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Ramification of stream networks

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The geometric complexity of stream networks has been a source of fascination for centuries. However, a comprehensive understanding of ramification—the mechanism of branching by which such networks grow—remains elusive. Here we show that streams incised by groundwater seepage branch at a characteristic angle of 2 x 5 = 72°. Our theory represents streams as a collection of paths growing and bifurcating in a diffusing field. Our observations of nearly 5,000 bifurcated streams growing in a 100 km² groundwater field on the Florida Panhandle yield a mean bifurcation angle of 71.9° ± 0.8°. This good accord between theory and observation suggests that the network geometry is determined by the external flow field but not, as classical theories imply, by the flow within the streams themselves.

Theory

We seek the exact value of α*. Our theory is built upon the following hypothesis: streams grow forward in the direction from which groundwater enters their tips (13).

To find the groundwater field around the bifurcated tip, we note that the height h of the water table above an impermeable layer is a solution of the Poisson equation (21, 22, 30)

\[ \nabla^2 h(x,y) = -\frac{2P}{K} \]

where P is the mean precipitation rate and K the hydraulic conductivity. Groundwater flows into the stream network along the paths of steepest descent down h and intersects the stream at the elevation of the channel profile. The network ramifies when new springs form as the result of a bifurcation of an existing spring.

We make two additional observations. First, rain falling immediately around a bifurcated spring represents a negligible fraction of the water draining into it. Specifically, the influence of precipitation P in Eq. 1 can be neglected whenever the length of the nascent branch is small relative to the typical linear dimension of a drainage area. The height of the water table near the stream is then a solution of Laplace’s equation, expressed here as

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\[ \nabla^2 h^2 = 0. \]  

Our second observation concerns boundary conditions: when the lengths of incipient streams are sufficiently small, the influence of the remainder of the network can be neglected and the branches feel an effectively infinite Laplacian field. The growth of the Florida network is therefore a natural manifestation of the growth of one-dimensional paths in a two-dimensional harmonic field (31, 32).

We proceed to solve Eq. 2 for the shape of the water table around a Y-shaped bifurcation that completely absorbs any groundwater that reaches it. The problem is straightforwardly analyzed in the complex plane (31, 33–36), where the coordinates of the point \((x,y)\) are represented by the complex number \(z = x + iy\), where \(i^2 = -1\). We describe the shape of a bifurcated head as follows. First, we choose a coordinate system in which the main channel lies on the negative imaginary axis \(z = iy\), where \(y < 0\). The nascent branches then lie on the lines

\[ \sigma = r e^{i(\pi \pm \alpha) / 2}, \quad 0 < \alpha < 2\pi, \]  

where \(r > 0\). These streams meet at the origin \(z = 0\). Notably, the only length scale needed to describe this bifurcation is the length of the nascent streams, which we choose to be unity. The growth process of an incipient bifurcation has no characteristic length scale, and therefore, we assume that during the first step of growth the bifurcation maintains its initial shape. Consequently,
the length of the streams can be rescaled in time to reveal an unchanging configuration, thus reducing the dynamical growth problem to a problem of geometry.

Given this parameterization of the bifurcated tip, we solve for the shape of the water table by conformal mapping (37). Using the Schwarz–Christoffel transformation (37), one can find the complex function

\[ g(w) = e^{i(x-\alpha)/2} \left( \frac{2\pi - \omega w^2}{2\pi} \right)^{1-\alpha/2}, \]

which maps the upper half-plane \( w = u + iv \) to the region around the bifurcated tips described in Eq. 3. This mapping, illustrated in Fig. 4, relates the solution \( h^2(z) \) of Eq. 2 around the bifurcated tip to the trivial solution \( \hat{h}^2(w) = \text{Re}(-iw) \) in the upper half of the mathematical plane, where the absorbing boundary is along the real axis. In particular, the shape of the water table around the bifurcated tip is

\[ h^2(z) = \text{Re}\left[ -ig^{-1}(z) \right]. \]

where \( g^{-1} \) is the inverse function of the mapping \( g \) given by Eq. 4.

Following our hypothesis that springs grow in the direction from which groundwater flows into them, we must find the streamlines flowing into the tips to determine the direction of growth. As illustrated in Fig. 4, the streamlines flowing into the springs in the mathematical (\( w \)) plane at \( w = \pm 1 \) are mapped from the vertical lines \( w(v) = \pm 1 + iv \). Consequently, in the physical (\( z \)) plane, water flows into the spring at \( w = +1 \) along the curve

\[ z(v) = g(1+iv) = R(v)e^{i(x-\alpha)/2}, \]

where

\[ R(v) = (1+iv)^{\alpha/2} \left( \frac{2\pi - \alpha(1+iv)^2}{2\pi - \alpha} \right)^{1-\alpha/2}. \]

The remaining streamline may be found by symmetry.

Eqs. 6 and 7 provide the shape of the streamline flowing into a spring for an arbitrary value of \( \alpha \). We identify the special angle \( \alpha^* \).
with the angle for which the streamline enters a tip without curvature. The curvature of $R(v)$ is given by $\kappa_\alpha(v) = \Im[R'/|R'|^2]$. After expanding around $v = 0$ up to first order, one obtains

$$\kappa_\alpha(v) = \left(\frac{5\alpha - 2\pi}{8\alpha}\right) \frac{1}{v} + O(v). \quad [8]$$

We find that the curvature vanishes at the origin only when $\alpha$ equals

$$\alpha^* = \frac{2\pi}{5} = 72^\circ. \quad [9]$$

In all other cases, the curvature diverges like $1/v$. A similar result has been obtained by Hastings for the dielectric breakdown model at the upper critical point (33). Related calculations may also be found in refs. 34–36.

**Observation**

We now return our attention to the real world. We analyze the valley networks of Fig. 1 and others immediately adjacent to it using a high-resolution LIDAR (Light Detection And Ranging) (38) map that extends over $\sim 100$ km$^2$ with a horizontal resolution of 1.2 m and a vertical resolution of about 5 cm. The combination of extensive spatial coverage and high resolution allows us to measure the angles of 4,966 channel junctions.

We first locate the streams. Streams stand out in a landscape as sharply incised cuts through the otherwise smooth topography. We thus find all grid points where the elevation contours are sharply curved (39–41). Rills in the ravine walls are then eliminated by requiring the slope to be moderate at the channel bottom. Finally, the resulting network is reduced to a collection of one-pixel-wide paths. The streams are then interpolated from the gridded data and indexed by their Horton–Strahler order (10). Fig. 5 shows the resulting network.

We determine the direction of valley growth by approximating each stream of a given Horton–Strahler order by a straight line using the reduced major axis (42, 43). We then define $\alpha$ as the angle between the two upstream segments of intersecting stream directions. Fig. 6 shows a histogram of the measured angles at all 4,966 junctions of our network. The mean angle $\langle \alpha \rangle = 71.9^\circ \pm 0.8^\circ$ (95% C.I.) is unambiguously consistent with the prediction of Eq. 9.

Although the error in the estimate of the mean angle is reasonably small, the standard deviation of the histogram, 27.7°, is considerable. Because the typical standard error associated with each angle measurement is only a few degrees, it would appear that deflections of streams due to localized material heterogeneities, landslides, vegetation, tree falls, etc., do not significantly contribute to the large sample variance. We are instead led to identify aspects of the bifurcation dynamics not addressed by our theory. First, we note that streams need not initially ramify at $72^\circ$; more generally, we expect that $72^\circ$ represents a stable fixed point for growth in an infinite Laplacian field (34, 44). However, groundwater is a finite Poissonian field that reflects interactions with the rest of the channel network. We therefore hypothesize that the wide variance of the angle distribution reflects a combination of the transient approach to $72^\circ$ at early stages of bifurcation and a departure at late stages as tips advance toward groundwater divides. In other words, the self-similarity of the $72^\circ$ bifurcation occurs in the limit of intermediate asymptotics (45, 46), whereas our measurements may not. Curiously, diffusion-limited aggregation (47) exhibits a similar large variance, but its angle histograms are asymmetric (48), unlike the histogram of Fig. 6.

**Discussion and Conclusion**

The coupling of our growth hypothesis—streams grow in the direction from which flow enters tips (13)—with our simple model of
groundwater flow (21, 22) appears to capture all of the mechanisms required to explain the average angle at which streams split in the Florida network. The ramification of the network is therefore entirely associated with the extrinsic groundwater field, without regard to the flow of water and sediment within the streams. This need not have been the case. For example, a widely held view of stream network formation suggests that the fractal geometry of stream networks is a consequence of the minimization of energy dissipation within streams (3). In analogy with Murray’s law for the growth of vascular networks (49), one could reason that the bifurcation angle itself is a consequence of such a minimization principle (50). If so, the angle would be a function of slope $s$ and discharge $Q$ (50). Minimizing dissipation within streams would then predict a bifurcation angle of 90° for two streams of equal discharge and slope (50), in disagreement with our measurements. In contrast, we find that the mechanism generating the Florida network’s geometric complexity is clearly independent of flow within streams. Moreover, our theory requires no assumption of optimality.

Distinctions between intrinsic and extrinsic control exist not only in studies of stream networks but also in investigations of neuron dendrites (52), fungal hyphae (53), and other branching problems. Because our results show how external and internal controls can lead to quantitatively distinct geometric features, we expect that similar analyses should be of use wherever the provenance of ramification remains controversial.

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