Stochastic relationships for periodic responses in randomly heterogeneous aquifers

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Stochastic relationships for periodic responses in randomly heterogeneous aquifers

M. G. Trefry,1,2,3 D. McLaughlin,4 D. R. Lester,5 G. Metcalfe,6 C. D. Johnston,1,3 and A. Ord3

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The aim of this work is to develop a theoretical framework for the analysis of groundwater head oscillations commonly observed in bores near boundaries of surface water bodies that are subject to periodic variations in stage height. Restricting attention to the linear groundwater flow equation, the dynamics of head variations induced by periodic modes acting at boundaries are governed by a complex-valued time-independent equation parameterized by the modal frequency of interest. For randomly heterogeneous aquifers the hydraulic conductivity field may be regarded as a spatial random variable. Stochastic relationships between the conductivity spectrum and the induced head oscillation spectrum are generated from a stochastic perturbation approach. Spatial correlative relationships are derived for several stochastic models incorporating up to three spatial dimensions. Explicit calculations of head oscillation autocovariances and spectral densities are parameterized by conductivity statistics, including integral scale and variance, and by modal frequency. The results show that time domain head responses to periodic boundary forcing are strongly dependent on multidimensional effects and on spatial correlation structure. Computational simulations show that the stochastic variance estimators match simulated head fluctuation variances for a range of modal frequencies and aquifer diffusivities and that joint inversion of conductivity integral scale and variance is possible with moderate numbers of sampling points.


1. Introduction

It is well known that the effects of geological heterogeneity on subsurface flow and transport can be profound. Structural heterogeneity may be apparent in the spatial distribution of porosity and permeability, giving rise to variability in flow velocities or in the spatial distribution of biogeochemical species, which may generate complex and challenging feedback behaviors in kinetic reaction processes and flow relaxation. Early attempts to construct theoretical stochastic frameworks for groundwater flow and solute transport in heterogeneous media have led to significant conceptual advances in groundwater science, and a range of tools is now available to help researchers deal with simulation and prediction for practical geologic settings.

In the 1970s the central role of spatial correlation of hydraulic conductivity in determining the statistics of steady groundwater flow was noted by Gelhar and coworkers [Bakr et al., 1978, and references therein]. Fourier-Stieltjes representations [Lumley and Panofsky, 1964] were employed to obtain stochastic perturbation solutions to the steady flow equation, thereby allowing the spatial spectrum and autocorrelation functions to be derived for the head solution. As was shown by Bakr et al. [1978], flow statistics are strongly influenced by multidimensional effects and by correlation scales inherent to the spatial structure of the supporting porous medium. In subsequent decades these concepts proved fundamental to the development of stochastic theories of solute transport [Gelhar and Axness, 1983; Dagan, 1989; Rubin, 1990] since the stochastic velocity field provides random deflections to the migrating solute plumes. For different theoretical velocity densities, ergodic or anomalous plume states variously apply. Velocity covariance functions are essential in defining channeling structures and other aspects of the flow topology important to solute transport.

Periodic head responses in aquifer systems are well studied, often in connection with barometric variations or with Earth tides [Ritzi et al., 1991; Trefry and Bekele,
2. Deterministic Solutions for Periodic Darcy Flows in Aquifers

Consider a one-dimensional, homogeneous aquifer system having transmissivity $T$ and storativity $S$. Assuming linear Darcy flow and in the absence of vertical recharge, the governing groundwater flow equation is

$$S \frac{\partial h}{\partial t} = T \frac{\partial^2 h}{\partial x^2}. \quad (1)$$

Here we assume the solution $h(x,t)$ extends over an infinite domain $x \in [0, \infty]$ and is subject to the periodic Dirichlet boundary condition

$$h(0, t) = g(t), \quad (2)$$

where $g(t)$ is finite and cyclic with period $P$, i.e., $g(t) = g(t + P)$. Following Trefry and Bekele [2004] and because of the linearity of (1), $g$ may be expanded in the uniformly convergent Fourier series

$$g(t) = \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{\infty} g_m(\omega_m) e^{i \omega_m t}, \quad (3)$$

where the constituent excitation modes $\omega_m$ (representing angular frequencies $\omega_m = 2\pi f_m$, where $f_m$ is the frequency of the $m$th mode) are determined from $P$ and $\omega_0 = 0$. The $g_m$ coefficients are the complex Fourier coefficients of $g$, with $g_0$ related to the mean of $g(t)$ over any period. The
linearity of (3) ensures that by superposition, \( h \) has the representation

\[
h(x, t) = \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{\infty} h_m(x, \omega_m) e^{i\omega_m t},
\]

where the Fourier coefficient \( h_m \) of \( h \) for mode \( m \) is the solution of

\[
T \frac{d^2 h_m(x, \omega_m)}{dx^2} - i\omega_m S h_m(x, \omega_m) = 0
\]

and satisfies the modal boundary conditions

\[
h_m(0, \omega_m) = g_m, \quad |h_m(x, \infty)| \to 0 \text{ as } x \to \infty.
\]

[8] We note that (5) has been derived by neglecting source terms that may commonly arise from recharge or pumping activities or from barometric forcing. This is not an unduly restrictive assumption because through the linearity of the governing equation (1) we are focusing on the modal head response for frequency mode \( \omega_m \) (see Trefry [1999] for a discussion of the formulation of the modal recharge problem). Recharge and barometric variations are typically driven at much lower frequencies than tidal boundaries, so these spectral influences are well separated and can be treated in isolation [Trefry and Bekele, 2004]. For pumping wells, again, it is usually possible to identify tidal modes that are distinct from pumping modes.

[9] The work of Jacob [1950] and Ferris [1951] showed that the Fourier coefficient solutions to (5) modulate sinusoidal head oscillations of frequency \( \omega_m \) at every point \( x \). These oscillations have amplitudes, denoted \( \alpha_m \), that decline exponentially with distance from the oscillating boundary and display phase shifts, denoted \( \Delta \tau_m \), that increase linearly with distance from the boundary according to

\[
\alpha_m = \exp \left( -x \sqrt{\frac{\omega_m}{2D}} \right), \quad \Delta \tau_m = x \left( 1 - \frac{1}{2\omega_m D} \right)
\]

where the aquifer diffusivity \( D \equiv T/S \). It is clear from (7) that low-frequency modes are attenuated less (have larger amplitudes) than high-frequency modes. Trefry and Bekele [2004] discuss some characteristics of the modal solutions in deterministic systems in further detail.

### 3. Stochastic Analysis of Periodic Darcy Flows

[10] Consider the linear flow equation given by (8), where \( \kappa \) is heterogeneous and is termed the transmissivity for one- and two-dimensional (in plan) models and the hydraulic conductivity for three-dimensional models. For three-dimensional models, \( S \) represents the specific storage.

\[
S h_t = \nabla \cdot (\kappa \nabla h).
\]

Henceforth, for simplicity, we will refer to \( \kappa \) as the conductivity and \( S \) as storativity without reference to the specific model dimensionality. We anticipate the imposition of periodic boundary forcing conditions and, as before, seek to focus attention on the oscillation frequency of interest, \( \omega \). Formally, we achieve this by taking the temporal Fourier transformation \( F_{\omega} \) of (8), where the action of the operator \( F_{\omega} \) on a function \( f(t) \) is defined as

\[
F_{\omega}[f(t)] = \hat{f}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt.
\]

It is easy to show that \( F_{\omega}[h] = i\omega \hat{h}(\omega) \), and hence, the dynamical equation for the Fourier coefficient \( \hat{h}(\omega) \) becomes

\[
\nabla \cdot \left( \kappa e^{i\omega t} \nabla h \right) - i\omega S \hat{h} = 0.
\]

The corresponding steady state equation is obtained by setting \( \omega = 0 \) in (10).

#### 3.1. Low-Order Spectral Relationships

[11] We now seek to represent the effects of geologic heterogeneity mathematically by viewing the conductivity \( \kappa \) as a spatial random process. It is also possible to represent the storativity \( S \) as a spatial random process, but here, for simplicity, we assume that \( S \) is uniform and constant throughout the domain. In this derivation we follow the general approach of Trefry et al. [2010] to deduce a spectral transfer function between the stochastic form of \( \kappa \) and the resulting head distribution. If \( \kappa \) is statistically stationary, then we can write \( \log \kappa = F = \kappa F > + \delta F \), where the effective mean of the conductivity \( \kappa_{\text{eff}} \) satisfies \( \log \kappa_{\text{eff}} = F > \). The effective mean \( \kappa_{\text{eff}} \) is identified with the harmonic mean of \( \kappa \) in one dimension and with the geometric mean in two dimensions. In the remainder of this paper it is assumed that \( \kappa_{\text{eff}} \) can be fixed independently and used as input to the stochastic method. The quantity \( \delta F \) is the random spatial fluctuation about the effective mean of the log conductivity field, and we may perform a similar decomposition for the Fourier coefficient \( \hat{h} = \hat{h} > + \delta \hat{h} \), where \( \delta h \) is the spatial fluctuation about the mean function \( <h> \). It is important to distinguish between oscillations (used here to describe temporal variations) and fluctuations (used here to describe spatial variations). Substituting these expansions in (10) yields

\[
\nabla \cdot \left( \kappa_{\text{eff}} e^{i\omega t} \nabla (<h> + \delta h) \right) - i\omega S (<h> + \delta h) = 0.
\]

Expanding the Laplacian term gives

\[
\nabla \kappa_{\text{eff}} e^{i\omega t} \nabla <h> + \kappa_{\text{eff}} \nabla e^{i\omega t} \nabla <h> + \kappa_{\text{eff}} e^{i\omega t} \nabla^2 <h> + \nabla \kappa_{\text{eff}} e^{i\omega t} \delta h + \kappa_{\text{eff}} e^{i\omega t} \nabla \delta h + \kappa_{\text{eff}} e^{i\omega t} \nabla^2 \delta h - i\omega S <h> - i\omega S \delta h = 0.
\]

The \( \nabla \kappa_{\text{eff}} \) quantities are zero. Assuming the conductivity fluctuations are small, we may linearize the exponent terms:

\[
e^{i\omega t} \approx 1 + \delta F; \quad \nabla e^{i\omega t} \approx \nabla \delta F,
\]

where
which leads to

\[
\begin{align*}
\kappa_{\text{eff}} \nabla^2 \delta F &< h > + \kappa_{\text{eff}} \delta F^2 < h > + \kappa_{\text{eff}}^2 \nabla^2 \delta h < h > \\
&+ \kappa_{\text{eff}} \nabla \delta F \nabla \delta h + \kappa_{\text{eff}} \nabla^2 \delta h > 0
\end{align*}
\]

\[\kappa_{\text{eff}} \nabla^2 \delta F < h > + \kappa_{\text{eff}} \nabla \delta F \nabla \delta h + \kappa_{\text{eff}}^2 \nabla^2 \delta h - i \omega S \delta h \approx 0.
\tag{14}
\]

This expression may be reduced by noting that the mean solution < h > satisfies

\[\kappa_{\text{eff}} \nabla^2 < h > - i \omega S < h > = 0.
\tag{15}
\]

Neglecting higher-order terms containing products of fluctuations results in

\[\kappa_{\text{eff}} \nabla \delta F < h > + \kappa_{\text{eff}} \delta F^2 < h > + \kappa_{\text{eff}}^2 \nabla^2 \delta h - i \omega S \delta h \approx 0.
\tag{16}
\]

Equation (16) is a low-order stochastic relationship between the head fluctuation \( \delta h \) and the log conductivity fluctuation \( \delta F \). To progress further, we consider the spatial spectral characteristics of \( \delta F \) determined by application of the spatial Fourier transform \( F_{x,k} \), defined as

\[F_{x,k}[f(x)] = f(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ikx} dx,
\tag{17}
\]

where \( k \) is the wave vector associated with coordinate vector \( x \). Gradients of the mean solution \( \langle h \rangle \) appear in (16), which complicates further reduction. The second-order term may be simplified by integrating by parts and employing a proximity assumption \cite{Trefry et al., 2010}, yielding

\[ik \kappa_{\text{eff}} \nabla h_0 F_{x,k}[\delta F] - 2ik \kappa_{\text{eff}} \nabla h_0 F_{x,k}[\delta F] - k^2 \kappa_{\text{eff}} F_{x,k}[\delta h] - i \omega S F_{x,k}[\delta h] \approx 0.
\tag{18}
\]

In (18), \( k^2 = |k|^2 \), and \( \nabla h_0 \) is a constant vector chosen to approximate the slope of the mean solution near the forcing boundary (the proximity assumption; see Appendix B for an estimation procedure for \( \nabla h_0 \)). This simplifying assumption allows the stationary spectral approach of Bakr et al. \cite{1978} to be employed. An alternative approach may be to employ a nonstationary spectral approach \cite{Li and McLaughlin, 1991} to cater for the spatial nonstationarity contributed by the gradients of \( \langle h \rangle \) in (16), but this is beyond the scope of this paper. Application of the proximity assumption is discussed further in sections 4 and 5.

[12] Following Bakr et al. \cite{1978}, we introduce Fourier-Stieltjes notation to write

\[\int_{-\infty}^{\infty} e^{ikx} dZ_{\delta F}(k) \approx (k^2 \kappa_{\text{eff}} + i \omega S) dZ_{\delta h}(k),
\tag{19}
\]

where the random increments \( dZ_{\delta F}, dZ_{\delta h} \) are defined by

\[\delta F = \int_{-\infty}^{\infty} e^{ikx} dZ_{\delta F}(k), \quad \delta h = \int_{-\infty}^{\infty} e^{ikx} dZ_{\delta h}(k).
\tag{20}
\]

A relationship between the (spatial) spectral densities of \( \delta F \) and \( \delta h \) (the spectral transfer function) can easily be constructed from (19) by noting the identity

\[E\{dZ_{\delta h}(k) dZ_{\delta h}^*(k')\} = \delta_{kk'} \Phi_{\delta h}(k).
\tag{21}
\]

where \( \Phi_{\delta h}(k) \) is the spectral density of function \( f(x) \), \( \delta_{kk'} \) is the Kronecker delta, and the asterisk denotes the complex conjugation operation. From (19) and (21) and performing the necessary conjugations, we can finally write the spectral transfer function in the form

\[\Phi_{\delta h \delta h}(k) \approx \frac{k^2 \eta^4 (\nabla h_0)^2}{(\eta^4 k^4 + 1)} \Phi_{\delta F \delta F}(k),
\tag{22}
\]

where the length scale of penetration of the modal oscillations from the boundary into the porous medium is given by \( \eta^4 \), termed the transmissibility and defined as

\[\eta^2 = \kappa_{\text{eff}} / \omega S.
\tag{23}
\]

This parameter is closely related to the length scales used by Townley \cite{1995}, Trefry \cite{1999}, and Slooten et al. \cite{2010}. For three-dimensional cases involving hydraulic conductivities, \( \eta^2 \) has dimensions of \( L \), while for vertically averaged two- and one-dimensional cases involving transmissivities \( \eta^2 \) has dimensions of \( L^2 \).

[13] Equation (22) can be compared with the corresponding spectral expression for steady flow in heterogeneous aquifers \cite{1978}, equation (9)), which is recovered from (22) in the steady limit \( \omega \to 0 \). The substantive difference is the presence of the transmissibility terms in (22), which for \( \omega > 0 \), remove the singularity present in the steady flow relationship noted by Bakr et al. \cite{1978}. The importance of this result should be stressed: the dissipative temporal dynamics lead to well-defined correlation functions for the induced head fluctuations for a wide range of input log \( \kappa \) correlation structures. This simplifies the development of covariance and variance measures as functions of problem dimensionality, making the time-dependent analysis most appealing from a theoretical standpoint.

[14] In following sections the covariance statistics of the periodic heads are derived using (22) and various forms of covariance functions of the conductivity fields; the latter quantities are regarded as stochastic inputs to the problem.

3.2. Covariance in One Dimension

3.2.1. One-Dimensional Exponential Conductivity Covariance

[15] Consider a log conductivity distribution in one dimension that has an exponential autocovariance function given by

\[R_{\delta F \delta F}(\xi) = \sigma^2_{\delta F} e^{-|\xi|/\lambda},
\tag{24}
\]

where \( \sigma^2_{\delta F} \) is the variance of \( \delta F \), \( \lambda > 0 \) is the integral scale, and \( \xi \) is the lag displacement coordinate (a scalar quantity
in this case). The corresponding spectral density is given by the inverse Fourier transformation of \( R_{EF,EF} \):

\[
\Phi_{EF,EF}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik\xi} R_{EF,EF}(\xi) \, d\xi
\]

\[
= \frac{\lambda \sigma_{F}^{2}}{\pi} \frac{1}{1 + k^{2}\lambda^{2}}.
\]

Inserting result (25) into (22) yields the low-order approximation to the spectral density of the head fluctuation \( \Phi_{hk,hk} \):

\[
\Phi_{hk,hk}(k) \approx \frac{\eta^{2} \left( \nabla h_{0} \right)^{2} k^{2} \lambda \sigma_{F}^{2} / \pi}{1 + k^{2}\lambda^{2}} \frac{1}{1 + k^{2}\lambda^{2}}.
\]

The head fluctuation autocovariance function \( R_{hk,hk} \) can be recovered from (26) by Fourier transformation:

\[
R_{hk,hk}(\xi) = \int_{-\infty}^{\infty} e^{ik\xi} \Phi_{hk,hk}(k) \, dk
\]

\[
= \frac{\lambda \sigma_{F}^{2}}{\pi} \eta^{2} \left( \nabla h_{0} \right)^{2} \int_{-\infty}^{\infty} e^{-ik\xi} \frac{k^{2}}{(1 + k^{2}\eta^{2})(1 + k^{2}\lambda^{2})} \, dk
\]

\[
= \frac{\lambda \sigma_{F}^{2}}{\sqrt{2}} \frac{\eta^{2} \left( \nabla h_{0} \right)^{2} e^{-i\xi/\sqrt{2}\eta} \cos(\xi/\sqrt{2}\eta) + \left( \lambda^{2} - \eta^{2} \right) \sin(\xi/\sqrt{2}\eta) - \sqrt{2}e^{i\xi/\sqrt{2}\eta}}{(\eta^{2} + \lambda^{2})}.
\]

The variance of the induced head fluctuations is measured by the zero-lag value of the autocovariance function, i.e.,

\[
\sigma_{hk}^{2} = R_{hk,hk}(0) = \frac{\left( \nabla h_{0} \right)^{2} \lambda \sigma_{F}^{2} \eta^{2} \left( \eta^{2} - \sqrt{2}\eta \lambda + \lambda^{2} \right)}{\sqrt{2} \left( \eta^{2} + \lambda^{2} \right)}. \tag{28}
\]

**3.2.2. One-Dimensional Gaussian Conductivity Covariance**

[16] For a Gaussian conductivity autocovariance,

\[
R_{EF,EF}(\xi) = \sigma_{F}^{2} e^{-\xi^{2}/2\lambda^{2}}, \tag{29}
\]

the conductivity spectrum is

\[
\Phi_{EF,EF}(k) = \frac{\lambda \sigma_{F}^{2}}{2\sqrt{\pi}} e^{-k^{2}\lambda^{2}/4}, \tag{30}
\]

and the low-order approximation to the head spectrum is thus

\[
\Phi_{hk,hk}(k) \approx \frac{\lambda \sigma_{F}^{2}}{2\sqrt{\pi}} \frac{\eta^{2} \left( \nabla h_{0} \right)^{2} k^{2}}{1 + k^{2}\eta^{2}} e^{-k^{2}\lambda^{2}/4}. \tag{31}
\]

The resulting Fourier integral to evaluate the head fluctuation autocovariance is difficult but may be addressed by expanding the denominator in a Maclaurin series \((k \approx 0)\) as

\[
(1 + k^{2}\eta^{2})^{-1} = \sum_{j=0}^{\infty} (-k\eta)^{j}. \tag{32}
\]

Hence, Fourier integration of (31) reduces to

\[
R_{hk,hk}(\xi) = \int_{-\infty}^{\infty} e^{-ik\xi} \Phi_{hk,hk}(k) \, dk = \frac{\lambda \sigma_{F}^{2}}{2\sqrt{\pi}} \frac{\eta^{2} \left( \nabla h_{0} \right)^{2}}{1 + k^{2}\eta^{2}} \sum_{j=0}^{\infty} Q_{j}(\xi), \tag{33}
\]

where the auxiliary functions \( Q_{j}(\xi) \) are defined by

\[
Q_{j}(\xi) = \int_{-\infty}^{\infty} (-\eta^{2}k^{2})^{j/2} e^{-ik\xi} \, dk
\]

\[
= \left( \frac{2}{\lambda} \right)^{j} \left( -1 \right)^{j} \left( \frac{2\eta}{\lambda} \right)^{j} \Gamma(2j+3/2) F_{j}(2j+3/2;1/2; -\xi^{2}/\lambda^{2}), \tag{34}
\]

where \( \Gamma(n + 1) = n! \) is the Gamma function and the \( F_{j} \) quantities are confluent hypergeometric functions [Abramowitz and Stegun, 1965]. The results of (33) and (34) are useful only for \( \eta \ll 1 \). Fortunately, we can calculate the head fluctuation variance directly by

\[
\sigma_{hk}^{2} = R_{hk,hk}(0) = \int_{-\infty}^{\infty} \Phi_{hk,hk}(k) \, dk
\]

\[
\approx \frac{\pi}{8} \lambda \eta \sigma_{F}^{2} \left( \nabla h_{0} \right)^{2} \left\{ \cos(\pi\varphi^{2}/2)[1 - 2C(\varphi)] + \sin(\pi\varphi^{2}/2)[1 - 2S(\varphi)] \right\}, \tag{35}
\]

where \( C \) and \( S \) are the cosine and sine Fresnel integrals [Abramowitz and Stegun, 1965], respectively, and \( \varphi \equiv \lambda / (\eta\sqrt{2\pi}) \).

**3.3. Covariance in Two Dimensions**

[17] In two dimensions it becomes necessary to specify the direction of the mean solution gradient \( \nabla h_{0} \). We will assume that this term applies in the direction of the \( x \) coordinate, i.e., that the periodic forcing applies only at the \( x = 0 \) boundary and the corresponding wave number component is \( k_{1} \). In this case the spectral relationship (22) becomes

\[
\Phi_{hk,hk}(k) \approx \frac{k_{1}^{2} \eta^{4} \left( \nabla h_{0} \right)^{2}}{\left( 1 + k^{2}\eta^{2} \right)^{2}} - \Phi_{EF,EF}(k). \tag{36}
\]

**3.3.1. Two-Dimensional Exponential Conductivity Covariance**

[18] Consider a log conductivity distribution in two dimensions that has an isotropic exponential autocovariance function given by

\[
R_{EF,EF}(\xi) = \sigma_{P}^{2} e^{-\xi^{2}/\lambda^{2}}. \tag{37}
\]
The low-order estimate of the two-dimensional head fluctuation spectrum is then

$$
\Phi_{nk}(k) \approx \frac{\eta^4 \left( \nabla h_0 \right)^2 k^2}{1 + k^4 \eta^4} \frac{\lambda^2 \sigma_{xy}^2}{2\pi (1 + k^2 \lambda_0^2)^{3/2}}.
$$

(40)

The head fluctuation autocovariance is the evaluated by taking the Fourier transformation of (40):

$$
R_{nk}(\xi, \chi) \approx \frac{\lambda^2 \sigma_{xy}^2 \eta^4 \left( \nabla h_0 \right)^2}{2\pi} \int_0^{2\pi} \int_0^{2\pi} e^{ik \xi \cos \theta} e^{-i\zeta \lambda_0} k d\theta d\zeta
$$

$$
= \frac{\lambda^2 \sigma_{xy}^2 \eta^4 \left( \nabla h_0 \right)^2}{\xi (1 + k^2 \lambda_0^2)^{3/2}} (1 + k^2 \lambda_0^2)^{3/2} dk.
$$

(41)

From Figure 1 it is clear that $k_1 = k \cos(\chi - \theta)$; hence, making the lag vector $\xi$ explicit on the polar coordinates ($\xi$, $\chi$), we may express

$$
R_{nk}(\xi, \chi) = \frac{\lambda^2 \sigma_{xy}^2 \eta^4 \left( \nabla h_0 \right)^2}{\xi (1 + k^2 \lambda_0^2)^{3/2}} (1 + k^2 \lambda_0^2)^{3/2} dk.
$$

(42)

Here $J_1$ and $J_2$ are Bessel functions of the first kind, of orders 0 and 1, respectively. The remaining integral over $k$ is not amenable to algebraic solution, so $R_{nk}(\xi, \chi)$ remains unevaluated analytically and must be pursued numerically. However, again, the variance may be expressed algebraically by using the $\xi \rightarrow 0$ identity:

$$
\sigma_{nk}^2 = R_{nk}(0) = \int_0^{\infty} \Phi_{nk}(k) dk
$$

$$
\approx \frac{\lambda^2 \eta^2 \sigma_{xy}^2 \left( \nabla h_0 \right)^2}{8 (\eta^4 + \lambda^4)^{1/4}} \left[ \frac{\pi d \eta^2 (2 \lambda^2 + \sqrt{\eta^4 + \lambda^4})}{\lambda^2 + \eta^4 + \lambda^4} \right.
$$

$$
- \frac{\sqrt{\eta^4 + \lambda^4}}{\lambda^2} \left\{ \cos \left[ \pi + 2 \arctan(\eta^2 / \lambda^2) \right] / 4 \right\} \left[ \sqrt{2} \eta (\eta^2 + \lambda^2) - 2(a - b) \eta \lambda + 2 \lambda^4 \log(g) \right]
$$

$$
+ \sin \left[ \pi + 2 \arctan(\eta^2 / \lambda^2) \right] / 4 \left\{ \sqrt{2} (\eta^2 + \lambda^2) + 2 (a + f) \eta \lambda - 2 \eta^2 \lambda^2 \log(g) \right\}
$$

$$
+ 4 \arccsc(h) (\eta^2 \lambda^2 \cos \left[ \pi + 2 \arctan(\eta^2 / \lambda^2) \right] / 4 \right) + \lambda^4 \sin \left[ \pi + 2 \arctan(\eta^2 / \lambda^2) \right] / 4 \right\}.
$$

(43)
where the quantities \(a, b, c, d, e, f, g,\) and \(h\) are as defined in Appendix A.

### 3.3.2. Two-Dimensional Gaussian Conductivity Covariance

[16] The same coordinate transformations are useful for the case of an isotropic Gaussian conductivity covariance in two dimensions:

\[
R_{SF, SF}(\xi) = \sigma_{SF}^2 e^{-\xi^2/\lambda^2}. \tag{44}
\]

The conductivity spectrum is

\[
\Phi_{SF, SF}(k) = \frac{\sigma_{SF}^2}{(2\pi)^2} \int_0^\infty \int_0^\infty e^{ik\xi \cos \theta} e^{-\xi^2/\lambda^2} \xi d\theta d\xi
\]

\[
\approx \frac{\lambda^2 \sigma_{SF}^2}{4\pi} e^{-k^2\lambda^2/4},
\]

and the low-order approximation to the head spectrum is thus

\[
\Phi_{hk,hk}(k) \approx \frac{\lambda^2 \sigma_{SF}^2}{4\pi} \frac{k^2}{1 + k^2 \lambda^2} e^{-k^2\lambda^2/4}.
\]

Equation (46) leads to the integral expression for \(R_{hk,hk} \colon \)

\[
R_{hk,hk}(\xi; \lambda) \approx \frac{\lambda^2 \sigma_{SF}^2 \eta^4}{(2\pi)^2} \int_0^\infty \int_0^\infty e^{i\xi \cos \theta} e^{-\xi^2/\lambda^2} \xi d\theta d\xi
\]

\[
= \frac{\lambda^2 \sigma_{SF}^2 \eta^4}{2\pi} \frac{k^2}{1 + k^2 \lambda^2} e^{-k^2\lambda^2/4} d\xi k^2 dk.
\]

This integral is also difficult to evaluate in closed form, but analytical approximations can be generated exactly by expanding the denominator as a Maclaurin series (32) and noting that the following component integrals \(P_{1,n}, P_{2,n}\) are available in closed form:

\[
P_{1,n} = \frac{1}{k^2} \frac{\Gamma(n + 1)}{\lambda^{2n+4}} F_1(1; n + 2; -\xi^2/\lambda^2).
\]

\[
P_{2,n} = \frac{1}{k^2} \frac{\Gamma(n + 1)}{\lambda^{2n+4}} F_1(1; n + 2; -\xi^2/\lambda^2).
\]

In (48), \(n \geq 0\) is an integer. Utilizing (32) and (48) in (47) results in an ascending series of component integral solutions for increasing powers of \(n\), which may be assembled to provide arbitrarily high order (but complicated) estimates of \(R_{hk,hk}\). The \(\xi \to 0\) variance limit can be recovered either by the power expansion approach or by integrating (47) directly for \(\xi = 0\) (since the integrand exists in this limit). The result is

\[
\sigma_{hk}^2 = R_{hk,hk}(0) = \int_0^\infty \Phi_{hk,hk}(k) dk = \frac{\lambda^2 \sigma_{SF}^2}{4\pi} \frac{k^2}{1 + k^2 \lambda^2}
\]

\[
\approx \frac{\lambda^2 \sigma_{SF}^2}{16} \left( -2 \cos \left( \frac{\lambda^2}{4\eta^2} \right) \sin \left( \frac{\lambda^2}{4\eta^2} \right) \right) + \sin \left( \frac{\lambda^2}{4\eta^2} \right) \left[ \pi - 2 \sin \left( \frac{\lambda^2}{4\eta^2} \right) \right].
\]

(49)

where \(Si\) and \(Ci\) are the sine and cosine integrals [Abramowitz and Stegun, 1965].

### 3.4. Covariance in Three Dimensions

[20] The main difference between the evaluation of isotropic covariance measures in two and three dimensions is the use of spherical coordinates rather than polar coordinates. The spherical coordinate system is discussed by Bakr et al. [1978], and we will use the corresponding transformations here.

#### 3.4.1. Three-Dimensional Exponential Conductivity Covariance

[21] Consider a log conductivity distribution in three dimensions that has an isotropic exponential autocovariance function given by (37). Again, the corresponding spectral density is given by the inverse Fourier transformation of \(R_{SF, SF} \colon \)

\[
\Phi_{SF, SF}(k) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{-ik \cdot \xi} R_{SF, SF}(\xi) d\xi,
\]

As before, we assume statistical isotropy and introduce spherical coordinates as defined by Bakr et al. [1978], leading to the expression \(k \cdot \xi = k \xi \cos \omega\), where vectors \(k\) and \(\xi\) have magnitudes \(k\) and \(\xi\), respectively. Then \(d\xi = \xi^2 \sin \omega d\omega d\phi d\phi\), and (50) becomes

\[
\Phi_{SF, SF}(k) = \frac{\sigma_{SF}^2}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{0}^{\pi} e^{-ik \xi \cos \phi} e^{-\xi^2/\lambda^2} \xi^2 \sin \phi d\phi d\xi.
\]

\[
\approx \frac{\lambda^2 \sigma_{SF}^2}{\pi^2} \frac{k^2}{1 + k^2 \lambda^2}.
\]

(51)

The low-order estimate of the two-dimensional head fluctuation spectrum is then

\[
\Phi_{hk,hk}(k) \approx \frac{\eta^4}{(2\pi)^2} \frac{k^2}{1 + k^2 \lambda^2} \frac{\lambda^2 \sigma_{SF}^2}{\pi^2 (1 + k^2 \lambda^2)}.
\]

(52)

The head fluctuation autocovariance is the evaluated by taking the Fourier transformation of (52):

\[
R_{hk,hk}(\xi) = \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{0}^{\pi} e^{ik \xi \cos \phi} \Phi_{hk,hk}(k) dk d\phi d\phi.
\]

\[
\approx \frac{\eta^4}{\pi^2} \int_{k=0}^{\infty} \int_{0}^{\infty} \int_{0}^{\pi} e^{ik \xi \cos \phi} \frac{k^2}{(1 + k^2 \lambda^2)} \sin \phi d\phi dk.
\]

(53)

From Bakr et al. [1978] we may express \(k_1 = k \cos \theta \cos \chi + \sin \theta \sin \chi \cos \phi\); hence, making the lag vector \(\xi\) explicit on the coordinates \((\xi, \chi)\), we may obtain
The head fluctuation variance may be calculated by using the \( \xi \to 0 \) identity:

\[
\sigma_{h}^2 = \lim_{\xi \to 0} R_{\text{sh,kh}}(\xi, \lambda) = \frac{\lambda^2 \eta^2 \sigma_{e}^2}{3} (\eta^2 - \sqrt{2}\eta^4 \lambda^4 + 2\sqrt{2}\eta^5 \lambda^3 - 3\eta^6 \lambda^4 + \sqrt{2}\lambda^5).
\]

(55)

### 3.4.2. Three-Dimensional Gaussian Conductivity Covariance

[22] Once more, this integral is difficult to evaluate in closed form, but analytical approximations can be generated exactly by expanding the \( \sin(k\xi) \) and \( \cos(k\xi) \) functions in Maclaurin series and noting that the following component integrals \( G_n \) which involve integrands with positive and even powers of \( k \), are available in closed form:

\[
G_n(\eta, \lambda) = \int_0^\infty \frac{e^{-k^4 \lambda^{1/4}/\eta^4} k^{2n} \mathrm{d}k}{1 + \eta^4 k^4}
\]

(59)

Utilizing (59) in (58) results in an ascending series of component integral solutions for increasing powers of \( k \), which may be assembled to provide arbitrarily high order (but algebraically complicated) estimates of \( R_{\text{sh,kh}} \). The \( \xi \to 0 \) variance limit can be recovered either by the power expansion approach or by integrating (58) directly for \( \xi = 0 \). The result is

\[
R_{\text{sh,kh}}(\xi, \chi) \approx \frac{\lambda^2 \sigma_{e}^2 \eta^4}{8 \pi^{3/2}} \int_0^\infty \int_0^\pi \int_0^{2\pi} e^{-k^4 \lambda^{1/4}/\eta^4} k^2 \sin \theta \sin \chi \cos \phi \left( \frac{\cos \theta \cos \chi + \sin \theta \sin \chi \cos \phi}{1 + k^4 \eta^4} \right) e^{-k^4 \lambda^{1/4}/\eta^4} k^4 \sin \theta \mathrm{d}\phi \mathrm{d}\theta \mathrm{d}k
\]

(58)

Equation (57) leads to the integral expression for \( R_{\text{sh,kh}}^2 \):
4. Discussion of Analytical Results

The head fluctuation variance estimates derived in section 3 for different log conductivity covariance and dimensionality cases represent the substantive results of this paper. The estimates are correct to first order in the stochastic perturbation expansion but also depend on the validity of the proximity assumption made in (18), where the constant \( r_0 \) is used to approximate the slope of the mean solution \( <h> \) near the oscillating boundary. In this section we examine the derived autocovariances, how dimensionality affects the head fluctuation statistics, and how the proximity assumption is best applied.

4.1. Autocovariances

Figure 2 presents induced head fluctuation autocovariances plotted along the longitudinal coordinate (perpendicular to the forcing boundary) for input exponential and Gaussian log \( \kappa \) autocovariance functions with unit integral scales (\( \lambda = 1 \)). In all cases the head fluctuation covariances display significant anticorrelation for lag displacements \( >1 \). For large \( \xi \) the anticorrelation declines to zero. Dimensional effects are modest in these autocovariances. That is, the autocovariance functions for one-, two-, and three-dimensional systems are quite similar, although not identical. These functions are plotted for unit values of the root transmissibility \( \eta \) (23). For lower \( \eta \) (higher forcing frequency), the autocovariance domain contracts to lower values of \( \xi \), while for higher \( \eta \) the autocovariance domain

\[
\sigma_{d h}^2 = \frac{R_{d h}(0)}{24 \eta} \left( \frac{\sqrt{2\pi \lambda}}{\kappa} \right)^2 \left( 1 - 2C \left( \frac{\lambda}{\sqrt{2\pi \eta}} \right) \sin \left( \frac{\lambda^2}{4\eta^2} \right) + 2S \left( \frac{\lambda}{\sqrt{2\pi \eta}} \right) - 1 \cos \left( \frac{\lambda^2}{4\eta^2} \right) \right) + 4\eta. \tag{60}
\]

![Figure 2. Normalized autocovariances of induced head fluctuation as a function of system dimension. Exponential (top) and Gaussian (bottom) log \( \kappa \) autocovariances are considered. Unit values are used for \( \eta \) and \( \lambda \).](image)

![Figure 3. Variation of the location of the maximum hole autocovariance \( \xi_{\text{hole}} \) for the exponential log \( \kappa \) autocovariance in (a) one dimension and (b) three dimensions. Root-transmissibility values \( \eta \) are numbered on the curves.](image)
expands. The contraction or expansion of the autocovariances can be assessed by considering the variation of the lag length at the maximum negative (hole) autocovariance $\xi_{\text{hole}}$ as a function of $\eta$. Figure 3 presents contours of $\xi_{\text{hole}}$ for one-dimensional and three-dimensional exponential $\log \kappa$ models, calculated over wide ranges of $\eta$ and integral scale $\lambda$. For a three-dimensional system the hole location varies more strongly with transmissibility and integral scale than in the one-dimensional case.

4.2. Variance Deflation

[26] For steady flows, variance deflation with increasing system dimensionality was quantified by Bakr et al. [1978], who showed an order of magnitude reduction in variance between one- and three-dimensional cases. This result was complicated by the singular nature of the steady flow spectral relationship for the exponential autocovariance (24), necessitating the use of different $\log \kappa$ autocovariance functions for the two different dimensional cases. In the present work the periodic flow spectral relationship (22) is nonsingular for $\omega > 0$, so comparison of variances is quantitative. Figure 4 shows normalized head fluctuation variances calculated for exponential and Gaussian $\log \kappa$ autocovariance models. It is clear that variances decrease significantly as system dimensionality increases. Algebraic changes in integral scale appear to have modest impacts on variance in two- and three-dimensional cases, although it is noted that one-dimensional variances decline with increasing integral scale, which is counter to the behavior of the two- and three-dimensional variances. This is most likely due to the inability of one-dimensional flows to bypass low-conductivity zones, as would be the case for flows in higher dimensions. Increasing the one-dimensional integral scale reduces the significance of low-conductivity zones removed from the forcing boundary and hence leads to a decline in induced head variance. In higher dimensions, there exists a more complex sampling of scales of heterogeneity along and perpendicular to the forcing boundary. It is also seen in Figure 4 that variances are shown to be strongly dependent on the root transmissibility $\eta$. The variance properties for the exponential and Gaussian autocovariance models are reasonably similar.

[27] Importantly, for the steady flow case, Bakr et al. [1987, p. 268] conclude that “significant errors could be introduced if results from a one-dimensional analysis are used to draw conclusions concerning the effect of spatial variability of flow properties which are inherently three-dimensional.” Here we emphasize this point for the periodic flow case since Figure 4 shows how the one-dimensional variances are significantly larger than the two- and three-dimensional counterparts, and that this difference is magnified greatly as the transmissibility increases (forcing frequency decreases). This is particularly relevant since the dissipative nature of the modal dynamics means that high-frequency (low $\eta$) applications are rare. Most applications for tidal analysis occur in sedimentary formations; let us consider a medium-sand

![Figure 4](image-url)

**Figure 4.** Deflation of the variance of induced head fluctuation as a function of transmissibility $\eta$ and system dimension. (a) Exponential and (b) Gaussian $\log \kappa$ autocovariances are considered. Integral scale values $\lambda$ are indicated in the legends.
T. Trefry and Muffels [2007] confined flow simulation of a domain length \( L = 1000 \) m and with a three-component (2Q1, O1, and K1 diurnal modes) tidal boundary condition run over 60 days to achieve solution stationarity. Model and statistical parameters for this simulation are supplied in Table 1. Solution heads were extracted at 50 evenly spaced observation points within the domain, and the spectral parameters of interest \( \alpha \) and \( \Phi(\Delta \tau) \) were calculated at each observation point by comparing local oscillatory responses during the 60th simulation day with the prescribed boundary condition (see Trefry and Bekele [2004] for a full description of the method for estimating spectral parameters). We note that the 50 observation points are insufficient to permit the spectral density of \( \Delta h \) to be constructed with precision, so we focus attention on estimating the log \( T \) integral scale and variance parameters from the observed fluctuation variance.

### 5. Example Analyses

[29] In order to consolidate the analysis presented in section 3 we examine applications to synthetic data sets for one- and two-dimensional confined flow problems, which are the systems most often encountered in practical tidal studies of aquifers. Accordingly, we will refer to aquifer transmissivity \( T \) within the following text.

#### 5.1. One-Dimensional Tidal Problem

[30] Consider a one-dimensional confined aquifer of length \( L \) subject to a periodic (in time) head oscillation at \( x = 0 \) and a zero-gradient condition at \( x = L \), with dynamics governed by equation (1). This arrangement simulates a simple aquifer tidal analysis problem where tidal oscillations at a coast induce periodic head responses within the aquifer. Because of the linearity of (1) we may focus purely on the time periodic flows, and for a homogeneous transmissivity field the pertinent solution to (10) subject to a unit Dirichlet condition at the origin is

\[
\hat{h}(x, \eta) = \cosh \left( \sqrt{\frac{1}{\eta}} \right) \sinh \left( \frac{\sqrt{i}x}{\eta} \right),
\]

where \( \eta \) is parameterized by the forcing mode \( \omega \) of interest according to (23). [31] For a heterogeneous system the aquifer transmissivity \( T \) was discretized into \( N_r = 1000 \) equal-sized elements with values drawn from a spatial random field for log \( T \) generated with Gaussian autocorrelation structure according to the method of Ruan and McLaughlin [1998]. This transmissivity field was incorporated into a Feflow v6.0 [see Trefry and Muffels, 2007] confined flow simulation with a domain length \( L = 1000 \) m and with a three-component (2Q1, O1, and K1 diurnal modes) tidal boundary condition run over 60 days to achieve solution stationarity. Model and statistical parameters for this simulation are supplied in Table 1. Solution heads were extracted at 50 evenly spaced observation points within the domain, and the spectral parameters of interest \( \alpha \) and \( \Phi(\Delta \tau) \) were calculated at each observation point by comparing local oscillatory responses during the 60th simulation day with the prescribed boundary condition (see Trefry and Bekele [2004] for a full description of the method for estimating spectral parameters). We note that the 50 observation points are insufficient to permit the spectral density of \( \Delta h \) to be constructed with precision, so we focus attention on estimating the log \( T \) integral scale and variance parameters from the observed fluctuation variance.

#### 5.1.1. One-Dimensional Numerical Results

[32] Figure 5 shows the autocovariance of the input log \( T \) distribution and plots the amplitude and phase lag profiles of the observed head oscillations induced by the K1 mode. Results for the 2Q1 and O1 modes are similar. The

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modal frequency ( \omega )</td>
<td>0.85605 ( \times 2\pi ) d(^{-1} )</td>
</tr>
<tr>
<td>( \omega ) (O1)</td>
<td>0.92955 ( \times 2\pi ) d(^{-1} )</td>
</tr>
<tr>
<td>( \omega ) (K1)</td>
<td>1.00272 ( \times 2\pi ) d(^{-1} )</td>
</tr>
<tr>
<td>Storativity ( S )</td>
<td>1.0E-04</td>
</tr>
</tbody>
</table>

[Table 1. Simulation Parameters for the One- and Two-Dimensional Examples]

| Dimension \( N_r \) | 1001 |
| Domain size \( L_x \) | 1000 m |
| Transmissivity properties | |
| Log \( T \) autocorrelation (isotropic) | Gaussian |
| Specified integral scale \( \lambda \) | 2.0 m |
| Harmonic mean of \( T \) | 3.28 m\(^2\) d\(^{-1} \) |
| Variance \( \sigma^2_{\text{log} T} \) | 1.076 |
| Mean root transmissibilities | |
| \( \mu^2_{\text{log} T \text{-K1}} \) | 78.04 m |
| \( \mu^2_{\text{log} T \text{-O1}} \) | 74.93 m |
| \( \mu^2_{\text{log} T \text{-2Q1}} \) | 72.14 m |

| Dimensions \( N_r \times N_x \) | 501 \times 501 |
| Domain size \( L_x \times L_y \) | 500 \times 500 m |
| Transmissivity properties | |
| Log \( T \) autocorrelation (isotropic) | Gaussian |
| Specified integral scale \( \lambda \) | 4.0 m |
| Geometric mean of \( T \) | 23.32 m\(^2\) d\(^{-1} \) |
| Variance \( \sigma^2_{\text{log} T} \) | 0.991 |
| Mean root transmissibilities | |
| \( \mu^2_{\text{log} T \text{-K1}} \) | 208.12 m |
| \( \mu^2_{\text{log} T \text{-O1}} \) | 199.83 m |
| \( \mu^2_{\text{log} T \text{-2Q1}} \) | 192.40 m |
amplitude and phase lag profiles for each mode are well represented by the respective mean profiles $\langle \hat{h} \rangle$ evaluated from (61) using $\eta = \eta_{\text{mean, } \omega}$ (see Table 1), which was established using the harmonic mean of the input $T$ values and the storativity and diurnal angular frequency values in Table 1. The heterogeneity of $T$ induces departures from the mean profile, which define the periodic head fluctuation $\delta h$ as discussed in section 3.1, that is,

$$\delta h(x, \omega) = \hat{h}_{\text{aut}}(x, \omega) - \hat{h}(x, \eta_{\text{mean, } \omega}) = \alpha(x, \omega) \exp(2 \pi \Phi(\Delta \tau(x, \omega))) - \hat{h}(x, \eta_{\text{mean, } \omega}).$$

(62)

In (62), $\delta h$ is a complex-valued spatial fluctuation whose real and imaginary parts are plotted for the K1 mode in Figure 5d. It is noticeable that the fluctuation decays rapidly with distance from the $x = 0$ boundary; the decay length scale is given by the transmissibility. At issue is whether we can recover a useful estimate of the input log $T$ statistics from the observed distribution of $\delta h$. For any forcing frequency $\omega$, equation (35) provides the estimated head fluctuation variance in terms of the log $T$ variance, the integral scale, and the mean transmissibility $\eta$. The effective slope $\nabla \hat{h}_0$ is also required and is calculated over the half domain width following Trefry et al. [2010]. Here the input log $T$ variance is regarded as a known input. Since $\delta h$ is complex valued, its variance is evaluated using the complex form $\sigma_{\delta h}^2 = (N - 1)^{-1} \sum_i (\delta h_i - \mu)^2 (\delta h_i - \mu)$, where $\mu$ is the sample mean of $\delta h$ and $i$ indexes the $N$ observation points.

5.1.2. One-Dimensional Proximity Assumption and Reduced Estimates

Table 2 lists slope estimates for the three modes and also compares variances of $\delta h$ calculated from the simulation (with 50 observation points) with the variance estimates calculated using equation (35). First, we note that the ratios of the variance estimates are close to unity, showing that for this modest log $T$ variance the stochastic variance estimator matches the numerical simulation well. Second, we note that the slope estimation performed over the domain half width seems to provide the correct weighting for the stochastic estimator for this case (where $L_x/\eta \approx 13$). The stochastic estimator scales with the square of the slope, so using a smaller averaging interval from the tidal boundary, where the slope is steeper, will likely lead to overestimates of variance. Finally, we note that the use of “reduced” variance

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Figure 5. One-dimensional example problem. (a) Autocorrelation of the heterogeneous transmissivity field. Solid curve is the input Gaussian correlation, and dots are the actual sample correlation. (b) Induced head oscillation attenuations (triangles) and mean response attenuation (solid curve). (c) Induced head oscillation lags (triangles) and mean response lags (solid curve). (d) Spatial fluctuation of the head response, $\delta h$, real part (dots) and imaginary part (circles).
Table 2. Variance Comparisons for the Example Problems

| $|\nabla h_0|$ | Simulated $\sigma^2_{\text{th}}$ | Predicted $\sigma^2_{\text{th}}$ | Ratio (Simulated/Predicted) |
|----------------|-------------------------------|-----------------------------|-----------------------------|
| **One-Dimensional Example** |
| $\sigma^2_{\log T} = 1.076, \lambda = 2.0$ |
| $\omega_{QO}$ | 1.978E-03 | 4.39E-04 | 4.03E-04 | 1.089 |
| $\omega_{O1}$ | 1.982E-03 | 4.22E-04 | 3.89E-04 | 1.058 |
| $\omega_{K1}$ | 1.985E-03 | 4.18E-04 | 3.74E-04 | 1.120 |
| **Two-Dimensional Example** |
| $\sigma^2_{\log T} = 0.991, \lambda = 4.0$ |
| $\omega_{QO}$ | 3.463E-03 | 1.98E-04 | 2.05E-04 | 0.966 |
| $\omega_{O1}$ | 3.514E-03 | 2.07E-04 | 2.09E-04 | 0.990 |
| $\omega_{K1}$ | 3.558E-03 | 2.18E-04 | 2.13E-04 | 1.023 |

*The mean slope is estimated over half the domain width. There were 50 observations for the 1-D example and 77 observations for the 2-D example. Equation (35) was used for the predictions for the 1-D example, and equation (49) was used for the predictions for the 2-D example.

estimates, i.e., based on either attenuations, lags, or real or imaginary parts alone, does not provide useful results. Thus, the full complex-valued structure of $\delta h$ must be resolved and utilized in order to apply the stochastic estimator.

5.1.3. Efficient Sampling Schemes in One Dimension

[34] In field applications the question arises of efficient sampling scheme design. It is beyond the scope of this paper to present an optimal sampling scheme solution; however, the following remarks may be useful in designing practical monitoring networks. First, it is unlikely that any practical study will see tens of observation locations emplaced along a one-dimensional tidal transect. Many literature studies use less than 10 observation locations to characterize aquifer tidal response [e.g., Trefry and Bekele, 2004, and references therein]. Information on the trade-off between numbers of observation locations and accuracies of the resulting variance estimates is presented for this one-dimensional example in Figure 6, which shows how the variance ratio changes under a variety of different sampling schemes. The horizontal axis shows the number of equally spaced observation points of the scheme. In general, as the number of observation locations increases, the variance ratio moves closer to unity. Figure 6 contains four curves, which differ according to the location of the first point used, i.e., the open squares are for the scheme (denoted 1) that commences with the first point (the tidal boundary condition), the solid squares are for the scheme (denoted 2) that starts on the second point (50 m in from the tidal boundary), and so on for curves 3 (starting 100 m in) and 4 (starting 150 m in). At large numbers of observation locations these schemes provide reasonably similar variance ratios; however, as the number of observation locations reduces below 10 some irregularity and divergence is seen in the variance ratios of the schemes. The scheme that uses the tidal boundary signal (open squares) is the most consistent, remaining approximately constant for more than 10 observation locations. Limiting the sampling region to the near-boundary zone can bias the variance estimation. Calculations for this numerical example (not shown) lead us to suggest that in order to provide a useful variance estimate the observation locations should span a distance into the aquifer of approximately eight root-transmissibility length scales measured from the tidal boundary. Increasing the number of (equally spaced) observation locations will always improve accuracy of the variance estimate.

5.1.4. One-Dimensional Structural Parameters

[35] The simultaneous estimation of $\log T$ integral scale and variance represents an important test of the stochastic theory. Again, we assume that the modal transmissibilities are known, and we construct the following simple least squares objective function with the integral scale $\lambda$ and $\sigma^2_{\log T}$ available as fitting parameters:

$$O_{1D} = 10^6 \left[ \sigma^2_{\text{th, sim}}(\omega_{QO}) - \sigma^2_{\text{th, 35}}(\omega_{QO}, \lambda, \sigma^2_{\log T}) \right]^2 + \left[ \sigma^2_{\text{th, sim}}(\omega_{O1}) - \sigma^2_{\text{th, 35}}(\omega_{O1}, \lambda, \sigma^2_{\log T}) \right]^2 + \left[ \sigma^2_{\text{th, sim}}(\omega_{K1}) - \sigma^2_{\text{th, 35}}(\omega_{K1}, \lambda, \sigma^2_{\log T}) \right]^2,$$

(63)

Here $\sigma^2_{\text{th, 35}}$ refers to the variance estimator in (35). Standard iterative techniques provide values that minimize $O_{1D}$, although the results can be sensitive to starting parameter values. Using the quasi-Newton technique in Mathematica and starting at $\lambda = 10$ m and $\sigma^2_{\log T} = 10$ minimizes $O_{1D}$ at $\lambda = 2.35$ m and $\sigma^2_{\log T} = 1.00$, while starting at $\lambda = 3$ m and $\sigma^2_{\log T} = 2$ yields a minimum at $\lambda = 2.27$ m and...
Two-dimensional example problem boundaries. The transmissivity field was incorporated into a Feflow v6.0 confined flow simulation with a domain of area $500 \times 500$ m and with a three-component (2Q1, O1, and K1 diurnal modes) tidal boundary condition specified along the left edge run over 60 days to achieve solution stationarity. We present results for the K1 mode; results for the 2Q1 and O1 modes are similar. Figure 8 shows the input tidal spectrum at the resolution of the observation sampling interval (0.01 day). Figure 8b plots all 77 observations of induced oscillation amplitude and phase lag for the K1 mode. These closely match the mean solution of equation (61) evaluated with the geometric mean transmissivity, although some dispersion of values is visible. Using equation (62), a regular two-dimensional grid of (complex) periodic head fluctuation values can be constructed. Figure 8c plots the real and imaginary parts of the constructed K1 modal fluctuation versus distance from the tidal boundary, aggregated over all transects. The real parts of the fluctuation are distributed evenly around zero, but the imaginary parts all lie below zero, other than the boundary values, which are synchronous with the tidal forcing signal and thus have zero imaginary parts. There is some evidence of declines in fluctuation magnitudes with penetration distance into the domain. Table 2 lists slope estimates and simulated and predicted fluctuation variances (equation (49)) for the full set of 77 observation points for all three tidal modes. As for the previous example, agreement between the stochastic estimates and the simulated variances is good. In this case the ratio $L_s/\eta \approx 2.5$, and again, the domain half-width average slope seems to be a useful proximity approximation.

5.2. Efficient Sampling Schemes in Two Dimensions

In practical studies it is likely that several transects of observation wells would be deployed to characterize the two-dimensional aquifer statistics. We consider this case here by calculating variances within each transect (perpendicular to the forcing boundary) of Figure 7 separately. Figure 8d shows variance ratios (simulated/predicted) for each transect (numbered curves) as increasing numbers of observations are included in each transect. The transect variances converge as five observations are included, and by the time the transects are fully established with 11 observation points each the variance ratios for all transects lie in the approximate range [0.6, 1.6]. As was seen from the one-dimensional example, even 50 points in a single transect will not achieve exact agreement, largely because of uncertainties associated with the proximity approximation. However, in line with the one-dimensional example results, it is apparent from Figure 8d that even 10 observations made along any transect may be sufficient to constrain the log $T$ variance to within approximately $\pm 50\%$ for moderately heterogeneous systems. We also remark that many other sampling schemes may be used in two dimensions, e.g., transects parallel to the forcing boundary, supersampling, etc. We have not pursued all possible methods; however, in our experience, methods based on perpendicular transects are likely to sample enough of the exponential decay of the fluctuation from the boundary to provide efficient estimates.

5.2.2. Two-Dimensional Structural Parameters

We turn to simultaneous estimation of integral scale and variance of the log $T$ field, assuming knowledge of the modal mean transmissivities. The objective function $O_{2-D}$ defined in equation (64) is used with standard iterative minimizers to obtain estimates of log $T$ statistical parameters.
Here $\sigma_{h,49}^2$ refers to the variance estimator in (49). In this example the optimization results were less consistent, possibly because of the special function representation of the estimator (49). Using the quasi-Newton technique in Mathematica and starting at $\lambda = 10$ m and $\sigma_{\log T}^2 = 10$ minimizes $O_{2,D}$ at $\lambda = 1.48$ m and $\sigma_{\log T}^2 = 5.68$, while starting at $\lambda = 3$ m and $\sigma_{\log T}^2 = 1$ yields a minimum at $\lambda = 3.24$ m and $\sigma_{\log T}^2 = 1.38$. For reference, the input values were 4.0 m and 0.997, respectively (Table 1). The estimates are less accurate than in the previous one-dimensional example. In comparison to the previous example, the two-dimensional sampling array employed here reduces the density of observations in any one transect. Therefore, in practical situations, this kind of two-dimensional estimation procedure would benefit from the use of independent estimates of statistical parameters as constraints. Alternatively, if the target transmissivity field is thought to be isotropic and statistically stationary, a single transect with a large number of observation points may be preferable to several or many transects, each with low numbers of observation points.

6. Conclusions

This work has focused on elucidating the spectral properties of groundwater head oscillations induced by temporal variations in boundary conditions, e.g., surface water stages. The approach used was the standard stationary spectral perturbation method of Gelhar and coworkers, with the point of difference being the use of sequential time-space transformations to resolve both temporal modes (induced by temporal variations of boundary conditions) and spatial modes (present in the spatial inhomogeneity of the aquifer porous medium). The spectral transfer function derived here, (22), unifies previous results for steady flows and pointwise temporal influences. The transfer function was used to generate explicit spatial spectral and autocovariance functions for induced head oscillations for one-, two-, and three-dimensional aquifers. These functions were parameterized by assumed $\log \kappa$ covariance models.
(exponential and Gaussian) and integral scales, and the dependence on the transmissibility length scale was made explicit. [41] This approach does not involve the imposition of specific boundary conditions, other than the requirement that the conditions themselves are linear (to permit spectral decomposition via Fourier transformation). Apart from the linearization inherent in this perturbation solution, one salient approximation is the use of a proximity assumption to estimate the magnitude of the forcing gradient near the boundary. Earlier work indicated that this may not be a sensitive assumption, at least for first-type boundary problems. General attributes of the spectral solutions include the nonsingular nature of the transfer function for nonzero modal frequencies, significant hole autocovariances for all nonzero values of log \( \kappa \) integral scale and aquifer transmissibility, and potentially large variance deflations with increasing aquifer dimensionality. As in the steady flow case, it is noted that the use of the simpler one-dimensional variance formulae may result in significant errors in making inferences about aquifer structures and properties that are inherently three-dimensional. For example, application of the one-dimensional theory with observations of periodic head fluctuation variance may yield transmissibility estimates significantly smaller than those found with higher-dimensional theories. Two-dimensional variances are always closer approximations to three-dimensional variances than one-dimensional variances for these periodic flow systems. [42] Through two computational examples presented in this work it was shown that the simple variance estimators derived here are consistent with the variances of spatial fluctuations of simulated periodic heads induced within heterogeneous conditions by tidal boundary conditions. Therefore, the application of the theories to inverse characterization of aquifer properties is possible. As is to be expected and as demonstrated here, the number and location of observation points is critical to the accuracy and precision of estimates of statistical parameters for the underlying conductivity field. If the linear flow approximation is adequate, identification of multiple forcing modes in the boundary condition spectrum provides an equal number of independent sampling equations for the inversion process. Here we showed how three boundary forcing modes could provide an overdetermined system for jointly estimating the integral scale and log variance of the conductivity field. Interestingly, using a domain half-width averaging interval for the effective mean slope at the boundary (proximity assumption) seemed to be a useful convention over a range of transmissibility-domain length ratios. This deserves further scrutiny as it is seems to be counter to the findings of Slooten et al. [2010], who showed that parameter sensitivity in inverse tidal estimations was tied strongly to the transmissibility length scale. [43] Future studies may usefully seek to clarify the suitability of the proximity assumption under likely practical scenarios including a greater range of log conductivity variances, to elucidate a more sophisticated statistical approach to estimate optimal placement of sampling locations for practical characterization studies, or to combine the present results with stochastic recharge inputs to improve the physical hydrological model.

**Appendix A**

[44] The quantities \( a, b, c, d, e, f, g, \) and \( h \) pertaining to (43) are defined as follows:

\[
A1: \quad a = \sqrt{\left( \eta^2 + \sqrt{2} \eta \lambda + \lambda^2 \right) \left( \eta^2 - \lambda^2 + \sqrt{\eta^4 + \lambda^4} \right)},
\]

\[
A2: \quad b = \sqrt{\left( \eta^2 - \sqrt{2} \eta \lambda + \lambda^2 \right) \left( \eta^2 - \lambda^2 + \sqrt{\eta^4 + \lambda^4} \right)},
\]

\[
A3: \quad c = \sqrt{\lambda^2 + \sqrt{\eta^4 + \lambda^4}},
\]

\[
A4: \quad d = \sqrt{\left( \eta^4 + \lambda^4 + \lambda^2 \sqrt{\eta^4 + \lambda^4} \right)}.
\]

\[
A5: \quad e = \sqrt{\left( \eta^2 - \sqrt{2} \eta \lambda + \lambda^2 \right) \left( -\eta^2 + \lambda^2 + \sqrt{\eta^4 + \lambda^4} \right)},
\]

\[
A6: \quad f = \sqrt{\left( \eta^2 + \sqrt{2} \eta \lambda + \lambda^2 \right) \left( -\eta^2 + \lambda^2 + \sqrt{\eta^4 + \lambda^4} \right)},
\]

\[
A7: \quad g = \frac{1}{\lambda^2} \left( \eta^2 + \eta \sqrt{\eta^4 + \lambda^4} + \sqrt{2} \eta \sqrt{\eta^4 + \lambda^4} \right),
\]

\[
A8: \quad h = \frac{\sqrt{2} \lambda}{\sqrt{\eta^2 + \lambda^2 - \sqrt{\eta^4 + \lambda^4}}},
\]

**Appendix B**

[45] Estimation of the effective slope quantity \( \nabla \hat{h}_0 \) is necessary before the stochastic estimators derived in this paper can be used. Guidance on the choice of the averaging interval for the effective slope is absent, so here we perform a slope estimation for a finite one-dimensional confined aquifer subject to periodic boundary forcing using a simple finite interval approach. The pertinent governing equation is (5), where the transmissivity \( T \) is assumed to be homogeneous. Assuming unit head oscillation amplitude at \( x = 0 \) and zero slope at \( x = L \), the complex solution for the oscillating head \( h \) for mode \( \omega \) is

\[
B1: \quad \hat{h}(x) = \cos(\varepsilon(\zeta - 1)\sqrt{\iota}) \sec h(\varepsilon \sqrt{\iota}),
\]

where, following (23), \( \iota^2 = T/\omega S \) and the nondimensionalized variables \( \zeta = x/L \) and \( \varepsilon = L/\eta \) have been employed. Here \( \zeta \) varies in the interval \([0, 1]\), and \( \varepsilon \) is a positive real number. We choose to estimate the effective slope \( \nu(\nu) \) of \( h \) from the \( \zeta = 0 \) boundary to any point \( \zeta \) within the aquifer domain by a finite interval gradient:

\[
B2: \quad \nu(\zeta) = \frac{\hat{h}(\zeta) - \hat{h}(0)}{\zeta} = \frac{\cosh(\varepsilon(\zeta - 1)\sqrt{\iota}) \sec h(\varepsilon \sqrt{\iota}) - 1}{\zeta}.
\]
The average value of $\nu$ over the one-dimensional domain $\zeta \in [0, 1]$ is denoted $\overline{\nu}$ and is determined by the simple integral form

$$\overline{\nu} = \int_0^1 \nu(\zeta) \, d\zeta. \quad (B3)$$

Figure B1a shows the relationship between $\nu$ and $\zeta$ for three different values of $\varepsilon$. The finite interval slope function displays curvature that increases with increasing $\varepsilon$. For each curve of constant $\varepsilon$ the location of the average value $\overline{\nu}$ is indicated by the dashed horizontal lines; the dashed vertical lines show at which $\zeta$ intercept each value of $\overline{\nu}$ is attained. As $\varepsilon$ declines, the associated $\zeta$ intercept tends monotonically toward 1/2. Trefry et al. [2010] showed that estimating $\nabla k_0$ by evaluating $\nu$ at $\zeta = 1/2$ (the domain half-width approximation) generated spectral density estimates in reasonable agreement with predictions of the Darcian spectral transfer function. Figure B1b evaluates the percentage relative discrepancy between $\overline{\nu}$ and $\nu(1/2)$, expressed as $100 \left( \overline{\nu} - \nu(1/2) \right) / \overline{\nu}$ for a range of values of $\varepsilon$. As implied by Figure B1a, as $\varepsilon$ decreases, the percentage discrepancy also decreases in magnitude, i.e., the domain half-width approximation becomes more accurate. On the basis of this analysis, the domain half-width approximation should provide accurate slope estimates for systems with $\varepsilon = L/\eta < 1$. For larger $\varepsilon$, percentage errors in

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**Figure B1.** Effective slope estimation for a finite, tidally forced aquifer. (a) The variation of $\nu(\zeta)$ (solid lines) throughout the aquifer domain for three different values of $\varepsilon$, together with the respective $\overline{\nu}$ values (dashed lines). (b) A log-log plot of the half-domain approximation error over a wide range of $\varepsilon$. The inset shows a linear plot of the error over a restricted range of $\varepsilon$. 

17 of 18
the slope estimates increase gradually, reaching approximately 75% at \( \varepsilon = 1000 \).

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References

Abramowitz, M., and I. A. Stegun (1965), Handbook of Mathematical Functions, Dover, Mineola, N. Y.


