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A UNIFIED ANALYSIS OF BALANCING DOMAIN DECOMPOSITION BY CONSTRAINTS FOR DISCONTINUOUS GALERKIN DISCRETIZATIONS

LASLO T. DIOSADY† AND DAVID L. DARMOFAL†

Abstract. The BDDC algorithm is extended to a large class of discontinuous Galerkin (DG) discretizations of second order elliptic problems. An estimate of \( C(1 + \log(H/h))^2 \) is obtained for the condition number of the preconditioned system where \( C \) is a constant independent of \( h \) or \( H \) or large jumps in the coefficient of the problem. Numerical simulations are presented which confirm the theoretical results. A key component for the development and analysis of the BDDC algorithm is a novel perspective presenting the DG discretization as the sum of elementwise “local” bilinear forms. The elementwise perspective allows for a simple unified analysis of a variety of DG methods and leads naturally to the appropriate choice for the subdomainwise local bilinear forms. Additionally, this new perspective enables a connection to be drawn between the DG discretization and a related continuous finite element discretization to simplify the analysis of the BDDC algorithm.

Key words. discontinuous Galerkin, domain decomposition, BDDC

AMS subject classifications. 65M55, 65M60, 65N30, 65N55

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1. Introduction. Domain decomposition (DD) methods provide efficient parallel preconditioners for solving large systems of equations arising from the discretization of partial differential equations. The development of DD methods for the solution of elliptic problems using conforming finite element methods has matured significantly over the past 20 years. Toselli and Widlund provide a detailed overview of DD methods in [22]. In this paper we consider a class of nonoverlapping DD methods based on the Neumann–Neumann methods originally introduced by Bourgat et al. [6]. These methods were improved by introducing a coarse space based on the null-space of the local Schur complement problems, leading to the balancing domain decomposition (BDD) method of Mandel [17]. Dohrmann extended the BDD method by selecting a coarse space formed by enforcing continuity of a small set of primal degrees of freedom [11]. This balancing domain decomposition by constraints (BDDC) method was later proven by Mandel and Dohrmann [18] to have a condition number bound of \( \kappa \leq C(1 + \log(H/h))^2 \) for a preconditioned system of a continuous finite element discretization of second order elliptic problems. Further analysis of BDDC methods as well as the relationship between BDDC methods and dual-primal finite element tearing and interconnecting (FETI-DP) methods has been presented in [16, 7, 19].

In this paper we present a BDDC method for the solution of a discontinuous Galerkin (DG) discretization of a second order elliptic problem. While DD methods have been widely studied for continuous finite element discretizations, relatively little work has been performed for DG discretizations. Feng and Karakashian presented a two-level Schwarz preconditioner for an interior penalty DG discretization
of the Poisson problem [15]. Feng and Karakashian considered both overlapping and nonoverlapping preconditioners and obtained condition number bounds of $O(H/\delta)$ and $O(H/h)$, respectively. Antonietti and Ayuso considered additive and multiplicative Schwarz preconditioners for a large class of DG discretizations of elliptic problems in [1, 2, 3]. Antonietti and Ayuso employed the unified framework of Arnold et al. [4] to analyze these DG methods and showed that condition number bounds of order $O(H/h)$ could be obtained with these preconditioners for symmetric DG schemes.

In the context of Neumann–Neumann type methods for DG discretizations, Dryja et al. employed a conforming finite element discretization on each subdomain while using an interior penalty method across nonconforming subdomain boundaries [14, 12, 13]. Using this discretization, Dryja et al. were able to leverage results from the continuous finite element analysis to obtain condition number bounds of $\kappa \leq C(1 + \log(H/h))^2$ for particular BDD and BDDC methods. In this work we present a BDDC method applied to a large class of DG methods considered in the unified analysis of Arnold et al. [4]. A key component for the development and analysis of the BDDC algorithm is a novel perspective presenting the DG discretization as the sum of elementwise “local” bilinear forms. The elementwise perspective leads naturally to the appropriate choice for the subdomainwise local bilinear forms. Additionally, this new perspective enables a connection to be drawn between the DG discretization and a related continuous finite element discretization. By exploiting this connection, we prove a condition number bound of $\kappa \leq C(1 + \log(H/h))^2$ for the BDDC preconditioned system for a large class of conservative and consistent DG methods.

In section 2 we give a classical presentation of the DG discretization. In section 3 we present our new perspective on the DG discretization. In sections 4 and 5, respectively, we discuss our DD strategy and present the BDDC algorithm. The analysis of the BDDC algorithm in presented in section 6, while in section 7 we present numerical results confirming the analysis.

2. DG discretization. We consider the following second order elliptic equation in a domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$,

$$
\begin{align*}
-\nabla \cdot (\rho \nabla u) &= f & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}
(2.1)
$$

with positive $\rho > 0 \in L^\infty(\Omega)$, $f \in L^2(\Omega)$. Following [4] we may rewrite (2.1) in mixed form in order to motivate the DG formulation. In practice, the fluxes are locally eliminated to obtain the DG discretization in primal form. The mixed form of (2.1) is given by

$$
\begin{align*}
\rho^{-1} q + \nabla u &= 0, & \text{in } \Omega, \\
\nabla \cdot q &= f & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega.
\end{align*}
(2.2)
$$

Prior to introducing the exact form of the discrete equations, we introduce the functional setting and notation. Denote by $T_h$ the family of triangulations obtained by partitioning $\Omega$ into triangles or quadrilaterals (if $n = 2$) or tetrahedra or hexahedra (if $n = 3$), with characteristic element size $h$. We make the usual assumption that the family of triangulations $T_h$ is shape-regular and quasiuniform [22]. Define $E$ to be the union of edges (if $n = 2$) or faces (if $n = 3$) of elements $\kappa$. Additionally, define $E^i \subset E$ and $E^\partial \subset E$ to be the set of interior, respectively, boundary edges. We note that
any edge $e \in \mathcal{E}^i$ is shared by two adjacent elements $\kappa^+$ and $\kappa^-$ with corresponding outward pointing normal vectors $n^+$ and $n^-$.

Let $P^p(\kappa)$ denote the space of polynomials of order at most $p$ on $\kappa$ and define $P^p(\kappa) := [P^p(\kappa)]^n$. Given the triangulation $\mathcal{T}_h$ define the following finite element spaces:

\begin{align}
W^P_h := \{ w_h \in L^2(\Omega) : w_h|_\kappa \in P^p(\kappa) \quad \forall \kappa \in \mathcal{T}_h \}, \\
V^P_h := \{ \nu_h \in L^2(\Omega) : \nu_h|_\kappa \in P^p(\kappa) \quad \forall \kappa \in \mathcal{T}_h \}.
\end{align}

Note that traces of functions $u_h \in W^P_h$ are, in general, double valued on each edge, $e \in \mathcal{E}^i$, with values $u^e_h$ and $u^e_h$ corresponding to traces from elements $\kappa^+$ and $\kappa^-$, respectively. On $e \in \mathcal{E}^\partial$, associate $u^e_h$ with the trace taken from the element, $\kappa^+ \in \mathcal{T}_h$, neighboring $e$. The DG discretization of (2.1) obtains a solution $u_h \in W^P_h$ such that for all $\kappa \in \mathcal{T}_h$,

\begin{align}
\left( \rho \nabla u_h, \nabla v_h \right)_\kappa - \left( \rho^+(u^+_h - \hat{u}_h) n^+, \nabla w^+_h \right)_{\partial\kappa} + \left( \hat{q}_h, w^+_h n^+ \right)_{\partial\kappa} = (f, w_h)_\kappa \quad \forall \kappa \in \mathcal{T}_h,
\end{align}

where $(\cdot, \cdot)_{\kappa} := \int_{\kappa}$ and $(\cdot, \cdot)_{\partial\kappa} := \int_{\partial\kappa}$. Superscript $+$ is used to explicitly denote values on $\partial\kappa$, taken from $\kappa$. For all $w_h \in W^P_h$, $\hat{w}_h = \hat{w}_h(u^+_h, w^-_h)$ is a single valued numerical trace on $e \in \mathcal{E}^i$, while $\hat{w}_h = 0$ for $e \in \mathcal{E}^\partial$. Note that $\hat{u}_h = 0$ on $e \in \mathcal{E}^\partial$, corresponds to weakly enforced homogeneous boundary conditions on $\partial\Omega$. Similarly, $\hat{q}_h = \hat{q}_h \left( \nabla u^+_h, \nabla u^-_h, u^+_h, u^-_h, \rho^+, \rho^- \right)$ is a single valued numerical flux on $e \in \mathcal{E}$. Summing (2.6) over all elements gives the complete DG discretization: Find $u_h \in W^P_h$ such that

\begin{align}
a(u_h, w_h) = (f, w_h)_\Omega \quad \forall w_h \in W^P_h.
\end{align}

Following [4], a piecewise discontinuous numerical approximation of the flux, $q_h$, may be evaluated locally as

\begin{align}
q_h = - \left( \rho^n u_h - \rho^{n+} r_\kappa(u^{n+}_h - \hat{u}_h) n^+ \right),
\end{align}

where $r_\kappa(\phi) \in P^p(\kappa)$ is defined by

\begin{align}
(r_\kappa(\phi), \nu_h)_\kappa = \left( \phi, \nu^+_h \right)_{\partial\kappa} \quad \forall \nu_h \in P^p(\kappa).
\end{align}

We note that while $\nabla u_h$ and $r_\kappa(r^{n+}(u^{n+}_h - \hat{u}_h) n^+)$ lie in the polynomial space $P^p(\kappa)$, $q_h$, in general, does not when $\rho$ varies within an element $\kappa$. The DG discretizations presented in this paper lift $\rho^n \nabla u$ (as opposed to $\nabla u$ or $\rho \nabla u$) to ensure that the discretization is symmetric for any $\rho \in L^\infty(\Omega)$. In the case of piecewise constant $\rho$ the DG formulations lifting $\nabla u$, $\rho^n \nabla u$, or $\rho \nabla u$ are identical.

The choice of the numerical trace $\hat{u}_h$ and flux $\hat{q}_h$ define the particular DG method considered. Table 2.1 lists the numerical traces and fluxes for the DG methods considered in this paper. In the definition of the different DG methods, the following average and jump operators are used to define the numerical trace and flux on $e \in \mathcal{E}^i$:

\begin{align}
\{ u_h \} = \frac{1}{2}(u^+_h + u^-_h) \quad \text{and} \quad \| u_h \| = u^+_h n^+ + u^-_h n^-.
\end{align}

Additionally we define a second set of jump operators involving the numerical trace $\hat{u}$:

\begin{align}
\| u_h \|^+ = u^+_h n^+ + \hat{u}_h n^- \quad \text{and} \quad \| u_h \|^-= \hat{u}_h n^+ + u^-_h n^-.
\end{align}
such that we may express \( \mathbf{q}_h \) as

\[
\mathbf{q}_h = - (\rho \nabla u_h - \rho \frac{e}{\kappa} r_c (\rho \frac{e}{\kappa} [u_h]^+) ) .
\]

We note that in the definition of the different DG methods, \( \eta_c \) is a penalty parameter defined on each edge in \( \mathcal{E} \), while \( r_c(\phi) \in \mathbf{P}^0(\kappa) \) is a local lifting operator defined by

\[
(r_c(\phi), \mathbf{v}_h)_\kappa = \langle \phi, \mathbf{v}_h^+ \rangle_c \quad \forall \mathbf{v}_h \in \mathbf{P}^2(\kappa).
\]

Additionally \( \mathbf{q}^r \) is given by

\[
\mathbf{q}^r_h = - (\rho \nabla u_h - \rho \frac{e}{\kappa} r_c (\rho \frac{e}{\kappa} [u]^+) ) .
\]

For the local discontinuous Galerkin (LDG) and compact discontinuous Galerkin (CDG) methods, \( \beta \) is a vector which is defined on each edge/face in \( \mathcal{E}^i \) as

\[
\beta = \frac{1}{2} \left( S^\kappa_{\kappa+} n^+ + S^\kappa_{\kappa-} n^- \right),
\]

where \( S^\kappa_{\kappa+} \in \{0, 1\} \) is a switch defined on each face of element \( \kappa^+ \) shared with element \( \kappa^- \), such that

\[
S^\kappa_{\kappa+} + S^\kappa_{\kappa-} = 1.
\]

3. The DG discretization from a new perspective. A key component, required for the development and analysis of the algorithms presented, is to express the global bilinear form \( a(u_h, w_h) \) as the sum of elementwise contributions \( a_\kappa(u_h, w_h) \) such that

\[
a(u_h, w_h) = \sum_{\kappa \in \Omega_h} a_\kappa(u_h, w_h),
\]

where \( a_\kappa(u_h, w_h) \) is a symmetric, positive semidefinite “local bilinear form.” In particular, we wish the local bilinear form to have a compact stencil, such that \( a_\kappa(u_h, w_h) \) is a function of only \( u_h, \nabla u_h \) in \( \kappa \), and \( u^+_h, \nabla u^+_h, \rho^+ \), and \( \hat{u}_h \) on \( \partial \kappa \). In particular, we note that in (2.6), which is summed over all elements to give \( a(u_h, w_h) \), \( \hat{q} \) depends, in general, upon \( u^+, u^-, \nabla u^+, \nabla u^- \), \( \rho^+ \), and \( \rho^- \). We write that local bilinear form as

\[
a_\kappa(u_h, w_h) = (\rho \nabla u_h, \nabla w_h)_\kappa - \langle \rho^+ (u_h^+ - \hat{u}_h) n^+, \nabla w_h^+ \rangle_{\partial \kappa} + \langle \begin{pmatrix} \mathbf{q}_h^+ \\ \mathbf{q}_h^- \end{pmatrix}^+, (w_h^+ - \hat{w}_h) n^+ \rangle_{\partial \kappa}
\]

\[
= (\rho \nabla u_h, \nabla w_h)_\kappa - \rho^+ [u^+ h, \nabla w^+_h]_{\partial \kappa} + \langle {\mathbf{q}}_h^+, [w^+_h]^+ \rangle_{\partial \kappa},
\]
where \( \hat{q}_h^+ = \hat{q}_h^+ (\nabla u_h^+, u_h^+, \rho^+) \) is a “local numerical flux.” In particular, in order to recover the original global bilinear form, \( \hat{q}_h^+ \) must satisfy the following relationship on each edge, \( e \):

(3.3) \[ \hat{q}_h^+ [w_h] = \hat{q}_h^+ [w_h]^+ + \hat{q}_h^+ [w_h]^− \quad \forall w_h \in W_h^p. \]

Table 3.2 lists the numerical traces and local fluxes for the DG methods considered, while Table 3.2 lists the corresponding local bilinear forms. It is simple to verify that (3.3) holds for each of the DG methods considered by using the following identities:

(3.4) \[
\begin{align*}
\left[ u_h \right] = \left[ u_h \right]^+ + \left[ u_h \right]^−
\end{align*}
\]

We now make an observation on the degrees of freedom involved in the local bilinear form, \( a_e(u_h, w_h) \). We consider using a nodal basis on each element \( \kappa \) to define \( W_h^p \). For the interior penalty (IP) method and the methods of Bassi and Rebay, and Brezzi et al., the numerical trace \( \hat{u}_h \) on an edge/face depends on both \( u_h^+ \) and \( u_h^- \). Hence the local bilinear form corresponds to all nodal degrees of freedom defining \( u_h \) on \( \kappa \) as well as nodal values on all edges/faces of \( \partial \kappa \cap E \) corresponding to the trace of \( u_h \) from elements neighboring \( \kappa \). On the other hand, for the LDG and CDG methods, the numerical trace \( \hat{u}_h \) takes on the value of \( u_h^+ \) if \( S_{\kappa}^- = 0 \) or \( u_h^- \) if \( S_{\kappa}^- = 1 \).

**Table 3.1**

<table>
<thead>
<tr>
<th>Method</th>
<th>( \hat{u}_h )</th>
<th>( \hat{q}_h^+ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interior penalty</td>
<td>{ ( u_h ) }</td>
<td>(-\rho^+ \nabla u_h^+ + \frac{\mu}{\rho^+} [u_h]^+ )</td>
</tr>
<tr>
<td>Bassi and Rebay</td>
<td>{ ( u_h ) }</td>
<td>(-\rho^+ \nabla u_h^+ + \eta_1 \rho^+ \frac{\mu}{\rho^+} [u_h]^+ )</td>
</tr>
<tr>
<td>Brezzi et al.</td>
<td>{ ( u_h ) }</td>
<td>( \hat{q}_h^+ + \eta_1 \rho^+ \frac{\mu}{\rho^+} [u_h]^+ )</td>
</tr>
<tr>
<td>LDG [9]</td>
<td>{ ( u_h ) } - \beta \cdot \left[ u_h \right]</td>
<td>( \hat{q}_h^+ + \frac{\mu}{\rho^+} [u_h]^+ )</td>
</tr>
<tr>
<td>CDG [20]</td>
<td>{ ( u_h ) } - \beta \cdot \left[ u_h \right]</td>
<td>( \hat{q}_h^+ + \frac{\mu}{\rho^+} [u_h]^+ )</td>
</tr>
</tbody>
</table>

**Table 3.2**

<table>
<thead>
<tr>
<th>Method</th>
<th>( a_e(u_h, w_h) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interior penalty</td>
<td>[ g + \sum_{\kappa \in \partial \kappa} \frac{\mu}{\rho^+} \left( \alpha [u_h]^+, [w_h]^+ \right) ]</td>
</tr>
<tr>
<td>Bassi and Rebay</td>
<td>[ g + \sum_{\kappa \in \partial \kappa} \left( r_\kappa (\rho^+ [u_h]^+), r_\kappa (\rho^+ [w_h]^+) \right) ]</td>
</tr>
<tr>
<td>Brezzi et al.</td>
<td>[ g + \sum_{\kappa \in \partial \kappa} \left( r_\kappa (\rho^+ [u_h]^+), r_\kappa (\rho^+ [w_h]^+) \right) ]</td>
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</tr>
<tr>
<td>CDG [20]</td>
<td>[ g + \sum_{\kappa \in \partial \kappa} \left( r_\kappa (\rho^+ [u_h]^+), r_\kappa (\rho^+ [w_h]^+) \right) ]</td>
</tr>
</tbody>
</table>

Where \( g = (\rho u_h, \nabla w_h)_{\partial \kappa} - \left( \rho^+ [u_h]^+, \nabla w_h^+ \right)_{\partial \kappa} - \left( \rho^+ \nabla u_h^+, [w_h]^+ \right)_{\partial \kappa} \).
Hence the local bilinear form corresponds only to degrees of freedom defining \( u_h \) on \( \kappa \) and nodal values corresponding to the trace of \( u_h \) on neighboring elements across edges/faces of \( \partial \kappa \cap \mathcal{E}^i \) for which \( S_{\kappa,x}^e = 1 \).

We denote by \( \rho_{\kappa} \) the average value of \( \rho(x) \) on each element \( \kappa \) and assume that the variation of \( \rho(x) \) within an element is uniformly bounded as

\[
(3.5) \quad c_{\rho} \rho_{\kappa} \leq \rho(x) \leq C_{\rho} \rho_{\kappa} \quad \forall x \in \kappa, \ \forall \kappa,
\]

where the constants \( c_{\rho} \) and \( C_{\rho} \) are independent of \( \rho_{\kappa} \).

We now give the following lemma regarding the local bilinear form \( a_\kappa(u_h, w_h) \).

**Lemma 3.1.** The elementwise bilinear form \( a_\kappa(u_h, u_h) \) satisfies

\[
(3.6) \quad a_\kappa(u_h, u_h) \geq 0
\]

with \( a_\kappa(u_h, u_h) = 0 \text{ if } u_h = \bar{u}_h = K \text{ for some constant } K. \)

**Proof.** We proceed to show that Lemma 3.1 holds for all of the DG methods considered. The proof of Lemma 3.1 closely follows the proof of boundedness and stability of the different DG methods presented in Arnold et al. [4], though here we consider the contribution of a single element.

For each of the DG methods considered we can show \( u_h = \bar{u}_h = K \Rightarrow a_\kappa(u_h, u_h) = 0 \) by recognizing \( u_h = K \Rightarrow \nabla u_h = 0 \) and substituting into the different bilinear forms.

It remains to prove \( a_\kappa(u_h, u_h) \geq 0 \) and \( a_\kappa(u_h, u_h) = 0 \Rightarrow u_h = \bar{u}_h = K. \)

In order to prove the result for the IP method we employ the following result from Arnold et al. [4]:

\[
(3.7) \quad c \left( r_e \left( \rho^{1/2} w \right), r_e \left( \rho^{1/2} w \right) \right)_\kappa \leq \frac{1}{h_{\kappa}} \left( \rho^{1/2}, w \right)_e \leq C \left( r_e \left( \rho^{1/2} w \right), r_e \left( \rho^{1/2} w \right) \right)_\kappa \quad \forall w \in W_h^p,
\]

where \( c \) and \( C \) are constants which depend only upon the minimum angle of \( \kappa \), the polynomial order \( p \), and the constants in (3.5). Hence, choosing \( \eta_e \) sufficiently large for the IP method we have

\[
(3.8) \quad a_{\kappa, \text{IP}}(u_h, u_h) \geq a_{\kappa, \text{BR2}}(u_h, u_h),
\]

and hence it is sufficient to show that Lemma 3.1 holds for the method of Bassi and Rebay [5]. Specifically, \( \eta_e \) may be chosen for the IP method as described in Shahbazi [21]. For the method of Bassi and Rebay,

\[
(3.9) \quad a_{\kappa, \text{BR2}}(u_h, u_h)
\]

\[
= (\rho \nabla u_h, \nabla u_h)_\kappa - 2 \left( \rho \nabla u_h, [u_h]^+ \right)_{\partial \kappa} + \sum_{e \in \partial \kappa} \eta_e \left( r_e \left( \rho^{1/2} \left[ u_h \right]^+ \right), r_e \left( \rho^{1/2} \left[ u_h \right]^+ \right) \right)_\kappa
\]

\[
= (\rho \nabla u_h, \nabla u_h)_\kappa - \sum_{e \in \partial \kappa} 2 \left( \rho^{1/2} \nabla u_h, r_e \left( \rho^{1/2} \left[ u_h \right]^+ \right) \right)_\kappa
\]

\[
+ \sum_{e \in \partial \kappa} \eta_e \left( r_e \left( \rho^{1/2} \left[ u_h \right]^+ \right), r_e \left( \rho^{1/2} \left[ u_h \right]^+ \right) \right)_\kappa
\]

\[
\geq \sum_{e \in \partial \kappa} \frac{1}{N_e} \left( \rho^{1/2} \nabla u_h - r_e \left( \rho^{1/2} \left[ u_h \right]^+ \right), \rho^{1/2} \nabla u_h - r_e \left( \rho^{1/2} \left[ u_h \right]^+ \right) \right)_\kappa
\]

\[
+ \sum_{e \in \partial \kappa} (\eta_e - N_e) \left( r_e \left( \rho^{1/2} \left[ u_h \right]^+ \right), r_e \left( \rho^{1/2} \left[ u_h \right]^+ \right) \right)_\kappa
\]

\[
\geq 0
\]
given $\eta_e > N_e$, where $N_e$ is the number of edges/faces of $\kappa$. In order to show $a_{\kappa,\text{BR2}}(u_h, u_h) = 0 \Rightarrow u_h = \hat{u}_h = K$, we note $a_{\kappa,\text{BR2}}(u_h, u_h) = 0$ implies

\begin{equation}
\sum_{e \in \partial \kappa} \frac{1}{N_e} \left( \rho^\frac{1}{2} \nabla u_h - r_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+ \right), \rho^\frac{1}{2} \nabla u_h - r_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+ \right) \right)_\kappa
\end{equation}

\begin{equation}
+ \sum_{e \in \partial \kappa} \left( \eta_e - N_e \right) \left( r_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+ \right), r_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+ \right) \right)_\kappa = 0.
\end{equation}

Hence $r_e(\rho^\frac{1}{2} [u_h]^+) = 0$ and $\rho^\frac{1}{2} \nabla u_h - r_e(\rho^\frac{1}{2} [u_h]^+) = 0$, which implies $\hat{u}_h = u_h^+$ on $\partial \kappa$ and $\nabla u_h = 0$ in $\kappa$.

The proof of the method of Brezzi et al. [8] follows in a similar manner. Namely

\begin{equation}
a_{\kappa,\text{Brezzi et al.}}(u_h, u_h)
\end{equation}

\begin{equation}
= \left( \rho \nabla u_h, \nabla u_h \right)_\kappa - 2 \left( \rho^\frac{1}{2} \nabla u_h, \left[ u_h \right]^+ \right)_{\partial \kappa} + \left( r_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+ \right), r_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+ \right) \right)_\kappa
\end{equation}

\begin{equation}
+ \sum_{e \in \partial \kappa} \eta_e \left( r_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+ \right), r_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+ \right) \right)_\kappa
\end{equation}

\begin{equation}
\geq \left( \rho^\frac{1}{2} \nabla u_h - r_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+ \right), \rho^\frac{1}{2} \nabla u_h - r_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+ \right) \right)_\kappa
\end{equation}

\begin{equation}
+ \sum_{e \in \partial \kappa} \eta_e \left( r_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+ \right), r_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+ \right) \right)_\kappa
\end{equation}

\begin{equation}
\geq 0
\end{equation}

provided $\eta_e > 0$. In order to show $a_{\kappa,\text{Brezzi et al.}}(u_h, u_h) = 0 \Rightarrow u_h = \hat{u}_h = K$, we note $a_{\kappa,\text{Brezzi et al.}}(u_h, u_h) = 0$ implies

\begin{equation}
\left( \rho^\frac{1}{2} \nabla u_h - r_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+ \right), \rho^\frac{1}{2} \nabla u_h - r_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+ \right) \right)_\kappa
\end{equation}

\begin{equation}
+ \sum_{e \in \partial \kappa} \eta_e \left( r_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+ \right), r_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+ \right) \right)_\kappa = 0.
\end{equation}

Hence $r_e(\rho^\frac{1}{2} [u_h]^+) = 0$ and $\rho^\frac{1}{2} \nabla u_h - r_e(\rho^\frac{1}{2} [u_h]^+) = 0$, which implies $\hat{u}_h = u_h^+$ on $\partial \kappa$ and $\nabla u_h = 0$ in $\kappa$.

For the LDG method we have

\begin{equation}
a_{\kappa,\text{LDG}}(u_h, u_h)
\end{equation}

\begin{equation}
= \left( \rho \nabla u_h, \nabla u_h \right)_\kappa - 2 \left( \rho^\frac{1}{2} \nabla u_h, \left[ u_h \right]^+ \right)_{e\partial \kappa} + \left( r_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+ \right), r_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+ \right) \right)_\kappa
\end{equation}

\begin{equation}
+ \sum_{e \in \partial \kappa} \eta_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+, \left[ u_h \right]^+ \right)_e
\end{equation}

\begin{equation}
= \left( \rho^\frac{1}{2} \nabla u_h - r_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+ \right), \rho^\frac{1}{2} \nabla u_h - r_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+ \right) \right)_\kappa
\end{equation}

\begin{equation}
+ \sum_{e \in \partial \kappa} \eta_e \left( \rho^\frac{1}{2} \left[ u_h \right]^+, \left[ u_h \right]^+ \right)_e
\end{equation}

\begin{equation}
\geq 0.
\end{equation}
that exists constants for the method of Bassi and Rebay. Hence, proof of Lemma 3.1 for the CDG method thus follows directly from the proof of Lemma 3.2 and a summation over all element \( \kappa \in \omega \). Namely, we have

\[
\sum_{\kappa \in \omega} \eta_{\kappa} \left( \rho_{B2}^{+} \| u_{h} \|^{+} \right)_{\kappa} = 0.
\]

Hence \( \| u_{h} \|^{+} = 0 \) and \( \rho_{B2}^{+} \nabla u_{h} + r_{\kappa} \left( \rho_{B2}^{+} \| u_{h} \|^{+} \right) = 0 \), which implies \( \nabla u_{h} = 0 \).

Finally for the CDG method, we again use (3.7) and note that if \( \eta_{c} \) is chosen sufficiently large for the CDG method, then we have

\[
a_{\kappa, CDG}(u_{h}, u_{h}) \geq a_{\kappa, BR2}(u_{h}, u_{h}).
\]

Hence, proof of Lemma 3.1 for the CDG method thus follows directly from the proof for the method of Bassi and Rebay.

We now parameterize the space \( W^{p}_{h} \) using a standard nodal basis defined at nodes \( x \) on each element \( \kappa \). The following lemmas show that the bilinear form is equivalent to a quadratic form based on the value of \( u_{h} \) at the nodes \( x \).

**Lemma 3.2.** There exist constants \( c \) and \( C \) independent of \( h \) and \( \rho_{\kappa} \) such that for all \( u_{h} \in W^{p}_{h} \),

\[
ca_{\kappa}(u_{h}, u_{h}) \leq \rho_{\kappa} h^{n-2} \sum_{x_{i}, x_{j} \in \kappa \cup \kappa'} (u_{h}(x_{i}) - u_{h}(x_{j}))^{2} \leq C a_{\kappa}(u_{h}, u_{h}),
\]

where \( x_{i}, x_{j} \) are the nodes on \( \kappa \) defining the basis for \( u_{h} \) and nodes on \( \partial \kappa' \) defining a basis for the trace \( u_{h}^{\Gamma}_{\kappa} \) from neighbors \( \kappa' \) of \( \kappa \). (We note that for the LDG and CDG methods nodes \( x_{i}, x_{j} \) include nodes defining a basis for \( u_{h}^{\Gamma}_{\kappa} \) only on faces for which \( S^{\kappa} = 1 \).)

**Proof.** Lemma 3.2 is a direct consequence of Lemma 3.1 and a scaling argument. See [10, Lemma 4.3] for the equivalent proof for a mixed finite element discretization.

We note that constants \( c \) and \( C \) in Lemma 3.2 depend, in general, on the polynomial order \( p \). Throughout this paper all generic constants will, unless explicitly stated otherwise, depend on the polynomial order \( p \).

**Lemma 3.3.** Consider a region \( \omega \subset \Omega \) composed of elements in \( T_{h} \). Denote by \( \rho_{\omega} \) the average value of \( \rho \) on \( \omega \), and suppose that \( \rho \) is uniformly bounded on \( \omega \) such that exists constants \( c_{\rho} \) and \( C_{\rho} \) independent of \( \rho_{\omega} \),

\[
c_{\rho} \rho_{\omega} \leq \rho \leq C_{\rho} \rho_{\omega}.
\]

Then there exist different constants \( c \) and \( C \) independent of \( h \), \( |\omega| \), and \( \rho_{\omega} \) such that for all \( u_{h} \in W^{p}_{h} \),

\[
ca_{\omega}(u_{h}, u_{h}) \leq \rho_{\omega} h^{n-2} \sum_{\kappa \in \omega} \left( u_{h}(x_{i}) - u_{h}(x_{j}) \right)^{2} \leq C a_{\omega}(u_{h}, u_{h}).
\]

**Proof.** Lemma 3.3 follows directly from Lemma 3.2 and a summation over all element \( \kappa \in \omega \). Note, we have used the assumption of a quasiform family of
triangulations (namely, \( h_k \leq C h \) for \( C \) independent of \( h \)) to replace \( h_k \) with \( h \) while ensuring that the constants in Lemma 3.3 are independent of \( h \). Similarly, the bound in (3.17) allows us to replace \( \rho_c \) with \( \rho_w \) while ensuring the constants are independent of \( \rho_w \). Clearly, the constant in Lemma 3.3 will depend, in general, upon \( C_h \), \( c_{\rho} \), and \( C_{\rho} \).

4. Domain decomposition. In this section we present a DD of the discrete form of the DG discretization and derive a Schur complement problem for the interfaces between subdomains. The presentation of the BDDC algorithm follows that presented in [16] for the case of continuous finite elements. We consider a partition of the domain \( \Omega \) into substructures \( \Omega_i \) such that \( \Omega = \bigcup_{i=1}^N \Omega_i \). The substructures \( \Omega_i \) are disjoint shape regular polygonal regions of diameter \( O(H) \), consisting of a union of elements in \( T_h \).

We denote by \( \rho_i \) the average value of \( \rho(x) \) on \( \Omega_i \). We assume that large jumps in \( \rho(x) \) are aligned with the subdomain interfaces such that \( \rho(x) \) and \( \rho_k \) may be uniformly bounded as

\[
\begin{align*}
(4.1) & \quad c_{\rho} \rho_i \leq \rho(x) \leq C_{\rho} \rho_i \quad \forall x \in \Omega_i, \quad \forall \Omega_i, \\
(4.2) & \quad c_{\rho} \rho_i \leq \rho_k \leq C_{\rho} \rho_i \quad \forall k \in \Omega_i, \quad \forall \Omega_i
\end{align*}
\]

with constants \( c_{\rho} \) and \( C_{\rho} \) independent of \( \rho_i \). We also make the following assumption.

Assumption 4.1. Each element \( \kappa \) in \( \Omega_i \) with an edge/face \( e \) on \( \partial \Omega_i \cap \partial \Omega_j \) has neighbors in \( \Omega_i \cup \Omega_j \) only.

We note that while this assumption may appear limiting, in practice it is always possible to locally split elements on corners/edges in 2D/3D, respectively, in order to satisfy this requirement.

We next define the local interface \( \Gamma_i = \partial \Omega_i \setminus \partial \Omega \) and global interface \( \Gamma \) by \( \Gamma = \bigcup_{i=1}^N \Gamma_i \). We denote by \( W^{(i)}_I \) the space of discrete nodal values on \( \Gamma_i \) which correspond to degrees of freedom shared between \( \Omega_i \) and neighboring subdomains \( \Omega_j \), while \( W^{(i)}_I \) denotes the space of discrete unknowns local to a single substructure \( \Omega_i \). In particular, we note that for the IP method, and the methods of Bassi and Rebay, and Brezzi et al., \( W^{(i)}_I \) includes for each edge/face \( e \in \Gamma_i \) degrees of freedom defining two sets of trace values \( u^+ \) from \( \kappa^+ \in \Omega_i \) and \( u^- \) for \( \kappa^- \in \Omega_j \). Thus, \( W^{(i)}_I \) corresponds to nodal values strictly interior to \( \Omega_i \) or on \( \partial \Omega_i \setminus \Gamma_i \). On the other hand, for the CDG and LDG methods, \( W^{(i)}_I \) includes for each edge/face \( e \in \Gamma_i \) degrees of freedom defining a single trace value corresponding to either \( u^+ \) from \( \kappa^+ \in \Omega_i \) if \( S_{\kappa^+}^\kappa = 0 \) or \( u^- \) from \( \kappa^- \in \Omega_j \) if \( S_{\kappa^-}^\kappa = 1 \). Hence, \( W^{(i)}_I \) corresponds to nodal values interior to \( \Omega_i \) and on \( \partial \Omega_i \setminus \Gamma_i \) as well as nodal values defining \( u^+ \) on \( e \in \Gamma_i \) for which \( S_{\kappa^+}^\kappa = 1 \).

Similarly, we define the spaces \( \hat{W}_I \) and \( W_I \) which correspond to the space of discrete unknowns associated with coupled degrees of freedom on \( \Gamma \) and local degrees of freedom on substructures \( \Omega_i \), respectively. We note that \( W_I \) is equal to the product of spaces \( W^{(i)}_I \) (i.e., \( W_I := \Pi_{i=1}^N W^{(i)}_I \)), while, in general, \( \hat{W}_I \subset W_I := \Pi_{i=1}^N W^{(i)}_I \). We define local operators \( R^{(i)}_I : \hat{W}_I \rightarrow W^{(i)}_I \) which extract the local degrees of freedom on \( \Gamma_i \) from those on \( \Gamma \). Additionally we define a global operator \( R_I : \hat{W}_I \rightarrow W_I \) which is formed by a direct assembly of \( R^{(i)}_I \).

We write the discrete form of (2.6) as

\[
\begin{bmatrix}
A_{II} & A^T_{II} \\
A_{III} & A_{III}
\end{bmatrix}
\begin{bmatrix}
u_I \\
u_T
\end{bmatrix} =
\begin{bmatrix}
b_I \\
b_T
\end{bmatrix},
\]
where \( u_I \) and \( u_I^r \) correspond to degrees of freedom associated with \( W_I \) and \( \hat{W}_I \), respectively. Since the degrees of freedom associated with \( W_I \) are local to a particular substructure, we may locally eliminate them to obtain a system

\[
\hat{S}_\Gamma u_I = g_I,
\]

where

\[
\hat{S}_\Gamma = A_{\Gamma I} - A_{\Gamma I} A_{I I}^{-1} A_{I I}^T,
\]

\[
g_I = b_{\Gamma I} - A_{\Gamma I} A_{I I}^{-1} b_{I I}.
\]

Additionally we note that \( \hat{S}_\Gamma \) and \( g_I \) may be formed by a direct assembly,

\[
\hat{S}_\Gamma = \sum_{i=1}^{N} R_I^{(i) T} S_I^{(i)} R_I^{(i)}, \quad g_I = \sum_{i=1}^{N} R_I^{(i) T} g_I^{(i)},
\]

where

\[
S_I^{(i)} = A_{I I}^{(i)} - A_{I I}^{(i)} A_{I I}^{-1} A_{I I}^{(i) T}, \quad g_I^{(i)} = b_I^{(i)} - A_{I I}^{(i)} A_{I I}^{-1} b_I^{(i)}.
\]

Here \( A_{I I}^{(i)} \) and \( b_I^{(i)} \) correspond to the contributions of a single substructure to the global system (4.3). We may also write \( \hat{S}_\Gamma \) as

\[
\hat{S}_\Gamma = R_I^T S_I R_I,
\]

where

\[
S_I = \begin{bmatrix} S_I^{(1)} \\ \vdots \\ S_I^{(N)} \end{bmatrix}.
\]

5. BDDC method. In this section we introduce the BDDC preconditioner for the Schur complement problem given in (4.4). In order to define the BDDC preconditioner we reparameterize \( W_I^{(i)} \) into two orthogonal spaces, \( W_{II}^{(i)} \) and \( W_{\Delta}^{(i)} \). The primal space \( W_{II}^{(i)} \) is the space of discrete unknowns corresponding to functions with a constant value of \( \hat{u} \) on each edge (face if \( n = 3 \)) \( F^{(i)} \) of substructure \( \Omega_i \). The dual space, \( W_{\Delta}^{(i)} \), is the space of discrete unknowns corresponding to functions which have zero mean value of \( \hat{u} \) on \( \Gamma_i \). We note that the reparameterization to obtain \( W_{II}^{(i)} \) and \( W_{\Delta}^{(i)} \) may be performed locally on each subdomain as described in [16]. We next define the partially assembled space

\[
\hat{W}_I = \hat{W}_{II} \oplus \left( \Pi_{i=1}^{N} W_{\Delta}^{(i)} \right),
\]

where \( \hat{W}_{II} \) is the assembled global primal space, single valued on \( \Gamma \), which is formed by assembling the local primal spaces, \( W_{II}^{(i)} \). We define additional local operators \( \tilde{R}_I^{(i)} : \hat{W}_I \rightarrow W_{II}^{(i)} \) which extract the degrees of freedom in \( \hat{W}_I \) corresponding to \( \Gamma_i \). The global operator \( \bar{R}_I : \hat{W}_I \rightarrow W_I \) is formed by a direct assembly of \( \tilde{R}_I^{(i)} \). We also define the global operator \( \bar{R}_I : \hat{W}_I \rightarrow \hat{W}_I \). We now define the partially assembled Schur complement matrix \( \tilde{S}_I \), given by

\[
\tilde{S}_I = \sum_{i=1}^{N} \tilde{R}_I^{(i) T} S_I^{(i)} \tilde{R}_I^{(i)}.
\]
We note that we may also write \( S_T \) as \( S_T = R_T^T S_T R_T \), where \( S_T \) is given in (4.9). In order to complete the definition of the BDDC preconditioner, we define a positive scaling factor \( \delta_i^x \) defined for each nodal degree of freedom on \( \partial \Omega_i \cap \partial \Omega_j \), corresponding to \( W^{(i)}_T \) by

\[
\delta_i^x = \frac{\rho_i^x}{\rho_i^x + \rho_j^x}, \quad \gamma \in [1/2, \infty),
\]

where \( \mathcal{N}_x \) is the set of indices of subdomains which share that particular degree of freedom. We define the scaled operator \( \tilde{R}_{D, T} : W_T \rightarrow \tilde{W}_T \), which is obtained by multiplying the entries of \( \tilde{R}_T \) corresponding to \( W^{(i)}_\Delta \) by \( \delta_i^x(x) \). Using \( \tilde{R}_T \) and \( \tilde{R}_{D, T} \) we define the interface averaging operator \( E_D : \tilde{W}_T \rightarrow \tilde{W}_T \) as

\[
E_D = \tilde{R}_T \tilde{R}_{D, T}^{-1}.
\]

The BDDC preconditioner \( M_{\text{BDDC}}^{-1} : \tilde{W}_T \rightarrow \tilde{W}_T \) is given by

\[
M_{\text{BDDC}}^{-1} = \tilde{R}_T^T \tilde{S}_T^{-1} \tilde{R}_{D, T}^{-1}.
\]

We note that this preconditioner can be efficiently implemented in parallel, as the only globally coupled degrees of freedom of \( \bar{S} \) are those associated with the primal space \( W_T \). Additionally, in the following section we will show that this preconditioner is quasi-optimal in that the condition number of the preconditioned system, \( M_{\text{BDDC}}^{-1} \bar{S} \), is independent of the number of subdomains and depends only weakly upon the number of degrees of freedom on each subdomain.

6. Analysis. In the following section we present the technical tools required to obtain the condition number bound. The analysis presented in the section closely follows that presented in [23] for mixed finite element methods, which in turn builds upon [10]. In particular, we note that all of the results presented in this section are simply the DG equivalents of similar results presented in [23] or [10]. The innovation which allows us to extend these results to DG discretizations is the new perspective presented in section 3.

The main tools developed in this section connect the DG discretization to a related continuous finite element discretization on a subtriangulation of \( \mathcal{T}_h \). Using these tools we are able to leverage the theory for continuous finite element to obtain the desired condition number bound. In order to define the related continuous finite element discretization we consider a special reparameterization of the space \( W_P^k \) on each subdomain \( \Omega_i \). Specifically, a nodal basis is employed on each element using a special set of nodal locations on each element \( \kappa \). Specifically, on elements \( \kappa \), which do not touch \( \partial \Omega_i \), nodal locations are chosen strictly interior to \( \kappa \). On elements \( \kappa \), which touch \( \partial \Omega_i \), nodal locations are chosen on \( \partial \kappa \cap \partial \Omega_i \) such that \( \bar{u}_{|\partial \kappa \cap \partial \Omega_i} \) is uniquely defined by nodal values on \( \partial \kappa \), while remaining nodal location are chosen interior to \( \kappa \). We use this reparameterization so that each node defining the basis corresponds to a unique coordinate \( \bar{x} \), and \( \bar{u}_{|\partial \kappa} \) is determined by nodal values on \( \partial \kappa \). The following lemma connects the two different parameterizations of the space \( W_P^k \).

**Lemma 6.1.** There exist constants \( c \) and \( C \) independent of \( h \) such that for each element \( \kappa \),

\[
\sum_{x_i \in \kappa} \phi(x_i)^2 \leq \sum_{\bar{x}_i \in \kappa} \phi(\bar{x}_i)^2 \leq C \sum_{x_j \in \kappa} \phi(x_j)^2 \quad \forall \phi \in P^p(\kappa)
\]
and

\[(6.2) \quad c \sum_{\mathbf{x}_i, \mathbf{x}_j \in \kappa} (\phi(\mathbf{x}_i) - \phi(\mathbf{x}_j))^2 \leq \sum_{\tilde{\mathbf{x}}_i, \tilde{\mathbf{x}}_j \in \kappa} (\phi(\tilde{\mathbf{x}}_i) - \phi(\tilde{\mathbf{x}}_j))^2 \leq C \sum_{\mathbf{x}_i, \mathbf{x}_j \in \kappa} (\phi(\mathbf{x}_i) - \phi(\mathbf{x}_j))^2
\]

\[\forall \phi \in P^p(\kappa).\]

**Proof.** The proof of Lemma 6.1 follows directly from the fact that by using either nodes \( \mathbf{x} \) or \( \tilde{\mathbf{x}} \) we can form a Lagrange basis for \( \phi \in P^p(\kappa) \), with the basis function bounded as in [22, Lemma B.5].

We now define the subtriangulation \( \tilde{T}_h \) of \( T_h \) by considering each element \( \kappa \in T_h \). The subtriangulation on each element \( \kappa \) consists of the primary vertices used to define \( W^p_h \), and secondary vertices, corresponding to nodes on \( \partial\kappa \setminus \partial\Omega \), required to form a quasi-uniform triangulation of \( \kappa \). We note that such a subtriangulation may be obtained on the reference element \( \hat{\kappa} \) then mapped to \( T_h \). As an example, Figure 6.1 shows the nodes defining the reparameterization as well as the subtriangulation for a \( p = 1 \) triangular element.

Define \( U_h(\Omega) \) to be the continuous linear finite element space defined on the triangulation \( \tilde{T}_h \). Additionally we define \( U_h(\Omega_i) \) and \( U_h(\partial\Omega_i) \) as the restriction of \( U_h(\Omega) \) to \( \Omega_i \) and \( \partial\Omega_i \), respectively. We now define a mapping \( I^\Omega_h \) from any function \( \phi \) defined at the primary vertices in \( \Omega_i \) to \( U_h(\Omega_i) \) as

\[
I^\Omega_h \phi(\mathbf{x}) = \begin{cases} 
\phi(\mathbf{x}) & \text{if } \mathbf{x} \text{ is a primary vertex}; \\
\text{the average of all adjacent primary vertices on } \partial\Omega_i, & \text{if } \mathbf{x} \text{ is a secondary vertex on } \partial\Omega_i; \\
\text{the average of all adjacent primary vertices on } \Omega_i, & \text{if } \mathbf{x} \text{ is a secondary vertex in the interior of } \Omega_i; \\
\text{the linear interpolation of the vertex values,} & \text{if } \mathbf{x} \text{ is not a vertex of } \tilde{T}_h. 
\end{cases}
\]

(6.3)
Since \((I_h^{\Omega_1}\phi)_{|_{\partial\Omega_1}}\) is uniquely defined by \(\phi_{|_{\partial\Omega_1}}\), we may define the map \(I_h^{\Omega_1}\) from a function defined on the primary vertices on \(\partial\Omega_1\) to \(U_h(\partial\Omega_1)\) such that \(I_h^{\Omega_1}\phi_{|_{\partial\Omega_1}} = (I_h^{\Omega_1}\phi)_{|_{\partial\Omega_1}}\). We define \(\hat{U}_h(\Omega_i) \subset U_h(\Omega_i)\) and \(\hat{U}_h(\partial\Omega_i) \subset U_h(\partial\Omega_i)\) as the range of \(I_h^{\Omega_i}\) and \(I_h^{\Omega_1}\), respectively.

We now connect the original DG discretization to the continuous finite element discretization on \(T_h\) by showing that both discretizations are equivalent to a quadratic form in terms of the nodal values on \(T_h\). The following lemmas and theorems are the equivalent of similar theorems for mixed finite element discretizations presented in [10] and [23]. These results are a direct consequence of Lemma 3.1, which is the DG equivalent of Lemma 4.2 of [10]. We list the relevant results from [10] and [23] and refer to these papers for the proofs.

**Lemma 6.2.** For \(\Omega_i\) composed of elements \(\kappa\) in \(T_h\), there exist constants \(c\) and \(C\) independent of \(h\), \(H\), and \(\rho_c\) such that for all \(u_h \in W_h^T\),

\[
(6.4)\quad c a_i(u_h, u_h) \leq \sum_{\kappa \in \Omega_i} \rho_c h^{n-2} \sum_{\bar{x}_i, \bar{x}_j \in \kappa, \kappa'} (u_h(\bar{x}_i) - u_h(\bar{x}_j))^2 \leq C a_i(u_h, u_h).
\]

**Proof.** Lemma 6.2 follows directly from Lemmas 3.3 and 6.1.

**Lemma 6.3.** There exists a constant \(C > 0\) independent of \(h\) and \(H\) such that

\[
(6.5)\quad \left| \hat{I}_h^{\Omega_1} \phi \right|_{H^1(\Omega_i)} \leq C \left| \phi \right|_{H^1(\Omega_i)} \quad \forall \phi \in \hat{U}_h(\Omega_i),
\]

\[
(6.6)\quad \| \hat{I}_h^{\Omega_1} \phi \|_{L^2(\Omega_i)} \leq C \| \phi \|_{L^2(\Omega_i)} \quad \forall \phi \in \hat{U}_h(\Omega_i).
\]

**Proof.** See [10, Lemma 6.1].

We define the following scaled norms:

\[
(6.7)\quad \| \phi \|_{H^1(\Omega_i)}^2 = \left| \phi \right|_{H^1(\Omega_i)}^2 + \frac{1}{H_i} \| \phi \|_{L^2(\Omega_i)},
\]

\[
(6.8)\quad \| \phi \|_{H^{1/2}(\partial\Omega_i)}^2 = \left| \phi \right|_{H^{1/2}(\partial\Omega_i)}^2 + \frac{1}{H_i} \| \phi \|_{L^2(\partial\Omega_i)}.
\]

**Lemma 6.4.** There exist constants \(c, C > 0\) independent of \(h\) and \(H\) such that for any \(\phi \in \hat{U}_h(\Omega_i),\)

\[
(6.9)\quad c \| \hat{\phi} \|_{H^{1/2}(\partial\Omega_i)} \leq \inf_{\phi \in U_h(\Omega_i)} \| \phi \|_{H^{1/2}(\partial\Omega_i)} \leq C \| \hat{\phi} \|_{H^{1/2}(\partial\Omega_i)},
\]

\[
(6.10)\quad c \left| \hat{\phi} \right|_{H^{1/2}(\partial\Omega_i)} \leq \inf_{\phi \in U_h(\Omega_i)} \| \phi \|_{H^1(\Omega_i)} \leq C \left| \hat{\phi} \right|_{H^{1/2}(\partial\Omega_i)}.
\]

**Proof.** See [10, Lemma 6.2].

**Lemma 6.5.** There exists a constant \(C > 0\) independent of \(h\) and \(H\) such that

\[
(6.11)\quad \| \hat{I}_h^{\Omega_1} \phi \|_{H^{1/2}(\partial\Omega_i)} \leq C \| \hat{\phi} \|_{H^{1/2}(\partial\Omega_i)} \quad \forall \phi \in U_h(\partial\Omega_i).
\]

**Proof.** See [10, Lemma 6.3].

**Lemma 6.6.** There exist constants \(c\) and \(C\) independent of \(h\), \(H\), and \(\rho_c\) such that for all \(u_i^{(T)} \in W_h^{T(1)}\),

\[
(6.12)\quad c \rho_c |I_h^{\Omega_1} u_i|_{H^{1/2}(\partial\Omega_i)}^2 \leq |u_i|_{S_i^{(1)}}^2 \leq C \rho_c |I_h^{\Omega_1} u_i|_{H^{1/2}(\partial\Omega_i)}^2,
\]

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Proof. See [10, Theorem 6.5].

**Lemma 6.7.** There exist constants $c$ and $C$ independent of $h$ and $H$ such that for all $u_\Gamma \in \tilde{W}_\Gamma$,

\begin{equation}
|E_D u_\Gamma|_{\tilde{S}_\Gamma}^2 \leq C (1 + \log(H/h))^2 |u_\Gamma|_{\tilde{S}_\Gamma}^2.
\end{equation}

Proof. The proof of Lemma 6.7 closely follows that of [23, Lemma 5.5]. We note that Assumption 4.1 is essential for this result. In particular, if Assumption 4.1 were not valid, then $(E_D u_\Gamma)_j$, the restriction of $E_D u_\Gamma$ to degrees of freedom on $\Omega_j$, would necessarily depend on degrees of freedom $u_k$ corresponding to a subdomain $\Omega_k$ which does not neighbor $\Omega_j$; however, are connected through the element $\kappa$ in $\Omega_i$ which has edges/faces on both $\partial \Omega_i \cap \partial \Omega_j$ and $\partial \Omega_i \cap \partial \Omega_k$.

Proof. Theorem 6.8 follows directly from Lemma 6.7. (See, for example, [23, Theorem 6.1]).

### 7. Numerical results

In this section we present numerical results for the BDDC preconditioner introduced in section 5. For each numerical experiment we solve the linear system resulting from the DG discretization using a preconditioned conjugate gradient (PCG) method. The PCG algorithm is run until the initial $l_2$ norm of the residual is decreased by a factor of $10^{10}$. We consider a domain $\Omega \in \mathbb{R}^2$ with $\Omega = (0,1)^2$. We divide $\Omega$ into $N \times N$ square subdomains $\Omega_i$ with side lengths $H$ such that $N = \frac{1}{H}$. Each subdomain is the union of triangular elements obtained by bisecting squares of side length $h$, ensuring that Assumption 4.1 is satisfied. Thus each subdomain has $n_i$ elements, where $n_i = 2 \left( \frac{H}{h} \right)^2$.

In the first set of numerical experiments we solve (2.1) on $\Omega$ with $f$ chosen such that the exact solution is given by $u = \sin(\pi x) \sin(\pi y)$. We discretize using each of the DG methods discussed in section 2 for polynomial orders $p = 1, 3,$ and $5$. Tables 7.1–7.5 show the corresponding number of PCG iteration required to converge for the considered DG methods. As predicted by the analysis, the number of iterations...

#### Table 7.1

**Iteration count for BDDC preconditioner using the IP Method.**

<table>
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is independent of the number of subdomains and only weakly dependent upon the number of elements per subdomain. In addition we note that the number of iterations also appears to be only weakly dependent on the solution order \( p \). Finally, we note that the number of iterations required for the solution of the LDG and CDG discretizations is smaller than those of the other DG methods. For the LDG and CDG methods the Schur complement problem has approximately half the number of degrees of freedom as for the other DG methods, hence it is not surprising that a smaller number of iterations is required to converge.

In the second numerical experiment we examine the behavior of the preconditioner for large jumps in the coefficient \( \rho \). We partition the domain in a checkerboard pattern and set \( \rho = 1 \) on half of the subdomains and set \( \rho = 1000 \) in the remaining subdomains. We solve (2.1) with \( f = 1 \). We discretize this problem using the CDG method. Initially we set \( \delta_i^\dagger = \frac{1}{|N_x|} \), where \( |N_x| \) is the number of elements in the set \( N_x \). We note that this choice of \( \delta_i^\dagger \) corresponds to setting \( \gamma = 0 \) in (5.3), which does not satisfy the assumption \( \gamma \in [1/2, \infty) \). Hence, we obtain poor convergence of the

### Table 7.2
Iteration count for BDDC preconditioner using the method of Bassi and Rebay.

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### Table 7.3
Iteration count for BDDC preconditioner using the method of Brezzi et al.

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Table 7.4

Iteration count for BDDC preconditioner using the LDG method.

(a) $p = 1$

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(b) $p = 3$

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(c) $p = 5$

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Table 7.5

Iteration count for BDDC preconditioner using the CDG method.

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(c) $p = 5$

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Table 7.6

Iteration count for BDDC preconditioner using the CDG method with $\rho = 1$ or 1000.

(a) $\delta_i^\dagger = \frac{1}{2}$

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(b) $\delta_i^\dagger = \frac{\rho_i}{\sum_j \rho_j}$

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BDDC algorithm as shown in Table 7.6(a). Next we set $\delta_i^\dagger$ as in (5.3) with $\gamma = 1$. With this choice of $\delta_i^\dagger$ the good convergence properties of the BDDC algorithm are recovered as shown in Table 7.6(b).
8. Conclusions. We have extended the BDDC preconditioner to a large class of DG discretizations for second order elliptic problems. The analysis shows that the condition number of the preconditioned system is bounded by $C(1 + \log(H/h))^2$, with constant $C$ independent of $h$, $H$ or large jumps in the coefficient $\rho$. Numerical results confirm the theory.

Acknowledgments. The authors would like to thank the anonymous reviewers for their suggestions to improve this paper.

REFERENCES


