THE NUMBER OF EDGES IN $k$-QUASI-PLANAR GRAPHS

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Abstract. A graph drawn in the plane is called $k$-quasi-planar if it does not contain $k$ pairwise crossing edges. It has been conjectured for a long time that for every fixed $k$, the maximum number of edges of a $k$-quasi-planar graph with $n$ vertices is $O(n)$. The best known upper bound is $n \log n \log \log k$. In the present paper, we improve this bound to $(n \log n)^{2 \alpha(n) \log k}$ in the special case where the graph is drawn in such a way that every pair of edges meet at most once. Here $\alpha(n)$ denotes the (extremely slowly growing) inverse of the Ackermann function. We also make further progress on the conjecture for $k$-quasi-planar graphs in which every edge is drawn as an $x$-monotone curve. Extending some ideas of Valtr, we prove that the maximum number of edges of such graphs is at most $2^{\alpha(n) \log n}$.

Key words. topological graphs, quasi-planar graphs, Turan-type problems

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1. Introduction. A topological graph is a graph drawn in the plane such that its vertices are represented by points and its edges are represented by nonself-intersecting arcs connecting the corresponding points. In notation and terminology, we make no distinction between the vertices and edges of a graph and the points and arcs representing them, respectively. No edge is allowed to pass through any point representing a vertex other than its endpoints. Any two edges can intersect only in a finite number of points. Tangencies between the edges are not allowed. That is, if two edges share an interior point, then they must properly cross at this point. A topological graph is simple if every pair of its edges intersect at most once: at a common endpoint or at a proper crossing. If the edges of a graph are drawn as straight-line segments, then the graph is called geometric.

Finding the maximum number of edges in a topological graph with a forbidden crossing pattern is a fundamental problem in extremal topological graph theory (see [2, 3, 4, 6, 10, 12, 16, 21, 23]). It follows from Euler’s polyhedral formula that every topological graph on $n$ vertices and with no two crossing edges has at most $3n - 6$ edges. A graph is called $k$-quasi-planar if it can be drawn as a topological graph with no $k$ pairwise crossing edges. A graph is 2-quasi-planar if and only if it is planar. According to an old conjecture (see Problem 1 in section 9.6 of [5]), for any fixed $k \geq 2$ there exists a constant $c_k$ such that every $k$-quasi-planar graph on $n$ vertices has at most $c_k n$ edges. Agarwal et al. [4] were the first to prove this conjecture for...
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For larger values of $k$, first Pach, Shahrokhi, and Szegedy [18] showed that every simple $k$-quasi-planar graph on $n$ vertices has at most $ckn(\log n)^{2k-4}$ edges. For $k \geq 3$ and for all (not necessarily simple) $k$-quasi-planar graphs, Pach, Radoičić, and Tóth [17] established the upper bound $ckn(\log n)^{4k-12}$. Plugging into these proofs the above mentioned result of Ackerman [1], for $k \geq 4$, we obtain the slightly better bounds $ckn(\log n)^{2k-8}$ and $ckn(\log n)^{4k-16}$, respectively. For large values of $k$, the exponent of the polylogarithmic factor in these bounds was improved by Fox and Pach [10], who showed that the maximum number of edges of a $k$-quasi-planar graph on $n$ vertices is $n(\log n)^{O(\log k)}$.

For the number of edges of geometric graphs, that is, graphs drawn by straight-line edges, Valtr [22] proved the upper bound $O(n \log n)$. He also extended this result to simple topological graphs whose edges are drawn as $x$-monotone curves [23].

The aim of this paper is to improve the best known bound, $n(\log n)^{O(\log k)}$, on the number of edges of a $k$-quasi-planar graph in two special cases: for simple topological graphs and for not necessarily simple topological graphs whose edges are drawn as $x$-monotone curves. In both cases, we improve the exponent of the polylogarithmic factor from $O(\log k)$ to $1 + o(1)$.

Theorem 1.1. Let $G = (V, E)$ be a $k$-quasi-planar simple topological graph with $n$ vertices. Then $|E(G)| \leq (n \log n)^{2\alpha(n)^{ck}}$, where $\alpha(n)$ denotes the inverse of the Ackermann function and $ck$ is a constant that depends only on $k$.

Recall that the Ackermann (more precisely, the Ackermann–Péter) function $A(n)$ is defined as follows. Let $A_1(n) = 2n$, and let $A_k(n) = A_{k-1}(A_k(n-1))$ for $k = 2, 3, \ldots$ In particular, we have $A_2(n) = 2^n$, and $A_3(n)$ is an exponential tower of $n$ two’s. Now let $A(n) = A_n(n)$, and let $\alpha(n)$ be defined as $\alpha(n) = \min\{k \geq 1 : A(k) \geq n\}$. This function grows much slower than the inverse of any primitive recursive function.

Theorem 1.2. Let $G = (V, E)$ be a $k$-quasi-planar (not necessarily simple) topological graph with $n$ vertices, whose edges are drawn as $x$-monotone curves. Then $|E(G)| \leq 2^{ckn \log n}$, where $c$ is an absolute constant.

In both proofs, we follow the approach of Valtr [23] and apply results on generalized Davenport–Schinzel sequences. An important new ingredient of the proof of Theorem 1.1 is a corollary of a separator theorem established in [9] and developed in [8]. Theorem 1.2 is not only more general than Valtr’s result, which applies only to simple topological graphs, but its proof gives a somewhat better upper bound: we use a result of Pettie [20], which improves the dependence on $k$ from double exponential to single exponential.

2. Generalized Davenport–Schinzel sequences. The sequence $u = a_1, a_2, \ldots, a_m$ is called $l$-regular if any $l$ consecutive terms are pairwise different. For integers $l, t \geq 2$, the sequence

$$S = s_1, s_2, \ldots, s_{lt}$$

of length $lt$ is said to be of type up$(l, t)$ if the first $l$ terms are pairwise different and

$$s_i = s_{i+l} = s_{i+2l} = \cdots = s_{i+(t-1)l}$$

for every $i$, $1 \leq i \leq l$. For example,

$$a, b, c, a, b, c, a, b, c, a, b, c$$
is a type \textit{up}(3, 4) sequence or, in short, an \textit{up}(3, 4) sequence. We need the following theorem of Klazar [13] on generalized Davenport–Schinzel sequences.

\textbf{Theorem 2.1} (Klazar). For \( l \geq 2 \) and \( t \geq 3 \), the length of any \( l \)-regular sequence over an \( n \)-element alphabet that does not contain a subsequence of type \textit{up}(l, t) has length at most

\[ n \cdot 2^{(l-3)} \cdot (10l)^{10\alpha(n)^t}. \]

For \( l \geq 2 \), the sequence

\[ S = s_1, s_2, \ldots, s_{3l-2} \]

of length \( 3l - 2 \) is said to be of type \textit{up-down-up}(l) if the first \( l \) terms are pairwise different and

\[ s_i = s_{2l-i} = s_{(2l-2)+i} \]

for every \( i, 1 \leq i \leq l \). For example,

\[ a, b, c, d, c, b, a, b, c, d \]

is an \textit{up-down-up}(4) sequence. Klazar and Valtr [14] showed that any \( l \)-regular sequence over an \( n \)-element alphabet, which contains no subsequence of type \textit{up-down-up}(l), has length at most \( 2^t n \) for some constant \( c \). This has been improved by Pettie [20], who proved the following.

\textbf{Lemma 2.2} (see Pettie [20]). For \( l \geq 2 \), the length of any \( l \)-regular sequence over an \( n \)-element alphabet, which contains no subsequence of type \textit{up-down-up}(l), has length at most \( 2^{O(t^2)} n \).

For more results on generalized Davenport–Schinzel sequences, see [15].

\textbf{3. On intersection graphs of curves.} In this section, we prove a useful lemma on intersection graphs of curves. It shows that every collection \( C \) of curves, no two of which intersect many times, contains a large subcollection \( C' \) such that in the partition of \( C' \) into its connected components \( C_1, \ldots, C_t \) in the intersection graph of \( C \), each component \( C_i \) has a vertex connected to all other \(|C_i| - 1\) vertices.

For a graph \( G = (V, E) \), a subset \( V_0 \) of the vertex set is said to be a separator if there is a partition \( V = V_0 \cup V_1 \cup V_2 \) with \(|V_1|, |V_2| \leq \frac{3}{4}|V|\) such that no edge connects a vertex in \( V_1 \) to a vertex in \( V_2 \). We need the following separator lemma for intersection graphs of curves, established in [9].

\textbf{Lemma 3.1} (see Fox and Pach [9]). There is an absolute constant \( c_1 \) such that every collection \( C \) of curves with \( x \) intersection points has a separator of size at most \( c_1 \sqrt{x} \).

Call a collection \( C \) of curves in the plane \textit{decomposable} if there is a partition \( C = C_1 \cup \cdots \cup C_t \) such that each \( C_i \) contains a curve which intersects all other curves in \( C_i \), and for \( i \neq j \), the curves in \( C_i \) are disjoint from the curves in \( C_j \). The following lemma is a strengthening of Proposition 6.3 in [8]. Its proof is essentially the same as that of the original statement. It is included here for completeness.

\textbf{Lemma 3.2}. There is an absolute constant \( c > 0 \) such that every collection \( C \) of \( m \geq 2 \) curves such that each pair of them intersect in at most \( t \) points has a decomposable subcollection of size at least \( \frac{cm}{t \log m} \).

\textbf{Proof of Lemma 3.2}. We prove the following stronger statement. There is an absolute constant \( c > 0 \) such that every collection \( C \) of \( m \geq 2 \) curves whose intersection graph has at least \( x \) edges, and each pair of curves intersects in at most \( t \) points
and has a decomposable subcollection of size at least \( cm_{\log m} + \frac{x}{m} \). Let \( c = \frac{1}{576c_1} \), where \( c_1 \geq 1 \) is the constant in Lemma 3.1. The proof is by induction on \( m \), noting that all collections of curves with at most three elements are decomposable. Define 
\[
d = d(m, x, t) := \frac{cm_{\log m}}{t} + \frac{x}{m}.
\]

Let \( \Delta \) denote the maximum degree of the intersection graph of \( C \). We have \( \Delta < d - 1 \). Otherwise, the subcollection consisting of a curve of maximum degree, together with the curves in \( C \) that intersect it, is decomposable and its size is at least \( d \), and we are done. Also, \( \Delta \geq 2\frac{m}{t} \), since \( 2\frac{m}{t} \) is the average degree of the vertices in the intersection graph of \( C \). Hence, if \( \Delta \geq 2\frac{cm_{\log m}}{t} \), then the desired inequality holds. Thus, we may assume \( \Delta < 2\frac{cm}{t\log m} \).

Applying Lemma 3.1 to the intersection graph of \( C \), we obtain that there is a separator \( V_0 \subset C \) with \( |V_0| \leq c_1 \sqrt{t} \), where \( c_1 \) is the absolute constant in Lemma 3.1. That is, there is a partition \( C = V_0 \cup V_1 \cup V_2 \) with \( |V_1|, |V_2| \leq 2|V|/3 \) such that no curve in \( V_1 \) intersects any curve in \( V_2 \). For \( i = 1, 2 \), let \( m_i = |V_i| \) and let \( x_i \) denote the number of pairs of curves in \( V_i \) that intersect, so that

\[
(1) \quad x_1 + x_2 \geq x - \Delta |V_0| \geq x - 2\frac{cm_{\log m}c_1 \sqrt{t}}{t}.
\]

As no curve in \( V_1 \) intersects any curve in \( V_2 \), the union of a decomposable subcollection of \( V_1 \) and a decomposable subcollection of \( V_2 \) is decomposable. Thus, by the induction hypothesis, \( C \) contains a decomposable subcollection of size at least

\[
d(m_1, x_1, t) + d(m_2, x_2, t) = \frac{cm_1}{\log m_1} + \frac{x_1}{m_1} + \frac{cm_2}{\log m_2} + \frac{x_2}{m_2} \geq \frac{c(m_1 + m_2)}{\log(2m/3)} + \frac{(x_1 + x_2)}{2m/3}.
\]

We split the rest of the proof into two cases.

Case 1. \( x \geq t^{-1}(12c_1c\frac{m}{\log m})^2 \). In this case, by (1), we have \( x_1 + x_2 \geq \frac{5}{6}x \), and hence there is a decomposable subcollection of size at least

\[
d(m_1, x_1, t) + d(m_2, x_2, t) \geq \frac{c(m_1 + m_2)}{\log m} + \frac{5x}{4m} = d + \frac{x}{4m} - \frac{c(m - (m_1 + m_2))}{\log m} \geq d + \frac{x}{4m} - \frac{c_1 \sqrt{t}x}{\log m} = d,
\]

completing the analysis.

Case 2. \( x < t^{-1}(12c_1c\frac{m}{\log m})^2 \). There is a decomposable subcollection of size at least

\[
d(m_1, x_1, t) + d(m_2, x_2, t) \geq \frac{c(m_1 + m_2)}{\log(2m/3)} \geq \frac{c}{t} \left( m - c_1 \sqrt{t}x \right) \left( \frac{1}{\log m} + \frac{1}{2\log^2 m} \right) \geq \frac{c}{t} \left( \frac{m}{\log m} + \frac{m}{2\log^2 m} \right) \geq \frac{c}{t} \left( \frac{m}{\log m} + \frac{m}{4\log^2 m} \right) \geq \frac{c}{t} \left( \frac{m}{\log m} + \frac{x}{m} \right) = d,
\]

where we used \( c = \frac{1}{4(12c_1)^2} = \frac{1}{576c_1^2} \). \( \square \)
4. Simple topological graphs. In this section, we prove Theorem 1.1. The following statement will be crucial for our purposes.

Lemma 4.1. Let $G = (V, E)$ be a $k$-quasi-planar simple topological graph with $n$ vertices. Suppose that $G$ has an edge that crosses every other edge. Then we have $|E| \leq n \cdot 2^{\alpha(n)k}$, where $\alpha(n)$ denotes the inverse Ackermann function and $c'_k$ is a constant that depends only on $k$.

Proof of Lemma 4.1. Let $k \geq 5$ and let $c'_k = 40 \cdot 2^{k^2+2k}$. To simplify the presentation, we do not make any attempt to optimize the value of $c'_k$. Label the vertices of $G$ from 1 to $n$, i.e., let $V = \{1, 2, \ldots, n\}$. Let $e = uv$ be the edge that crosses every other edge in $G$. Note that $d(u) = d(v) = 1$.

Let $E'$ denote the set of edges that cross $e$. Suppose, without loss of generality, that no two of elements of $E'$ cross $e$ at the same point. Let $e_1, e_2, \ldots, e_{|E'|}$ denote the edges in $E'$ listed in the order of their intersection points with $e$ from $u$ to $v$. We create two sequences of vertices $S_1 = p_1, p_2, \ldots, p_{|E'|}$ and $S_2 = q_1, q_2, \ldots, q_{|E'|} \subset V$, as follows. For each $e_i \in E'$, as we move along edge $e$ from $u$ to $v$ and arrive at the intersection point with $e_i$, we turn left and move along edge $e_i$ until we reach its endpoint $u_i$. Then we set $p_i = u_i$. Likewise, as we move along edge $e$ from $u$ to $v$ and arrive at edge $e_i$, we turn right and move along edge $e_i$ until we reach its other endpoint $w_i$. Then we set $q_i = w_i$. Thus, $S_1$ and $S_2$ are sequences of length $|E'|$ over the alphabet $\{1, 2, \ldots, n\}$. See Figure 1 for a small example.

![Figure 1](https://via.placeholder.com/150)

**Fig. 1.** In this example, $S_1 = v_1, v_3, v_4, v_3, v_2$ and $S_2 = v_2, v_2, v_1, v_5, v_5$.

We need two lemmas. The first one is due to Valtr [23].

Lemma 4.2 (Valtr). For $l \geq 1$, at least one of the sequences $S_1, S_2$ defined above contains an $l$-regular subsequence of length at least $|E'|/(4l)$.

Since each edge in $E'$ crosses $e$ exactly once, the proof of Lemma 4.2 can be copied almost verbatim from the proof of Lemma 4 in [23]. Indeed, the only fact about the sequences $S_1$ and $S_2$ it uses is that the edges $e_{j_1}, e_{j_1+1}, \ldots, e_{j_2}$ are spanned by the vertices $p_{j_1}, \ldots, p_{j_2}$ and $q_{j_1}, \ldots, q_{j_2}$, for each pair $j_1 < j_2$.

For the rest of this section, we set $l = 2^{k^2+k}$ and $t = 2^k$.

Lemma 4.3. Neither of the sequences $S_1$ and $S_2$ has a subsequence of type $up(l, t)$.

Proof. By symmetry, it suffices to show that $S_1$ does not contain a subsequence of type $up(l, t)$. The argument is by contradiction. We will prove by induction on $k$ that the existence of such a sequence would imply that $G$ has $k$ pairwise crossing
edges. The base cases $k = 1, 2$ are trivial. Now assume the statement holds up to $k - 1$. Let

$$S = s_1, s_2, \ldots, s_{lt}$$

be our up($l, t$) sequence of length $lt$ such that the first $l$ terms are pairwise distinct and for $i = 1, 2, \ldots, l$ we have

$$s_i = s_{i+l} = s_{i+2l} = s_{i+3l} = \cdots = s_{i+(t-1)l}.$$  

For each $i = 1, 2, \ldots, l$, let $v_i \in V$ denote the vertex $s_i$. Moreover, let $a_{i,j}$ be the arc emanating from vertex $v_i$ to the edge $e$ corresponding to $s_{i+jl}$ for $j = 0, 1, 2, \ldots, t - 1$. We will think of $s_{i+jl}$ as a point on $a_{i,j}$ very close but not on edge $e$. For simplicity, we will let $s_{i+jl} = s_q$ for all $q \in N$ and $a_{i,j} = a_{i,j'}$ for all $j \in Z$, where $j' \in \{0, 1, 2, \ldots, t - 1\}$ is such that $j \equiv j' \pmod{t}$. Hence there are $l$ distinct vertices $v_1, \ldots, v_l$, each vertex of which has $t$ arcs emanating from it to the edge $e$.

Consider the arrangement formed by the $t$ arcs emanating from $v_1$ and the edge $e$. Since $G$ is simple, these arcs partition the plane into $t$ regions. By the pigeonhole principle, there is a subset $V' \subset \{v_1, \ldots, v_l\}$ of size

$$\frac{l - 1}{t} = \frac{2k^2 + k - 1}{2k}$$

such that all of the vertices of $V'$ lie in the same region. Let $j_0 \in \{0, 1, 2, \ldots, t - 1\}$ be an integer such that $V'$ lies in the region bounded by $a_{1,j_0}$, $a_{1,j_0+1}$, and $e$. See Figure 2. In the case $j_0 = t - 1$, the set $V'$ lies in the unbounded region.

**Fig. 2. Vertices of $V'$ lie in the region enclosed by $a_{1,j_0}$, $a_{1,j_0+1}$, $e$.**

Let $v_i \in V'$ and $a_{i,j_0+j_1}$ be an arc emanating from $v_i$ for $j_1 \geq 1$. Notice that $a_{i,j_0+j_1}$ cannot cross both $a_{1,j_0}$ and $a_{1,j_0+1}$. Indeed, as $a_{i,j_0+j_1}$ can cross each of $a_{1,j_0}$ and $a_{1,j_0+1}$ at most once; had it crossed both of them, its endpoint $s_{1,j_0+j_1}$ would be in the shaded region on Figure 2. Suppose that $a_{i,j_0+j_1}$ crosses $a_{1,j_0+1}$. Then all arcs emanating from $v_1$,

$$A = \{a_{i,j_0+1}, a_{i,j_0+2}, \ldots, a_{i,j_0+j_1-1}\}$$

must also cross $a_{1,j_0+1}$. Indeed, let $\gamma$ be the simple closed curve created by the arrangement

$$a_{i,j_0+j_1} \cup a_{1,j_0+1} \cup e.$$
Since \( a_{i,j_0 + j_1}, a_{i,j_0 + 1}, \) and \( e \) pairwise intersect at precisely one point, \( \gamma \) is well defined. We define points \( x = a_{i,j_0 + j_1} \cap a_{1,j_0 + 1} \) and \( y = a_{1,j_0 + 1} \cap e \), and orient \( \gamma \) in the direction from \( x \) to \( y \) along \( \gamma \).

In view of the fact that \( a_{i,j_0 + j_1} \) intersects \( a_{1,j_0 + 1} \), the vertex \( v_i \) must lie to the right of \( \gamma \). Moreover, since the arc from \( x \) to \( y \) along \( a_{1,j_0 + 1} \) is a subset of \( \gamma \), the points corresponding to the subsequence

\[
S' = \{ s_q \in S \mid 2 + (j_0 + 1)l \leq q \leq (i - 1) + (j_0 + j_1)l \}
\]

must lie to the left of \( \gamma \). Hence, \( \gamma \) separates vertex \( v_i \) and the points of \( S' \). Therefore, using again that \( G \) is simple, each arc from \( A \) must cross \( a_{1,j_0 + 1} \) (these arcs cannot cross \( a_{i,j_0 + j_1} \)). See Figure 3.

![Diagram](image)

**Fig. 3. Defining \( \gamma \) and its orientation.**

By the same argument, if the arc \( a_{i,j_0 - j_1} \) crosses \( a_{1,j_0} \) for \( j_1 \geq 1 \), then the arcs emanating from \( v_i \),

\[
a_{i,j_0 - 1}, a_{i,j_0 - 2}, \ldots, a_{i,j_0 - j_1 + 1},
\]

must also cross \( a_{1,j_0} \). Since \( a_{i,j_0 + 1/2} = a_{i,j_0 - 1/2} \), we have the following observation.

**Observation 4.4.** For half of the vertices \( v_i \in V' \), the arcs emanating from \( v_i \) satisfy that either

1. \( a_{i,j_0 + 1}, a_{i,j_0 + 2}, \ldots, a_{i,j_0 + t/2} \) all cross \( a_{1,j_0 + 1} \), or
2. \( a_{i,j_0 - 1}, a_{i,j_0 - 2}, \ldots, a_{i,j_0 - t/2} \) all cross \( a_{1,j_0} \).
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Since $t/2 = 2^{k-1}$ and

$$\frac{|V'|}{2} \geq \frac{l - 1}{2t} = \frac{2^{k^2 + k} - 1}{2 \cdot 2^k} \geq 2^{(k-1)^2 + (k-1)},$$

by Observation 4.4, we obtain an $up(2^{(k-1)^2 + (k-1)}, 2^{k-1})$ sequence such that the corresponding arcs all cross either $a_{1,j_0}$ or $a_{1,j_0+1}$. By the induction hypothesis, it follows that there exist $k$ pairwise crossing edges.

Now we are ready to complete the proof of Lemma 4.1. By Lemma 4.2 we know that, say, $S_1$ contains an $l$-regular subsequence of length $|E'|/(4l)$. By Theorem 2.1 and Lemma 4.3, this subsequence has length at most

$$n \cdot l 2^{((l-3) \cdot (10l)^{10\alpha(n)/l}},$$

which implies

$$|E'| \leq 4 \cdot n \cdot l 2^{(l-3) \cdot (10l)^{10\alpha(n)/l}},$$

Since $c'_k = 40 \cdot l t = 40 \cdot 2^{k^2 + 2k}$, $\alpha(n) \geq 2$, and $k \geq 5$, we have

$$|E| = |E'| + 1 \leq n \cdot 2^{\alpha(n)^{c_k}},$$

which completes the proof of Lemma 4.1.

Now we are in position to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $G = (V, E)$ be a $k$-quasi-planar simple topological graph on $n$ vertices. By Lemma 3.2, there is a subset $E' \subset E$ such that $|E'| \geq c|E|/\log |E|$, where $c$ is an absolute constant and $E'$ is decomposable. Hence, there is a partition

$$E' = E_1 \cup E_2 \cup \cdots \cup E_t$$

such that each $E_i$ has an edge $e_i$ that intersects every other edge in $E_i$, and for $i \neq j$, the edges in $E_i$ are disjoint from the edges in $E_j$. Let $V_i$ denote the set of vertices that are the endpoints of the edges in $E_i$, and let $n_i = |V_i|$. By Lemma 4.1, we have

$$|E_i| \leq n_i 2^{\alpha(n)^k} + 2n_i,$$

where the $2n_i$ term accounts for the edges that share a vertex with $e_i$. Hence,

$$\frac{c|E|}{\log |E|} \leq \sum_{i=1}^{t} n_i 2^{\alpha(n)^k} + 2n_i \leq n 2^{\alpha(n)^k} + 2n.$$

Therefore, we obtain

$$|E| \leq (n \log n) 2^{\alpha(n)^{c_k}}$$

for a sufficiently large constant $c_k$. \qed
5. \textit{x-monotone topological graphs.} The aim of this section is to prove Theorem 1.2.

\textit{Proof of Theorem 1.2.} For \( k \geq 2 \), let \( g_k(n) \) be the maximum number of edges in a \( k \)-quasi-planar topological graph whose edges are drawn as \( x \)-monotone curves. We will prove by induction on \( n \) that

\[
g_k(n) \leq 2^{ck^6} n \log n,
\]

where \( c \) is a sufficiently large absolute constant.

The base case is trivial. For the inductive step, let \( G = (V, E) \) be a \( k \)-quasi-planar topological graph whose edges are drawn as \( x \)-monotone curves, and let the vertices be labeled \( 1, 2, \ldots, n \). Let \( L \) be a vertical line that partitions the vertices into two parts, \( V_1 \) and \( V_2 \), such that \( |V_1| = \lfloor n/2 \rfloor \) vertices lie to the left of \( L \), and \( |V_2| = \lceil n/2 \rceil \) vertices lie to the right of \( L \). Furthermore, let \( E_1 \) denote the set of edges induced by \( V_1 \), let \( E_2 \) denote the set of edges induced by \( V_2 \), and let \( E' \) be the set of edges that intersect \( L \). Clearly, we have

\[
|E_1| \leq g_k(\lfloor n/2 \rfloor) \quad \text{and} \quad |E_2| \leq g_k(\lceil n/2 \rceil).
\]

It suffices to show that

\[
|E'| \leq 2^{ck^6/2} n,
\]

since this would imply

\[
g_k(n) \leq g_k(\lfloor n/2 \rfloor) + g_k(\lceil n/2 \rceil) + 2^{ck^6/2} n \leq 2^{ck^6} n \log n.
\]

In the rest of the proof, we consider only the edges belonging to \( E' \). For each vertex \( v_i \in V_1 \), consider the graph \( G_i \), whose vertices are the edges with \( v_i \) as a left endpoint, and two vertices in \( G_i \) are adjacent if the corresponding edges cross at some point to the left of \( L \). Since \( G_i \) is an incomparability graph (see [7, 11]) and does not contain a clique of size \( k \), \( G_i \) contains an independent set of size \( |E(G_i)|/(k-1) \). We keep all edges that correspond to the elements of this independent set, and discard all other edges incident to \( v_i \). After repeating this process for all vertices in \( V_1 \), we are left with at least \( |E'|/(k-1) \) edges.

Now we continue this process on the other side. For each vertex \( v_j \in V_2 \), consider the graph \( G_j \), whose vertices are the edges with \( v_j \) as a right endpoint, and two vertices in \( G_j \) are adjacent if the corresponding edges cross at some point to the right of \( L \). Since \( G_j \) is an incomparability graph and does not contain a clique of size \( k \), \( G_j \) contains an independent set of size \( |E(G_j)|/(k-1) \). We keep all edges that corresponds to this independent set, and discard all other edges incident to \( v_j \). After repeating this process for all vertices in \( V_2 \), we are left with at least \( |E'|/(k-1)^2 \) edges.

We order the remaining edges \( e_1, e_2, \ldots, e_m \) in the order in which they intersect \( L \) from bottom to top. (We assume, without loss of generality, that any two intersection points are distinct.) Define two sequences, \( S_1 = p_1, p_2, \ldots, p_m \) and \( S_2 = q_1, q_2, \ldots, q_m \), such that \( p_i \) denotes the left endpoint of edge \( e_i \) and \( q_i \) denotes the right endpoint of \( e_i \). We need the following lemma.

\textsc{Lemma 5.1.} Neither of the sequences \( S_1 \) and \( S_2 \) has subsequence of type \( \text{up-down-up}(k^3 + 2) \).
The number of edges in $k$-quasi-planar graphs

Proof. By symmetry, it suffices to show that $S_1$ does not have a subsequence of type up-down-up$(k^3 + 2)$. Suppose for contradiction that $S_1$ does contain such a subsequence. Then there is a sequence

$$S = s_1, s_2, \ldots, s_{3(k^3+2)−2}$$

such that the integers $s_1, \ldots, s_{k^3+2}$ are pairwise distinct and

$$s_i = s_{2(k^3+2)−i} = s_{2(k^3+2)−2+i}$$

for $i = 1, 2, \ldots, k^3+2$.

For each $i \in \{1, 2, \ldots, k^3+2\}$, let $v_i \in V_1$ denote the label (vertex) of $s_i$ and let $x_i$ denote the $x$-coordinate of the vertex $v_i$. Moreover, let $a_i$ be the arc emanating from vertex $v_i$ to the point on $L$ that corresponds to $s_{2(k^3+2)−i}$. Let $A = \{a_2, a_3, \ldots, a_{k^3+1}\}$. Note that the arcs in $A$ are enumerated downwards with respect to their intersection points with $L$, and they correspond to the elements of the “middle” section of the up-down-up sequence. We define two partial orders on $A$ as follows:

$$a_i \prec_1 a_j \text{ if } i < j, \text{ } x_i < x_j \text{ and the arcs } a_i, a_j \text{ do not intersect, }$$

$$a_i \prec_2 a_j \text{ if } i < j, \text{ } x_i > x_j \text{ and the arcs } a_i, a_j \text{ do not intersect.}$$

Clearly, $\prec_1$ and $\prec_2$ are partial orders. If two arcs are not comparable by either $\prec_1$ or $\prec_2$, then they cross. Since $G$ does not contain $k$ pairwise crossing edges, by Dilworth’s theorem [7], there exist $k$ arcs $\{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\}$ such that they are pairwise comparable by either $\prec_1$ or $\prec_2$. Now the proof falls into two cases.

Case 1. Suppose that $a_{i_1} \prec_1 a_{i_2} \prec_1 \cdots \prec_1 a_{i_k}$. Then the arcs emanating from $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ to the points corresponding to $s_{2(k^3+2)−2+i_1}, s_{2(k^3+2)−2+i_2}, \ldots, s_{2(k^3+2)−2+i_k}$ are pairwise crossing. See Figure 4.

![Figure 4. Case 1 of the proof of Lemma 5.1.](image)

Case 2. Suppose that $a_{i_1} \prec_2 a_{i_2} \prec_2 \cdots \prec_2 a_{i_k}$. Then the arcs emanating from $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ to the points corresponding to $s_{i_1}, s_{i_2}, \ldots, s_{i_k}$ are pairwise crossing. See Figure 5.

![Figure 5. Case 2 of the proof of Lemma 5.1.](image)

We are now ready to complete the proof of Theorem 1.2. By Lemma 4.2, we know that, $S_1$, say, contains a $(k^3 + 2)$-regular subsequence of length

$$\frac{|E'|}{4(k^3+2)(k-1)^2}.$$
By Lemmas 2.2 and 5.1, this subsequence has length at most $2^{c'}k^6n$, where $c'$ is an absolute constant. Hence, we have

$$\frac{|E'|}{4(k^3 + 2)(k - 1)^2} \leq 2^{c'}k^6n,$$

which implies that

$$|E'| \leq 4k^52^{c'}k^n \leq 2^{ck^6/2}n$$

for a sufficiently large absolute constant $c$.  

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**REFERENCES**


