The Number of Edges in $k$-Quasi-planar Graphs

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THE NUMBER OF EDGES IN $k$-QUASI-PLANAR GRAPHS

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Abstract. A graph drawn in the plane is called $k$-quasi-planar if it does not contain $k$ pairwise crossing edges. It has been conjectured for a long time that for every fixed $k$, the maximum number of edges of a $k$-quasi-planar graph with $n$ vertices is $O(n)$. The best known upper bound is $n(\log n)^{O(\log k)}$. In the present paper, we improve this bound to $(n \log n)^{2^k n \log n}$ in the special case where the graph is drawn in such a way that every pair of edges meet at most once. Here $\alpha(n)$ denotes the (extremely slowly growing) inverse of the Ackermann function. We also make further progress on the conjecture for $k$-quasi-planar graphs in which every edge is drawn as an $x$-monotone curve. Extending some ideas of Valtr, we prove that the maximum number of edges of such graphs is at most $2^{ck} n \log n$.

Key words. topological graphs, quasi-planar graphs, Turan-type problems

AMS subject classifications. 05C35, 05C10, 68R10, 52C10

DOI. 10.1137/110858586

1. Introduction. A topological graph is a graph drawn in the plane such that its vertices are represented by points and its edges are represented by nonself-intersecting arcs connecting the corresponding points. In notation and terminology, we make no distinction between the vertices and edges of a graph and the points and arcs representing them, respectively. No edge is allowed to pass through any point representing a vertex other than its endpoints. Any two edges can intersect only in a finite number of points. Tangencies between the edges are not allowed. That is, if two edges share an interior point, then they must properly cross at this point. A topological graph is simple if every pair of its edges intersect at most once: at a common endpoint or at a proper crossing. If the edges of a graph are drawn as straight-line segments, then the graph is called geometric.

Finding the maximum number of edges in a topological graph with a forbidden crossing pattern is a fundamental problem in extremal topological graph theory (see [2, 3, 4, 6, 10, 12, 16, 21, 23]). It follows from Euler’s polyhedral formula that every topological graph on $n$ vertices and with no two crossing edges has at most $3n - 6$ edges. A graph is called $k$-quasi-planar if it can be drawn as a topological graph with no $k$ pairwise crossing edges. A graph is 2-quasi-planar if and only if it is planar. According to an old conjecture (see Problem 1 in section 9.6 of [5]), for any fixed $k \geq 2$ there exists a constant $c_k$ such that every $k$-quasi-planar graph on $n$ vertices has at most $c_k n$ edges. Agarwal et al. [4] were the first to prove this conjecture for...

For larger values of $k$, first Pach, Shahrokhi, and Szegedy [18] showed that every simple $k$-quasi-planar graph on $n$ vertices has at most $c_k n (\log n)^{2k-4}$ edges. For $k \geq 3$ and for all (not necessarily simple) $k$-quasi-planar graphs, Pach, Radoičić, and Tóth [17] established the upper bound $c_k n (\log n)^{4k-12}$. Plugging into these proofs the above mentioned result of Ackerman [1], for $k \geq 4$, we obtain the slightly better bounds $c_k n (\log n)^{2k-8}$ and $c_k n (\log n)^{4k-16}$, respectively. For large values of $k$, the exponent of the polylogarithmic factor in these bounds was improved by Fox and Pach [10], who showed that the maximum number of edges of a $k$-quasi-planar graph on $n$ vertices is $n (\log n)^{O(\log k)}$.

For the number of edges of geometric graphs, that is, graphs drawn by straight-line edges, Valtr [22] proved the upper bound $O(n \log n)$. He also extended this result to simple topological graphs whose edges are drawn as $x$-monotone curves [23].

The aim of this paper is to improve the best known bound, $n (\log n)^{O(\log k)}$, on the number of edges of a $k$-quasi-planar graph in two special cases: for simple topological graphs and for not necessarily simple topological graphs whose edges are drawn as $x$-monotone curves. In both cases, we improve the exponent of the polylogarithmic factor from $O(\log k)$ to $1 + o(1)$.

**Theorem 1.1.** Let $G = (V, E)$ be a $k$-quasi-planar simple topological graph with $n$ vertices. Then $|E(G)| \leq (n \log n)^2^{o(n)^c}$, where $o(n)$ denotes the inverse of the Ackermann function and $c_k$ is a constant that depends only on $k$.

Recall that the Ackermann (more precisely, the Ackermann–Péter) function $A(n)$ is defined as follows. Let $A_1(n) = 2n$, and let $A_k(n) = A_{k-1}(A_k(n-1))$ for $k = 2, 3, \ldots$. In particular, we have $A_2(n) = 2^n$, and $A_3(n)$ is an exponential tower of $n$ two’s. Now let $A(n) = A_n(n)$, and let $o(n)$ be defined as $o(n) = \min\{k \geq 1: A(k) \geq n\}$. This function grows much slower than the inverse of any primitive recursive function.

**Theorem 1.2.** Let $G = (V, E)$ be a $k$-quasi-planar (not necessarily simple) topological graph with $n$ vertices, whose edges are drawn as $x$-monotone curves. Then $|E(G)| \leq 2^{ck^8} n \log n$, where $c$ is an absolute constant.

In both proofs, we follow the approach of Valtr [23] and apply results on generalized Davenport–Schinzel sequences. An important new ingredient of the proof of Theorem 1.1 is a corollary of a separator theorem established in [9] and developed in [8]. Theorem 1.2 is not only more general than Valtr’s result, which applies only to simple topological graphs, but its proof gives a somewhat better upper bound: we use a result of Pettie [20], which improves the dependence on $k$ from double exponential to single exponential.

2. **Generalized Davenport–Schinzel sequences.** The sequence $u = a_1, a_2, \ldots, a_m$ is called $l$-regular if any $l$ consecutive terms are pairwise different. For integers $l, t \geq 2$, the sequence

$$S = s_1, s_2, \ldots, s_{lt}$$

of length $lt$ is said to be of type $u$ if the first $l$ terms are pairwise different and

$$s_i = s_{i+l} = s_{i+2l} = \cdots = s_{i+(t-1)l}$$

for every $i$, $1 \leq i \leq l$. For example,

$$a, b, c, a, b, c, a, b, c$$
is a type \( \text{up}(3, 4) \) sequence or, in short, an \( \text{up}(3, 4) \) sequence. We need the following theorem of Klazar [13] on generalized Davenport–Schinzel sequences.

**Theorem 2.1 (Klazar).** For \( l \geq 2 \) and \( t \geq 3 \), the length of any \( l \)-regular sequence over an \( n \)-element alphabet that does not contain a subsequence of type \( \text{up}(l, t) \) has length at most

\[
 n \cdot 2^{(l-3)} \cdot (10l)^{10\alpha(n)^t}.
\]

For \( l \geq 2 \), the sequence

\[
 S = s_1, s_2, \ldots, s_{3l-2}
\]

of length \( 3l - 2 \) is said to be of type \( \text{up-down-up}(l) \) if the first \( l \) terms are pairwise different and

\[
 s_i = s_{2l-i} = s_{(2l-2)+i}
\]

for every \( i, 1 \leq i \leq l \). For example,

\[
a, b, c, d, c, b, a, b, c, d
\]

is an \( \text{up-down-up}(4) \) sequence. Klazar and Valtr [14] showed that any \( l \)-regular sequence over an \( n \)-element alphabet, which contains no subsequence of type \( \text{up-down-up}(l) \), has length at most \( 2^l n \) for some constant \( c \). This has been improved by Pettie [20], who proved the following.

**Lemma 2.2.** For \( l \geq 2 \), the length of any \( l \)-regular sequence over an \( n \)-element alphabet, which contains no subsequence of type \( \text{up-down-up}(l) \), has length at most \( 2^\Omega(l^2) n \).

For more results on generalized Davenport–Schinzel sequences, see [15].

### 3. On intersection graphs of curves

In this section, we prove a useful lemma on intersection graphs of curves. It shows that every collection \( C \) of curves, no two of which intersect many times, contains a large subcollection \( C' \) such that in the partition of \( C' \) into its connected components \( C_1, \ldots, C_t \) in the intersection graph of \( C \), each component \( C_i \) has a vertex connected to all other \( |C_i| - 1 \) vertices.

For a graph \( G = (V, E) \), a subset \( V_0 \) of the vertex set is said to be a separator if there is a partition \( V = V_0 \cup V_1 \cup V_2 \) with \( |V_1|, |V_2| \leq \frac{1}{4}|V| \) such that no edge connects a vertex in \( V_1 \) to a vertex in \( V_2 \). We need the following separator lemma for intersection graphs of curves, established in [9].

**Lemma 3.1.** There is an absolute constant \( c_1 \) such that every collection \( C \) of curves with \( x \) intersection points has a separator of size at most \( c_1 \sqrt{x} \).

Call a collection \( C \) of curves in the plane decomposable if there is a partition \( C = C_1 \cup \cdots \cup C_t \) such that each \( C_i \) contains a curve which intersects all other curves in \( C_i \), and for \( i \neq j \), the curves in \( C_i \) are disjoint from the curves in \( C_j \). The following lemma is a strengthening of Proposition 6.3 in [8]. Its proof is essentially the same as that of the original statement. It is included here for completeness.

**Lemma 3.2.** There is an absolute constant \( c > 0 \) such that every collection \( C \) of \( m \geq 2 \) curves such that each pair of them intersect in at most \( t \) points has a decomposable subcollection of size at least \( \frac{cm}{t \log m} \).

**Proof of Lemma 3.2.** We prove the following stronger statement. There is an absolute constant \( c > 0 \) such that every collection \( C \) of \( m \geq 2 \) curves whose intersection graph has at least \( x \) edges, and each pair of curves intersects in at most \( t \) points.
and has a decomposable subcollection of size at least $\frac{cm}{t \log m} + \frac{x}{m}$. Let $c = \frac{1}{576 c_1^2}$, where $c_1 \geq 1$ is the constant in Lemma 3.1. The proof is by induction on $m$, noting that all collections of curves with at most three elements are decomposable. Define $d = d(m, x, t) := \frac{cm}{t \log m} + \frac{x}{m}$.

Let $\Delta$ denote the maximum degree of the intersection graph of $C$. We have $\Delta < d - 1$. Otherwise, the subcollection consisting of a curve of maximum degree, together with the curves in $\Delta$ of $V$, is decomposable and its size is at least $d$, and we are done. Also, $\Delta \geq 2 \frac{m}{x}$, since $2 \frac{m}{x}$ is the average degree of the vertices in the intersection graph of $C$. Hence, if $\Delta \geq 2 \frac{cm}{t \log m}$, then the desired inequality holds. Thus, we may assume $\Delta < 2 \frac{cm}{t \log m}$.

Applying Lemma 3.1 to the intersection graph of $C$, we obtain that there is a separator $V_0 \subset C$ with $|V_0| \leq c_1 \sqrt{tx}$, where $c_1$ is the absolute constant in Lemma 3.1. That is, there is a partition $C = V_0 \cup V_1 \cup V_2$ with $|V_1|, |V_2| \leq 2|V|/3$ such that no curve in $V_1$ intersects any curve in $V_2$. For $i = 1, 2$, let $m_i = |V_i|$ and let $x_i$ denote the number of pairs of curves in $V_i$ that intersect, so that

$$x_1 + x_2 \geq x - \Delta |V_0| \geq x - 2 \frac{cm}{t \log m} c_1 \sqrt{tx}.$$  

As no curve in $V_1$ intersects any curve in $V_2$, the union of a decomposable subcollection of $V_1$ and a decomposable subcollection of $V_2$ is decomposable. Thus, by the induction hypothesis, $C$ contains a decomposable subcollection of size at least

$$d(m_1, x_1, t) + d(m_2, x_2, t) \geq \frac{cm_1}{t \log m_1} + \frac{x_1}{m_1} + \frac{cm_2}{t \log m_2} + \frac{x_2}{m_2} \geq c \frac{m_1 + m_2}{t \log (2m/3)} + \frac{(x_1 + x_2)}{2m/3}.$$  

We split the rest of the proof into two cases.

Case 1. $x \geq t^{-1} (12c_1 \frac{m}{c \log m})^2$. In this case, by (1), we have $x_1 + x_2 \geq \frac{5}{6} x$, and hence there is a decomposable subcollection of size at least

$$d(m_1, x_1, t) + d(m_2, x_2, t) \geq \frac{cm_1 + m_2}{t \log m} + \frac{5x}{4m} = d + \frac{x}{4m} - \frac{c(m - (m_1 + m_2))}{t \log m} \geq d + \frac{x}{4m} - \frac{c_1 \sqrt{tx}}{t \log m} > d,$$

completing the analysis.

Case 2. $x < t^{-1} (12c_1 \frac{m}{c \log m})^2$. There is a decomposable subcollection of size at least

$$d(m_1, x_1, t) + d(m_2, x_2, t) \geq \frac{cm_1 + m_2}{t \log (2m/3)} \geq \frac{c}{t} \left( m_1 + m_2 \right) \left( \frac{1}{\log m} + \frac{1}{2 \log^2 m} \right) \geq \frac{c}{t} \left( \frac{m}{\log m} + \frac{m}{2 \log^2 m} \right) \geq \frac{c}{t} \left( \frac{m}{\log m} + \frac{m}{4 \log^2 m} \right) \geq \frac{cm}{t \log m} + \frac{x}{m} = d,$$

where we used $c = \frac{1}{4 (12c_1)^2} = \frac{1}{576 c_1^2}$. \qed
4. Simple topological graphs. In this section, we prove Theorem 1.1. The following statement will be crucial for our purposes.

**Lemma 4.1.** Let $G = (V, E)$ be a $k$-quasi-planar simple topological graph with $n$ vertices. Suppose that $G$ has an edge that crosses every other edge. Then we have $|E| \leq n \cdot 2^{\alpha(n)k}$, where $\alpha(n)$ denotes the inverse Ackermann function and $c_k'$ is a constant that depends only on $k$.

**Proof of Lemma 4.1.** Let $k \geq 5$ and let $c_k' = 40 \cdot 2^{k^2+2k}$. To simplify the presentation, we do not make any attempt to optimize the value of $c_k'$. Label the vertices of $G$ from 1 to $n$, i.e., let $V = \{1, 2, \ldots, n\}$. Let $e = uv$ be the edge that crosses every other edge in $G$. Note that $d(u) = d(v) = 1$.

Let $E'$ denote the set of edges that cross $e$. Suppose, without loss of generality, that no two of elements of $E'$ cross $e$ at the same point. Let $e_1, e_2, \ldots, e_{|E'|}$ denote the edges in $E'$ listed in the order of their intersection points with $e$ from $u$ to $v$. We create two sequences of vertices $S_1 = p_1, p_2, \ldots, p_{|E'|}$ and $S_2 = q_1, q_2, \ldots, q_{|E'|} \subset V$, as follows. For each $e_i \in E'$, as we move along edge $e$ from $u$ to $v$ and arrive at the intersection point with $e_i$, we turn left and move along edge $e_i$ until we reach its endpoint $u_i$. Then we set $p_i = u_i$. Likewise, as we move along edge $e$ from $u$ to $v$ and arrive at edge $e_i$, we turn right and move along edge $e_i$ until we reach its other endpoint $w_i$. Then we set $q_i = w_i$. Thus, $S_1$ and $S_2$ are sequences of length $|E'|$ over the alphabet $\{1, 2, \ldots, n\}$. See Figure 1 for a small example.

![Figure 1](image-url)\(\text{Fig. 1. In this example, } S_1 = v_1, v_3, v_4, v_3, v_2 \text{ and } S_2 = v_2, v_2, v_1, v_5, v_5.\)

We need two lemmas. The first one is due to Valtr [23].

**Lemma 4.2** (Valtr). For $l \geq 1$, at least one of the sequences $S_1, S_2$ defined above contains an $l$-regular subsequence of length at least $|E'|/(4l)$. \(\Box\)

Since each edge in $E'$ crosses $e$ exactly once, the proof of Lemma 4.2 can be copied almost verbatim from the proof of Lemma 4 in [23]. Indeed, the only fact about the sequences $S_1$ and $S_2$ it uses is that the edges $e_{j_1}, e_{j_1+1}, \ldots, e_{j_2}$ are spanned by the vertices $p_{j_1}, \ldots, p_{j_2}$ and $q_{j_1}, \ldots, q_{j_2}$, for each pair $j_1 < j_2$.

For the rest of this section, we set $l = 2^{k^2+k}$ and $t = 2^k$.

**Lemma 4.3.** Neither of the sequences $S_1$ and $S_2$ has a subsequence of type $up(l, t)$.

**Proof.** By symmetry, it suffices to show that $S_1$ does not contain a subsequence of type $up(l, t)$. The argument is by contradiction. We will prove by induction on $k$ that the existence of such a sequence would imply that $G$ has $k$ pairwise crossing
edges. The base cases $k = 1, 2$ are trivial. Now assume the statement holds up to $k - 1$. Let

$$S = s_1, s_2, \ldots, s_{lt}$$

be our $up(l, t)$ sequence of length $lt$ such that the first $l$ terms are pairwise distinct and for $i = 1, 2, \ldots, l$ we have

$$s_i = s_{i+l} = s_{i+2l} = s_{i+3l} = \cdots = s_{i+(l-1)t}.$$

For each $i = 1, 2, \ldots, l$, let $v_i \in V$ denote the vertex $s_i$. Moreover, let $a_{i,j}$ be the arc emanating from vertex $v_i$ to the edge $e$ corresponding to $s_{i+jl}$ for $j = 0, 1, 2, \ldots, t-1$. We will think of $s_{i+jl}$ as a point on $a_{i,j}$ very close but not on edge $e$. For simplicity, we will let $s_{lt+q} = s_q$ for all $q \in \mathbb{N}$ and $a_{i,j} = a_{i,j'}$ for all $j, j' \in \mathbb{Z}$, where $j' \in \{0, 1, 2, \ldots, t-1\}$ is such that $j \equiv j' \pmod{t}$. Hence there are $l$ distinct vertices $v_1, \ldots, v_l$, each vertex of which has $t$ arcs emanating from it to the edge $e$.

Consider the arrangement formed by the $t$ arcs emanating from $v_1$ and the edge $e$. Since $G$ is simple, these arcs partition the plane into $t$ regions. By the pigeonhole principle, there is a subset $V' \subset \{v_1, \ldots, v_l\}$ of size

$$\frac{l-1}{t} = \frac{2^{k+2+k} - 1}{2^k}$$

such that all of the vertices of $V'$ lie in the same region. Let $j_0 \in \{0, 1, 2, \ldots, t-1\}$ be an integer such that $V'$ lies in the region bounded by $a_{1,j_0}$, $a_{1,j_0+1}$, and $e$. See Figure 2. In the case $j_0 = t - 1$, the set $V'$ lies in the unbounded region.

![Figure 2](image)

**Fig. 2.** Vertices of $V'$ lie in the region enclosed by $a_{1,j_0}$, $a_{1,j_0+1}$, $e$.

Let $v_i \in V'$ and $a_{i,j_0+j_1}$ be an arc emanating from $v_i$ for $j_1 \geq 1$. Notice that $a_{i,j_0+j_1}$ cannot cross both $a_{1,j_0}$ and $a_{1,j_0+1}$. Indeed, as $a_{i,j_0+j_1}$ can cross each of $a_{1,j_0}$ and $a_{1,j_0+1}$ at most once; had it crossed both of them, its endpoint $s_{1,j_0+j_1}$ would be in the shaded region on Figure 2. Suppose that $a_{i,j_0+j_1}$ crosses $a_{1,j_0+1}$. Then all arcs emanating from $v_1$,

$$A = \{a_{i,j_0+1}, a_{i,j_0+2}, \ldots, a_{i,j_0+j_1-1}\}$$

must also cross $a_{1,j_0+1}$. Indeed, let $\gamma$ be the simple closed curve created by the arrangement

$$a_{i,j_0+j_1} \cup a_{1,j_0+1} \cup e.$$
Since \(a_{i,j_0 + j_1}, a_{i,j_0 + 1}\), and \(e\) pairwise intersect at precisely one point, \(\gamma\) is well defined. We define points \(x = a_{i,j_0 + j_1} \cap a_{1,j_0 + 1}\) and \(y = a_{1,j_0 + 1} \cap c\), and orient \(\gamma\) in the direction from \(x\) to \(y\) along \(\gamma\).

In view of the fact that \(a_{i,j_0 + j_1}\) intersects \(a_{1,j_0 + 1}\), the vertex \(v_i\) must lie to the right of \(\gamma\). Moreover, since the arc from \(x\) to \(y\) along \(a_{1,j_0 + 1}\) is a subset of \(\gamma\), the points corresponding to the subsequence 

\[S' = \{s_q \in S \mid 2 + (j_0 + 1)l \leq q \leq (i - 1) + (j_0 + j_1)l\}\]

must lie to the left of \(\gamma\). Hence, \(\gamma\) separates vertex \(v_i\) and the points of \(S'\). Therefore, using again that \(G\) is simple, each arc from \(A\) must cross \(a_{1,j_0 + 1}\) (these arcs cannot cross \(a_{i,j_0 + j_1}\)). See Figure 3.

![Diagram](image)

Fig. 3. Defining \(\gamma\) and its orientation.

By the same argument, if the arc \(a_{i,j_0 - j_1}\) crosses \(a_{1,j_0}\) for \(j_1 \geq 1\), then the arcs emanating from \(v_i\),

\[a_{i,j_0 - 1}, a_{i,j_0 - 2}, \ldots, a_{i,j_0 - j_1 + 1}\]

must also cross \(a_{1,j_0}\). Since \(a_{i,j_0 + t/2} = a_{i,j_0 - t/2}\), we have the following observation.

**Observation 4.4.** For half of the vertices \(v_i \in V'\), the arcs emanating from \(v_i\) satisfy that either

1. \(a_{i,j_0 + 1}, a_{i,j_0 + 2}, \ldots, a_{i,j_0 + t/2}\) all cross \(a_{1,j_0 + 1}\), or
2. \(a_{i,j_0 - 1}, a_{i,j_0 - 2}, \ldots, a_{i,j_0 - t/2}\) all cross \(a_{1,j_0}\).
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Since $t/2 = 2^{k-1}$ and

$$\frac{|V'|}{2} \geq \frac{l - 1}{2t} = \frac{2^{k^2 + k} - 1}{2 \cdot 2^k} \geq 2^{(k-1)^2 + (k-1)},$$

by Observation 4.4, we obtain an up$(2^{(k-1)^2 + (k-1)} \cdot 2^{k-1})$ sequence such that the corresponding arcs all cross either $a_{1,j_0}$ or $a_{1,j_0+1}$. By the induction hypothesis, it follows that there exist $k$ pairwise crossing edges.

Now we are ready to complete the proof of Lemma 4.1. By Lemma 4.2 we know that, say, $S_1$ contains an $l$-regular subsequence of length $|E'|/(4t)$. By Theorem 2.1 and Lemma 4.3, this subsequence has length at most

$$n \cdot l 2^{(l-3)} \cdot (10l)^{10\alpha(n)l}.$$

Therefore, we have

$$\frac{|E'|}{4 \cdot l} \leq n \cdot l 2^{(l-3)} \cdot (10l)^{10\alpha(n)l},$$

which implies

$$|E'| \leq 4 \cdot n \cdot l 2^{(l-3)} \cdot (10l)^{10\alpha(n)l}.$$

Since $c'_k = 40 \cdot l = 40 \cdot 2^{k^2 + 2k}$, $\alpha(n) \geq 2$, and $k \geq 5$, we have

$$|E| = |E'| + 1 \leq n \cdot 2^{\alpha(n)k},$$

which completes the proof of Lemma 4.1.

Now we are in position to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $G = (V,E)$ be a $k$-quasi-planar simple topological graph on $n$ vertices. By Lemma 3.2, there is a subset $E' \subset E$ such that $|E'| \geq c|E|/\log |E|$, where $c$ is an absolute constant and $E'$ is decomposable. Hence, there is a partition

$$E' = E_1 \cup E_2 \cup \cdots \cup E_t$$

such that each $E_i$ has an edge $e_i$ that intersects every other edge in $E_i$, and for $i \neq j$, the edges in $E_i$ are disjoint from the edges in $E_j$. Let $V_i$ denote the set of vertices that are the endpoints of the edges in $E_i$, and let $n_i = |V_i|$. By Lemma 4.1, we have

$$|E_i| \leq n_i 2^{\alpha(n_i)k} + n_i,$$

where the $2n_i$ term accounts for the edges that share a vertex with $e_i$. Hence,

$$\frac{c|E|}{\log |E|} \leq \sum_{i=1}^t n_i 2^{\alpha(n_i)k} + n_i \leq n 2^{\alpha(n)k} + 2n.$$

Therefore, we obtain

$$|E| \leq (n \log n) 2^{\alpha(n)k}$$

for a sufficiently large constant $c_k$. \qed
5. \(x\)-monotone topological graphs. The aim of this section is to prove Theorem 1.2.

Proof of Theorem 1.2. For \(k \geq 2\), let \(g_k(n)\) be the maximum number of edges in a \(k\)-quasi-planar topological graph whose edges are drawn as \(x\)-monotone curves. We will prove by induction on \(n\) that

\[
g_k(n) \leq 2^{ck^6} n \log n,
\]

where \(c\) is a sufficiently large absolute constant.

The base case is trivial. For the inductive step, let \(G = (V, E)\) be a \(k\)-quasi-planar topological graph whose edges are drawn as \(x\)-monotone curves, and let the vertices be labeled 1, 2, \ldots, \(n\). Let \(L\) be a vertical line that partitions the vertices into two parts, \(V_1\) and \(V_2\), such that \(|V_1| = \lceil n/2 \rceil\) vertices lie to the left of \(L\), and \(|V_2| = \lfloor n/2 \rfloor\) vertices lie to the right of \(L\). Furthermore, let \(E_1\) denote the set of edges induced by \(V_1\), let \(E_2\) denote the set of edges induced by \(V_2\), and let \(E'\) be the set of edges that intersect \(L\). Clearly, we have

\[
|E_1| \leq g_k(\lceil n/2 \rceil) \quad \text{and} \quad |E_2| \leq g_k(\lfloor n/2 \rfloor).
\]

It suffices to show that

\[
(2) \quad |E'| \leq 2^{ck^6/2} n,
\]

since this would imply

\[
g_k(n) \leq g_k(\lceil n/2 \rceil) + g_k(\lfloor n/2 \rfloor) + 2^{ck^6/2} n \leq 2^{ck^6} n \log n.
\]

In the rest of the proof, we consider only the edges belonging to \(E'\). For each vertex \(v_i \in V_1\), consider the graph \(G_i\) whose vertices are the edges with \(v_i\) as a left endpoint, and two vertices in \(G_i\) are adjacent if the corresponding edges cross at some point to the left of \(L\). Since \(G_i\) is an incomparability graph (see [7, 11]) and does not contain a clique of size \(k\), \(G_i\) contains an independent set of size \(|E(G_i)|/(k - 1)|\). We keep all edges that correspond to the elements of this independent set, and discard all other edges incident to \(v_i\). After repeating this process for all vertices in \(V_1\), we are left with at least \(|E'|/(k - 1)|\) edges.

Now we continue this process on the other side. For each vertex \(v_j \in V_2\), consider the graph \(G_j\) whose vertices are the edges with \(v_j\) as a right endpoint, and two vertices in \(G_j\) are adjacent if the corresponding edges cross at some point to the right of \(L\). Since \(G_j\) is an incomparability graph and does not contain a clique of size \(k\), \(G_j\) contains an independent set of size \(|E(G_j)|/(k - 1)|\). We keep all edges that corresponds to this independent set, and discard all other edges incident to \(v_j\). After repeating this process for all vertices in \(V_2\), we are left with at least \(|E'|/(k - 1)^2|\) edges.

We order the remaining edges \(e_1, e_2, \ldots, e_m\) in the order in which they intersect \(L\) from bottom to top. (We assume, without loss of generality, that any two intersection points are distinct.) Define two sequences, \(S_1 = p_1, p_2, \ldots, p_m\) and \(S_2 = q_1, q_2, \ldots, q_m\), such that \(p_i\) denotes the left endpoint of edge \(e_i\) and \(q_i\) denotes the right endpoint of \(e_i\). We need the following lemma.

Lemma 5.1. Neither of the sequences \(S_1\) and \(S_2\) has subsequence of type up-down-up\((k^3 + 2)\).
Proof. By symmetry, it suffices to show that $S_1$ does not have a subsequence of type up-down-up($k^3 + 2$). Suppose for contradiction that $S_1$ does contain such a subsequence. Then there is a sequence

$$S = s_1, s_2, \ldots, s_{3(k^3+2)-2}$$

such that the integers $s_1, \ldots, s_{k^3+2}$ are pairwise distinct and

$$s_i = s_{2(k^3+2)-i} = s_{2(k^3+2)-2+i}$$

for $i = 1, 2, \ldots, k^3 + 2$.

For each $i \in \{1, 2, \ldots, k^3+2\}$, let $v_i \in V_i$ denote the label (vertex) of $s_i$ and let $x_i$ denote the $x$-coordinate of the vertex $v_i$. Moreover, let $a_i$ be the arc emanating from vertex $v_i$ to the point on $L$ that corresponds to $s_{2(k^3+2)-i}$. Let $A = \{a_2, a_3, \ldots, a_{k^3+1}\}$. Note that the arcs in $A$ are enumerated downwards with respect to their intersection points with $L$, and they correspond to the elements of the “middle” section of the up-down-up sequence. We define two partial orders on $A$ as follows:

$$a_i \prec_1 a_j \quad \text{if} \quad i < j, \quad x_i < x_j \quad \text{and the arcs} \ a_i, a_j \ \text{do not intersect},$$

$$a_i \prec_2 a_j \quad \text{if} \quad i < j, \quad x_i > x_j \quad \text{and the arcs} \ a_i, a_j \ \text{do not intersect}.$$ 

Clearly, $\prec_1$ and $\prec_2$ are partial orders. If two arcs are not comparable by either $\prec_1$ or $\prec_2$, then they cross. Since $G$ does not contain $k$ pairwise crossing edges, by Dilworth’s theorem [7], there exist $k$ arcs $\{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\}$ such that they are pairwise comparable by either $\prec_1$ or $\prec_2$. Now the proof falls into two cases.

Case 1. Suppose that $a_{i_1} \prec_1 a_{i_2} \prec_1 \cdots \prec_1 a_{i_k}$. Then the arcs emanating from $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ to the points corresponding to $s_{2(k^3+2)-2+i_1}, s_{2(k^3+2)-2+i_2}, \ldots, s_{2(k^3+2)-2+i_k}$ are pairwise crossing. See Figure 4.

Case 2. Suppose that $a_{i_1} \prec_2 a_{i_2} \prec_2 \cdots \prec_2 a_{i_k}$. Then the arcs emanating from $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ to the points corresponding to $s_{2(k^3+2)-2+i_1}, s_{2(k^3+2)-2+i_2}, \ldots, s_{2(k^3+2)-2+i_k}$ are pairwise crossing. See Figure 5.

We are now ready to complete the proof of Theorem 1.2. By Lemma 4.2, we know that, $S_1$, say, contains a $(k^3 + 2)$-regular subsequence of length

$$|E'| \leq \frac{4(k^3 + 2)(k - 1)^2}{k^3 (k^3 + 1)}.$$
By Lemmas 2.2 and 5.1, this subsequence has length at most $2c'k^n$, where $c'$ is an absolute constant. Hence, we have

$$\frac{|E'|}{4(k^3 + 2)(k - 1)^2} \leq 2c'k^n,$$

which implies that

$$|E'| \leq 4k^52c'k^n \leq 2ck^n/2n$$

for a sufficiently large absolute constant $c$.  \[\square\]

**Acknowledgment.** We would like to thank the referee for helpful comments.

**REFERENCES**


THE NUMBER OF EDGES IN $k$-QUASI-PLANAR GRAPHS


