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THE NUMBER OF EDGES IN $k$-QUASI-PLANAR GRAPHS

JACOB FOX†, JÁNOS PACH‡, AND ANDREW SUK†

Abstract. A graph drawn in the plane is called $k$-quasi-planar if it does not contain $k$ pairwise crossing edges. It has been conjectured for a long time that for every fixed $k$, the maximum number of edges of a $k$-quasi-planar graph with $n$ vertices is $O(n)$. The best known upper bound is $n(\log n)^{O(\log k)}$. In the present paper, we improve this bound to $(n \log n)^{2^{\alpha(n)/k}}$ in the special case where the graph is drawn in such a way that every pair of edges meet at most once. Here $\alpha(n)$ denotes the (extremely slowly growing) inverse of the Ackermann function. We also make further progress on the conjecture for $k$-quasi-planar graphs in which every edge is drawn as an $x$-monotone curve. Extending some ideas of Valtr, we prove that the maximum number of edges of such graphs is at most $2^c k^6 n \log n$.

Key words. topological graphs, quasi-planar graphs, Turan-type problems

AMS subject classifications. 05C35, 05C10, 68R10, 52C10

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1. Introduction. A topological graph is a graph drawn in the plane such that its vertices are represented by points and its edges are represented by nonself-intersecting arcs connecting the corresponding points. In notation and terminology, we make no distinction between the vertices and edges of a graph and the points and arcs representing them, respectively. No edge is allowed to pass through any point representing a vertex other than its endpoints. Any two edges can intersect only in a finite number of points. Tangencies between the edges are not allowed. That is, if two edges share an interior point, then they must properly cross at this point. A topological graph is simple if every pair of its edges intersect at most once: at a common endpoint or at a proper crossing. If the edges of a graph are drawn as straight-line segments, then the graph is called geometric.

Finding the maximum number of edges in a topological graph with a forbidden crossing pattern is a fundamental problem in extremal topological graph theory (see [2, 3, 4, 6, 10, 12, 16, 21, 23]). It follows from Euler’s polyhedral formula that every topological graph on $n$ vertices and with no two crossing edges has at most $3n - 6$ edges. A graph is called $k$-quasi-planar if it can be drawn as a topological graph with no $k$ pairwise crossing edges. A graph is 2-quasi-planar if and only if it is planar. According to an old conjecture (see Problem 1 in section 9.6 of [5]), for any fixed $k \geq 2$ there exists a constant $c_k$ such that every $k$-quasi-planar graph on $n$ vertices has at most $c_k n$ edges. Agarwal et al. [4] were the first to prove this conjecture for

For larger values of $k$, first Pach, Shahrokhi, and Szegedy [18] showed that every simple $k$-quasi-planar graph on $n$ vertices has at most $c_k n (\log n)^{2k-4}$ edges. For $k \geq 3$ and for all (not necessarily simple) $k$-quasi-planar graphs, Pach, Radoiˇci´c, and T´oth [17] established the upper bound $c_k n (\log n)^{4k-12}$. Plugging into these proofs the above mentioned result of Ackerman [1], for $k \geq 4$, we obtain the slightly better bounds $c_k n (\log n)^{2k-8}$ and $c_k n (\log n)^{4k-16}$, respectively. For large values of $k$, the exponent of the polylogarithmic factor in these bounds was improved by Fox and Pach [10], who showed that the maximum number of edges of a $k$-quasi-planar graph on $n$ vertices is $n (\log n)^{O(\log k)}$.

For the number of edges of geometric graphs, that is, graphs drawn by straight-line edges, Valtr [22] proved the upper bound $O(n \log n)$. He also extended this result to simple topological graphs whose edges are drawn as $x$-monotone curves [23].

The aim of this paper is to improve the best known bound, $n (\log n)^{O(\log k)}$, on the number of edges of a $k$-quasi-planar graph in two special cases: for simple topological graphs and for not necessarily simple topological graphs whose edges are drawn as $x$-monotone curves. In both cases, we improve the exponent of the polylogarithmic factor from $O(\log k)$ to $1 + o(1)$.

**Theorem 1.1.** Let $G = (V, E)$ be a $k$-quasi-planar simple topological graph with $n$ vertices. Then $|E(G)| \leq (n \log n)^{2^{o(\sqrt{n})} k}$, where $\alpha(n)$ denotes the inverse of the Ackermann function and $c_k$ is a constant that depends only on $k$.

Recall that the Ackermann (more precisely, the Ackermann–P´eter) function $A(n)$ is defined as follows. Let $A_1(n) = 2n$, and let $A_k(n) = A_{k-1}(A_k(n-1))$ for $k = 2, 3, \ldots$. In particular, we have $A_2(n) = 2^n$, and $A_3(n)$ is an exponential tower of $n$ two’s. Now let $A(n) = A_n(n)$, and let $\alpha(n)$ be defined as $\alpha(n) = \min \{k \geq 1 : A(k) \geq n\}$. This function grows much slower than the inverse of any primitive recursive function.

**Theorem 1.2.** Let $G = (V, E)$ be a $k$-quasi-planar (not necessarily simple) topological graph with $n$ vertices, whose edges are drawn as $x$-monotone curves. Then $|E(G)| \leq 2^{c_k n \log n}$, where $c$ is an absolute constant.

In both proofs, we follow the approach of Valtr [23] and apply results on generalized Davenport–Schinzel sequences. An important new ingredient of the proof of Theorem 1.1 is a corollary of a separator theorem established in [9] and developed in [8]. Theorem 1.2 is not only more general than Valtr’s result, which applies only to simple topological graphs, but its proof gives a somewhat better upper bound: we use a result of Pettie [20], which improves the dependence on $k$ from double exponential to single exponential.

**2. Generalized Davenport–Schinzel sequences.** The sequence $u = a_1, a_2, \ldots, a_m$ is called $l$-regular if any $l$ consecutive terms are pairwise different. For integers $l, t \geq 2$, the sequence

$$S = s_1, s_2, \ldots, s_{lt}$$

of length $lt$ is said to be of type up$(l, t)$ if the first $l$ terms are pairwise different and

$$s_i = s_{i+l} = s_{i+2l} = \cdots = s_{i+(t-1)l}$$

for every $i, 1 \leq i \leq l$. For example,

$$a, b, c, a, b, c, a, b, c, a, b, c$$
is a type up(3, 4) sequence or, in short, an up(3, 4) sequence. We need the following theorem of Klazar [13] on generalized Davenport–Schinzel sequences.

**Theorem 2.1 (Klazar).** For \( l \geq 2 \) and \( t \geq 3 \), the length of any \( l \)-regular sequence over an \( n \)-element alphabet that does not contain a subsequence of type up\((l,t)\) has length at most
\[
n \cdot 2^{(l-3)} \cdot (10l)^{10\alpha(n)^t}.
\]

For \( l \geq 2 \), the sequence
\[
S = s_1, s_2, \ldots, s_{3l-2}
\]
of length \( 3l - 2 \) is said to be of type \( \text{up-down-up}(l) \) if the first \( l \) terms are pairwise different and
\[
s_i = s_{2l-i} = s_{(2l-2)+i}
\]
for every \( i, 1 \leq i \leq l \). For example,
\[
a, b, c, d, c, b, a, b, c, d
\]
is an \( \text{up-down-up}(4) \) sequence. Klazar and Valtr [14] showed that any \( l \)-regular sequence over an \( n \)-element alphabet, which contains no subsequence of type up-down-up\((l)\), has length at most \( 2^l n \) for some constant \( c \). This has been improved by Pettie [20], who proved the following.

**Lemma 2.2 (see Pettie [20]).** For \( l \geq 2 \), the length of any \( l \)-regular sequence over an \( n \)-element alphabet, which contains no subsequence of type up-down-up\((l)\), has length at most \( 2^{O(l^2)} n \).

For more results on generalized Davenport–Schinzel sequences, see [15].

3. **On intersection graphs of curves.** In this section, we prove a useful lemma on intersection graphs of curves. It shows that every collection \( C \) of curves, no two of which intersect many times, contains a large subcollection \( C' \) such that in the partition of \( C' \) into its connected components \( C_1, \ldots, C_t \) in the intersection graph of \( C \), each component \( C_i \) has a vertex connected to all other \( |C_i| - 1 \) vertices.

For a graph \( G = (V, E) \), a subset \( V_0 \) of the vertex set is said to be a *separator* if there is a partition \( V = V_0 \cup V_1 \cup V_2 \) with \( |V_1|, |V_2| \leq \frac{n}{4} \) such that no edge connects a vertex in \( V_1 \) to a vertex in \( V_2 \). We need the following separator lemma for intersection graphs of curves, established in [9].

**Lemma 3.1 (see Fox and Pach [9]).** There is an absolute constant \( c_1 \) such that every collection \( C \) of curves with \( x \) intersection points has a separator of size at most \( c_1 \sqrt{x} \).

Call a collection \( C \) of curves in the plane *decomposable* if there is a partition \( C = C_1 \cup \cdots \cup C_t \) such that each \( C_i \) contains a curve which intersects all other curves in \( C_i \), and for \( i \neq j \), the curves in \( C_i \) are disjoint from the curves in \( C_j \). The following lemma is a strengthening of Proposition 6.3 in [8]. Its proof is essentially the same as that of the original statement. It is included here for completeness.

**Lemma 3.2.** There is an absolute constant \( c > 0 \) such that every collection \( C \) of \( m \geq 2 \) curves such that each pair of them intersect in at most \( t \) points has a decomposable subcollection of size at least \( \frac{cm}{t \log m} \).

**Proof of Lemma 3.2.** We prove the following stronger statement. There is an absolute constant \( c > 0 \) such that every collection \( C \) of \( m \geq 2 \) curves whose intersection graph has at least \( x \) edges, and each pair of curves intersects in at most \( t \) points...
and has a decomposable subcollection of size at least \( \frac{cm}{t \log m} + \frac{x}{m} \). Let \( c = \frac{1}{576c_1} \), where \( c_1 \geq 1 \) is the constant in Lemma 3.1. The proof is by induction on \( m \), noting that all collections of curves with at most three elements are decomposable. Define 
\[ d = d(m, x, t) := \frac{cm}{t \log m} + \frac{x}{m}. \]

Let \( \Delta \) denote the maximum degree of the intersection graph of \( C \). We have \( \Delta < d - 1 \). Otherwise, the subcollection consisting of a curve of maximum degree, together with the curves in \( \Delta \), is decomposable and its size is at least \( d \), and we are done. Also, \( \Delta \geq 2\frac{m}{t} \) since \( 2\frac{m}{t} \) is the average degree of the vertices in the intersection graph of \( C \). Hence, if \( \Delta \geq 2\frac{cm}{t \log m} \), then the desired inequality holds. Thus, we may assume \( \Delta < 2\frac{cm}{t \log m} \).

Applying Lemma 3.1 to the intersection graph of \( C \), we obtain that there is a separator \( V_0 \subset C \) with \( |V_0| \leq c_1 \sqrt{tx} \), where \( c_1 \) is the absolute constant in Lemma 3.1. That is, there is a partition \( C = V_0 \cup V_1 \cup V_2 \) with \( |V_1|, |V_2| \leq 2|V|/3 \) such that no curve in \( V_1 \) intersects any curve in \( V_2 \). For \( i = 1, 2 \), let \( m_i = |V_i| \) and let \( x_i \) denote the number of pairs of curves in \( V_i \) that intersect, so that

\[ x_1 + x_2 \geq x - \Delta |V_0| \geq x - 2\frac{cm}{t \log m} c_1 \sqrt{tx}. \]

As no curve in \( V_1 \) intersects any curve in \( V_2 \), the union of a decomposable subcollection of \( V_1 \) and a decomposable subcollection of \( V_2 \) is decomposable. Thus, by the induction hypothesis, \( C \) contains a decomposable subcollection of size at least

\[ d(m_1, x_1, t) + d(m_2, x_2, t) = \frac{cm_1}{t \log m_1} + \frac{x_1}{m_1} + \frac{cm_2}{t \log m_2} + \frac{x_2}{m_2} \]

\[ \geq \frac{c(m_1 + m_2)}{t \log (2m/3)} + \frac{(x_1 + x_2)}{2m/3}. \]

We split the rest of the proof into two cases.

**Case 1.** \( x \geq t^{-1}(12c_1 c\frac{m}{t \log m})^2 \). In this case, by (1), we have \( x_1 + x_2 \geq \frac{5}{6}x \), and hence there is a decomposable subcollection of size at least

\[ d(m_1, x_1, t) + d(m_2, x_2, t) \geq \frac{c(m_1 + m_2)}{t \log m} + \frac{5x}{4m} = d + \frac{x}{4m} - \frac{c(m - (m_1 + m_2))}{t \log m} \]

\[ \geq d + \frac{x}{4m} - \frac{c_1 c \sqrt{tx}}{t \log m} > d, \]

completing the analysis.

**Case 2.** \( x < t^{-1}(12c_1 c\frac{m}{t \log m})^2 \). There is a decomposable subcollection of size at least

\[ d(m_1, x_1, t) + d(m_2, x_2, t) \geq \frac{c(m_1 + m_2)}{t \log (2m/3)} \geq \frac{c}{t} \left( m - c_1 \sqrt{tx} \right) \left( \frac{1}{\log m} + \frac{1}{2 \log^2 m} \right) \]

\[ \geq \frac{c}{t} \left( \frac{m}{\log m} + \frac{m}{2 \log^2 m} - \frac{2c_1 \sqrt{tx}}{\log m} \right) \geq \frac{c}{t} \left( \frac{m}{\log m} + \frac{m}{4 \log^2 m} \right) \]

\[ \geq \frac{c}{t} \left( \frac{m}{\log m} + \frac{m}{4 \log^2 m} \right) \geq \frac{cm}{t \log m} + \frac{x}{m} = d, \]

where we used \( c = \frac{1}{4(12c_1)^2} = \frac{1}{576c_1^2} \). \( \square \)
4. Simple topological graphs. In this section, we prove Theorem 1.1. The following statement will be crucial for our purposes.

**Lemma 4.1.** Let \( G = (V, E) \) be a \( k \)-quasi-planar simple topological graph with \( n \) vertices. Suppose that \( G \) has an edge that crosses every other edge. Then we have
\[
|E| \leq n \cdot 2^{\alpha(n) k},
\]
where \( \alpha(n) \) denotes the inverse Ackermann function and \( c'_k \) is a constant that depends only on \( k \).

**Proof of Lemma 4.1.** Let \( k \geq 5 \) and let \( c'_k = 40 \cdot 2^{k^2 + 2k} \). To simplify the presentation, we do not make any attempt to optimize the value of \( c'_k \). Label the vertices of \( G \) from 1 to \( n \), i.e., let \( V = \{1, 2, \ldots, n \} \). Let \( e = uv \) be the edge that crosses every other edge in \( G \). Note that \( d(u) = d(v) = 1 \).

Let \( E' \) denote the set of edges that cross \( e \). Suppose, without loss of generality, that no two of elements of \( E' \) cross \( e \) at the same point. Let \( e_1, e_2, \ldots, e_{|E'|} \) denote the edges in \( E' \) listed in the order of their intersection points with \( e \) from \( u \) to \( v \). We create two sequences of vertices \( S_1 = p_1, p_2, \ldots, p_{|E'|} \) and \( S_2 = q_1, q_2, \ldots, q_{|E'|} \subseteq V \), as follows. For each \( e_i \in E' \), as we move along edge \( e \) from \( u \) to \( v \) and arrive at the intersection point with \( e_i \), we turn left and move along edge \( e_i \) until we reach its endpoint \( u_i \). Then we set \( p_i = u_i \). Likewise, as we move along edge \( e \) from \( u \) to \( v \) and arrive at edge \( e_i \), we turn right and move along edge \( e_i \) until we reach its other endpoint \( w_i \). Then we set \( q_i = w_i \). Thus, \( S_1 \) and \( S_2 \) are sequences of length \( |E'| \) over the alphabet \( \{1, 2, \ldots, n\} \). See Figure 1 for a small example.

![Figure 1](image-url)

We need two lemmas. The first one is due to Valtr [23].

**Lemma 4.2 (Valtr).** For \( l \geq 1 \), at least one of the sequences \( S_1, S_2 \) defined above contains an \( l \)-regular subsequence of length at least \( |E'|/(4l) \). \( \square \)

Since each edge in \( E' \) crosses \( e \) exactly once, the proof of Lemma 4.2 can be copied almost verbatim from the proof of Lemma 4 in [23]. Indeed, the only fact about the sequences \( S_1 \) and \( S_2 \) it uses is that the edges \( e_j, e_{j+1}, \ldots, e_{j+l} \) are spanned by the vertices \( p_{j_1}, \ldots, p_{j_2} \) and \( q_{j_1}, \ldots, q_{j_2} \), for each pair \( j_1 < j_2 \).

For the rest of this section, we set \( l = 2^{k^2 + k} \) and \( t = 2^k \).

**Lemma 4.3.** Neither of the sequences \( S_1 \) and \( S_2 \) has a subsequence of type \( up(l, t) \).

**Proof.** By symmetry, it suffices to show that \( S_1 \) does not contain a subsequence of type \( up(l, t) \). The argument is by contradiction. We will prove by induction on \( k \) that the existence of such a sequence would imply that \( G \) has \( k \) pairwise crossing...
edges. The base cases \( k = 1, 2 \) are trivial. Now assume the statement holds up to \( k - 1 \). Let

\[
S = s_1, s_2, \ldots, s_{lt}
\]

be our \( up(l, t) \) sequence of length \( lt \) such that the first \( l \) terms are pairwise distinct and for \( i = 1, 2, \ldots, l \) we have

\[
s_i = s_{i+l} = s_{i+2l} = \cdots = s_{i+(t-1)l}.
\]

For each \( i = 1, 2, \ldots, l \), let \( v_i \in V \) denote the vertex \( s_i \). Moreover, let \( a_{i,j} \) be the arc emanating from vertex \( v_i \) to the edge \( e \) corresponding to \( s_{i+jl} \) for \( j = 0, 1, 2, \ldots, t-1 \). We will think of \( s_{i+jl} \) as a point on \( a_{i,j} \) very close but not on edge \( e \). For simplicity, we will let \( s_{lt+q} = s_q \) for all \( q \in \mathbb{N} \) and \( a_{i,j} = a_{i,j'} \) for all \( j, j' \in \mathbb{Z}, \) where \( j' \in \{0, 1, 2, \ldots, t-1\} \) is such that \( j \equiv j' \) (mod \( t \)). Hence there are \( l \) distinct vertices \( v_1, \ldots, v_l \), each vertex of which has \( t \) arcs emanating from it to the edge \( e \).

Consider the arrangement formed by the \( t \) arcs emanating from \( v_1 \) and the edge \( e \). Since \( G \) is simple, these arcs partition the plane into \( t \) regions. By the pigeonhole principle, there is a subset \( V' \subset \{ v_1, \ldots, v_l \} \) of size

\[
\frac{l - 1}{t} = \frac{2^{k^2+k} - 1}{2^k}
\]

such that all of the vertices of \( V' \) lie in the same region. Let \( j_0 \in \{0, 1, 2, \ldots, t-1\} \) be an integer such that \( V' \) lies in the region bounded by \( a_{1,j_0}, a_{1,j_0+1}, \) and \( e \). See Figure 2. In the case \( j_0 = t - 1 \), the set \( V' \) lies in the unbounded region.

Let \( v_1 \in V' \) and \( a_{i,j_0+j_1} \) be an arc emanating from \( v_1 \) for \( j_1 \geq 1 \). Notice that \( a_{i,j_0+j_1} \) cannot cross both \( a_{1,j_0} \) and \( a_{1,j_0+1} \). Indeed, as \( a_{i,j_0+j_1} \) can cross each of \( a_{1,j_0} \) and \( a_{1,j_0+1} \) at most once; had it crossed both of them, its endpoint \( s_{1,j_0+j_1} \) would be in the shaded region on Figure 2. Suppose that \( a_{i,j_0+j_1} \) crosses \( a_{1,j_0+1} \). Then all arcs emanating from \( v_1 \),

\[
A = \{a_{i,j_0+1}, a_{i,j_0+2}, \ldots, a_{i,j_0+j_1-1}\}
\]

must also cross \( a_{1,j_0+1} \). Indeed, let \( \gamma \) be the simple closed curve created by the arrangement

\[
a_{i,j_0+j_1} \cup a_{1,j_0+1} \cup e.
\]
Since $a_{i,j_0} + 1$, $a_{i,j_0} + 1$, and $e$ pairwise intersect at precisely one point, $\gamma$ is well defined. We define points $x = a_{i,j_0} + 1 \cap a_{1,j_0} + 1$ and $y = a_{1,j_0} + 1 \cap e$, and orient $\gamma$ in the direction from $x$ to $y$ along $\gamma$.

In view of the fact that $a_{i,j_0} + 1$ intersects $a_{1,j_0} + 1$, the vertex $v_i$ must lie to the right of $\gamma$. Moreover, since the arc from $x$ to $y$ along $a_{1,j_0} + 1$ is a subset of $\gamma$, the points corresponding to the subsequence

$$S' = \{s_q \in S \mid 2 + (j_0 + 1)l \leq q \leq (i - 1) + (j_0 + j_1)l\}$$

must lie to the left of $\gamma$. Hence, $\gamma$ separates vertex $v_i$ and the points of $S'$. Therefore, using again that $G$ is simple, each arc from $A$ must cross $a_{1,j_0} + 1$ (these arcs cannot cross $a_{i,j_0} + 1$). See Figure 3.

![Diagram](image)

**Fig. 3. Defining $\gamma$ and its orientation.**

By the same argument, if the arc $a_{i,j_0} - j_1$ crosses $a_{1,j_0}$ for $j_1 \geq 1$, then the arcs emanating from $v_i$,

$$a_{i,j_0} - 1, a_{i,j_0} - 2, \ldots, a_{i,j_0} - j_1 + 1,$$

must also cross $a_{1,j_0}$. Since $a_{i,j_0} + 1 / 2 = a_{i,j_0} - j_1 / 2$, we have the following observation.

**Observation 4.4.** For half of the vertices $v_i \in V'$, the arcs emanating from $v_i$ satisfy that either

1. $a_{i,j_0} + 1, a_{i,j_0} + 2, \ldots, a_{i,j_0} + 1 / 2$ all cross $a_{1,j_0} + 1$, or
2. $a_{i,j_0} - 1, a_{i,j_0} - 2, \ldots, a_{i,j_0} - 1 / 2$ all cross $a_{1,j_0}$. 

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Since \( t/2 = 2^{k-1} \) and
\[
\frac{|V'|}{2} \geq \frac{l - 1}{2t} = \frac{2^{k^2+k} - 1}{2 \cdot 2^k} \geq 2^{(k-1)^2 + (k-1)},
\]
by Observation 4.4, we obtain an up\((2^{(k-1)^2 + (k-1)} , 2^{k-1})\) sequence such that the corresponding arcs all cross either \(a_{1, j_0}\) or \(a_{1, j_0+1}\). By the induction hypothesis, it follows that there exist \(k\) pairwise crossing edges. \(\square\)

Now we are ready to complete the proof of Lemma 4.1. By Lemma 4.2 we know that, say, \(S_1\) contains an \(l\)-regular subsequence of length \(|E'|/(4l)|\). By Theorem 2.1 and Lemma 4.3, this subsequence has length at most
\[
n \cdot l 2^{(lt-3)} \cdot (10l)^{10\alpha(n)^lt},
\]
Therefore, we have
\[
\frac{|E'|}{4 \cdot l} \leq n \cdot l 2^{(lt-3)} \cdot (10l)^{10\alpha(n)^lt},
\]
which implies
\[
|E'| \leq 4 \cdot n \cdot l 2^{(lt-3)} \cdot (10l)^{10\alpha(n)^lt}.
\]
Since \(\epsilon_k = 40 \cdot lt = 40 \cdot 2^{k^2+2k} \), \(\alpha(n) \geq 2\), and \(k \geq 5\), we have
\[
|E| = |E'| + 1 \leq n \cdot 2^{\alpha(n)^{\epsilon_k}},
\]
which completes the proof of Lemma 4.1. \(\square\)

Now we are in position to prove Theorem 1.1.

\textit{Proof of Theorem 1.1.} Let \(G = (V, E)\) be a \(k\)-quasi-planar simple topological graph on \(n\) vertices. By Lemma 3.2, there is a subset \(E' \subset E\) such that \(|E'| \geq c|E|/\log |E|\), where \(c\) is an absolute constant and \(E'\) is decomposable. Hence, there is a partition
\[
E' = E_1 \cup E_2 \cup \cdots \cup E_t
\]
such that each \(E_i\) has an edge \(e_i\) that intersects every other edge in \(E_i\), and for \(i \neq j\), the edges in \(E_i\) are disjoint from the edges in \(E_j\). Let \(V_i\) denote the set of vertices that are the endpoints of the edges in \(E_i\), and let \(n_i = |V_i|\). By Lemma 4.1, we have
\[
|E_i| \leq n_i 2^{\alpha(n_i)^{\epsilon_k}} + 2n_i,
\]
where the \(2n_i\) term accounts for the edges that share a vertex with \(e_i\). Hence,
\[
\frac{c|E|}{\log |E|} \leq \sum_{i=1}^t n_i 2^{\alpha(n_i)^{\epsilon_k}} + 2n_i \leq n 2^{\alpha(n)^{\epsilon_k}} + 2n.
\]
Therefore, we obtain
\[
|E| \leq (n \log n) 2^{\alpha(n)^{\epsilon_k}}
\]
for a sufficiently large constant \(c_k\). \(\square\)
5. x-monotone topological graphs. The aim of this section is to prove Theorem 1.2.

Proof of Theorem 1.2. For \( k \geq 2 \), let \( g_k(n) \) be the maximum number of edges in a \( k \)-quasi-planar topological graph whose edges are drawn as \( x \)-monotone curves. We will prove by induction on \( n \) that

\[
g_k(n) \leq 2^{ck^6} n \log n,
\]

where \( c \) is a sufficiently large absolute constant.

The base case is trivial. For the inductive step, let \( G = (V, E) \) be a \( k \)-quasi-planar topological graph whose edges are drawn as \( x \)-monotone curves, and let the vertices be labeled \( 1, 2, \ldots, n \). Let \( L \) be a vertical line that partitions the vertices into two parts, \( V_1 \) and \( V_2 \), such that \( |V_1| = \lfloor n/2 \rfloor \) vertices lie to the left of \( L \), and \( |V_2| = \lceil n/2 \rceil \) vertices lie to the right of \( L \). Furthermore, let \( E_1 \) denote the set of edges induced by \( V_1 \), let \( E_2 \) denote the set of edges induced by \( V_2 \), and let \( E' \) be the set of edges that intersect \( L \). Clearly, we have

\[
|E_1| \leq g_k(\lfloor n/2 \rfloor) \quad \text{and} \quad |E_2| \leq g_k(\lceil n/2 \rceil).
\]

It suffices to show that

\[
|E'| \leq 2^{ck^6/2} n,
\]

since this would imply

\[
g_k(n) \leq g_k(\lfloor n/2 \rfloor) + g_k(\lceil n/2 \rceil) + 2^{ck^6/2} n \leq 2^{ck^6} n \log n.
\]

In the rest of the proof, we consider only the edges belonging to \( E' \). For each vertex \( v_i \in V_1 \), consider the graph \( G_i \) whose vertices are the edges with \( v_i \) as a left endpoint, and two vertices in \( G_i \) are adjacent if the corresponding edges cross at some point to the left of \( L \). Since \( G_i \) is an incomparability graph (see [7, 11]) and does not contain a clique of size \( k \), \( G_i \) contains an independent set of size \( |E(G_i)|/(k-1) \). We keep all edges that correspond to the elements of this independent set, and discard all other edges incident to \( v_i \). After repeating this process for all vertices in \( V_1 \), we are left with at least \(|E'|/(k-1)| \) edges.

Now we continue this process on the other side. For each vertex \( v_j \in V_2 \), consider the graph \( G_j \) whose vertices are the edges with \( v_j \) as a right endpoint, and two vertices in \( G_j \) are adjacent if the corresponding edges cross at some point to the right of \( L \). Since \( G_j \) is an incomparability graph and does not contain a clique of size \( k \), \( G_j \) contains an independent set of size \( |E(G_j)|/(k-1) \). We keep all edges that correspond to this independent set, and discard all other edges incident to \( v_j \). After repeating this process for all vertices in \( V_2 \), we are left with at least \(|E'|/(k-1)^2| \)
edges.

We order the remaining edges \( e_1, e_2, \ldots, e_m \) in the order in which they intersect \( L \) from bottom to top. (We assume, without loss of generality, that any two intersection points are distinct.) Define two sequences, \( S_1 = p_1, p_2, \ldots, p_m \) and \( S_2 = q_1, q_2, \ldots, q_m \), such that \( p_i \) denotes the left endpoint of edge \( e_i \) and \( q_i \) denotes the right endpoint of \( e_i \). We need the following lemma.

Lemma 5.1. Neither of the sequences \( S_1 \) and \( S_2 \) has subsequence of type up-down-up(\( k^3 + 2 \)).
The number of edges in \( k \)-quasi-planar graphs

Proof. By symmetry, it suffices to show that \( S_1 \) does not have a subsequence of type up-down-up\((k^3 + 2)\). Suppose for contradiction that \( S_1 \) does contain such a subsequence. Then there is a sequence

\[
S = s_1, s_2, \ldots, s_{3(k^3 + 2) - 2}
\]
such that the integers \( s_1, \ldots, s_{k^3 + 2} \) are pairwise distinct and

\[
s_i = s_{2(k^3 + 2) - i} = s_{2(k^3 + 2) - 2 + i}
\]
for \( i = 1, 2, \ldots, k^3 + 2 \).

For each \( i \in \{1, 2, \ldots, k^3 + 2\} \), let \( v_i \in V_1 \) denote the label (vertex) of \( s_i \) and let \( x_i \) denote the \( x \)-coordinate of the vertex \( v_i \). Moreover, let \( a_i \) be the arc emanating from vertex \( v_i \) to the point on \( L \) that corresponds to \( s_{2(k^3 + 2) - i} \). Let \( A = \{a_2, a_3, \ldots, a_{k^3 + 1}\} \). Note that the arcs in \( A \) are enumerated downwards with respect to their intersection points with \( L \), and they correspond to the elements of the “middle” section of the up-down-up sequence. We define two partial orders on \( A \) as follows:

\[
a_i \prec_1 a_j \quad \text{if} \quad i < j, \quad x_i < x_j \quad \text{and the arcs} \ a_i, a_j \ \text{do not intersect},
\]

\[
a_i \prec_2 a_j \quad \text{if} \quad i < j, \quad x_i > x_j \quad \text{and the arcs} \ a_i, a_j \ \text{do not intersect}.
\]

Clearly, \( \prec_1 \) and \( \prec_2 \) are partial orders. If two arcs are not comparable by either \( \prec_1 \) or \( \prec_2 \), then they cross. Since \( G \) does not contain \( k \) pairwise crossing edges, by Dilworth’s theorem [7], there exist \( k \) arcs \( \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\} \) such that they are pairwise comparable by either \( \prec_1 \) or \( \prec_2 \). Now the proof falls into two cases.

Case 1. Suppose that \( a_{i_1} \prec_1 a_{i_2} \prec_1 \cdots \prec_1 a_{i_k} \). Then the arcs emanating from \( v_{i_1}, v_{i_2}, \ldots, v_{i_k} \) to the points corresponding to \( s_{2(k^3 + 2) - 2 + i_1}, s_{2(k^3 + 2) - 2 + i_2}, \ldots, s_{2(k^3 + 2) - 2 + i_k} \) are pairwise crossing. See Figure 4.

Case 2. Suppose that \( a_{i_1} \prec_2 a_{i_2} \prec_2 \cdots \prec_2 a_{i_k} \). Then the arcs emanating from \( v_{i_1}, v_{i_2}, \ldots, v_{i_k} \) to the points corresponding to \( s_{i_1}, s_{i_2}, \ldots, s_{i_k} \) are pairwise crossing. See Figure 5.

We are now ready to complete the proof of Theorem 1.2. By Lemma 4.2, we know that, \( S_1 \), say, contains a \((k^3 + 2)\)-regular subsequence of length

\[
\frac{|E'|}{4(k^3 + 2)(k - 1)^2}.
\]
By Lemmas 2.2 and 5.1, this subsequence has length at most $2^{c'}k^6n$, where $c'$ is an absolute constant. Hence, we have

$$\frac{|E'|}{4(k^3 + 2)(k - 1)^2} \leq 2^{c'}k^6n,$$

which implies that

$$|E'| \leq 4k^52^{c'}k^6n \leq 2^{ck^6/2}n$$

for a sufficiently large absolute constant $c$. □

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REFERENCES


