Low Dimensional Geometry and Topology Special Feature: Tour of bordered Floer theory

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Detailed Terms
Tour of bordered Floer theory

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Heegaard Floer theory is a kind of topological quantum field theory (TQFT), assigning graded groups to closed, connected, oriented 3-manifolds and group homomorphisms to smooth, oriented four-dimensional cobordisms. Bordered Heegaard Floer homology is an extension of Heegaard Floer homology to 3-manifolds with boundary, with extended-TQFT-type gluing properties. In this survey, we explain the formal structure and construction of bordered Floer homology and sketch how it can be used to compute some aspects of Heegaard Floer theory.

Heegaard Floer homology | 4-manifolds

Heegaard Floer homology, introduced in a series of papers (1–3) of Szabó and the second author has become a useful tool in three- and four-dimensional topology. The Heegaard Floer invariants contain subtle topological information, allowing one to detect the genera of knots and homology classes (4); detect fiberedness for knots (5–9) and 3-manifolds (10–13); bound the slice genus (14) and unknotting number (15, 16); prove tightness and obstruct Stein fillability of contact structures (5, 17); and more. It has been useful for resolving a number of conjectures, particularly related to questions about Dehn surgery (18, 19); see also ref. 20. It is either known or conjectured to be equivalent to several other gauge-theoretic or holomorphic curve invariants in low-dimensional topology, including monopole Floer homology (21), embedded contact homology (22), and the Lagrangian matching invariants of 3- and 4-manifolds (23–25). Heegaard Floer homology is known to relate to Khovanov–Rozansky-type homologies are conjectured (29).

Heegaard Floer homology has several variants; the technically simplest is \( \tilde{HF} \), which is sufficient for most of the three-dimensional applications discussed above. Bordered Heegaard Floer homology, the focus of this paper, is an extension of \( \tilde{HF} \) to 3-manifolds with boundary (30). This extension gives a conceptually satisfying way to compute essentially all aspects of the Heegaard Floer package related to \( \tilde{HF} \). [There are also other algorithms for computing many parts of Heegaard Floer theory (31–39).]

We will start with the formal structure of bordered Heegaard Floer homology. Most of the paper is then devoted to sketching its definition. We conclude by explaining how bordered Floer homology can be used for calculations of Heegaard Floer invariants.

Formal Structure

Review of Heegaard Floer Theory. Heegaard Floer theory has many components. Most basic among them, it associates:

- To a closed, connected, oriented 3-manifold \( Y \), an abelian group \( \tilde{HF}(Y) \) and \( \mathbb{Z}[U] \) modules \( HF^+(Y), HF^-(Y), \) and \( HF^\infty(Y) \). These are the homologies of chain complexes \( \tilde{CF}(Y) \), \( CF^+(Y) \), \( CF^-(Y) \), and \( CF^\infty(Y) \), respectively. The chain complexes (and their homology groups) decompose into spin^c structures, \( CF(Y) = \bigoplus_{\text{spin}^c(Y)} CF(Y, \mathfrak{s}) \), where \( CF(Y, \mathfrak{s}) \) is any of the four chain complexes. Each \( CF(Y, \mathfrak{s}) \) has a relative grading modulo the divisibility of \( c_1(\mathfrak{s}) \) (1). The chain complex \( \tilde{CF}(Y) \) is the \( U = 0 \) specialization of \( CF^-(Y) \).

- To a smooth, compact, oriented cobordism \( W \) from \( Y_1 \) to \( Y_2 \), maps \( F_W: HF(Y_1) \to HF(Y_2) \) induced by chain maps \( f_W: CF(Y_1) \to CF(Y_2) \). These maps decompose according to spin^c structures on \( W \).

The maps \( f_W \) satisfy a topological quantum field theory (TQFT) composition law:

- If \( W' \) is another cobordism, from \( Y_2 \) to \( Y_3 \), then \( F_{W'} \circ F_W = F_{W' \cup W} \).

The Heegaard Floer invariants are defined by counting pseudoholomorphic curves in symmetric products of Heegaard surfaces. The Heegaard Floer groups were conjectured to be equivalent to the monopole Floer homology groups (defined by counting solutions of the Seiberg–Witten equations), via the correspondence: \( HF^+(Y) \leftrightarrow HM(Y), HF^-(Y) \leftrightarrow HM(Y), HF^\infty(Y) \leftrightarrow TM(Y) \), and similarly for the corresponding cobordism maps. A proof of this conjecture has recently been announced by Kutluhan et al. (40–42). Colin et al. have announced an independent proof for the \( U = 0 \) specialization (43).

In particular, the Heegaard Floer package contains enough information to detect exotic smooth structures on 4-manifolds (10, 44). For closed 4-manifolds, this information is contained in \( HF^+ \) and \( HF^- \); the weaker invariant \( \tilde{HF} \) is not useful for distinguishing smooth structures on closed 4-manifolds.

The Structure of Bordered Floer Theory. Bordered Floer homology is an extension of \( \tilde{HF} \) to 3-manifolds with boundary, in a TQFT form. Bordered Floer homology associates:

- To a closed, oriented, connected surface \( F \), together with some extra markings (see Definition 1), a differential graded \( (dg) \) algebra \( \mathcal{A}(F) \).

- To a compact, oriented 3-manifold \( Y \) with connected boundary, together with a diffeomorphism \( \phi: F \to \partial Y \) marking the boundary, a module over \( \mathcal{A}(F) \). Actually, there are two different invariants for \( Y: \tilde{CFD}(Y) \), a left \( df \) module over \( \mathcal{A}(F) \), and \( CFA(Y) \), a right \( df \) module over \( \mathcal{A}(F) \), each well-defined up to quasi-isomorphism. We sometimes refer to a 3-manifold \( Y \) with \( \partial Y = F \); we actually mean \( Y \) together with an identification \( \phi \) of \( \partial Y \) with \( F \). We call these data a \emph{bordered 3-manifold}.

- More generally, to a 3-manifold \( Y \) with two boundary components \( \partial_b Y \) and \( \partial_b Y \), diffeomorphisms \( \phi_L: F_L \to \partial_b Y \) and \( \phi_R: F_R \to \partial_b Y \) and a framed arc \( \gamma \) from \( \partial_b Y \) to \( \partial_b Y \) (compatible with \( \phi_L \) and \( \phi_R \) in a suitable sense), a \( df \) bi-module \( \tilde{CFD}(Y) \) with commuting left actions of \( \mathcal{A}(F_L) \) and \( \mathcal{A}(F_R) \); an \( df \) bi-module \( CFA(Y) \) with commuting right actions of \( \mathcal{A}(F_L) \) and \( \mathcal{A}(F_R) \). Each of \( \tilde{CFD}(Y) \), \( CFA(Y) \), and \( \tilde{CF}(Y) \) is assigned a graded (topological) invariance.

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*For \( CF^+ \) and \( CF^\infty \), we mean the completions with respect to the formal variable \( U \).

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$\text{CFDA}(Y)$, and $\overline{\text{CFAA}}(Y)$ is well-defined up to quasi-isomorphism. We call the data $(Y, \phi_1, \phi_2, \gamma)$ a strongly bordered 3-manifold with two boundary components.

To keep the sidedness straight, note that type $D$ boundaries are always on the left, and type $A$ boundaries are always on the right; and for $[0,1] \times F$, the boundary component on the left side, $\{0\} \times F$, is oriented as $-F$, whereas the one on the right side is oriented as $F$.

Gluing 3-manifolds corresponds to tensoring invariants; for any valid tensor product (necessarily matching $D$ sides with $A$ sides) there is a corresponding gluing. More concretely (30, 45):

- Given 3-manifolds $Y_1$ and $Y_2$ with $\partial Y_1 = F = -\partial Y_2$, there is a quasi-isomorphism
  \[
  \overline{\text{CF}}(Y_1 \cup_F Y_2) \cong \overline{\text{CF}}(Y_1) \otimes_{\text{CF}} \overline{\text{CF}}(Y_2),
  \]
  where $\otimes$ denotes the derived tensor product. So, $HF(Y_1 \cup_F Y_2) \cong \text{Tor}_{\otimes \text{CF}}(\overline{\text{CF}}(Y_1), \overline{\text{CF}}(Y_2))$.

- More generally, given 3-manifolds $Y_1$ and $Y_2$ with $\partial Y_1 = -F_1 \sqcup F_2$ and $\partial Y_2 = -F_2 \sqcup F_3$, there are quasi-isomorphisms of bimodules corresponding to any valid tensor product. For instance,
  \[
  \text{CFDD}(Y_1 \cup_{F_1} F_2 Y_2) \cong \text{CFDD}(Y_1) \otimes_{\text{CFDD}(F_2)} \text{CFDD}(Y_2)
  \]
  Also, $F_1$ or $F_3$ may be $S^2$ (or empty), in which case these statements reduce to pairing theorems for a module and a bimodule.

We refer to theorems of this kind as pairing theorems. There is also a self-pairing theorem. Let $Y$ be a 3-manifold with $\partial Y = -F \sqcup F$ and $\gamma$ be a framed arc connecting corresponding points in the boundary components of $Y$. The self-pairing theorem relates the Hochschild homology of the bimodule $\text{CFDA}(Y)$ with the knot Floer homology of a generalized open book decomposition associated to $Y$ and $\gamma$.

These invariants satisfy a number of duality properties (46); e.g.:
- The algebra $\mathcal{A}(F)$ is the opposite algebra of $\mathcal{A}(-F)$. (There are also more subtle duality properties of the algebras; see Remark 4.)
- The module $\text{CFD}(Y)$ is dual [over $\mathcal{A}(F)$] to $\text{CFA}(Y)$:
  \[
  \text{CFD}(Y) \cong \text{Mor}_{\mathcal{A}(F)}(\text{CFD}(Y), \mathcal{A}(F))
  \]
  \[
  \text{CFD}(Y) \cong \text{Mor}_{\mathcal{A}(F)}(\text{CFD}(Y), \mathcal{A}(F)).
  \]

- The module $\text{CFDD}(-Y)$ is the one-sided dual of $\text{CFDD}(Y)$:
  \[
  \text{CFDD}(-Y) \cong \text{Mor}_{\mathcal{A}(F)}(\text{CFDD}(Y), \mathcal{A}(F)) \cong \text{CFDD}(Y).
  \]

As a consequence of these dualities, one can give pairing theorems using the Hom functor rather than the tensor product (46); e.g.:
- Let $Y_1$ and $Y_2$ be 3-manifolds with $\partial Y_1 = \partial Y_2 = F$. Then
  \[
  \overline{\text{CF}}(-Y_1 \cup_Y Y_2) \cong \text{Mor}_{\mathcal{A}(F)}(\overline{\text{CF}}(Y_1), \overline{\text{CF}}(Y_2)).
  \]
  Similar statements hold for $\overline{\text{CF}}$ and for bimodules.
- Given 3-manifolds $Y_1$ and $Y_2$ with $\partial Y_1 = F = -\partial Y_2$,
  \[
  \overline{\text{CF}}(Y_1 \cup_F Y_2) \cong \text{Mor}_{\mathcal{A}(F) \otimes \mathcal{A}(F)}(\overline{\text{CF}}(Y_1), \overline{\text{CF}}(Y_2)).
  \]

Similarly, if $Y_2$ has another boundary component $F'$, then
  \[
  \overline{\text{CF}}(Y_1 \cup_{F'} F_2 Y_2) \cong \text{Mor}_{\mathcal{A}(F) \otimes \mathcal{A}(F)}(\overline{\text{CF}}(Y_1), \overline{\text{CF}}(Y_2)).
  \]

(If both $Y_1$ and $Y_2$ had two boundary components, then the left-hand side would pick up a change of framing.)

**Remark 1:** Some of the duality properties discussed above can also be seen from the Fukaya-categorical perspective (47).

**Remark 2:** It is natural to expect that to a 4-manifold with corners one would associate a map of bimodules, satisfying certain gluing axioms. We have not done this; however, as discussed below, even without this bordered Floer homology allows one to compute the maps $F\triangleright$ associated to cobordisms $W$ between closed 3-manifolds.

**The Algebras**

As mentioned earlier, the bordered Floer algebras are associated to surfaces together with some extra markings. We encode these markings as *pointed matched circles* $\mathcal{I}$, which we discuss next. We then introduce a simpler algebra, $\mathcal{A}(n)$, depending only on an integer $n$, of which the bordered Floer algebras $\mathcal{A}(\mathcal{I})$ are subalgebras. The definition of $\mathcal{A}(\mathcal{I})$ itself is given in the last subsection.

**Pointed Matched Circles.**

**Definition 1:** A pointed matched circle $\mathcal{I}$ consists of an oriented circle $Z$; 4k points $a = \{a_1, \ldots, a_{4k}\}$ in $Z$, a matching of $M$ of the points in $a$ in pairs, which we view as a fixed-point free involution $M: a \rightarrow a$; and a basepoint $z \in Z \setminus a$. We require that performing surgery on $Z$ along the matched pairs of points yields a connected 1-manifold. A pointed matched circle $\mathcal{I}$ with $|a| = 4k$ specifies:
- A closed surface $F(Z)$ of genus $k$, as follows: Fill $Z$ with a disk $D$. Attach a two-dimensional 1-handle to each pair of points in $a$ matched by $M$. By hypothesis, the result has connected boundary; fill that boundary with a second disk.
- A distinguished disk in $F(Z)$: the disk $D$ (say).
- A basepoint $z$ in the boundary of the distinguished disk.

**Remark 3:** Matched circles can be seen as a special case of fat graphs (48). They are also dual to the typical representation of a genus $g$ surface as a $4g$-gon with sides glued together.

**The Strands Algebra.**

We next define a differential algebra $\mathcal{A}(n)$, depending only on an integer $n$; the algebra $\mathcal{A}(\mathcal{I})$ associated to a pointed matched circle with $|a| = 4k$ will be a subalgebra of $\mathcal{A}(4k)$. The algebra $\mathcal{A}(n)$ has an $F_2$ basis consisting of all triples $(S, T, \phi)$, where $S$ and $T$ are subsets of $\{1, \ldots, n\}$ and $\phi: S \rightarrow T$ is a bijection such that for all $s \in S$, $\phi(s) \geq s$. Given such a map $\phi$, let $\text{Inv}(\phi) = \{(s_1, s_2) \in S \times S \mid s_1 < s_2, \phi(s_1) = \phi(s_2)\}$ and $\text{Inv}(\phi)$ is the number of inversions of $\phi$.

The product $(S, T, \phi) \cdot (U, V, \psi) = (S \cup U, T \cup V, \psi \circ \phi)$ is defined to be 0 if $U \neq T$ or if $U = T$, but $\text{Inv}(\psi \circ \phi) \neq \text{Inv}(\psi) + \text{Inv}(\phi)$. If $U = T$ and $\text{Inv}(\psi \circ \phi) = \text{Inv}(\psi) + \text{Inv}(\phi)$, then let $(S, T, \phi) \cdot (U, V, \psi) = (S \cup U, T \cup V, \psi \circ \phi)$. In particular, the elements $(S, \emptyset)$ (where $\emptyset$ denotes the identity map) are the indecomposable idempotents in $\mathcal{A}(n)$.

Given a generator $(S, T, \phi) \in \mathcal{A}(n)$ and an element $\sigma = (s_1, s_2) \in \text{Inv}(\phi)$, let $\phi_\sigma: S \rightarrow T$ be the map defined by $\phi_\sigma(s) = \phi(s)$ if $s \neq s_1, s_2$; $\phi_\sigma(s_1) = \phi(s_2)$; and $\phi_\sigma(s_2) = \phi(s_1)$. Define a differential on $\mathcal{A}(n)$ by
The Algebra Associated to a Pointed Matched Circle. Fix a pointed matched circle \( \mathcal{Z} = (Z, a, M, z) \) with \( |a| = 4k \). After cutting \( Z \) at \( z \), the orientation of \( Z \) identifies \( a \) with \( \overline{a} \), so we can view \( M \) as a matching of \( 4k \).

Call a basis element \((S, T, \phi)\) of \( \mathcal{A}(4k) \) equitable if no two elements of \( 4k \) that are matched (with respect to \( M \) to occur in \( S \), and no two elements of \( 4k \) that are matched both occur in \( T \).

Given equitable basis elements \( x = (S, T, \phi) \) and \( y = (S', T', \psi) \) of \( \mathcal{A}(4k) \), we say that \( x \) and \( y \) are related by horizontal strand swapping, and write \( x \sim y \), if there is a subset \( U \subset S \) such that \( S' = (S \setminus U) \cup M(U), \phi|_{SU} = \psi|_{SU}, \phi|_{U} = I_{U}, \) and \( \psi|_{M(U)} = \overline{I}_{M(U)} \).

Given an equitable basis element \( x \) of \( \mathcal{A}(4k) \), let \( a(x) = \sum_{\phi_i \sim \psi} y \).

The algebra associated to the pointed matched circle \( \mathcal{Z} \) for a genus 2 surface with matching \((1 \leftrightarrow 3, 2 \leftrightarrow 4, 5 \leftrightarrow 7, 6 \leftrightarrow 8) \) has Poincaré polynomial (45, Sect. 4)

\[
\sum_{i} \dim_{F_{2}} H_{i} (\mathcal{A}(\mathcal{Z}, i))T^{i} = T^{-2} + 32T^{-1} + 98 + 32T + 2T^{2}.
\]

The algebra associated to the pointed matched circle \( \mathcal{Z}' \) for a genus 2 surface with matching \((1 \leftrightarrow 5, 2 \leftrightarrow 6, 3 \leftrightarrow 7, 4 \leftrightarrow 8) \) has Poincaré polynomial

\[
\sum_{i} \dim_{F_{2}} H_{i} (\mathcal{A}(\mathcal{Z}', i))T^{i} = T^{-2} + 32T^{-1} + 70 + 32T + 2T^{2}.
\]

The ranks in the genus two examples which are equal are explained by the observations that for any pointed matched circle, \( \mathcal{A}(\mathcal{Z}, -k) \cong F_{2}; \mathcal{A}(\mathcal{Z}, -k + 1) \) has no differential; the dimension of \( \mathcal{A}(\mathcal{Z}, -k + 1) \) is independent of the matching; and the following:

Remark 4: The algebras \( \mathcal{A}(\mathcal{Z}, i) \) and \( \mathcal{A}(\mathcal{Z}, -i) \) are Koszul dual. (Here, \( -\mathcal{Z} \) denotes the pointed matched circle obtained by reversing the orientation on \( Z \).) Also, given a pointed matched circle \( \mathcal{Z} \) for \( F, \) let \( \mathcal{Z} \) denote the pointed matched circle corresponding to the dual handle decomposition of \( F \). Then \( \mathcal{A}(\mathcal{Z}, i) \) and \( \mathcal{A}(\mathcal{Z}, -i) \) are Koszul dual.

Remark 5: In Zarev’s bordered-sutured extension of the theory (49), the strands algebra \( \mathcal{A}(n, k) \) has a topological interpretation as the algebra associated to a disk with boundary sutures.

Combinatorial Representations of Bordered 3-Manifolds

A bordered 3-manifold \( Y \) together with an orientation-preserving homeomorphism \( \phi: F(\mathcal{Z}) \to \partial Y \) for some pointed matched circle \( \mathcal{Z} \). Two bordered 3-manifolds \( (Y_1, \phi_1; F(\mathcal{Z}_1)) \to \partial Y_1 \) and \( (Y_2, \phi_2; F(\mathcal{Z}_2)) \to \partial Y_2 \) are called equivalent if there is an orientation-preserving homeomorphism \( \psi: Y_1 \to Y_2 \) such that \( \phi_1 = \phi_2 \circ \psi; \) in particular, this implies that \( \mathcal{Z}_1 = \mathcal{Z}_2 \). Bordered Floer theory associates homotopy equivalence classes of modules to equivalence classes of bordered 3-manifolds. Just as the bordered Floer algebras are associated to combinatorial representations of surfaces, not directly to surfaces, the bordered Floer modules are associated to combinatorial representations of bordered 3-manifolds.
The Closed Case. Recall that a three-dimensional handlebody is a regular neighborhood of a connected graph in $\mathbb{R}^3$. According to a classical result of Heegaard (50), every closed, orientable 3-manifold can be obtained as a union of two such handlebodies, $H_a$ and $H_b$. Such a representation is called a Heegaard splitting. A Heegaard splitting along an orientable surface $\Sigma$ of genus $g$ can be represented by a Heegaard diagram: a pair of $g$ tuples of pairwise disjoint, homologically linearly independent, embedded circles $\alpha = (\alpha_1, \ldots, \alpha_g)$ and $\beta = (\beta_1, \ldots, \beta_g)$ in $\Sigma$. These curves are chosen so that each $\alpha_i$ (respectively, $\beta_i$) bounds a disk in the handlebody $H_a$ (respectively, $H_b$). Any two Heegaard diagrams for the same manifold $Y$ are related by a sequence of moves, called Heegaard moves; see, for instance, ref. 51 or ref. 1, Sect. 2.1.

Representing 3-Manifolds with Boundary. The story extends easily to 3-manifolds with boundary, using a slight generalization of handlebodies. A compression body (with both boundaries connected) is the result of starting with a connected orientable surface $\Sigma_2$ times $[0,1]$ and then attaching thickened disks (three-dimensional 2-handles) along some number of homologically linearly independent, disjoint circles in $\Sigma_2 \times \{0\}$. A compression body has two boundary components, $\Sigma_1$ and $\Sigma_2$, with genera $g_1 \geq g_2$. Up to homeomorphism, a compression body is determined by its boundary.

A Heegaard decomposition of a 3-manifold $Y$ with nonempty, connected boundary is a decomposition $Y = H_a \cup_{\Sigma} H_b$, where $H_a$ is a compression body and $H_b$ is a handlebody. Let $g$ be the genus of $\Sigma$, and let $\partial Y$. A Heegaard diagram for $Y$ is got by choosing $g$ pairs of disjoint circles $\beta_1, \ldots, \beta_g$ in $\Sigma$ and $g-k$ disjoint circles $\alpha_1', \ldots, \alpha_g'$ in $\Sigma_1$ so that

- The circles $\beta_1, \ldots, \beta_g$ bound disks $D_{\beta_1}, \ldots, D_{\beta_g}$ in $H_b$ such that $H_b \setminus \left( \bigcup D_{\beta_i} \right)$ is topologically a ball, and
- The circles $\alpha_1', \ldots, \alpha_g'$ bound disks $D_{\alpha_1'}, \ldots, D_{\alpha_g'}$ in $H_a$ such that $H_a \setminus \left( \bigcup D_{\alpha_i'} \right)$ is topologically the product of a surface and an interval.

To specify a parametrization, or bordering, of $\partial Y$, we need a little more data. A bordered Heegaard diagram for $Y$ is a tuple

$$\mathcal{H} = (\Sigma, \alpha, \beta, \rho),$$

where

- $\Sigma$ is an oriented surface with a single boundary component;
- $(\Sigma \cup \partial \Sigma, \alpha' \cup \alpha, \beta)$ is a Heegaard diagram for $Y$;
- $\alpha_1', \ldots, \alpha_g'$ are pairwise disjoint, embedded arcs in $\Sigma$ with boundary on $\partial \Sigma$, and are disjoint from $\alpha'$;
- $\Sigma \setminus (\alpha' \cup \cdots \cup \alpha_1' \cup \cdots \cup \alpha_g')$ is a disk with $2(g-k)$ holes; and
- $z$ is a point in $\partial \Sigma_1$ disjoint from all of the $\alpha_i'$.

Let $\alpha = \alpha' \cup \alpha$. A bordered Heegaard diagram $\mathcal{H}$ specifies a pointed matched circle $\mathcal{F}(\mathcal{H}) = (Z = \partial \Sigma, \mathbf{a} = (\alpha^x \cap \partial \Sigma), M, z)$, where two points in $\mathbf{a}$ are matched in $M$ if they lie on the same $\alpha^x$. A bordered Heegaard diagram for $Y$ also specifies an identification $\phi: \mathcal{F}(\mathcal{H}) \to \partial Y$, well-defined up to isotopy.

There are moves analogous to Heegaard moves, relating any two bordered Heegaard diagrams for equivalent bordered 3-manifolds.

The Modules and Bimodules

As discussed above, there are two invariants of a 3-manifold $Y$ with boundary $F(Y)$. $C\bar{F}D(Y)$ has a straightforward module structure but a differential that counts holomorphic curves, whereas $\tilde{CF}A(Y)$ uses holomorphic curves to define the module structure itself.

Fix a bordered Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, \rho)$ for $Y$. Let $\mathfrak{G}(\mathcal{H})$ be the set of $g$ tuples $x = (x_i)_{i=1}^g \in \alpha \cap \beta$ so that there is exactly one point $x_i$ on each $\beta$-circle and on each $\alpha$-circle, and there is at most one $x_i$ on each $\alpha$-arc. The invariant $\tilde{CF}A(Y)$ is a direct sum of copies of $F_2$, one for each element of $\mathfrak{G}(\mathcal{H})$, whereas $C\bar{F}D(Y)$ is a direct sum of elementary projective $\mathcal{A}(\mathcal{H})$ modules, one for each element of $\mathfrak{G}(\mathcal{H})$. Let $X(\mathcal{H})$ be the $F_2$-vector space generated by $\mathfrak{G}(\mathcal{H})$, which is also the vector space underlying $\tilde{CF}A(Y)$.

Each generator $x \in \mathfrak{G}(\mathcal{H})$ determines a spin$^c$ structure $\mathcal{S}(x) \in \text{spin}^c(Y)$; the construction (30) is an easy adaptation of the closed case (1, Sect. 2.6).

Before continuing to describe the bordered Floer modules, we digress to briefly discuss the moduli spaces of holomorphic curves.

Moduli Spaces of Holomorphic Curves

Fix a bordered Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, \rho)$. Let $\Sigma = \Sigma \setminus \partial \Sigma$. Choose a symplectic form $\omega$ on $\Sigma$ giving it a cylindrical end and a complex structure $j_z$ compatible with $\omega$, making $\Sigma$ into a punctured Riemann surface. Let $p$ denote the puncture in $\Sigma$. We choose the $\alpha_i'$ so their intersections with $\Sigma$ (also denoted $\alpha_i'$) are cylindrical ($\mathbb{R}$-invariant) in a neighborhood of $p$.

We consider curves $u: (\Sigma, \partial \Sigma) \to (\Sigma \times [0,1] \times \mathbb{R}, \alpha \times \{1\} \times \mathbb{R} \cup \beta \times \{0\} \times \mathbb{R})$.

holomorphic with respect to an appropriate almost complex structure $J$, satisfying conditions spelled out in ref. 30. The reader may wish to simply think of a product complex structure $j_z \times j_y$, though these complex structures may not be general enough to achieve transversality.

Such holomorphic curves $u$ have asymptotics in three places:

- $\Sigma \times [0,1] \times \{-\infty, +\infty\}$. We consider curves asymptotic to $g$-tuples of strips $x \times [0,1] \times \mathbb{R}$ at $-\infty$ and $y \times [0,1] \times \mathbb{R}$ at $+\infty$, where $x, y \in \mathfrak{G}(\mathcal{H})$.
- $\{p\} \times [0,1] \times \mathbb{R}$, which we denote $\infty$. We consider curves asymptotic to chords $\rho_i$ in $(\partial \Sigma, \alpha)$ at a point $(1, t_i) \in [0,1] \times \mathbb{R}$. [These are chords for thecotisorfoliation of $\partial \Sigma \times [0,1] \times \mathbb{R}$, whose leaves are the circles $\partial \Sigma \times \{(s_i, t_i)\}$.]
- We impose the condition that these chords $\rho_i$ not cross $z \in \partial \Sigma$.

Topological maps of this form can be grouped into homology classes. Let $\pi_1(x, y)$ denote the set of homology classes of maps asymptotic to $x \times [0,1] \times \{-\infty\}$ and $y \times [0,1] \times \{+\infty\}$. Then $\pi_1(x, y)$ is canonically isomorphic to $H_2(Y, \partial Y); \pi_1(x, y)$ is nonempty if and only if $\sigma(x) = \sigma(y)$, and if $\sigma(x) = \sigma(y)$ then $\pi_1(x, y)$ is an affine copy of $H_2(Y, \partial Y)$, under concatenation by elements of $\pi_1(x, y) \times [\sigma(x, y)] (30)$. [Again, these results are easy adaptations of the corresponding results in the closed case (1, Sect. 2).] Note that our usage of $\pi_1(x, y)$ differs from the usage in ref. 1, where homology classes are allowed to cross $z$, but agrees with the usage in ref. 30.

Given generators $x, y \in \mathfrak{G}(\mathcal{H})$, a homology class $B \in \pi_1(x, y)$, and a sequence $\rho = (\rho_1, \ldots, \rho_m)$ of sets $\rho_i = \{\rho_{i1}, \ldots, \rho_{im_i}\}$ of Reeb chords, let $\mathfrak{M}^{\rho}(x, y; \rho)$ denote the moduli space of embedded holomorphic curves $u$ in the homology class $B$, asymptotic to $x \times [0,1] \times \{+\infty\}$, $y \times [0,1] \times \{-\infty\}$, and $\rho_{ij} \times (1, t_i)$ at $\infty$, for some sequence of heights $t_i < \cdots < t_m$. There is an action of $\mathbb{R}$ on $\mathfrak{M}^{\rho}(x, y; \rho)$, by translation. Let $\mathfrak{M}^{\rho}(x, y; \rho) = \mathfrak{M}^{\rho}(x, y; \rho)/\mathbb{R}$.

The modules $C\bar{F}D(\mathcal{H})$ and $\tilde{CF}A(\mathcal{H})$ will be defined using counts of zero-dimensional moduli spaces $\mathfrak{M}^{\rho}(x, y; \rho)$. Proving that these modules satisfy $\partial^2 = 0$ and the $d_{\mathcal{A}}$ relations, respectively, involves studying the ends of one-dimensional moduli spaces. These ends correspond to the following four kinds of degenerations:
1. Breaking into a two-story holomorphic building. That is, the \( R \) coordinate of some parts of the curve go to \( +\infty \) with respect to other parts, giving an element of \( \mathcal{M}_B(x, y; \widetilde{p}_1) \times \mathcal{M}_B(x, y; \widetilde{p}_2) \), where \( B \) is the concatenation \( B_1 \ast B_2 \) and \( \widetilde{p} \) is the concatenation \( \widetilde{p}_1 \ast \widetilde{p}_2 \).

2. Degenerations in which a boundary branch point of the projection \( \pi_2 \ast u \) approaches \( \infty \), in such a way that some chord \( \rho_{ij} \) splits into a pair of chords \( \rho_{ij}^+ \) and \( \rho_{ij}^- \). This degeneration results in a curve at \( \infty \), a join curve, and an element of \( \mathcal{M}_B(x, y; \widetilde{p}) \), where \( \widetilde{p} \) is obtained by replacing the chord \( \rho_{ij} \in p_i \in \rho \) with two chords, \( \rho_{ij}^+ \) and \( \rho_{ij}^- \).

3. The difference in \( R \) coordinates is \( t_2 - t_1 \), between two consecutive sets of chords \( \rho_i \) and \( \rho_{i+1} \) in \( \rho \) going to 0. In the process, some boundary branch points of \( \pi_2 \ast u \) may approach \( \infty \), degenerating a split curve, along with an element of \( \mathcal{M}_B(x, y; \widetilde{p}) \), where \( \widetilde{p} \) is obtained from \( \rho_{ij} \cup \rho_{ij+1} \) by gluing together any pairs of chords \( \rho_{ij}^+ \rho_{ij+1} \) where \( \rho_{ij} \) ends at the starting point of \( \rho_{ij+1} \).

4. Degenerations in which a pair of branch boundary points of \( \pi_2 \ast u \) approach \( \infty \), causing a pair of chords \( \rho_{ij}^+ \) and \( \rho_{ij}^- \) in some \( \rho \), whose endpoints \( \rho_{ij}^+ \) and \( \rho_{ij}^- \) are nested, say \( \rho_{ij}^+ \leq \rho_{ij}^- \leq \rho_{ij}^+ \), to break apart and recombine into a pair of chords \( \rho_{ij} = (\rho_{ij}^+ \rho_{ij}^-)^{\ast} \) and \( \rho_{ij} = (\rho_{ij}^+ \rho_{ij}^-) \). This gives an odd shuffle curve at \( \infty \) and an element of \( \mathcal{M}_B(x, y; \widetilde{p}) \), where \( \widetilde{p} \) is obtained from \( \rho_{ij} \) replacing \( \rho_{ij}^+ \) and \( \rho_{ij}^- \) in \( \rho \) with \( \rho_{ij}^+ \) and \( \rho_{ij}^- \).

See Fig. 3 for examples of the first three kinds of degenerations.

**Remark 6:** This analytic setup builds on the “cylindrical reformulation” of Heegaard Floer theory (52). It relates to the original formulation of Heegaard Floer theory, in terms of holomorphic disks in \( \text{Sym}^n(\Sigma) \), by thinking of a map \( D \to \text{Sym}^n(\Sigma) \) as a multivalued map \( D \to \Sigma \) and then taking the graph. See, for instance, ref. 52, Sect. 13. Some of the results were previously proved in ref. 53.

**Type D Modules.** Fix a bordered Heegaard diagram \( \mathcal{H} \) and a suitable almost complex structure \( J \). Let \( F = -J^\ast \mathcal{H} \) be the orientation reverse of the pointed matched circle given by \( \mathcal{H} \). Given a generator \( x \in \mathcal{H} \), let \( I_F(x) \) denote the indecomposable idempotent of \( \mathcal{A}(F,0) \subset \mathcal{A}(F) \) corresponding to the set of \( a \)-arcs intersecting \( x \). Fix another \( x \) and let \( \mathcal{A}(F,Y) \) be a module over \( \mathcal{A}(F) \). Define a differential \( \mathcal{D}(F,Y) \) on \( \mathcal{A}(F,Y) \) by

\[
\mathcal{D}(F,Y)(\sum y_i) = \sum \mathcal{D}B(x, y_i; \{ \{ p_i \} \} , \{ \{ p_i \} \} ) \cdot y_i,
\]

and extending multilinearly. As for \( \mathcal{C}(F,Y) \), to ensure finiteness of these sums, we need to assume that \( \mathcal{H} \) is provably admissible.

**Theorem 4.** The operations \( m_{n+1} \) satisfy the \( \mathcal{A}_\infty \) module relation.

**Proof sketch:** Because \( \mathcal{A}(F) \) is a differential algebra, the \( \mathcal{A}_\infty \) relation for \( CFA(F) \) takes the form

\[
0 = \sum_{i+j=n+1} m_i m_j (x, a_1, \ldots, a_{i-1}, a_i) \cdot y + \sum_{\ell=1}^n m_{n+1} (x, a_1, \ldots, a_{\ell}, \ldots, a_n) + \sum_{\ell=1}^{n-1} m_{n+1} (x, a_1, \ldots, a_{\ell}, a_{\ell+1}, \ldots, a_n).
\]

The first term in Eq. 4 corresponds to degenerations of type 1. The second term corresponds to degenerations of types 2 and 4, depending on whether one of the strands in the crossing being resolved is horizontal (2) or not (4). The third term corresponds to degenerations of type 3. This proves the result.

**Theorem 5.** (31) Up to homotopy equivalence, the \( \mathcal{A}_\infty \) module \( CFA(F) \) is independent of the (provably admissible) bordered Heegaard diagram \( \mathcal{H} \) representing the bordered 3-manifold \( Y \).
Given a surface $F$, let $G(F)$ be the $Z$-central extension of $H_{1}(F)$, $Z \to G(F) \to H_{1}(F)$, where $1 \in Z$ maps to $\lambda \in G(Z)$. Explicitly, $G(F) \subset \frac{1}{2}Z \times H_{1}(F)$ with

$$(k_1, a_1) \cdot (k_2, a_2) = (k_1 + k_2 + a_1 \cap a_2, a_1 + a_2).$$

Here, $\cap$ denotes the intersection pairing on $H_{1}(F)$. It turns out that $\mathcal{A}(\mathcal{X})$ has a grading by $(G(F(\mathcal{X})), \lambda)$ (30, Sect-3.3). Similarly, given a $3$-manifold $Y$ bounded by $F(\mathcal{X})$, one can construct $G(F(\mathcal{X}))\text{-}set$ gradings on $\mathcal{CFD}(Y)$ and $\mathcal{CFD}(\mathcal{Y})$ (30).

Even in the closed case, the grading on Heegaard Floer homology has a somewhat nonstandard form: a partial relative cyclic grading. That is, generators do not have well-defined gradings, but only well-defined grading differences $gr(x, y)$; the grading difference $gr(x, y)$ is defined only for generators representing the same spin$^c$ structure; and $gr(x, y)$ is well-defined only modulo the divisibility of $c_1(\mathbf{s}(x))$. A partial relative cyclic grading is precisely a grading by a $Z$ set. This leads naturally to a graded version of the pairing theorems, including Eq. I (30).

**Deforming the Diagonal and the Pairing Theorems.**

The tensor product pairing theorems are the main motivation for the definitions of the modules and bimodules. We will sketch the proof of the archetype, Eq. I. Fix bordered Heegaard diagrams $\mathcal{H}_1$ and $\mathcal{H}_2$ for $Y_1$ and $Y_2$, respectively, with $\partial Y_1 = -\partial Y_2$. It is easy to see that $\mathcal{H} = \mathcal{H}_1 \cup_\partial \mathcal{H}_2$ is a Heegaard diagram for $Y_1 \cup_\partial Y_2$.

There are two sides to the proof, one algebraic and one analytic. We start with the algebra, typically, the $\mathcal{A}_\mathcal{D}$ tensor product $M \otimes \mathcal{N}$ of $\mathcal{A}_\mathcal{D}$ modules $M$ and $\mathcal{N}$ is defined using a chain complex whose underlying vector space is $M \otimes \mathcal{N} \otimes T^* \mathcal{A}_\mathcal{D} \mathcal{N}$ (where $T^* \mathcal{A}_\mathcal{D}$ is the tensor algebra of $\mathcal{A}_\mathcal{D}$ and $\mathcal{K}$ is the ground ring of $\mathcal{A}_\mathcal{D}$—for us, the ring of idempotents). This complex is typically infinite-dimensional, and so is unlikely to align easily with $\mathcal{C}_\mathcal{F}$.

However, the differential module $\mathcal{CFD}(\mathcal{R})$ has a special form: It is given as $\mathcal{A}(\mathcal{X}) \otimes_{\mathcal{J}(\mathcal{X})} \mathcal{X}(\mathcal{X}_2)$, so the differential is encoded by a map $\delta': \mathcal{X}(\mathcal{X}_2) \to \mathcal{A}(\mathcal{X}) \otimes_{\mathcal{J}(\mathcal{X})} \mathcal{X}(\mathcal{X}_2)$. This allows us to define a smaller model for the $\mathcal{A}_\mathcal{D}$ tensor product. Let $\delta': \mathcal{X}(\mathcal{X}_2) \to \mathcal{A}(\mathcal{X}) \otimes_{\mathcal{J}(\mathcal{X})} \mathcal{X}(\mathcal{X}_2)$ be the result of iterating $\delta'$ $n$ times. For notational convenience, let $M = \mathcal{CFD}(\mathcal{H}_1)$ and $X = \mathcal{X}(\mathcal{X}_2)$. Define $\mathbf{CF}(\mathcal{H}_1) \otimes_{\mathcal{J}(\mathcal{X})} \mathbf{CF}(\mathcal{X}_2)$ to be the $F_2$-vector space $M \otimes_{\mathcal{J}(\mathcal{X})} X$, with differential (graphically depicted in Fig. 4)

$$d = \sum_{i=0}^{\infty} (m_{i+1} \otimes i_{X}) \circ (i_{M} \otimes \delta').$$

The sum in Eq. 5 is not a priori finite. To ensure that it is finite, we need to assume an additional boundedness condition on either $\mathcal{CFD}(\mathcal{H}_1)$ or $\mathcal{CFD}(\mathcal{X})$. These boundedness conditions correspond to an admissibility hypothesis for $\mathcal{H}_1$, which in turn guarantees that $\partial Y_1 \cup_\partial Y_2$ is (weakly) admissible.

**Lemma 1.** There is a canonical homotopy equivalence

$$\mathcal{CFD}(\mathcal{H}_1) \otimes_{\mathcal{J}(\mathcal{X})} \mathbf{CF}(\mathcal{X}_2) \cong \mathcal{CFD}(\mathcal{H}_1) \otimes_{\mathcal{J}(\mathcal{X})} \mathbf{CF}(\mathcal{X}_2)$$

The proof is straightforward.

We turn to the analytic side of the argument next. Because of how the idempotents act on $\mathcal{CFD}(\mathcal{H}_1)$ and $\mathbf{CF}(\mathcal{X}_2)$, there is an obvious identification between generators $x_0 \otimes x_0$ of $\mathcal{CFD}(\mathcal{H}_1) \otimes_{\mathcal{J}(\mathcal{X})} \mathbf{CF}(\mathcal{X}_2)$ and generators $x$ of $\mathbf{CF}(\mathcal{H}_1)$.

Let $Z = \mathcal{H}_1 \otimes \mathcal{H}_2 \subset \mathcal{H}$. The differential on $\mathcal{CF}(\mathcal{H}_1)$ counts rigid $J$-holomorphic curves in $X \times [0,1] \times R$. For a sequence of almost complex structures $J$, with longer and longer necks at $Z$, such curves degenerate to pairs of curves $(u_{1}, u_{2})$ for $\mathcal{H}_1$ and $\mathcal{H}_2$, with matching asymptotics at $\infty$. More precisely, in the limit as we stretch the neck, the moduli space degenerates to a fibered product.
The operation $\boxdot$. (A) Graphical representation of $\delta^!$ for $\text{CFD}$. (B) Graphical representation of $m_{0,1}$ (n = 3) for $\text{CF}. (C)$ Graphical representation of the sum from Eq. 5. In all cases, dashed lines represent module elements and solid lines represent algebra elements.

$$\bigoplus_{(\rho_1, \ldots, \rho_l)} \mathcal{M}(x_L, y_L; (\rho_1, \ldots, \rho_l))_{\text{ev}_x \text{ev}_y} \mathcal{M}(x_R, y_R; (\rho_1, \ldots, \rho_l)).$$

Here, $\text{ev}_x, \text{ev}_y: \mathcal{M}(x, y; (\rho_1, \ldots, \rho_l)) \rightarrow \mathbb{R}/\mathbb{R}$ is given by taking the successive height differences (in the $\mathbb{R}$ coordinate) of the chords $\rho_1, \ldots, \rho_l$, and similarly for $\text{ev}_y$. Also, we are suppressing homology classes from the notation.

Because we are taking a fiber product over a large-dimensional space, the moduli spaces in Formula 6 are not conducive to defining invariants of $\mathcal{F}_1$ and $\mathcal{F}_2$. To deal with this, we deform the matching condition, considering instead the fiber products,

$$\mathcal{M}(x_L, y_L; (\rho_1, \ldots, \rho_l))_{\text{ev}_x \text{ev}_y} \mathcal{M}(x_R, y_R; (\rho_1, \ldots, \rho_l)),$$

and sending $R \rightarrow \infty$. In the limit, some of the chords on the left collide, whereas some of the chords on the right become infinitely far apart. The result exactly recaptures the definitions of $\text{CF}A(\mathcal{F}_L)$ and $\text{CFD}(\mathcal{F}_R)$ and the algebra of Eq. 5 (30).

**Remark 11:** In ref. 30, we also give another proof of the pairing theorem (Eq. 1), using the technique of nice diagrams (31).

**Computing with Bordered Floer Homology**

**Computing $\text{CF}$**. Let $Y$ be a closed 3-manifold. As discussed earlier, $Y$ admits a Heegaard splitting into two handlebodies, glued by some homeomorphism $\phi$ between their boundaries. Via the pairing theorems (Eqs. 2 and 3), this reduces computing $\text{HF}(Y)$ to computing $\text{CFD}(H_g)$ for some particular bordered handlebody $H_g$ of each genus $g$ and $\text{CFDD}(\phi)$ for arbitrary $\phi$ in the strongly based mapping class group. For an appropriate $H_g$, $\text{CFD}(H_g)$ is easy to compute. Moreover we do not need to compute $\text{CFDD}(\phi)$ for every mapping class, just for generators for the mapping class groupoid. This groupoid has a natural set of generators: arc-sides (compare refs. 57 and 58). It turns out that the type $DD$ invariants of arc-sides are determined by a small amount of geometric input (essentially, the set of generators and a nondegeneracy condition for the differential) and the condition that $\mathbf{d} \phi = 0$ (59).

These techniques also allow one to compute all types of the bordered invariants for any bordered 3-manifold.

**Cobordism Maps.** Next, we discuss how to compute the map $\hat{f}_W: \text{CF}(Y_1) \rightarrow \text{CF}(Y_2)$ associated to a 4-dimensional cobordism $W$ from $Y_1$ to $Y_2$. The cobordism $W$ can be decomposed into three cobordisms $W_1W_2W_3$, where $W_i: Y_i \rightarrow Y_{i+1}$ consists of $i$-handle attachments and $\hat{f}_W$ is a corresponding composition $\hat{f}_{W_3} \circ \hat{f}_{W_2} \circ \hat{f}_{W_1}$.

The maps $\hat{f}_{W_1}$ and $\hat{f}_{W_2}$ are simple to describe: $Y_3 \cong Y_i \# (S^2 \times S^1)$, whereas $Y_2 \cong Y_i \# (S^2 \times S^1); \text{CF}(S^2 \times S^1)$ is (homotopy equivalent to) $F_2 \oplus F_2 = H_1(S^2; F_2)$; and the invariant $\text{HF}(Y)$ satisfies a Künneth theorem for connect sums, so

$$\text{CF}(Y_i \# (S^2 \times S^1)) \cong \text{CF}(Y_i) \otimes_{F_2} H_1(T^k; F_2)$$

(with respect to appropriate Heegaard diagrams), where $T^k = (S^1)^k$ is the $k$-dimensional torus. Let $\theta$ be the top-dimensional generator of $H_1(T^k; F_2)$ and $\eta$ the bottom-dimensional generator of $H_1(T^k; F_2)$. Then $\hat{f}_W$ is $x \mapsto x \oplus \theta$, whereas $\hat{f}_W$ takes $x \oplus \eta \mapsto x$ and $x \oplus \epsilon \mapsto 0$ if $gt(e) > gr(\eta)$.

By contrast, $f_W$ is defined by counting holomorphic triangles in a suitable Heegaard triple diagram. Two additional properties of bordered Floer theory allow us to compute $f_W$:

- The invariant $\text{CFD}(H)$ of a handlebody $H$ is rigid, in the sense that it has no nontrivial graded automorphisms. This allows one to compute the homotopy equivalences between the results of making different choices in the computation of $\text{CF}(Y)$.

- There is a pairing theorem for holomorphic triangles.

Given these, one can compute $\hat{f}_W$ as follows: Using results from the previous section, we can compute $\text{CF}(Y_2)$ [respectively, $\text{CF}(Y_3)]$ using a Heegaard decomposition making the decomposition $Y_3 \cong Y_1 \# (S^2 \times S^1)$ [respectively, $Y_3 \cong Y_1 \# (S^2 \times S^1)$] manifest. With respect to this decomposition, the map $\hat{f}_W$ (respectively, $\hat{f}_W$) is easy to read off.

To compute $\hat{f}_W$, one works with Heegaard decompositions of $Y_2$ and $Y_3$ with respect to which the cobordism $W_2$ takes a particularly simple form, replacing one of the handlebodies $H$ of a Heegaard decomposition of $Y_2$ with a differently framed handlebody $H$. It is easy to compute the triangle map $\text{CFD}(H) \rightarrow \text{CFD}(H')$. By the pairing theorem for triangles, extending this map by the identity map on the rest of the decomposition gives the map $\hat{f}_W$. Finally, the rigidity result allows one to write down the isomorphisms between $\text{CF}(Y_2)$ [and $\text{CF}(Y_3)]$ computed in the two different ways. The map $\hat{f}_W$ is then the composition of the three maps $\hat{f}_W$ and the equivalences connecting the two different models of $\text{CF}(Y_2)$ and of $\text{CF}(Y_3)$.

The details will appear in forthcoming work.

**Polygon Maps and the Oszváth–Szabó Spectral Sequence.** Khovanov introduced a categorification of the Jones polynomial (60). This categorification associates to an oriented link $L$ a bigraded abelian group $K_{hj}(L)$, the Khovanov homology of $L$, whose graded Euler characteristic is $(q + q^{-1})$ times the Jones polynomial $J(L)$. There is also a reduced version $\tilde{K}_{h}(L)$, whose graded Euler characteristic is simply $J(L)$. The skein relation for $J(L)$ is replaced by a skein exact sequence. Given a link $L$ and a crossing $c$ of $L$, let $L_0$ and $L_1$ be the two resolutions of $c$. Then there is a long exact sequence relating the (reduced) Khovanov homology groups of $L$, $L_0$, and $L_1$.

Szabó and the second author observed that the Heegaard Floer group $\text{HF}(L)$ of the double cover of $S^3$ branched over $L$ satisfies a similar skein exact triangle to (reduced) Khovanov homology and takes the same value on an $n$-component unlink (with some collapse of gradings). Using these observations, they produced a spectral sequence from Khovanov homology (with $F_2$ coefficients) to $\text{HF}(L)$ (26). Because of a difference in conventions, one must take the Khovanov homology of the mirror $\tilde{r}(L)$ of $L$. Baldwin recently showed (61) that the entire spectral sequence $\tilde{K}_{h}(r(L)) \Rightarrow \text{HF}(L(D))$ is a knot invariant.

Bordered Floer homology can be used to compute this spectral sequence (62). Write $L$ as the plat closure of some braid $B$, and decompose $B$ as a product of braid generators $b_1 \cdots b_n$. The branched double cover of a braid generator $b_i$ is the mapping cylinder of a Dehn twist, and the branched double covers of the plats closing $B$ is a handlebody $H$. So $\text{CF}(D(K))$ is quasi-isomorphic to...
The bordered invariant of a Dehn twist $\tau$, along $\gamma \subset F(X)$, can be written as a mapping cone of a map between the identity cobordism $I = [0,1] \times F(X)$ and the manifold $Y_{\partial_0 I}$ obtained by $0$-surgery on $[0,1] \times F(X)$ along $\gamma$:

$$\text{CFDA}(\tau) \cong \text{cone}(\text{CFDA}(Y_{\partial_0 I}) \to \text{CFDA}(I)).$$

The key ingredient is another pairing theorem identifying the algebra of tensor products of mapping cones with counts of holomorphic polygons.

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