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Kalman Filter for Inhomogeneous Population Markov Chains with Application to Stochastic Recruitment Control of Muscle Actuators

Lael Odhner and H. Harry Asada, Member, IEEE

Abstract—A population of stochastic agents, as seen in swarm robots and some biological systems, can be modeled as a population Markov chain where the transition probability matrix is time-varying, or inhomogeneous. This paper presents a Kalman filter approach to estimating the population state, i.e., the headcount of the number of agents in each possible agent-state. The probabilistic state transition formalism originated in Markov chain modeling is recast as a standard state transition equation perturbed by an additive random process with a multinomial distribution. An optimal linear filter is derived for the recast state equation; the resultant optimal filter is a type of Kalman filter with a modified covariance propagation law. Convergence properties are examined, and the state estimation error covariance is guaranteed to converge. The state estimation method is applied to stochastic control of muscle actuators, where individual artificial muscle fibers are stochastically recruited with probabilities broadcasted from a central controller. The system output is the resultant force generated by the population of muscle fibers, each of which takes a discrete level of output force. The linear optimal filter estimates the population state (the headcount of agents producing each level of force) from the aggregate output alone. Experimental results demonstrate that stochastic recruitment control works effectively with the linear optimal filter.

I. INTRODUCTION

There is an increasing interest across the control community in controlling a population of agents. Emergent behaviors of a cluster of cells in morphogenesis and regenerative medicine [1], a colony of insects, and swarm robots [2] are just a few examples of amazing behaviors as a result of collective efforts of individual agents.

Modeling and control of a population of agents has been an active research field in the last decade. Stochasticity is one of the key elements characterizing the population of agents. Endothelial cells in angiogenesis, for example, are known to be stochastic; each cell makes a stochastic decision based on probabilities determined by inputs and the environment conditions [3]. Clusters of bacteria, too, are modeled as a population of stochastic agents [4]. In these systems treating individual cells and bacteria as stochastic agents is a rational choice, in particular, for dealing with a large scale population. Effective methodologies and theories can be applied when the system can be treated as an ensemble of stochastic agents.

Stochastic broadcast control is among the stochastic control methods recently developed for population control [5,6]. Instead of dictating each single agent to take a deterministic action, a central controller broadcasts state transition probabilities with which each agent makes a state transition. In this control architecture, control commands are broadcasted to all the agents by treating them as anonymous members and thereby no address and individual control commands are necessary. Yet, the aggregate emergent behavior described as ensemble outputs can be regulated effectively and stably with broadcast control. The plant regulated by broadcast control is modeled as an inhomogeneous population Markov chain, where the transition probability matrix is time-varying or inhomogeneous. Provably stable broadcast controls have been developed and applied to several processes where the population output, a function of agents’ state distribution, is regulated towards a given goal [7,8].

Broadcast control performance will be improved significantly if the population state, i.e. a vector representing the distribution of headcounts, can be estimated from aggregate population outputs. This paper presents a type of Kalman filter for estimating the population state from observable aggregate outputs. Stochastic estimation, such as Bayesian estimation and Maximum Likelihood estimation, are computationally heavy for real-time applications as the number of agents increases. The recursive linear optimal filter this paper presents is much more efficient and useful.

To our knowledge, the Kalman filter for population Markov chains has been developed for fishery population monitoring in the ocean engineering area [9]. However, no profound theoretical issues have been addressed in the oceanography literature. There is a significant gap between the standard state equation for which the Kalman filter has been formulated and the probabilistic state transition equation governed by inhomogeneous Markov chains. This paper presents a formal methodology for converting the latter to the former formalism, and obtains a liner optimal filter, which is a type of Kalman filter with a modified covariance propagation law. Fundamental properties of the new filter are examined. The basic properties of the Markov chain’s probability matrix provides us with favorable characteristics needed for guaranteeing convergence. The method will be applied to an artificial muscle control system to demonstrate the validity and usefulness of the method.
II. PROBLEM FORMULATION

Figure 1 shows the control system to be discussed in this paper. The plant is a population of \( N \) agents, described as \( N \) independent and identically distributed Markov chains. Each agent assumes a state among \( m \) discrete states \( S_i \in \{1, \cdots, m\} \) at discrete time \( t \). The state of the entire population is given by the number of agents in each state.

\[
X(t) = [N_1(t), \cdots, N_m(t)]^T
\]  

where \( N_i(t) \) is the number of agents in state \( S_i = i \). To differentiate \( S_i \) and \( X(t) \), the former is called individual “agent state”, while the latter is referred to as “population state”.

With this random vector, the state equation is given by

\[
X(t) = X(t-1)A + z(t)
\]

where \( A \) is a deterministic state transition matrix, \( z(t) \) is a random vector, and \( X(t) \) is the state of the entire population.

Our interest in this paper is to regulate the population state rather than agent states. A central controller does not have access to individual agent states; it is immaterial whether each and every agent takes a particular agent state. Their aggregate head count distribution among the \( m \) states, that is, the population state, is of our interest. The individual agents are totally anonymous.

The input to the plant is state transition probabilities broadcasted by a central controller. The population Markov chains are therefore inhomogeneous, because the transition probabilities are modulated by the broadcast input. Let \( A_{ij} \) be the probability of agent state transition from state \( j \) to state \( i \) defined by

\[
A_{ij}(t) = \Pr(S_{i+1} = i | S_i = j)
\]

Collectively we denote all the state transition probabilities at time \( t \) in matrix form:

\[
A(t) = \{A_{ij}(t)\} \in \mathbb{R}^{m \times m}
\]

Since \( A \) is a probability matrix, the following conditions apply:

\[
\sum_{i} A_{ij}(t) = 1, \quad 0 \leq A_{ij}(t) \leq 1, \quad \forall i, j \in \{1, \cdots, m\}
\]

Given \( X(t) \) and \( A(t) \), the population state transition is predicted as

\[
\tilde{X}(t+1) = E[\tilde{X}(t+1) | X(t), A(t)] = A(t)X(t)
\]

in the absence of process disturbances.

The outputs from the plant, i.e. the population Markov chains, are assumed to be linear functions of the population state:

\[
Y(t) = H(t)X(t)
\]

where \( Y(t) \in \mathbb{R}^{r \times 1} \) and \( H(t) \in \mathbb{R}^{r \times m} \).

The main problem of this paper is to estimate the population state \( X(t) \) from aggregate output \( Y(t) \) and state transition matrix \( A(t) \) with unknown initial population state. Stochastic estimation techniques, such as Bayesian estimation and Maximum Likelihood estimation, are available, but they are too computationally complex to apply to real-time control problems, as the number of agents, \( N \), as well as the number of discrete agent states, \( m \), increase. This paper presents an alternative to those stochastic estimation methods. Based on Kalman filter formalism, an efficient recursive algorithm will be developed and applied to a practical problem.

III. KALMAN FILTER FORMALISM OF POPULATION STATE ESTIMATION

A. Converting the Population Markov Chain to a State Equation

In an attempt to apply Kalman Filter to the above population state estimation, we first consider the standard state equation with an additive process disturbance:

\[
x(t+1) = F(t)x(t) + G(t)w(t)
\]

where \( F(t) \) is a deterministic state transition matrix, \( w(t) \) is uncorrelated process noise, and the vectors and matrices involved have consistent dimensions. In the absence of additive process noise, the state transition is completely deterministic: \( x(t+1) = F(t)x(t) \). In comparison, the Markov state transition, \( X(t+1) = A(t)X(t) \), is stochastic since the matrix \( A(t) \) is a probability matrix. To convert this to the normal state equation, we introduce a random variable vector \( z(t) \in \mathbb{R}^{r \times 1} \):

\[
z(t) \triangleq X(t+1) - \tilde{X}(t+1) = X(t+1) - A(t)X(t)
\]

With this random vector, the state equation is given by
\[ \mathbf{X}(t+1) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{z}(t) \quad (9) \]

It is important to note that the above matrix \( \mathbf{A}(t) \) no longer means a probability matrix, although all the elements of \( \mathbf{A}(t) \) take the same values as \( \mathbf{A}(t) \). In other words, \( \mathbf{A}(t)\mathbf{X}(t) \) is not a random variable, given \( \mathbf{A}(t) \) and \( \mathbf{X}(t) \).

The randomness of population state \( \mathbf{X}(t+1) \) has been separated out and manifested in random vector \( \mathbf{z}(t) \). As a result \( \mathbf{A}(t) \) has the same physical sense as the deterministic state transition matrix \( \mathbf{F}(t) \). Random vector \( \mathbf{z}(t) \) has a probability mass function:

\[ f_z(\mathbf{z}(t); \mathbf{X}(t), \mathbf{A}(t)) \triangleq \Pr(\mathbf{Z} = \mathbf{z}(t) | \mathbf{X}(t), \mathbf{A}(t)) \quad (10) \]

Note that the probability mass function \( f_z(\mathbf{z}(t); \mathbf{X}(t), \mathbf{A}(t)) \) is a function of population state \( \mathbf{X}(t) \), and is given by a complex multinomial distribution. See more details in [10].

In addition to the state dependent noise \( \mathbf{z}(t) \), a real plant is likely to be disturbed by uncorrelated process noise \( \mathbf{w}(t) \). Combining these yields the full population state equation given by

\[ \mathbf{X}(t+1) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{z}(t) + \mathbf{G}(t)\mathbf{w}(t) \quad (11) \]

Likewise, the population output observed by physical sensors may contain measurement noise \( \mathbf{v}(t) \):

\[ \mathbf{y}(t) = \mathbf{H}(t)\mathbf{X}(t) + \mathbf{v}(t) \quad (12) \]

Process noise \( \mathbf{w}(t) \) and measurement noise \( \mathbf{v}(t) \) are uncorrelated, and each has covariance \( \mathbf{Q}_n(t) \) and \( \mathbf{R}(t) \), respectively:

\[ C_{n_n}(t, s) = \begin{cases} 0, & \forall t \neq s \\ \mathbf{Q}_n(t), & \forall t = s \end{cases} \]
\[ C_{n_m}(t, s) = \begin{cases} 0, & \forall t \neq s \\ \mathbf{R}(t), & \forall t = s \end{cases} \]
\[ C_{m_m}(t, s) = 0, & \forall t, \forall s \quad (13) \]

Furthermore, the state dependent noise \( \mathbf{z}(t) \) is uncorrelated with both process and measurement noise terms:

\[ C_{z_n}(t, s) = 0, C_{z_m}(t, s) = 0, & \forall t, \forall s \quad (14) \]

**B. Linear Optimal Filter**

Based on the converted state equation (11) we now obtain a linear optimal filter that minimizes the expected squared error of population state estimation:

\[ J_t = E[(\mathbf{z}(t) - \mathbf{X}(t))^T (\mathbf{z}(t) - \mathbf{X}(t))] \quad (15) \]

where population state estimate \( \hat{\mathbf{X}}(t) \) is corrected by assimilating new output \( \mathbf{y}(t) \) recursively by

\[ \hat{\mathbf{X}}(t) = \hat{\mathbf{X}}(t|t-1) + \mathbf{K}(t)[\mathbf{y}(t) - \hat{\mathbf{y}}(t|t-1)] \quad (16) \]

Derivation of this linear filter is similar to the standard discrete Kalman Filter except for the state dependent nature of the random variable \( \mathbf{z}(t) \) that stems from stochasticity of the population Markov chain.

The a priori estimate of population state \( \hat{\mathbf{X}}(t|t-1) \) is given

\[ \hat{\mathbf{X}}(t|t-1) = \mathbf{A}(t-1)\hat{\mathbf{X}}(t-1) \quad (17) \]

and the output prediction \( \hat{\mathbf{y}}(t|t-1) \) by

\[ \hat{\mathbf{y}}(t|t-1) = \mathbf{H}(t)\hat{\mathbf{X}}(t|t-1) \quad (18) \]

Substituting (16), (17), and (18) into (15) and differentiating (15) with respect to Kalman gain \( \mathbf{K}(t) \) yields the same result as the standard Kalman Filter:

\[ \mathbf{K}(t) = \mathbf{P}(t|t-1)\mathbf{H}(t)^T (\mathbf{H}(t)\mathbf{P}(t|t-1)\mathbf{H}(t)^T + \mathbf{R}(t))^{-1} \quad (19) \]

Also, the update law of a priori state estimation error covariance \( \mathbf{P}(t|t-1) \) is obtained as:

\[ \mathbf{P}(t) = (\mathbf{I} - \mathbf{K}(t)\mathbf{H}(t))\mathbf{P}(t|t-1) \quad (20) \]

which is the same as the standard Kalman Filter, since the derivation does not require the population state equation (11).

The propagation of a posteriori error covariance \( \mathbf{P}(t) \), on the other hand, differs from the standard Kalman Filter, since it exploits the process model, i.e. the population state equation, for predicting \( \mathbf{P}(t+1|t) \). Let \( \varepsilon(t) \) be a priori state estimation error. From (17) and (11),

\[ \varepsilon(t+1) \triangleq \hat{\mathbf{X}}(t+1|t) - \mathbf{X}(t+1) = \mathbf{A}(t)\varepsilon(t) - \mathbf{z}(t) - \mathbf{G}(t)\mathbf{w}(t) \quad (21) \]

where \( \varepsilon(t) = \hat{\mathbf{X}}(t) - \mathbf{X}(t) \) is a posteriori estimation error.

From (13) and (14),
This method resembles the Extended Kalman Filter, but estimate $\hat{\chi}(t)$ is applied to computing covariance rather than linearizing a nonlinear system.

### ii) Conservative Propagation Covariance

Replacing the auto-covariance $Q_z(t; \chi(t))$ by its most conservative one. (See (35) below.) Since each component of population state is bounded by the total number of units, $0 \leq N_i \leq N$, there exists the worst case, i.e. the largest variance, for the auto-covariance of $\chi(t)$.

Using these approximate $\hat{Q}_z(t)$, the propagation law of estimation error covariance is given by

$$P(t+1|t) = \hat{A}(t)P(t)\hat{A}(t)^T + G(t)Q_m(t)G(t)^T + \hat{Q}_z(t)$$

(27)

### IV. CONVERGENCE PROPERTIES OF THE FILTER

In this section we examine convergence of estimation error covariance $P(t)$. If we assume no process noise $\{w(t)\}$ and a constant probability matrix $A$, we can apply the Perron-Frobenius Theorem to obtain the error covariance in equilibrium. Assuming that matrix $A$ is a matrix of positive elements whose columns sum to one, the Theorem guarantees:

- Matrix $A$ has a simple eigenvalue of 1, $\lambda_1 = 1$,
- The corresponding eigen vector $\pi_1 = [\pi_1^1, \ldots, \pi_1^n]^T$ has all positive elements, and
- All other eigenvalues are smaller than 1: $|\lambda_i| < 1, \forall i \neq 1$. (28)

This implies that the equilibrium probability for an agent to take agent state $i$ is given by $\pi_i^1$, and the expected population state converges to

$$\lim_{t \to \infty} E[\chi(t) | A] = N\pi_1$$

(29)

and the error covariance converges to

$$\lim_{t \to \infty} \text{Cov}[\chi(t) | A] = \{N\pi_i^1(\delta_i - \pi_i^1)\} \in \mathbb{R}^{m \times m}$$

(30)

When process noise $\{w(t)\}$ is present, it is more complex. However, the stable nature of $A$ matrix as well as the conservation of the total number of agents, $N$, allow us to guarantee that the error covariance is bounded. Consider that no assimilation of data is performed, namely, $P(t) = P(t | t-1)$. Then the covariance propagation and update laws reduce to

$$P(t+1) = \hat{A}P(t)\hat{A}^T + Q(t)$$

(31)
where \( \mathbf{Q}(t) = \mathbf{Q}_n(t) + \mathbf{G}(t)\mathbf{Q}_w(t)\mathbf{G}(t)^T \) is the covariance of the total process noise and transition stochasticity.

Let \( e_i \) be the \( i-th \) right eigenvector of matrix \( \mathbf{\Lambda}^T \):

\[
\mathbf{\Lambda}^T e_i = \lambda_i e_i, \quad i = 1, \cdots, m
\]

Transforming the covariance matrix using \( \mathbf{V} = [e_1, \cdots, e_m] \) yields,

\[
\Gamma(t) = \mathbf{V}^T \mathbf{P}(t) \mathbf{V}.
\]

The \( ij \) element of the transformed covariance matrix is given by

\[
\Gamma_{ij}(t+1) = e_i^T \mathbf{P}(t+1) e_j
= e_i^T \mathbf{\Lambda} \mathbf{P}(t) \mathbf{\Lambda}^T e_j + e_i^T \mathbf{Q}(t) e_j
= \lambda_i \lambda_j \Gamma_{ii}(t) + e_i^T \mathbf{Q}(t) e_j
\]

from (31) and (32). If the total process noise covariance is bounded as

\[
|e_i^T \mathbf{Q}(t) e_j| \leq \tilde{Q}_{ij}, \quad \forall t \geq 0,
\]

then all the elements of \( \Gamma(t) = \mathbf{V}^T \mathbf{P}(t) \mathbf{V} \), other than \( \Gamma_{11} \), converges to a bounded value:

\[
\lim_{t \to \infty} |\Gamma_{ij}(t)| \leq \frac{\tilde{Q}_{ij}}{1 - \lambda_i \lambda_j}
\]

For the special case, we can show that \( \Gamma_{11}(t) = 0, t \geq 1 \), assuming that the total number of agents is kept constant, \( N \). Therefore, \( \mathbf{P}(t) = \mathbf{V}^{-1} \Gamma(t) \mathbf{V}^{-1} \) is bounded. See more details in [10].

In summary the estimation error covariance is bounded despite perturbations by both process noise \( \mathbf{w}(t) \) and randomness of state transition \( \mathbf{z}(t) \).

V. APPLICATION TO STOCHASTIC RECRUITMENT CONTROL OF MUSCLE ACTUATORS

To demonstrate the application of Kalman filter estimators to ensembles of finite state machines, we will present a case study from the authors' work on recruitment-based shape memory alloy (SMA) actuators. Figure 2 shows an actuator composed of 60 individual SMA springs. Each SMA spring contracts to produce force when heated. The individual elements are arranged into functional units, as shown in Fig. 3, having a small finite state machine that governs whether or not the SMA material is heated. These finite state machines have two states, on and off. The force or stiffness of this actuator can be controlled by “recruiting” enough units into the on state that the desired output is produced.

Fig. 2. A 60 unit SMA actuator. Each unit is composed of a single helical wire that contracts when heated to produce force.

Fig. 3. Each unit is driven by Joule heating when the finite state machine controlling the unit is in the on state. State transitions from off to on and on to off are determined by a central controller.

Fig. 4. The transient response of the actuator is slow, taking approximately 5 seconds for the output of a unit to settle fully into the on state.

Control over this kind of actuator has been achieved by intentionally randomizing the behavior of the two-state machines governing each unit. A central controller measures the error between the present output and the desired output, and chooses state transition probabilities for the units that guarantee that the expected rate of state transitions is proportional to the number of units that need to be recruited with some constant of proportionality, \( \lambda \). This process of measuring the system output and broadcasting new commands occurs at regular time intervals, whose length is dominated by the characteristic response time of the SMA wires. Figure 4 illustrates the response rate of the actuator force, when the actuator is held at constant length. When a unit transitions from the off state to the on state, it takes approximately 5 seconds to reach steady state. When the time interval is long enough that 75% or so of the steady state force has been produced, the control system is relatively predictable and stable. However, it becomes difficult to control the actuator output when the control time...
interval is shortened much past that point. Figure 5 compares a plot of the actuator control system regulating the isometric force at a desired reference that changes with time. For a sampling time of 2.0 seconds, the response is smooth and repeatable. However, if the sampling time is set to 0.5 seconds, the tracking behavior becomes oscillatory and unreliable.

The root cause of the instability exhibited by the actuator lies with the inability of the controller to estimate the number of units presently in the on state based on the output. If the output of a unit that has just transitioned is one fifth of the steady state output, it is impossible to distinguish between a single unit that is at steady state and five units that have just begun to transition. As a result, the controller will either overcompensate for the error, causing the observed oscillatory behavior. In order to increase the sampling rate while avoiding instability, the control system must incorporate some dynamic model of the units' behavior, including the time history of commands sent to the actuator. The Kalman filtering framework presented here can be used to accomplish this.

Fig. 5. The top plot shows the feedback control system operating at a sampling interval of 2 s. The discretized model matches the actual output very well. The bottom plot shows the control system operating at a sampling interval of 0.5 s. The system dynamics are too slow, so the model predicts the true output poorly. The overshoot is a result of this error.

Fig. 6. A plot of the training data showing agreement between the augmented state output prediction and the measured force produced.

Fig. 7: The identified output model, $H$. The horizontal axis corresponds to each of the 20 discrete states in the augmented model.

Figure 9 shows the performance of the controller, which, as before, is choosing the transition probabilities so that the expected rate of transitions is proportional to the error. The sampling interval has been set to 0.5 seconds, identical to the conditions for the lower plot of Fig. 5. When the Kalman filter is used to estimate the number of on units, the performance is improved dramatically. The lower plot of Fig. 9 shows the estimated number of units in the on state, compared with ground truth that was available through a debugging interface for this testbed actuator. The estimate tracks the actual number of on units, except toward the end of the experiment when the actuator “sticks” due to static friction. Despite this error in estimation, the approximate
estimates of the actuator's ensemble state are good enough to ensure convergent behavior.

Fig. 8. This figure shows the empirically identified state transition graph. The solid lines correspond to the state transitions made if no active transitions are made.

Fig. 9. The top plot shows the tracking response of the system compensated with a Kalman filter. The bottom plot shows the filter's estimate of the number of on units, compared to ground truth. The constant error in the estimate (shown in red) reflects a constant disturbance due to friction. Note that the output converges in spite of this estimation error.

In summary, this experiment has demonstrated that it is possible to apply the Kalman filter to estimate the behavior of a population of finite state machines, whose actual, hybrid behavior is approximated using a series of discrete transient states. The Kalman filter successfully provides estimates of the number of units in the on state, which in turn are used to choose centralized commands that guarantee an expected rate of state transitions proportional to the number of units to be recruited. Using the better estimates of the population state, the rate of controller feedback sampling can be increased, allowing the control system to respond more quickly to measured disturbances.

CONCLUSION

This paper presented a Kalman filter approach to estimating the population state of inhomogeneous population Markov chains. The probabilistic state transition formalism originated in Markov chain modeling was recast as a standard state transition equation perturbed by an additive random process with a known distribution. An optimal linear filter was derived for the converted state equation; the resultant optimal filter is a type of Kalman filter with a modified covariance propagation law. Convergence properties were examined, and the state estimation error covariance is bounded despite both process noise and randomness of state transition. The state estimation method is applied to stochastic control of muscle actuators, where individual artificial muscle fibers are stochastically recruited with probabilities broadcasted from a central controller.

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