqrt{N}$. Exchanging dislocations can produce topologically protected projective non-Abelian geometric phases. Therefore, we discover a kind of (projective) non-Abelian anyon that appears as the dislocations in an Abelian $Z_N$ rotor model. These types of non-Abelian anyons can be viewed as a generalization of the Majorana zero modes.

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Introduction. Searching for Majorana fermions (or more precisely, Majorana zero modes) in condensed matter systems has attracted increasing research interests recently.\textsuperscript{1-10} But what is really the Majorana zero mode? In fact, the so-called “Majorana zero mode” is actually a phenomenon of topologically protected degeneracy\textsuperscript{11,12} in the presence of certain topological defects [such as vortices in two-dimensional (2D) $p_x + ip_y$ superconductors].\textsuperscript{2,3} In the race for finding Majorana zero modes, much attention has been paid to the fermion systems.\textsuperscript{4-10} However the boson/spin systems also have topologically protected degeneracies,\textsuperscript{11-20} which may also be ascribed to Majorana zero modes or their generalizations.

A 2D bosonic example of emergent Majorana zero modes is found in the $Z_2$ plaquette model,\textsuperscript{19} where lattice dislocations are braided and fused as if they were Majorana zero modes\textsuperscript{21,22} which resemble non-Abelian anyons\textsuperscript{23-26} of quantum dimension $\sqrt{2}$. The $Z_2$ plaquette model can be generalized to a $Z_N$ rotor model, whose low energy effective theory is a self-dual $Z_N$ gauge theory,\textsuperscript{26,27} which could be realized in the $Z_N$ spin liquid states.\textsuperscript{28-31} In this Rapid Communication, we study the topologically protected degeneracy associated with the extrinsic topological defects, namely, lattice dislocations in the $Z_N$ rotor model, and found that these defects are of quantum dimension $\sqrt{N}$, which can be viewed as a generalization of the “Majorana zero mode.” Braiding topological defects with protected degeneracy will lead to a topologically protected projective non-Abelian geometric phase,\textsuperscript{32} which may allow us to perform decoherence-free quantum computations.\textsuperscript{25}

We would like to remark that the dislocations in our $Z_N$ rotor model are not non-Abelian anyons, since the non-Abelian anyons must be excitations of the Hamiltonian, while the dislocations are not the excitations in this sense. The dislocations do not really carry non-Abelian statistics since the non-Abelian geometric phase from exchanging dislocations is topologically protected only up to a total phase. We say the dislocations carry a projective non-Abelian statistics.\textsuperscript{33,34} Another example of projective non-Abelian statistics for dislocations in fractional quantum Hall states on lattice can be found in Ref. 35.

To summarize, the finite energy quasiparticles in our $Z_N$ rotor model have only Abelian statistics. However, the dislocation defects in the model have a (projective) non-Abelian statistics, which generalize the (projective) non-Abelian statistics of Majorana zero modes.

$Z_N$ plaquette model. The $Z_N$ plaquette model is a rotor model on a two-dimensional square lattice (see Fig. 1). On each site $i$, define a $Z_N$ rotor with $N$ basis states $|m_i\rangle$, labeled by the angular momentum $m_i = 0, 1, \ldots, (N - 1)$. For each rotor, introduce $U_i$ to measure the angular momentum $U_i|m_i\rangle = e^{i\theta_{m_i}} |m_i\rangle$ with $\theta_{N} \equiv 2\pi/N$, and $V_i$ to lower the angular momentum by one $V_i|m_i\rangle = |(m_i - 1)_{mod N}\rangle$. Both $U_i$ and $V_i$ are unitary operators $U_i^\dagger U_i = V_i^\dagger V_i = 1$, satisfying $V_i U_i = e^{i\theta_{m_i}} U_i V_i$.

The $Z_N$ plaquette model is given by the Hamiltonian

$$H = - \sum_P O_p + \text{H.c.},$$

(1)

where the operator $O_p$ describes a kind of ring coupling among the rotors on the corner sites of each plaquette $p$,

$$O_p = \frac{4}{\sqrt{N}} \sum_i U_i V_i^\dagger V_{i+1}^\dagger V_{i+1} V_i = U_1 V_2 U_3^\dagger V_4^\dagger.$$

(2)

Here we adopt the graphical representation for the operators, $U_i = \chi$, $V_i = \chi$, $U_i^\dagger = \chi$, $V_i^\dagger = \chi$, by drawing directed strings going through the site. Because these operators only connect diagonal plaquettes, a string starting from the even plaquette will never enter the odd plaquette (and vice versa). So we can locally distinguish two different types of strings: $e$-string ($m$-string) if it lives in the even (odd) plaquettes (see Fig. 1). The assignment of even or odd to the plaquettes can be reversed under the translation of one lattice spacing, so the interchange of $e$- and $m$-strings could be realized by the curvature of the lattice, as will be seen later. This duality related translation symmetry is a unique property of the $Z_N$ plaquette model (self-dual $Z_N$ gauge theory), and is not presented in Kitaev’s quantum double model\textsuperscript{16} for $Z_N$ (generic $Z_N$ gauge theory).

The $Z_N$ plaquette model Eq. (1) is exactly solvable, as evidenced from the commutation relation $[O_p, O_p'] = 0$, as $O_p O_p' = e^{i\theta_{m_p}} e^{-i\theta_{m_p'}} O_p' O_p$ for adjacent
p and p', where the overlay of strings indicates the ordering of the operators, such as \( V_i U_i = X \) and \( U_i V_i = X \), with the algebra \( X = e^{iθX} \).

Every \( O_p \) operator has \( N \) distinct eigenvalues \( e^{iθq_p} \) labeled as \( q_p = 0, 1, \ldots, (N - 1) \), as implied from \( O_p^N = 1 \). \( q_p \) denotes the (generalized) \( Z_N \) charge hosted by the plaquette \( p \). If the plaquette is even (odd), we may call it \( e \)-charge (\( m \)-charge).

The energy will be minimized if all \( O_p \)'s take the eigenvalue 1 (\( q_p = 0 \)). Therefore the ground states are the common eigenstates that satisfy \( O_p \cdot \text{[ground]} = \text{[ground]} \) for all \( p \)'s, and are free of any \( Z_N \) charges. They can be obtained from the projection \( \text{[ground]} = \prod_p p_p \cdot \text{[anystate]} \), where the projection operator \( p_p = \sum_{q=0}^{N-1} O_p^q \) is defined for each plaquette \( p \). The projective construction enumerates all degenerated ground states as the starting [anystate] transverses all superselection sectors.

**Intrinsic Abelian anyon excitations.** The excited states can be obtained by applying open-string operators to the ground state, which create opposite \( Z_N \) charge excitations in pairs at both ends of the string. These excitations and can be detected by the closed-string operator (as \( O_p \)) surrounding them in the counterclockwise direction. Take \( S \) in Fig. 1, for example, \( O_p S \cdot \text{[ground]} = \bigotimes q \cdot \text{[ground]} = e^{iθq_1} \cdots e^{iθq_p} \cdot \text{[ground]} = e^{iθS} \cdot \text{[ground]} \), showing that a charge \( q_p = +1 \) is created at the end of the open-string by the action of \( S \). One can show that the opposite charge \( q_p = -1 \) is created at the other end.

Because \( Z_N \) charge excitations are the ends of open-strings, their statistics are inherited from the algebra of the string operators. According to \( X = e^{iθX} \), braiding a \( q_1 \) \( e \)-charge with a \( q_m \) \( m \)-charge would acquire a phase \( e^{iθq_1q_m/2} \). In this sense, these excitations are Abelian anyons. However, we must stress that these anyons are **intrinsic**, as they are collective motions of rotors, described by the excited state within the rotor Hilbert space. This is to be distinguished from the **extrinsic** anyons introduced later as lattice dislocations, which do not belong to the rotor Hilbert space. Note that both the phase \( e^{iθq_1q_m/2} \) and the excitation energy are invariant under the exchange of \( e \) - and \( m \). This manifests the self-duality of the \( Z_N \) plaquette model, and can be realized by lattice translation.

**Ground state degeneracy.** The degeneracy of the ground state depends on the topology of the lattice. Let us consider the torus topology by setting the model on a \( L_x \times L_y \) sized lattice with periodic boundary conditions in both directions. The total number of states is \( N^{N_{\text{site}}} \), with \( N_{\text{site}} = L_x L_y \) being the number of sites. To count the ground states, we note that they are constrained by \( \forall p: O_p = 1 \). Consider a particular \( O_p \) operator and the subspaces labeled by its different eigenvalues. Those subspaces all have the same dimension, because any open-string operator that ends in the plaquette \( p \) (with the other end in a fixed particular plaquette) can be used to perform a unitary transform that rotates these subspaces into each other. So each time imposing \( O_p = 1 \) on a particular plaquette will reduce the available Hilbert space dimension by a factor of \( N \). However, the \( O_p \) operators are not independent. Because \( e \)-charges (\( m \)-charges) are created in opposite pairs, summing over the lattice, \( e \)-charges and \( m \)-charges must be neutralized, respectively, i.e., \( \prod_{p \text{ even}} O_p = \prod_{p \text{ odd}} O_p = 1 \). This is true on an even \( \times \) even lattice (i.e., both \( L_x \) and \( L_y \) are even), which reduces the number of independent \( O_p \) constraints to \( (N_{\text{plaq}} - 2) \), with \( N_{\text{plaq}} = L_x L_y \) being the number of plaquettes. So after restricting the full Hilbert space to the ground state subspace, the remaining dimension is \( N^{N_{\text{site}} - N_{\text{plaq}} + 2} = N^2 \), meaning the ground state degeneracy of the \( Z_N \) plaquette model is \( N^2 \) on the even \( \times \) even lattice. However, for the even \( \times \) odd or odd \( \times \) odd lattices (i.e., \( L_x \) or \( L_y \) is odd), \( e \)-strings and \( m \)-strings can be continued into each other by going along the odd direction, thus \( e \)-charges and \( m \)-charges are made identical. So they are no longer required to be neutralized respectively, but only neutralized as a whole. Therefore we only have one relation \( \prod_p O_p = 1 \), which reduces the number of independent \( O_p \) constraints to \( (N_{\text{plaq}} - 1) \), and the resulting ground state degeneracy will be \( N^{N_{\text{site}} - N_{\text{plaq}} + 1} = N \).

To summarize, the ground state degeneracy \( D_{\text{GS}} \) of the \( Z_N \) plaquette model on a torus follows from the general formula (which holds for any arbitrary large lattice with or without dislocations)

\[
D_{\text{GS}} = N^{N_{\text{site}} - N_{\text{plaq}}},
\]

where \( N \) denotes the number of species \( 37 \) of the intrinsic excitations that are supported by the lattice topology. On the even \( \times \) even lattice, we have a total of \( N = N^2 \) distinct excitations by a combination of \( e \) - and \( m \)-charges. When it comes to the even \( \times \) odd or odd \( \times \) odd lattice, \( e \)- and \( m \)-charges are no longer distinct, and the number of excitation species is reduced to \( N = N \). The topological order in the ground state is now evidenced from the protected ground state degeneracy on the torus \( 11,12 \) and from the dependence of the ground state degeneracy on the parity of the lattice periodicity.

**Dislocations.** One can change the lattice periodicity by first generating a pair of edge dislocations with opposite unit length Burger’s vectors, and moving them in the direction perpendicular to their Burger’s vectors all the way around the lattice, then annihilating them as they meet again at the periodic boundary. During this process, the ground state degeneracy must have changed. This motivates us to introduce dislocations as shown in Fig. 2(a) to probe the topological order by looking at the degeneracy associated with them. With dislocations, one can no longer globally color the plaquettes consistently. Branch cuts must be left behind between each pair of dislocations. Going around a dislocation will exchange the \( e \) - and \( m \)-charges, as \( e \)- and \( m \)-strings are transmuted into each
FIG. 2. (Color online) (a) Lattice with a pair of dislocations, marked out by ⊿ and ⊿. Plaquettes on the branch cut are colored in violet. Periodic boundary conditions are assumed in both directions by sticking the dashed edges with the solid edges on the opposite side. The ring operators $O_p$ are redefined around the pentagonal plaquette. $C$ operators denote large closed-strings looping around the lattice. (b) Plaquette to site mapping. The site that is not mapped to is marked by a black dot.

FIG. 3. (Color online) Diffeomorphism of the torus with a pair of dislocations at $c$ and $d$. Expand the branch cut (violet dashed line) between $c$ and $d$ into a hole, with two edges marked by $\alpha$ and $\beta$. Separate the $e$- and $m$ layers. Unwrap both layers by cutting along the large loops around the torus. Rotate one layer to glue the $\beta$ edges together along the marked direction. Glue the other edges and rewrap into a double torus.

FIG. 4. (Color online) Diffeomorphism of string operators on the torus (a) with a pair of dislocations, or (b) with three pairs of dislocations.
following commutation relations, \[ [C_{ex}, C_{ey}] = [C_{mx}, C_{my}] = C_{ex} C_{mx} - C_{mx} C_{ex} = e^{i\theta_a} C_{mx} C_{ex}, \]
where \( \theta_a \) is the Aharonov-Bohm phase, and \( C_{ex}, C_{ey}, C_{mx}, C_{my} \) are the operators corresponding to the different directions. Each algebra requires an \( N \)-dimensional representation space, so the four cycle operators together require \( 2N \)-dimensional representation space, which must have completed the ground state subspace, since all the noncontractable cycles can be generated by these four basis cycles. Therefore the ground states are \( N^2 \)-fold degenerated, and each of them corresponds to a basis in the representation space. Any perturbation of the Hamiltonian that is nonzero only in a local region will not change the ground state degeneracy, since the noncontractable cycle operators that avoid the local region still commute with the Hamiltonian.

The above can be generalized to the case with any number of dislocations. Consider \( n \) dislocations with \( n/2 \) branch cuts. Following the similar cut-and-glue procedures in Fig. 3, the topological space will be a genus \( g = n/2 + 1 \) surface as in Fig. 4(b), on which one can choose \( g \) pairs of noncontractable cycle operators \( C_a \) and \( C_b \) \((a = 1, \ldots, g)\), such that \( [C_a, C_b] = [C_a, C_b^\prime] = 0 \) and \( C_a C_b^\prime = e^{i\theta_a} C_b C_a \). These operators span an \( N^2 \)-dimensional representation space isomorphic to the ground state subspace. Therefore the ground state degeneracy of the \( \mathbb{Z}_N \) plaquette model with \( n \) dislocations is \( D_{GS} = N^{n/2+1} \) which is consistent with our previous result. Each dislocation contributes to the ground state degeneracy by a factor of \( \sqrt{N} \). Thus the dislocations resemble non-Abelian anyons of quantum dimension \( \sqrt{N} \), as described in Ref. 38. Braiding the dislocations leads to topologically protected projective non-Abelian geometric phases.\(^{32}\) We see that projective non-Abelian anyons can emerge from an Abelian model as the extrinsic topological defects, such as lattice dislocations. Those projective non-Abelian anyons can be used to perform topological quantum computations, but not universally since the square of the quantum dimension is not universally since the square of the quantum dimension is not an integer.\(^{40}\)

Parton approach. For the \( N = 2 \) case, the quantum dimension \( \sqrt{2} \) implies that the extrinsic anons are Majorana fermions.\(^{21}\) To expose the Majorana fermion explicitly, we evoke the parton constructive construction, in which four Majorana fermions \( \eta^a_i \) \((a = 1, 2, 3, 4)\) are introduced on each site \( i \), obeying the anticommutation relation \( \{\eta^a_i, \eta^b_j\} = \delta_{ij} \delta_{ab} \). Under the constraint \( \eta^a_i \eta^a_i \eta^b_j \eta^b_j = 1/4 \), the rotor operators can be expressed as \( U_i = i\eta^a_i \eta^a_i, \ V_i = i\eta^a_i \eta^a_i \). Then the \( \mathbb{Z}_2 \) plaquette model can be mapped to an interacting fermion model, which has a “mean-field” description given by \( H_{\text{mea}} = -\sum_{ij} (s_i j \Delta_{ij} + \text{H.c.}) \) with the ansatz \( s_i j = \pm 1 \) on each bound, where \( \Delta_{i,j} = i\eta^a_i \eta^a_j \) and \( \Delta_{i,j}^\dagger = i\eta^a_i \eta^a_j \). Let \( \{s_i j\} \) be a free fermion ground state of \( H_{\text{mea}} \), and \( P = \prod_{i,j} \frac{1}{2} (1 + 2\eta^a_i \eta^a_j \eta^b_j \eta^b_j) \) be the projection operator to the physical Hilbert space of rotors. All the eigenstates of the \( \mathbb{Z}_N \) plaquette model can be obtained by the projective construction as \( P\{s_i j\} \). To obtain the ground states, \( \{s_i j\} \) must satisfy the flux configuration given by \( Q = 1 \). There are altogether four gauge inequivalent solutions for \( \{s_i j\} \) on a torus. Given a particular \( \{s_i j\} \), all the Majorana fermions will be paired up across the bound, except for the dangling Majorana fermion at the dislocation site. If there are \( n \) dislocations in the system, there will be \( n \) dangling Majorana zero modes, which leads to a \( 2^{n/2} \)-fold degeneracy in the free fermion ground states. So altogether we have \( 4 \times 2^{n/2} \) fermion states to be projected from, half of which will be projected to nothing due to their odd fermion parity. Therefore the resulting physical ground states add up to \( 4 \times 2^{n/2} / 2 = 2^{n+1} \), consistent with our previous formula. The above discussion has shown that the \( \sqrt{N} \)-dimension of the extrinsic anons actually originated from the dangling Majorana fermion, or the Majorana zero mode, at the dislocation site. It has been shown that exchanging Majorana zero modes will lead to a non-Abelian geometric phase. In Ref. 32, it is shown that exchanging dislocations in our \( \mathbb{Z}_N \) plaquette model will also lead to a protected (projective) non-Abelian geometric phase.

Conclusion. We studied the phenomenon of topologically protected degeneracy and topologically protected projective non-Abelian geometric phases produced by extrinsic topological defects (such as dislocations) in a \( \mathbb{Z}_N \) rotor model. We find that these dislocations are projective non-Abelian anyons with quantum dimension \( \sqrt{N} \). For \( N = 2 \), such a result can be derived from a parton construction where the dislocations can be identified as Majorana zero modes. For higher \( N \) \((N > 2)\), the projective non-Abelian anyons \((i.e., \text{the dislocations) can be viewed as a generalization of the Majorana zero modes.}

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