Fermionic measurement-based quantum computation

Yu-Ju Chiu, Xie Chen, and Isaac L. Chuang

Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA
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Fermions, as a major class of quantum particles, provide platforms for quantum information processing beyond the possibilities of spins or bosons, which have been studied more extensively. One particularly interesting model to study, in view of recent progress in manipulating ultracold fermion gases, is the fermionic version of measurement-based quantum computation (MBQC), which implements full quantum computation with only single-site measurements on a proper fermionic many-body resource state. However, it is not known which fermionic states can be used as the resource states for MBQC and how to find them. In this paper, we generalize the framework of spin MBQC to fermions. In particular, we provide a general formalism to construct many-body entangled fermion resource states for MBQC based on the fermionic projected entangled pair state representation. We give a specific fermionic state which enables universal MBQC and demonstrate that the nonlocality inherent in fermion systems can be properly taken care of with suitable measurement schemes. Such a framework opens up possibilities of finding MBQC resource states which can be more readily realized in the laboratory.

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I. INTRODUCTION

Quantum computation can be realized with different quantum mechanical particles, for example photons and spins. Fermions, as another major class of quantum particles, have been relatively less explored for their application in quantum computation and can lead to new possibilities. Although it is expected [1] that fermions have polynomially equivalent quantum computation power as spins or bosons, it is possible that subexponential speedups can be achieved with fermions over spins or bosons in certain computational tasks. For example, the quantum simulation [2,3] of fermionic many-body systems can be much more easily implemented with fermionic degrees of freedom due to the intrinsic sign issue in the simulation of fermions with spins or bosons.

The possibility of using fermions for quantum computation has been studied in a few contexts. It has been shown that the circuit-model quantum computation can be implemented with fermions, which efficiently simulates quantum circuits with spins [1]. On the other hand, topological quantum computation can be realized using certain two-dimensional fermion states with strong correlations [4]. In particular, it is known that the fractional quantum Hall state with filling fraction $\nu = 5/2$ can support universal topological quantum computation [5]. Moreover, quantum teleportation [6], an important quantum protocol for both quantum communication and quantum computation, has also been generalized to fermion systems [7].

The measurement-based quantum computation model [8] has been extensively studied in spin or boson systems [9,10] where quantum computation is implemented with only single spin or boson measurements on a proper many-body entangled resource state. Many spin or bosonic resource states are known [8,11–19], but a large-scale experimental realization has not been achieved. With the recent exciting experimental progress in manipulating ultracold fermion gases [20–22], it is then interesting to ask whether a similar computational scheme could be implemented in fermion systems, with only single-site measurements on a fermionic resource state which ideally can be realized in a controlled way with ultracold fermionic atoms. With the large variety of quantum states that exist in simple free fermion systems, like Fermi liquids, quantum Hall states, and topological insulators, a fermionic version of measurement-based quantum computation (MBQC) may provide new platforms for quantum information processing with reduced experimental complexity while at the same time enjoying the same advantage as in the spin MBQC model that no coherent quantum operations are needed to carry out the whole computation. In circuit model QC, it was found that charge detection in a free fermion system would enable universal quantum computation [23]. Similar construction for MBQC is highly desired.

However, no theory exists for fermionic MBQC, which studies which fermionic resource states are useful and which single-site measurement patterns are necessary to achieve universal quantum computation. Naively, one might expect that a direct Jordan Wigner mapping of spin resource states to fermions would give a useful fermionic resource state for MBQC, but this is not true as the mapping is nonlocal and local spin measurements on the resource state can no longer be implemented with local fermion measurements after the mapping. Moreover, one of the key properties wanted for a MBQC resource state is lost during this mapping. It is highly desirable to have the MBQC resource states be the ground states of local Hamiltonians, and many spin resource states are designed to have this property [14–19]. Unfortunately, this property is not preserved by the nonlocal mapping to fermions and the resulting fermion states can no longer be generated by engineering the appropriate local Hamiltonian terms in the system and then lowering the temperature. Therefore, different approaches are needed to construct a useful fermionic MBQC model.

In this paper, we establish a general framework for studying fermionic MBQC based on the fermionic projected entangled pair states (fPEPS) representation [24,25], which is known to describe ground states in local fermion systems [26], by generalizing the idea of designing spin MBQC resource states based on the spin PEPS representation [12,13,27,28] to fermion systems. We present a simple explicit construction of a fermionic resource state together with the single-site...
measurement patterns necessary to realize universal quantum computation. By encoding the quantum information to be processed into pairs of local fermion modes, we demonstrate how universal quantum computation can be achieved on a fermionic state with only single-site measurements. Starting from this explicit construction, we discuss how general fPEPS can be used as resource states for fermionic measurement-based quantum computation (fMBQC) with all possible kinds of encoding schemes and entanglement structures of the state. The fMBQC formalism is the central result of this paper and provides the basis for the search of fermionic resource states with simpler measurement schemes and as the ground states of more easily realizable local Hamiltonians than current MBQC schemes.

The paper is organized as follows: In Sec. II, we review how MBQC is done in spin systems. In particular, we focus on the interpretation of the measurement operation on the resource state as teleportation in the virtual space in the PEPS representation, which is crucial for our generalization to fermion systems. In Sec. III, we start from the basic building block of MBQC teleportation and show how it can be realized in fermion systems. Putting the teleportation steps together, we obtain a simple fermionic resource state and demonstrate in detail how each step in MBQC can be realized on such a state. In Sec. IV, we present the general formalism of fMBQC based on fPEPS with all possible types of encoding schemes and entanglement structures of the state. The discussion in this section goes beyond the possibilities offered by the simple example constructed in Sec. III. Finally, we conclude and discuss future directions in Sec. V.

II. REVIEW: MEASUREMENT-BASED QUANTUM COMPUTATION

In this section, we review the MBQC scheme, which claims that universal quantum computation can be achieved with only single-site measurement operations on a many-body entangled spin state. We focus on the interpretation by Refs. [12,27,28] of the computational power of resource states in terms of their projected entangled pair state (PEPS) representation [29,30]. In such a representation, measurements on the physical lattice sites in the resource state correspond to teleportation steps in the virtual space of entangled pairs, which can then be composed and designed to simulate full quantum circuits.

Following this logic, we first review how teleportation with entangled pairs can perform a universal set of unitary gates while transmitting the information. We then discuss following Ref. [12] how such teleportation steps occur in the virtual space of the PEPS representation when measurement operations are performed on individual spins in a many-body entangled resource state. This line of thought allows us to generalize the MBQC scheme from spins to fermions.

A. Teleportation

Teleportation is a way of achieving quantum computation by doing multispin measurements and it can be thought of as the basic building block for MBQC which realizes the universal quantum computation with single-spin measurements. Originally, teleportation was discovered as a way of transmitting information [6], that when a measurement is done in the Bell basis, information stored at one place can flow to another. Later on, it was found that not only the information but also some extra gates can be teleported as well if measurements are done in a modified Bell basis [31].

First, to see how teleportation transmits information, consider the following setup that involves an input qubit $|\psi\rangle$ to be teleported and an entangled pair $|E\rangle = |00\rangle + |11\rangle$ (suppressing normalization) shared between the input end and the output end. The information in $|\psi\rangle$ is transmitted to the output end when one measures $|\psi\rangle$ and half of the entangled pair jointly. Furthermore, by choosing different measurement bases, different gates can be performed on $|\psi\rangle$ while it is teleported to the output end.

To see this, consider an input qubit

$$|\psi_1\rangle = |m_0\rangle |0\rangle + |m_1\rangle |1\rangle,$$  

with $|m_0|^2 + |m_1|^2 = 1$ and an entangled pair $|E_{23}\rangle$, as shown in Fig. 1. Qubits 1 and 2 belong to the input end. Qubit 3 belongs to the output end.

The total wave function of the system is

$$|\psi_{123}\rangle = (m_0 |0\rangle + m_1 |1\rangle) \otimes (|00\rangle + |11\rangle).$$  

If we measure qubits 1 and 2 in the Bell basis $|\phi_1\rangle = \sigma_\alpha \otimes I(|00\rangle + |11\rangle)$, where $\sigma_\alpha$ are Pauli matrices, the wave function of the unmeasured qubit 3 results in

$$|\phi_{12}\rangle = \sigma_\alpha |m_0\rangle |0\rangle + m_1 |1\rangle).$$

Thus, one can see that the original information is teleported from qubit 1 to qubit 3 with possible extra Pauli operations.

In general, teleportation not only transmits information, but also implements gates at the same time, as the Pauli gates seen in the example above. Measuring in a basis of the generic form

$$|\phi_{12}\rangle = (U^\dagger \sigma_\alpha) \otimes I(|00\rangle + |11\rangle),$$

with $U$ is any one-qubit unitary gate, yields at the output

$$|\phi_{12}\rangle = \sigma_\alpha U |m_0\rangle |0\rangle + m_1 |1\rangle.$$

For example, we can choose $U$ as the Hadamard operation $H = |0\rangle \langle + | + |1\rangle \langle - |$, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, and the phase gate $Z(\theta) = e^{-i\theta \sigma_z} |0\rangle \langle 0| + e^{i\theta} |1\rangle \langle 1|$ to implement the corresponding one-qubit gates during teleportation.

In addition to one-qubit gates, the requirement of universal quantum computation also involves certain two-qubit gates, for example, the controlled-Z gate together with the Hadamard gates on the two qubits [32], which we denote as $U_{ph}$. Specifically,

$$U_{ph} = |00\rangle \langle + + | + |01\rangle \langle + - | + |10\rangle \langle - + | - |11\rangle \langle - - |.$$  

FIG. 1. (Color online) Teleportation of one-qubit unitary gates. The qubits are denoted by dots and the entangled pair is depicted by a dashed line. The big circle represents the input end and qubits inside it are measured together. After measuring qubits 1 and 2 together, information in qubit 1 flows to qubit 3.
The schematic for teleporting $U_{ph}$ is depicted in Fig. 2. It can be interpreted as a generalized version of teleportation which involves a two-qubit input and three entangled pairs [27].

It can be checked that measuring qubits 1, 2, and 3 in
\[
(\sigma_x^1)^y \otimes (\sigma_x^2)^y \otimes I((0++) \pm |1-\rangle)
\]
and measuring qubits 4, 5, and 6 in
\[
(\sigma_x^4)^z \otimes (\sigma_x^5)^z \otimes I((00+) \pm |11-\rangle),
\]
with $i,j,k,l = 0,1$, teleports $U_{ph}$ to qubits 7 and 8 up to Pauli operations on 7 and 8 separately. Hereby, one can see that teleportation is indeed a way of realizing universal quantum computation with multiqubit measurements.

**B. Projected entangled pair states and spin measurement based quantum computation**

A PEPS [30] is a way of expressing a many-body entangled state as a projection from a product of maximally entangled pairs. PEPS was first invented to study many-body systems in condensed-matter theory; however, it turned out to be useful for understanding the power of resource states in MBQC [12,27]. More explicitly, if we imagine the maximally entangled pairs in PEPS as in a virtual space where teleportation can be achieved and interpret the physical Hilbert space of spins as a projection from a product of maximally entangled states. PEPS was first invented to study many-body systems in condensed-matter theory; however, it turned out to be useful for understanding the power of resource states in MBQC [12,27].

For simplicity, let us first consider PEPS on a one-dimensional chain. As depicted in Fig. 3, a spatially one-dimensional (1D) virtual space is a chain consisting of maximally entangled pairs shared between nearest-neighbor sites. With $D$-dimensional virtual spins, the maximally entangled pairs are in state $\sum_{i=0}^{D-1} |ii\rangle$. At the left and right end of the chain, there are boundary states $|L\rangle$ and $|R\rangle$. On every site, there are two virtual spins, each being half of an entangled pair connecting neighboring sites. Shortly, we discuss how virtual and physical spins are related via a projection on each site.

The wave function for the virtual chain follows
\[
|\psi_{\text{PEPS}}\rangle = |L\rangle \prod_{k=1}^{N-1} \left( \sum_{i=0}^{D-1} |ii\rangle_{k,k+1} \right) |R\rangle,
\]
where $k$ labels different sites in the chain.

A PEPS with $d$-dimensional physical spins is obtained by a local projection $P$ on virtual spins located on each site in the virtual space which maps the virtual space to physical space. In Fig. 3, the projections are presented as circles. Note that the projection here is only a map between two Hilbert spaces and is not the usual sense of projection that needs to obey $P^2 = P$. More specifically,
\[
|\psi_{\text{PEPS}}\rangle = \prod_{k=1}^{N} P_k |L\rangle \prod_{k=1}^{N} \left( \sum_{i=0}^{D-1} |ii\rangle_{k,k+1} \right) |R\rangle,
\]
with local projection operators on each site defined as
\[
P_k = \sum_{l=0}^{d-1} \sum_{r=0}^{d-1} |\tilde{l}\rangle A^i_{k,\ell r} \langle \ell r|.
\]

Note that each $|\tilde{l}\rangle$ is a state in the $d$-dimensional physical space and $A^i_{\ell r}$’s are coefficients of $P$ that depend on $|\tilde{l}\rangle$ in the physical space that one wants to project onto. Also note that the physical dimension refers to the internal degrees of freedom of spins and is different from the spatial dimension of the lattice.

Using Eqs. (9) and (10), one can show that the wave function of a 1D PEPS with $N$ sites can be expressed as
\[
|\psi_{\text{PEPS}}\rangle = \sum_{\tilde{i}0}^{d-1} (L|A^i_{1,\tilde{i}0} A^i_{2,\tilde{i}0} \cdots A^i_{N,\tilde{i}0} |R\rangle |\tilde{i}_1\tilde{i}_2\cdots\tilde{i}_N\rangle.
\]

The PEPS construction provides a perspective to see the relation between MBQC and teleportation [12]. Here we give an explicit example that illustrates this idea. Consider measuring site 1 and projecting it in state $|\phi\rangle = a|0\rangle + b|1\rangle$ ($d=2$). After the measurement, the wave function of the unmeasured physical spins becomes
\[
\langle \phi | \psi_{\text{PEPS}} \rangle = \sum_{\tilde{i}0}^{d-1} \langle L|(a^* A^0 + b^* A^1) A^i_{2,\tilde{i}0} \cdots A^i_{N,\tilde{i}0} |R\rangle \\
\times |\tilde{i}_2\tilde{i}_3\cdots\tilde{i}_N\rangle.
\]

The form of the state remains unchanged while the left boundary gets teleported to site 2 and is changed into $\langle L|(a^* A^0 + b^* A^1)$, which can be viewed as a gate acting on the old boundary. From Eq. (10), we learned that the physical and virtual space are related by the projection. Given a certain $P$, we can relate measurements in the physical space to teleportation in the virtual space as well as changes in lattice boundary to unitary operations teleportated in the virtual space. Quantum computation could therefore be achieved in the virtual space by choosing appropriate measurement bases.
in the physical space that correspond to bases for teleporting a universal set of gates in the virtual space.

In the above, we have seen that unitary operations on one spin could be realized with MBQC on a 1D PEPS. With suitably chosen projection \( P \), arbitrary single-spin operations could be implemented. Yet, for the universality of quantum computation, entangling operations between two spins must also be feasible; thus, a more general 2D lattice is required for MBQC. For this purpose, a 2D PEPS as shown in Fig. 4 can be constructed similarly. The only differences are that the virtual space now contains both vertical [between sites \((i, j)\) and \((i + 1, j)\)] and horizontal [between \((i, j)\) and \((i, j + 1)\)] entangled pairs \( \sum_{i=0}^{D-1} |ii\rangle \), and the on-site projection \( P_{ij} \) becomes

\[
P_{ij} = \sum_{l=0}^{D-1} \sum_{r,d=0}^{D-1} |\tilde{i}, \tilde{j}\rangle A_{ij,ulrd}^\dagger |ulrd\rangle, \tag{12}
\]

which is almost the same as the 1D projection in Eq. \((10)\), except that the number of virtual indices is doubled. Note that \( \tilde{i} \) denotes the state of a physical spin as defined in Eq. \((10)\).

One of the best-known universal resource states for MBQC is the 2D cluster state, with on-site projection \( P_{ulrd} = |0\rangle\langle 0| + |+\rangle\langle +| + |\tilde{1}\rangle\langle \tilde{1}| \). It can be checked that measurement on this projected space can be used to input information, implement the universal set of gates discussed in Sec. II A, correct possible computation by-products due to measurement randomness, and finally read out the computational result \([27] \). Therefore, the 2D cluster state can serve as a resource state for MBQC. Various other resource states can be constructed using this framework \([12, 13, 33]\), but a general rule is still missing for determining whether a PEPS can be used as a resource state or not.

III. A SIMPLE RESOURCE STATE FOR FERMIONIC MBQC

Many-body fermion systems exist naturally in available experimental settings and it would be nice to generalize the mechanism of MBQC to fermions. Our goal in the next sections is to establish a general framework for studying fMBQC based on the fermionic PEPS representation \([24, 25]\), an analog of PEPS in a spin system. In achieving such a goal, we first build a simple resource state in this section for fermionic MBQC, starting from simple building blocks of individual fermionic teleportation steps. This example serves as a proof of principle that universal fermionic resource state for MBQC exists and as an intuitive introduction to fermionic MBQC in many-body fermion state. In the next section, we move on to the discussion of the more general settings of fermionic MBQC based on the fermionic PEPS formalism, which goes beyond the construction presented for the simple example in this section.

We build the proof-of-principle example with a similar procedure as that used in spin MBQC: We first construct fermionic teleportation steps for implementing a universal set of gates and then use the fPEPS representation to map teleportation in the virtual space to single-site measurements in a physical state. However, fermions are very different from spins in two specific ways: \((1)\) fermion operators anticommute with each other and fermion wave functions are antisymmetric; \((2)\) the total parity of a fermionic system is always preserved. This is the so-called “superselection” rule of fermions and it also restricts measurement basis states to have fixed fermion parity. Therefore, care must be taken in mapping from spins to fermions and the generalization is far from direct. In this section, we discuss a fermionic version of teleportation, give a way to achieve universal quantum computation with fermionic teleportation, and then combine them into a full fermionic MBQC scheme. We would like to note that while the construction scheme of this example illustrates the general idea of processing quantum information through measurements on a fermionic state, many of the tricks (like the encoding scheme) used here are not necessary in the most general settings discussed in the next section.

A. Fermionic teleportation

In quantum computation with spins, the analogy to the bits 0 and 1 in classical computation is the two-level spin up and down states \(|0\rangle, |1\rangle\), or qubits. In the fermionic case, the information is encoded in the wave function of local fermionic modes. It seems straightforward to define the two-level states by the occupation number of the modes. Namely, the analogy of \(|0\rangle\) is a state with no fermion in a mode, or vacuum \(|\Omega\rangle\), and that of \(|1\rangle\) is \(a^\dagger |\Omega\rangle\), where \(a^\dagger\) is the creation operator for a fermion mode. The maximally entangled spin state \(|00\rangle + |11\rangle\) can be replaced, accordingly, by two entangled modes defined as \((1 + a^\dagger a) |\Omega\rangle\).

However, this naive mapping fails as one attempts to do fermionic teleportation with the configuration shown in Fig. 1, where each dot now represents a fermion mode. As fermion parity of a system is always preserved, the input mode 1 cannot be in a superposition state of \(|m_0 + m_1 a^\dagger\rangle |\Omega\rangle\). In order to deal with this problem, we take a route similar to that in Refs. \([1, 7]\) by adding extra modes and encoding information in a fixed-parity sector. As depicted in Fig. 5, we add an extra mode and a second pair of entangled modes, such that the input defined in Eq. \((1)\) becomes \(|\psi\rangle = (m_0 |\Omega\rangle + m_1 a^\dagger |\Omega\rangle_{13}\), which has a definite even parity.

To check the feasibility of this strategy, we show in the following that the input in modes 1 and 3 can be teleported to
modes 5 and 6: The total wave function of the system is
\[
|\psi\rangle = \frac{1}{m_0} \sum_{a=0}^{1} m_a (a_1^\dagger a_2^\dagger)^{a_1 a_2} (a_3^\dagger a_4^\dagger + 1) (a_5^\dagger a_6^\dagger + 1) |\Omega\rangle. \tag{13}
\]

A measurement of modes 1–4 which projects them in state \(|\phi\rangle = (\Omega (1 + a_1 a_2 a_3 a_4)) |\phi\rangle\) results in state \(|\psi\rangle = (m_0 + m_1 a_1 a_2 a_3 a_4) |\Omega\rangle\) on modes 5 and 6, which is exactly what is desired.

This is the simplest case of a teleportation circuit and illustrates the general strategy we take to deal with the special property of fermions: (1) a proper ordering of all fermion modes needs to be given at the beginning and carried throughout the whole scheme; (2) information is encoded in a fixed-parity sector and all operations have fixed parities. In the following, we apply these strategies to the general cases. First we need to specify the encoding scheme of general quantum circuits into fermion modes.

1. \(n \rightarrow 2n\) encoding scheme

Due to the parity constraint discussed above, extra modes are needed when encoding spin states into fermion modes to preserve the total fermion parity of a system. Various encoding schemes have been proposed \cite{1} which satisfy this constraint. For discussion in this section, we choose to encode one qubit into two fermionic modes, or more generally, \(n\) qubits into \(2n\) fermionic modes. Similarly, a “dual-rail” encoding scheme has also been widely utilized in previous studies of bosonic or fermionic \cite{9,10,23,34} quantum computations where each qubit is encoded in the occupation of either one of two spatial, spin, or polarization modes. We would like to note that this choice of encoding scheme is arbitrary and we use it here so that the fermionic model can be better understood from the usual spin or bosonic MBQC model. In Sec. IV, we discuss general encoding schemes and fMBQC iPEPS with general entanglement structures.

As illustrated in Figs. 5 and 6, in our scheme, a “parity” mode is assigned to every “info” mode containing the real information to ensure that the total parity of an info mode and the auxiliary parity mode is always fixed. Thus, a spin system with \(n\) qubits in state \(|\psi_n\rangle = \sum_{\{a_i\}} m_{\{a\}} a_1 a_2 \ldots a_n |\Omega\rangle\), where \(\{a\} = \{a_1, a_2, \ldots, a_n\}\), is encoded into a fermionic state with \(2n\) modes, \(|\psi_f\rangle = \sum_{\{a\}} m_{\{a\}} (a_1^\dagger a_2^\dagger)^{a_1 a_2} \ldots (a_5^\dagger a_6^\dagger + 1) |\Omega\rangle\), where \(i\) is the parity mode of the info mode. Note that here we have chosen the order of the modes such that fermionic operators \(a_i^\dagger\) always appear in front of \(a_j^\dagger\) for \(i < j\), and the fermion parity of the state shall always be even. It might seem that, because of the dual-rail pairing, the fermion degrees of freedom become bosonic and we can implement MBQC as we did with spin or boson systems. However, this is not true, because measurement randomness can change fermion parity and result in computational by-products which are fermionic. We address the issue of fermionic by-products in Sec. III C, where we show that such by-products can be properly taken care of by keeping a fermionic “frame” together with the Pauli frame for the by-products as in the spin or bosonic MBQC schemes.

As a spin state with \(n\) qubits is encoded into a fixed-parity fermionic state with \(2n\) modes, spin gates must also be redesigned accordingly so that an \(n\)-qubit spin operator is encoded into a \(2n\)-mode parity-preserving fermion operator and the universal set of spin gates are mapped to a set of fermionic gates which possess the same universality.

A generic one-qubit unitary spin operator
\[
U = U_{00}|0\rangle\langle 0| + U_{10}|1\rangle\langle 0| + U_{01}|0\rangle\langle 1| + U_{11}|1\rangle\langle 1| \tag{14}
\]
is encoded into a two-mode fermionic gate (where mode 1 is the info mode and mode 2 is the parity mode of mode 1):
\[
U_f = U_{00} a_1^\dagger a_2^\dagger a_2 a_1^\dagger + U_{10} a_1^\dagger a_2^\dagger a_1 + U_{01} a_1 a_2 + U_{11} a_1^\dagger a_1 a_2^\dagger a_2.
\]

With this encoding, we have the two-mode fermionic phase gate,
\[
Z_f(\theta) = a_1 a_2 a_2^\dagger a_1^\dagger + e^{i\theta} a_1^\dagger a_1 a_2 a_2^\dagger, \tag{15}
\]
and the fermionic Hadamard gate,
\[
H_f = a_1 a_2 a_2^\dagger a_1^\dagger + a_1^\dagger a_2 a_2 + a_1 a_2 - a_1 a_1 a_2^\dagger a_2^\dagger, \tag{16}
\]
which can be composed to simulate arbitrary unitary gates on a single qubit.

Similarly, the two-qubit \(U_{ph}\) is mapped to \(U_{ph,f}\) on four consecutive fermionic modes, modes 1, 1p, 2, and 2p, where 1 and 2 are control and target modes and 1p and 2p are parity modes of 1 and 2, respectively. For simplicity, here we denote fermionic states \(|\Omega\rangle, a_1^\dagger a_2^\dagger |\Omega\rangle\), and \((1 + a_1^\dagger a_2^\dagger)|\Omega\rangle\) as \(|\Omega_f\rangle, |1_f\rangle, \) and \(|00\rangle + |11\rangle\), where we always order the modes as 1, 1p, 2, 2p, etc.,
\[
U_{ph,f} = |0000\rangle f(00) + |11\rangle f(00) + |11\rangle f(10) + |0011\rangle f(00) + |11\rangle f(00) - |11\rangle f(11) + |1100\rangle f(00) - |11\rangle f(00) + |11\rangle f(11) - |1111\rangle f(00) - |11\rangle f(00) - |11\rangle f(11). \tag{17}
\]
Therefore, with Eqs. (15), (16), and (17), a universal set of fermionic gates for fermionic quantum computation is constructed and can be used to simulate the universal set of spin gates for the original spin quantum computation. Note that the gates discussed here only act on the even fermion parity sector and are unitary only within this sector. However, as information is encoded fully in this sector, these gates are sufficient for quantum computation and are implemented in the MBQC scheme described below. Unlike in the fermionic circuit model of quantum computation [1], where fully unitary fermionic gates are necessary, in fermionic MBQC simulating such quasunitary operations are sufficient and can be readily realized. The odd fermion parity sector contributes to the fermionic MBQC scheme as computational by-products when the measurement result falls into this sector. As we show below, such computational by-products can be properly dealt without destroying the universality of the computation scheme.

2. Fermionic teleportation for a universal set of gates

Now that we have defined the encoding of states and the mapping between gates, in the following, we show that the universal set of fermionic gates can be implemented by measuring entangled fermionic states in certain bases, thus achieving universal quantum computation with fermionic teleportation. Note that in the discussion of this section, we always assume that each pair of info mode and parity mode always have even parity. The occurrence of odd parity pairs is considered as computational by-products later.

The schematic for teleporting an arbitrary two-mode parity-preserving fermionic gate $U_f$, the equivalent of a one-qubit spin gate, is shown in Fig. 5. By comparing Figs. 1 and 5, we can see that the number of inputs as well as that of entangled pairs are both doubled in the fermionic case. The wave function of the state that corresponds to Eq. (2) in the spin case is given in Eq. (13). It can be checked that the measurement of modes 1–4 which projects them in state

$$|\psi\rangle_f = (U_{00} + U_{10}a_4^\dagger a_4^\dagger + U_{01}a_3^\dagger a_3^\dagger + U_{11}a_2^\dagger a_2^\dagger a_3^\dagger a_4^\dagger)|\Omega\rangle$$

(18)
teleports $U_f$ to modes 5 and 6.

As for teleporting the four-mode controlled operation $U_{ph:f}$, the setup depicted in Fig. 6 is utilized.

The wave function of this state is

$$|\psi\rangle_f = \sum_{\alpha, \beta} m_{\alpha\beta} (a_3^\dagger a_3^\dagger) \langle \beta | (1 + a_3^\dagger a_4^\dagger) | (1 + a_3^\dagger a_4^\dagger) \rangle_{\Omega}$$

$$\times (1 + a_3^\dagger a_4^\dagger) (1 + a_3^\dagger a_4^\dagger) |(1 + a_3^\dagger a_4^\dagger) \rangle_{\Omega}.$$  

(19)

One can check that $U_{ph:f}$ can be teleported by measuring modes 1–6 of the top site and projecting them in

$$|\phi\rangle_f = (1 - a_4^\dagger a_4^\dagger - a_3^\dagger a_3^\dagger + a_2^\dagger a_2^\dagger a_3^\dagger a_4^\dagger + a_3^\dagger a_4^\dagger a_4^\dagger a_5^\dagger + a_3^\dagger a_4^\dagger a_5^\dagger a_6^\dagger + a_3^\dagger a_4^\dagger a_5^\dagger a_6^\dagger) |\Omega\rangle$$

(20)

and modes 7–12 of the bottom site in

$$|\phi\rangle_f = (1 - a_4^\dagger a_4^\dagger + a_3^\dagger a_3^\dagger a_4^\dagger a_5^\dagger - a_3^\dagger a_4^\dagger a_5^\dagger a_6^\dagger + a_3^\dagger a_4^\dagger a_5^\dagger a_6^\dagger) |\Omega\rangle.$$  

(21)

Hereby, we have successfully found a measurement basis corresponding to a universal set of gates and have shown that universal quantum computation can be achieved by teleportation with fermions.

However, our consideration so far is oversimplified as we have assumed that the computation always occurs in the even fermion parity sector and the measurements always result in the basis we want. In fact, measurement errors always occur as we cannot choose which particular basis among a complete set to measure in. Measurement errors in teleportation steps lead to unwanted by-product operations being teleported. For fermion states, it is also possible to change the parity sector of the states, which seems to pose a serious problem for our scheme. We address these issues in the following sections and show that they can be properly taken care of and will not impede our ability to do MBQC. We refer to the extra operations teleported as “by-products” instead of “errors” to emphasize that the former is due to the intrinsic randomness of quantum mechanics and cannot be avoided while the latter is due to noise and perturbation and can, in principle, be reduced.

B. A simple example of fermionic resource state

Even though we have demonstrated the viability of fermionic teleportation for individual gates in the previous section, our ultimate goal is to show that a circuit consisting of multiple operations can be simulated with local measurements or, in other words, to achieve fMBQC. Thus, it is necessary to have a fermionic lattice state similar to the spin lattice state in Fig. 4 which allows multiple steps of measurements as the information flows from one place to another.

In this section, we first assemble the teleportation steps and give a simple yet universal example of resource state for fMBQC. We examine in detail the possible by-product operations that occur in the measurement process and show how they can be taken care of with proper measurement schemes. The construction in this section is in close analogy to the bosonic or spin model discussed in Ref. [27]. We then discuss the more general PEPs formalism which, like PEPS for spin, allows more possibilities for finding novel fermionic resource states.

Here we demonstrate that fMBQC can be achieved using a special fermionic resource state on the lattice shown in Fig. 7. Like what we have seen in fermionic teleportation, the number of input and entangled pairs are doubled in the fermionic case compared to the spin case; thus, the spin lattice for MBQC (shown in Fig. 4), which has three or four qubits on every site corresponds to the fermion lattice in Fig. 7, which has six or eight modes per site and two entangled pairs connecting neighboring sites.

The lattice consists of input mode pairs $\alpha\alpha$’s on the left boundary and entangled pairs ($\beta^i_j, \alpha^i_j, a_{i,j+1} + 1)\Omega$ and ($\beta^i_j, \alpha^i_j, a_{i,j+1} + 1)\Omega$ for horizontal bonds, and ($\gamma^i_j, \delta^i_j, a_{i,j+1} + 1)\Omega$ and ($\gamma^i_j, \delta^i_j, a_{i,j+1} + 1)\Omega$ for vertical bonds. The labeling of modes on a site is shown in Fig. 8. When writing the bonds, we define the ordering of sites on the lattice as left ($i,j$) to right ($i,j+1$) and top ($i,j$) to down ($i+1,j$). This state can be thought of as a fermionic PEPS with a trivial projection on each site. In the following we think of all the modes on each
site as one big degree of freedom and discuss how MBQC can be implemented with single-site measurements on this state.

To see the feasibility of fMBQC in this example, we give in detail the procedure to implement each necessary step in fMBQC on this state.

(i) Assume without loss of generality that the input modes on the left boundary are all initialized in $|\Omega\rangle$ and that measurements are performed on the sites in the first column from top to bottom and then column by column from left to right so that the information flows to the right.

(ii) Just like in MBQC for spins, the lattice is initially entirely entangled. As one wants to achieve certain operations, for example, two-mode gates or the four-mode controlled operation $U_{ph-f}$, which involve only one or two entangled rows, one needs to isolate the rows and decouple them from other rows. To isolate a row, we remove its entanglement with the neighboring upper and lower rows by measuring modes on the sites of the upper and lower rows in the occupation number basis

$$\delta |n_1\delta' \delta'\rangle \alpha |n_2\alpha' \alpha'\rangle \beta |n_3\beta' \beta'\rangle \gamma |n_4\gamma' \gamma'\rangle |\Omega\rangle$$

for all $n = 0, 1$. Apply this measurement wherever necessary to prepare the lattice for the implementation of a particular circuit.

(iii) After partially decoupling the lattice when necessary in the way introduced above and using the results from Eqs. (18), (20), and (21), we see that we can implement a universal set of fermion gates by single-site measurements on this state.

However, this is not enough to claim universality for MBQC as we have not considered the effect of measuring in a basis other than the desired one. We discuss how to deal with the computational by-products introduced by the randomness in fermionic measurements in the next section.

(iv) We finally read out the output on the right boundary by measuring the sites in the occupation number bases. Therefore, for fermions, we just measure the rightmost column in $(\alpha' \alpha'')^n |\Omega\rangle$ or $\alpha |n_\alpha \alpha' |\Omega\rangle$, with $n_\alpha = 0, 1$ to yield the results.

C. Dealing with measurement randomness in the simple model

In this section, we address the effect of measurement randomness in our scheme. As discussed in the previous section, fMBQC could, in principle, be achieved with measurements which project in certain states; however, measurement results in orthogonal bases that span the rest of the Hilbert space may lead to by-products to the simulated operation. This can be viewed as the fermionic analog of the Pauli by-products that emerge when a Bell measurement is performed. In general, we cannot choose which basis state results from the measurement and whenever a measurement is done, a by-product occurs. Therefore, dealing with by-products becomes a necessity to make sure that there is a finite probability of simulating the wanted operation in order to achieve efficient quantum computation.

So far in our discussion, we have used two important assumptions: (1) We required that the pair of input modes on a site always have even parity as we designed one mode as the parity mode of another and (2) we only mentioned the projection basis state that gives rise to the desired answer without discussing other orthogonal bases that would potentially produce by-products. In general, the fermion parity constraint only requires that the total parity of modes be fixed. Therefore, the input mode pairs could also have odd parity, for example, $m_\alpha \alpha' + m_\beta \beta' |\Omega\rangle$. Similarly, there are no other constraints on the measurement bases as long as the total parity is fixed. As a result, it may seem that the choice of bases is arbitrary, leading to all kinds of by-products in the simulated operation. Yet, for the consistency of our scheme, we choose a complete set of measurement bases in which the mode pairs $(\alpha\alpha', \beta\beta', \gamma\gamma')$ each have a fixed parity, which could be either odd or even. Therefore, depending on the parity of the measurement bases, we could characterize the by-products into two categories.

(i) Parity-preserving by-products: by-products that come from measurement results which preserve the parity of the input (i.e., with an even parity measurement basis state), such that the parity of the output is the same as the input. These parity-preserving local fermionic operations can be mapped to local spin operations and could be corrected locally as spins. To see this more explicitly, we assume on a 1D chain with input of even parity that measurement which projects in the state $(1 + \alpha' \alpha'')^n |\Omega\rangle$ simulates the desired operation. Other orthogonal bases that preserve the parity are $(1 - \alpha' \alpha'')^{n_\alpha} |\Omega\rangle$, $(\alpha' \alpha'')^{n_\beta} |\Omega\rangle$, and $(\alpha' \alpha'' - \beta' \beta'') |\Omega\rangle$. It is obvious that the by-products on the output for these bases are the fermionic equivalent of
Pauli $Z$, $X$, and $Y$, respectively. In our simple example, such by-products can all be incorporated into the next operation to be teleported and hence get corrected.

(ii) Parity-violating by-products: by-products that emerge in measurement results which change the parity of the input (i.e., with an odd measurement basis state), such that the parity of the output is the opposite of the input. Using the example above, the orthogonal bases in the odd sector are $(\alpha^\dagger \alpha \beta^\dagger \beta \pm \beta^\dagger \beta)^\Omega$ and $(\alpha^\dagger \alpha \beta^\dagger \beta \pm \beta^\dagger \beta)^\Omega$. To keep the information flow, a corresponding encoding of spin states into the odd parity sector needs to be defined, for example, by requiring that the parity mode always has the opposite parity to that of the info mode. Moreover, this type of by-products are not the typical spin by-products. Instead they implement odd fermionic operations on the input, which maps back to nonlocal spin operations. Nevertheless, since in our scheme we have required a fixed parity on each mode pair, the nonlocal part of an odd operation only contributes an overall ($\pm 1$) to the total state, and therefore we only need to worry about the local part which is correctable by local measurements. Note that this is a special property of this example. Generally, by-products from odd measurement basis states are nonlocal, and we discuss the general case in the next section.

IV. GENERAL FPEPS CONSTRUCTION FOR FMBQC

We now present a general formalism for fMBQC by building on the fPEPS representation and using ideas illustrated by the simple examples of the previous sections. In the previous section, we demonstrated that fMBQC is feasible, in principle, on a 2D lattice. While under the $n \to 2n$ encoding scheme the simple example is in close analogy to the spin or bosonic MBQC resource state known previously [8–10] and follows closely the construction in Ref. [27]; in general, fMBQC does not have to take this restricted form. Moreover, in this model, the on-site measurements involve many degrees of freedom and the resource state may not be readily realizable. Our ultimate goal is to find a resource state which contains few modes per site and is the unique gapped ground state of a simple Hamiltonian, for example, a free fermion Hamiltonian.

This simple model provides a good illustration of the connection between fPEPS and fMBQC, and in this section we present the general formalism to study fMBQC based on fPEPS by understanding how measurements on many-body fermion states simulate unitary quantum circuits in the virtual space of the fPEPS representation. Note that the general construction we present in this section goes beyond the discussion in the previous section for the simple example. For example, we consider a fPEPS state with arbitrary number of bonds in each direction and more general encoding schemes can be used as long as they satisfy the fermion parity constraint.

First, we review the fPEPS formalism, based on which we discuss general information flow and processing in fPEPS with single-site measurements and how to deal with measurement randomness during the process.

A. Review of fPEPS formalism

First, we review the fPEPS formalism [24,25]. fPEPS, like PEPS for spin, represents many-body fermion states as projections from entangled virtual fermion pairs. A 2D fPEPS is obtained from a lattice of fermionic entangled pairs (for example, as shown in Figs. 9 or 7) by projecting the fermion modes on each site to a smaller physical Hilbert space. For the simple example given above, the projection is trivial on each site. In a general fPEPS state, the boundary modes and the entangled modes between sites $\alpha$, $\beta$, $\gamma$, $\delta$ are only virtual and we denote the physical modes as $c$ to distinguish them from the virtual ones. The virtual entangled mode pairs are again ordered from left $(i,j)$ to right $(i,j+1)$ and top $(i,j)$ to bottom $(i+1,j)$. The virtual boundaries and mode pairs between sites are denoted as $B_{i1}$ and $H_{ij}^k = (\alpha_i^\dagger \beta_i^\dagger \gamma_i \delta_i + 1\mid_k \Omega)$ for horizontal bonds and $V_{ij}^k = (\gamma_i^\dagger \beta_i^\dagger \gamma_i \delta_i + 1\mid_k \Omega)$ for vertical bonds, respectively, where the integer $k$ labels the number of bonds per direction per site. Figures 7 and 9 represent models with $k = 2$ and $k = 1$, respectively.

The wave function for the virtual space can be expressed as [24]

$$\ket{\psi} = \prod_{i,j,k} B_{i1} H_{ij}^k V_{ij}^k \Omega_v.$$  

An on-site projection $P_{ij}$ that maps the virtual space to the physical space with physical modes $c_i$ on every site is defined as [24]

$$P_{ij} = \sum_{[n]} A_{ij} \prod_{l,k} (c_l^\dagger \alpha_k^\dagger \beta_k^\dagger \gamma_k n_{ij})_{ij},$$  

where $[n]$ is the set of occupation numbers for every mode, and $A_{ij}([n])$ depends on the intrinsic properties of the physical state one wants to project to. $P_{ij}$ is constrained to have a fixed parity for the resulting state to be physical, or

$$\sum_{l,k} (n_c + n_{\beta c} + n_{\gamma c} + n_{\alpha c} + n_{\delta c}) \mod 2 = c,$$  

where $c$ is constant for each site.

To yield a physical state, one applies the projection operator $P_{ij}$'s to the virtual state $\ket{\psi}_v$ together with the physical vacuum.
state $|\Omega\rangle_p$ and then takes the vacuum expectation value on the virtual space as all the virtual modes must be annihilated and only physical modes are left,

$$|\psi\rangle_p = e^{(\Omega)} \prod_{i,j} P_{ij} |\psi\rangle_v |\Omega\rangle_p$$

$$= e^{(\Omega)} \prod_{i,j} P_{ij} \sum_{i,j,k} B_{i,j} H_{k,j} V_{i,j} (|\Omega\rangle_v |\Omega\rangle_p). \quad (25)$$

This form of many-body fermion state is the starting point for a more general construction of fermionic resource states for MBQC.

### B. Information flow in fPEPS

Similar to MBQC based on PEPS, by measuring single sites in the fPEPS, information stored in the left boundaries $B_{1,1}$ gets transmitted to the right and gets processed at the same time. Before discussing how information is transformed under a particular measurement pattern, an information encoding scheme needs to be specified. In a general fMBQC scheme, any encoding scheme can be used as long as it satisfies the fermion parity constraint. For example, instead of encoding $n$ qubits of information into $2n$ fermion modes as we did in the last section, one can encode $n$ qubits into $n + 1$ fermion modes where the last fermion mode compensates for the total parity of the state. Such an encoding scheme has also been discussed in Ref. [1] for fermionic circuit model quantum computation which has much lower overhead than the $n \rightarrow 2n$ scheme.

After specifying an encoding scheme, one can analyze what transformation is applied on the encoded information if measurements are performed on the fPEPS state. We can consider fPEPS with any number $k$ of bonds between neighboring sites and simpler fermionic resource state and measurement schemes may be found beyond the simple example constructed in the previous section. For example, we use the $n \rightarrow n + 1$ encoding and consider the 2D fPEPS state with $k = 1$ shown in Fig. 9 and no projection on each site. If the first row is detached from the rest of the lattice, then measuring the two modes in site (1, 1) and projecting them in state $(I + e^{-\frac{i\pi}{4}} \alpha^1 |\beta^1\rangle^\dagger) |\Omega\rangle$ implements the unitary gate $e^{i\frac{\pi}{4} \alpha^1 \beta^1}$ on the input mode, which is among the set of universal gates given by Ref. [1] for the $n \rightarrow n + 1$ encoding scheme. If the first two rows are detached from the rest of the lattice, it can be checked that measuring the three modes in site (1, 1) and projecting them in state

$$(i \alpha^1 + \gamma^1 - i \beta^1 - \alpha^1 \gamma^1 |\beta^1\rangle^\dagger) |\Omega\rangle \quad (26)$$

and projecting the three modes in site (2, 1) in state

$$(\alpha^1 + \delta^1 + \beta^1 - \alpha^1 \delta^1 |\beta^1\rangle^\dagger) |\Omega\rangle \quad (27)$$

applies the unitary gate

$$e^{i\frac{\pi}{2}(\alpha_{(1,1)} - \alpha_{(2,1)}) \alpha_{(2,1)} + \alpha_{(2,1)}} \quad (28)$$

to the input modes, which is one of the two-mode gates in the universal set given by Ref. [1]. The measurement pattern necessary for achieving universality and for dealing with measurement randomness can be similarly analyzed as for the simple example in the previous section.

Therefore, the fPEPS formalism allows us to go beyond the simple model and search for new possibilities. In general, there can be an arbitrary number of $k$ bonds between sites and there can be nontrivial projection on each site. In the following, we discuss how information flows and gets processed in a general fPEPS with arbitrary encoding and measurement patterns. For simplicity of notation, we still assume $k = 1$, as shown in Fig. 9, but the discussion applies, in general, for any $l$. The measurements are performed site by site from top to bottom and then from left to right, starting from site (1, 1) in column 1.

To illustrate the flow of information, we first look at the measurement on site (1, 1). Suppose site (1, 1) is measured and projected in state $|\phi_{11}\rangle = |\Omega_{11}\rangle |\Omega_{11}\rangle_p$. To see what the total state becomes after the measurement, we first rearrange $|\psi\rangle_p$ in Eq. (25) and commute the terms containing modes on site (1, 1) together. We get

$$|\psi\rangle_p = \sum_{a} e^{(\Omega)} \prod_{(i,j)\neq (1,1)} P_{ij} Q^a_{i,j,k} R^a_{1,1} H_{i,j} V_{i,j} (|\Omega\rangle_v |\Omega\rangle_p) \quad (29)$$

where $a$ denotes different terms in $B_{1,1}$ if input modes on site (1, 1) are entangled with other modes of $B_{1,1}$, which could possibly happen in a generic state as long as the total parity is fixed. Note that no extra signs are produced in this procedure as we are free to move the entangled pairs and the projections because they have fixed parities.

Using Eqs. (25) and (29), it is obvious to see that after the measurement, the state becomes

$$|\phi_{11}\rangle (|\phi_{11}\rangle |\psi\rangle_p)$$

$$= |\phi_{11}\rangle \sum_{a} e^{(\Omega)} \prod_{(i,j)\neq (1,1)} P_{ij} R^a_{1,1} B^a_{i,j} H_{i,j} V_{i,j} (|\Omega\rangle_v |\Omega\rangle). \quad (30)$$

with

$$R^a_{1,1} = \langle \Omega_{11} | P_{11} B^a_{1,1} H_{1,1} V_{1,1} (|\Omega_{11}\rangle v |\Omega_{11}\rangle_p)$$

$R^a_{1,1}$ is an operator on the $\alpha$ virtual mode of site (1, 2) and the $\delta$ virtual mode of site (2, 1). We can hence interpret the effect of measuring site (1, 1) as information flow in the virtual space from site (1, 1) to site (1, 2) and (2, 1), as shown in Fig. 10. The encoded state changes from $B_{1,1}$ to $R_{1,1}$ and, correspondingly,
a certain operation is implemented. This is similar to the picture we had with spin MBQC, where measurements on physical sites correspond to operations implemented on the information flow in the virtual space.

This formalism provides a general framework to study MBQC based on a many-body fermionic state. The simple example we studied before falls into this framework with $k = 2$ and trivial projection $P_{ij} = I$. Based on the general formulation, it is possible to find a physically more feasible fPEPS resource state for MBQC. Extra care needs to be taken when dealing with measurement randomness on a general fPEPS and we discuss briefly possible difficulties in the next section.

### C. Dealing with measurement randomness for general fPEPS

In dealing with measurement randomness for the simple model, we classified the by-products into two categories depending on whether the output contains the same parity as the input. Yet, we showed that the by-products are locally and locally correctable as all the mode pairs $(\alpha \alpha', \beta \beta', \text { etc.})$ have a fixed parity. However, in a general fPEPS state with possibly an odd number of bonds per direction ($k = 2n + 1, n \in \mathbb{N}$) and a more general encoding scheme, nonlocal by-products could occur and special attention is needed when designing MBQC schemes based on such states. Note that here the nonlocality of the by-product is reflected in its nontrivial action on all the quantum information encoded in fermionic modes before it, even though the operator itself is of a local (fermionic) form. As we show in the following, the nonlocality of such by-products can be properly taken care of with careful design and is not a fundamental difficulty in using fPEPS states as MBQC resource states.

We use the $k = 1$ model to illustrate the basic idea. Assume that we are measuring the sites in a columnwise order; that is, we first measure the first column from first row to last row and then second column from first row to last row, etc. Let us look more closely at the measurements on column 1, starting from site $(1, 1)$, and moving downward. Suppose that the boundary modes are always ordered from up to down. So after measurement on site $(1, 1)$, they are ordered from site $(1, 2)$ to site $(2, 1)$, to site $(3, 1)$, etc. The parity constraint with a general encoding scheme is that the whole boundary chain has a fixed total parity, but each boundary mode may not. In particular, the boundary mode on site $(1, 2)$ might not have a fixed parity. This leads to extra sign effect when site $(2, 1)$ is measured. In particular, if site $(2, 1)$ is measured in an odd basis which corresponds to an odd operation on the boundary, it applies a nontrivial sign factor $(-1)^{n_{1,2}}$ to the boundary mode on site $(1, 2)$. Similarly, measuring site $(i, 1)$ in an odd basis causes a nontrivial sign factor on sites $(i', 2)$ for $i' < i$. Therefore, the by-product induced is indeed nonlocal.

In general, after finishing measurements on the $j$th column, the overall sign $S_{(i, j+1)}$ accumulated on site $(i, j+1)$ in column $j+1$ is determined by the number of odd measurement basis states below site $(i, j)$ in column $j$ and the occupation number operator $n_{a(i, j+1)}$ of a mode on site $(i, j+1)$. Define

$$N_{(i, j+1)} = \sum_{i', j > i} f_{i', j},$$

where $f_{i, j}$ is the parity of the measurement basis state on site $(i, j)$. Then, we obtain

$$S_{(i, j+1)} = (-1)^{n_{a(i, j+1)} N_{(i, j+1)}}.$$  

Even though the by-products are nonlocal, they can be dealt with in a local way. Note that as long as one keeps track of all the measurement results, the total by-products that happen to the boundary modes can be determined after one finishes the measurements of one column. Moreover, the by-products factorize into a product form, of individual operators on each boundary mode separately, for example as given in Eq. (32). Such by-products can be incorporated into the operation to be implemented when measuring the column $j + 1$ and can be corrected locally just like correcting $Z$ by-product in spin systems.

To sum up, in a general fMBQC scheme based on fPEPS, nonlocal by-products do occur. However, as they factorize into a product form, they can be corrected locally.

### V. CONCLUSION

In this paper we generalize the MBQC scheme from spin systems to fermion systems. We give a simple example of many-body fermion states and demonstrate how it could be used as a universal resource state for MBQC. More generally, we provide a framework for constructing fermionic resource states for MBQC based on the fPEPS representation of fermion states and discuss ways to deal with the nonlocal by-products that might come up in the general scheme.

This framework provides a general starting point for the construction of new MBQC schemes. The ultimate goal is to find resource states that are easy to realize experimentally, for example in a free fermion system where particles move around but do not interact with each other. Unlike spin systems, which factorize into total product states and lose all computational power without interaction, the hopping of fermions in the lattice and their nontrivial mutual statistics can generate entanglement among different sites in space, which can subsequently provide the basis for the power of quantum computation.

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