Low Dimensional Geometry and Topology Special Feature: Tour of bordered Floer theory

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Tour of bordered Floer theory

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Heegaard Floer theory is a kind of topological quantum field theory (TQFT), assigning graded groups to closed, oriented 3-manifolds and group homomorphisms to smooth, oriented four-dimensional cobordisms. Bordered Heegaard Floer homology is an extension of Heegaard Floer homology to 3-manifolds with boundary, with extended-TQFT-type gluing properties. In this survey, we explain the formal structure and construction of bordered Floer homology and sketch how it can be used to compute some aspects of Heegaard Floer theory.

Heegaard Floer homology | 4-manifolds

Heegaard Floer homology, introduced in a series of papers (1–3) of Szabó and the second author has become a useful tool in three- and four-dimensional topology. The Heegaard Floer invariants contain subtle topological information, allowing one to detect the genera of knots and homology classes (4); detect fiberedness for knots (5–9) and 3-manifolds (10–13); bound the slice genus (14) and unknotting number (15, 16); prove tightness and obstruct Stein fillability of contact structures (5, 17); allow one to detect the genera of knots and homology classes (4); detect fiberedness for knots (5–9) and 3-manifolds (10–13); bound the slice genus (14) and unknotting number (15, 16); prove tightness and obstruct Stein fillability of contact structures (5, 17); and more. It has been useful for resolving a number of conjectures, particularly related to questions about Dehn surgery (18, 19); see also ref. 20. It is either known or conjectured to be equivalent to several other gauge-theoretic or holomorphic curve invariants in low-dimensional topology, including monopole Floer homology (21), embedded contact homology (22), and the Lagrangian matching invariants of 3- and 4-manifolds (23–25). Heegaard Floer homology is known to relate to Khovanov homology (26–28), and more relations with Khovanov–Rozansky-type homologies are conjectured (29).

Heegaard Floer homology has several variants; the technically simplest is \( \hat{HF} \), which is sufficient for most of the three-dimensional applications discussed above. Bordered Heegaard Floer homology, the focus of this paper, is an extension of \( \hat{HF} \) to 3-manifolds with boundary (30). This extension gives a conceptually satisfying way to compute essentially all aspects of the Heegaard Floer package related to \( \hat{HF} \). [There are also other algorithms for computing many parts of Heegaard Floer theory (31–39).]

We will start with the formal structure of bordered Heegaard Floer homology. Most of the paper is then devoted to sketching its definition. We conclude by explaining how bordered Floer homology can be used for calculations of Heegaard Floer invariants.

Formal Structure

Review of Heegaard Floer Theory. Heegaard Floer theory has many components. Most basic among them, it associates:

- To a closed, connected, oriented 3-manifold \( Y \), an abelian group \( \hat{HF}(Y) \) and \( \mathbb{Z}[U] \) modules \( HF^+(Y) \), \( HF^-(Y) \), and \( HF^\infty(Y) \). These are the homologies of chain complexes \( CF(Y) \), \( CF^+(Y) \), \( CF^-(Y) \), and \( CF^\infty(Y) \), respectively. The chain complexes (and their homology groups) decompose into spin structures \( CF(Y) = \bigoplus_{\text{spin} \text{ structures}} CF(Y, \mathfrak{s}) \), where \( CF(Y, \mathfrak{s}) \) is any of the four chain complexes. Each \( CF(Y, \mathfrak{s}) \) has a relative grading modulo the divisibility of \( c_1(\mathfrak{s}) \) (1). The chain complex \( CF(Y) \) is the \( U = 0 \) specialization of \( CF^-(Y) \).

- To a smooth, compact, oriented cobordism \( W \) from \( Y_1 \) to \( Y_2 \), maps \( F_w: HF(Y_1) \to HF(Y_2) \) induced by chain maps \( f_w: CF(Y_1) \to CF(Y_2) \). These maps decompose according to spin structures on \( W \).

The maps \( f_w \) satisfy a topological quantum field theory (TQFT) composition law:

- If \( W' \) is another cobordism, from \( Y_2 \) to \( Y_3 \), then \( F_{w'} \circ F_w = F_{w \circ w'} \).

The Heegaard Floer invariants are defined by counting pseudo-holomorphic curves in symmetric products of Heegaard surfaces. The Heegaard Floer groups were conjectured to be equivalent to the monopole Floer homology groups (defined by counting solutions of the Seiberg–Witten equations), via the correspondence: \( HF^+(Y) \leftrightarrow HM(Y), HF^-(Y) \leftrightarrow HM(Y), HF^\infty(Y) \leftrightarrow TM(Y) \), and similarly for the corresponding cobordism maps. A proof of this conjecture has recently been announced by Kutluhan et al. (40–42). Colin et al. have announced an independent proof for the \( U = 0 \) specialization (43).

In particular, the Heegaard Floer package contains enough information to detect exotic smooth structures on 4-manifolds (10, 44). For closed 4-manifolds, this information is contained in \( HF^+ \) and \( HF^- \); the weaker invariant \( HF \) is not useful for distinguishing smooth structures on closed 4-manifolds.

The Structure of Bordered Floer Theory. Bordered Floer homology is an extension of \( HF \) to 3-manifolds with boundary, in a TQFT form. Bordered Floer homology associates:

- To a closed, oriented, connected surface \( F \), together with some extra markings (see Definition 1), a differential graded (dg) algebra \( \mathcal{A}(F) \).

- To a compact, oriented 3-manifold \( Y \) with connected boundary, together with a diffeomorphism \( \varphi: F \to \partial Y \) marking the boundary, a module over \( \mathcal{A}(F) \). Actually, there are two different invariants for \( Y: \mathcal{CFD}(Y) \), a left dg module over \( \mathcal{A}(\mathcal{F}^-) \), and \( \mathcal{CFA}(Y) \), a right \( \mathcal{A}(\mathcal{F}^-) \) module over \( \mathcal{A}(F) \), each well-defined up to quasi-isomorphism. We sometimes refer to a 3-manifold \( Y \) with \( \partial Y = F \); we actually mean \( Y \) together with an identification \( \varphi \) of \( \partial Y \) with \( F \). We call these data a bordered 3-manifold.

- More generally, to a 3-manifold \( Y \) with two boundary components \( \partial_1 Y \) and \( \partial_2 Y \), diffeomorphisms \( \varphi_1: F_1 \to \partial_1 Y \) and \( \varphi_2: F_2 \to \partial_2 Y \) and a framed arc \( \gamma \) from \( \partial_1 Y \) to \( \partial_2 Y \) (compatible with \( \varphi_1 \) and \( \varphi_2 \) in a suitable sense), a dg module \( \mathcal{CFDD}(Y) \) with commuting left actions of \( \mathcal{A}(\mathcal{F}^-) \) and \( \mathcal{A}(\mathcal{F}^-) \); an \( \mathcal{A}(\mathcal{F}^-) \) module \( \mathcal{CFDA}(Y) \) with left action of \( \mathcal{A}(\mathcal{F}^-) \) and a right action of \( \mathcal{A}(\mathcal{F}^-) \); and an \( \mathcal{A}(\mathcal{F}^-) \) module \( \mathcal{CFAA}(Y) \) with commuting right actions of \( \mathcal{A}(\mathcal{F}^-) \) and \( \mathcal{A}(\mathcal{F}^-) \).

Each of \( \mathcal{CFDD}(Y) \), (For \( CF^+ \) and \( CF^- \), we mean the completions with respect to the formal variable \( U \).

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*For \( CF^+ \) and \( CF^- \), we mean the completions with respect to the formal variable \( U \).

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Given 3-manifolds $Y_1$ and $Y_2$ with $\partial Y_1 = \partial Y_2 = F = -F$, there is a quasi-isomorphism

$$\tilde CF(Y_1 \cup_F Y_2) \cong \tilde CF(A(Y_1) \otimes_{\tilde CF(F)} \tilde CF(Y_2)).$$

where $\otimes$ denotes the derived tensor product. So,

$$HF(Y_1 \cup_Y Y_2) \cong \text{Tor}_{\tilde CF(F)}(\tilde CF(Y_1), \tilde CF(Y_2)).$$

More generally, given 3-manifolds $Y_1$ and $Y_2$ with $\partial Y_1 = -F \cup F$ and $\partial Y_2 = -F \cup F$, there are quasi-isomorphisms of bimodules corresponding to any valid tensor product. For instance,

$$\tilde CFDD(Y_1 \cup_F Y_2) \cong \tilde CFDD(Y_1) \otimes_{\tilde CFDD([0,1] \times F)} \tilde CFDD(Y_2).$$

Also, $F_1$ or $F_3$ may be $S^2$ (or empty), in which case these statements reduce to pairing theorems for a module and a bimodule.

We refer to theorems of this kind as pairing theorems. There is also a self-pairing theorem. Let $Y$ be a 3-manifold with $\partial Y = -F \cup F$ and $\gamma$ be a framed arc connecting corresponding points in the boundary components of $Y$. The self-pairing theorem relates the Hochschild homology of the bimodule $\tilde CFDD(Y)$ with the knot Floer homology of a generalized open book decomposition associated to $Y$ and $\gamma$.

These invariants satisfy a number of duality properties (46); e.g.:

- The algebra $\mathcal{A}(F)$ is the opposite algebra of $\mathcal{A}(-F)$. (There are also more subtle duality properties of the algebra; see Remark 4.)
- The module $\tilde CFDD(Y)$ is dual [over $\mathcal{A}(F)$] to $\tilde CFDD(\gamma)$:

$$\tilde CFDD(Y) \cong \text{Mor}_{\mathcal{A}(F)}(\tilde CFDD(\gamma), \mathcal{A}(F)).$$

- The module $\tilde CFDD(\gamma)$ is the one-sided dual of $\tilde CFDD(Y)$:

$$\text{Mor}_{\mathcal{A}(F)}(\tilde CFDD(\gamma), \mathcal{A}(F)) \cong \tilde CFDD(Y).$$

The symmetric statement also holds, as does the corresponding statement for $\tilde CFDD(Y)$. As a consequence of these dualities, one can give pairing theorems using the $\text{Hom}$ functor rather than the tensor product (46); e.g.:

- Let $Y_1$ and $Y_2$ be 3-manifolds with $\partial Y_1 = \partial Y_2 = F$. Then

$$\tilde CF(-Y_1 \cup_F Y_2) \cong \text{Mor}_{\mathcal{A}(F)}(\tilde CFDD(Y_1), \tilde CFDD(Y_2)).$$

Similar statements hold for $\tilde CFDD(Y)$ and for bimodules.

- Given 3-manifolds $Y_1$ and $Y_2$ with $\partial Y_1 = F = -\partial Y_2$, $\tilde CF(Y_1 \cup_F Y_2) \cong \text{Mor}_{\mathcal{A}(F)}(\tilde CFDD(Y_1 \cup_F Y_2), \tilde CFDD(Y_2 \cup_F Y_1)).$

Similarly, if $Y_3$ has another boundary component $F'$, then

$$\tilde CF(Y_1 \cup_{F'} Y_3) \cong \text{Mor}_{\mathcal{A}(F')}(\tilde CFDD(Y_1 \cup_{F'} Y_3), \tilde CFDD(Y_3 \cup_{F'} Y_1)).$$

If both $Y_1$ and $Y_2$ had two boundary components, then the left-hand side would pick up a change of framing.)

Remark 1: Some of the duality properties discussed above can also be seen from the Fukaya-categorical perspective (47).

Remark 2: It is natural to expect that to a 4-manifold with corners one would associate a map of bimodules, satisfying certain gluing axioms. We have not done this; however, as discussed below, even without this bordered Floer homology allows one to compute the maps $F_W$ associated to cobordisms $W$ between closed 3-manifolds.

The Algebras
As mentioned earlier, the bordered Floer algebras are associated to surfaces together with some extra markings. We encode these markings as pointed matched circles $\Xi$, which we discuss next. We then introduce a simpler algebra, $\mathcal{A}(n)$, depending only on an integer $n$, of which the bordered Floer algebras $\mathcal{A}(\Xi)$ are subalgebras. The definition of $\mathcal{A}(\Xi)$ itself is given in the last subsection.

Pointed Matched Circles

Definition 1: A pointed matched circle $\Xi$ consists of an oriented circle $Z$; 4k points $a = \{a_1, \ldots, a_{4k}\}$ in $Z$, a matching $M$ of the points in $a$ in pairs, which we view as a fixed-point free involution $M: a \to a$; and a basepoint $z \in Z \setminus a$. We require that performing surgery on $Z$ along the matched pairs of points yields a connected 1-manifold. A pointed matched circle $\Xi$ with $|a| = 4k$ specifies:

- A closed surface $F(Z)$ of genus $k$, as follows: Fill $Z$ with a disk $D$. Attach a two-dimensional 1-handle to each pair of points in $a$ matched by $M$. By hypothesis, the resulting space is connected, with boundary consisting of a single disk $D$ (say).
- A distinguished disk in $F(Z)$: the disk $D$ (say).
- A basepoint $z$ in the boundary of the distinguished disk.

Remark 3: Matched circles can be seen as a special case of fat graphs (48). They are also dual to the typical representation of a genus $g$ surface as a 4g-gon with sides glued together.

The Strands Algebras
We next define a differential algebra $\mathcal{A}(n)$, depending only on an integer $n$; the algebra $\mathcal{A}(\Xi)$ associated to a pointed matched circle with $|\Xi| = 4k$ will be a subalgebra of $\mathcal{A}(4k)$. The algebra $\mathcal{A}(n)$ has an $F_2$ basis consisting of all triples $(S, T, \phi)$, where $S$ and $T$ are subsets of $g = \{1, \ldots, n\}$ and $\phi: S \to T$ is a bijection such that for all $S \subseteq S$, $\phi(s) \geq s$. Given such a map $\phi$, let $\text{Inv}(\phi) = \{(s_1, s_2) \in S \times S | s_1 < s_2, \phi(s_1) < \phi(s_2)\}$ and $\text{inv}(\phi) = |\text{Inv}(\phi)|$, so $\text{Inv}(\phi)$ is the number of inversions of $\phi$.

The product $(S, T, \phi) \cdot (U, V, \psi)$ in $\mathcal{A}(n)$ is defined to be 0 if $U \neq T$ or $U = T$, but $\text{inv}(\psi + \phi) \neq \text{inv}(\psi) + \text{inv}(\phi)$. If $U = T$ and $\text{inv}(\psi + \phi) = \text{inv}(\psi) + \text{inv}(\phi)$, then let $(S, T, \phi) \cdot (U, V, \psi) = (S \cup U, V, \psi + \phi)$. In particular, the elements $(S, \emptyset)$ (where $\emptyset$ denotes the identity map) are the indecomposable idempotents in $\mathcal{A}(n)$.

Given a generator $(S, T, \phi) \in \mathcal{A}(n)$ and an element $\sigma = (s_1, s_2) \in \text{Inv}(\phi)$, let $\phi_\sigma: S \to T$ be the map defined by $\phi_\sigma(s) = (s_1, s_2)$ if $s \neq s_1, s_2$; $\phi_\sigma(s_1) = \phi(s_2)$; and $\phi_\sigma(s_2) = (s_1, s_2)$. Define a differential on $\mathcal{A}(n)$ by
The strands algebra. A product, two vanishing products, and a differential. In this notation, the restrictions on the number of inversions means that elements with double crossings in the product or differential are set to 0.

\[
\partial(S, T, \phi) = \sum_{\text{inv}(\phi)} (S, T, \phi_i).
\]

See Fig. 1 for a graphical representation.

Given a generator \((S, T, \phi) \in \mathcal{A}(n)\), define the weight of \((S, T, \phi)\) to be the cardinality of \(S\). Let \(\mathcal{A}(n, i)\) be the subalgebra of \(\mathcal{A}(n)\) generated by elements of weight \(i\), so \(\mathcal{A}(n) = \bigoplus_{i=0}^{n} \mathcal{A}(n, i)\).

**The Algebra Associated to a Pointed Matched Circle.** Fix a pointed matched circle \(\mathcal{Z} = (Z, a, M, z)\) with \(|a| = 4k\). After cutting \(Z\) at \(z\), the orientation of \(Z\) identifies \(a\) with \(4k\), so we can view \(M\) as a matching of \(4k\).

Call a basis element \((S, T, \phi)\) of \(\mathcal{A}(4k)\) equitable if no two elements of \(4k\) that are matched (with respect to \(M\)) occur in \(S\), and no two elements of \(4k\) that are matched both occur in \(T\). Given equitable basis elements \(x = (S, T, \phi)\) and \(y = (S', T', \psi)\) of \(\mathcal{A}(4k)\), we say that \(x\) and \(y\) are related by horizontal strand swapping, and write \(x \sim y\), if there is a subset \(U \subseteq S\) such that \(S' = (S \setminus U) \cup M(U)\), \(\phi_{|SU} = \psi_{|SU}\), \(\phi_{|U} = \psi_{|U}\), and \(\psi_{|M(U)} = \lambda_{M(U)}\).

Given an equitable basis element \(x\) of \(\mathcal{A}(4k)\), let \(a(x) = \sum_{\phi \in \phi_{\text{inv}}(y)} y\). See Fig. 2 for an example. Define \(\mathcal{A}(\mathcal{Z}) \subset \mathcal{A}(4k)\) to be the \(F_2\) subspace with basis \(\{a(x) | x \text{ is equitable}\}\). It is straightforward to verify that \(\mathcal{A}(\mathcal{Z})\) is a differential subalgebra of \(\mathcal{A}(4k)\). We call the elements \(a(x)\) basic generators of \(\mathcal{A}(\mathcal{Z})\). If \(x\) is not equitable, set \(a(x) = 0\) and extend \(a\) linearly to a map \(a: \mathcal{A}(4k) \to \mathcal{A}(\mathcal{Z})\).

This is not an algebra homomorphism.

Indecomposable idempotents of \(\mathcal{A}(\mathcal{Z})\) correspond to subsets of the set of matched pairs in \(a\). These generate a subalgebra \(\mathcal{F}(\mathcal{Z})\) where all strands are horizontal. The algebra \(\mathcal{A}(\mathcal{Z})\) decomposes as \(\bigoplus_{i=0}^{k} \mathcal{A}(\mathcal{Z}, i)\), where \(\mathcal{A}(\mathcal{Z}, i) = \mathcal{A}(\mathcal{Z}) \cap \mathcal{A}(4k, k + i)\).

As the figures suggest, we often think of elements of \(\mathcal{A}(\mathcal{Z})\) in terms of sets of chords in \((Z, a)\), i.e., arcs in \(Z\) with endpoints in \(a\), with orientations induced from \(Z\). Given a chord \(\rho\) in \((Z \setminus \{a\}, a)\), let \(\rho^+\) (respectively, \(\rho^-\)) be the initial (respectively, terminal) endpoint of \(\rho\). Given a set \(\rho = \{\rho_i\}\) of chords in \((Z \setminus \{a\}, a)\) such that no two \(\rho_i\) have the same initial (respectively, terminal) endpoint, let \(\rho^- = \{\rho_i^-\}\) and \(\rho^+ = \{\rho_i^+\}\); we can think of \(\rho\) as a map \(\rho^+ \to \rho^-\). Let

\[
a(\rho) = \sum_{S: J \rho^- \to S \in \phi^{-} - \phi^{+}} (S_0 \cup \rho^-; (S_0 \cup \rho^+; \phi_0), \phi_0^{H_{\phi_0}}).
\]

**Example 1:** The algebra associated to the unique pointed match circle for \(S^2\) is \(\mathbb{F}_2\). The algebra \(\mathcal{A}(T^2, 0)\) associated to the unique pointed match circle for \(T^2\), with \((1 \leftrightarrow 3, 2 \leftrightarrow 4)\), is given by the path algebra with relations:

\[
\rho_{1,2}; \rho_{3,4} \\
I_1 \xrightarrow{\rho_{0,2}} I_2 / (\rho_{2,3} \rho_{1,2} = \rho_{3,4} \rho_{2,3} = 0).
\]

The algebra associated to the pointed match circle \(\mathcal{Z}\) for a genus 2 surface with matching \((1 \leftrightarrow 3, 2 \leftrightarrow 4, 5 \leftrightarrow 7, 6 \leftrightarrow 8)\) has Poincaré polynomial (45, Sect. 4)

\[
\sum \dim_i H_i(\mathcal{A}(\mathcal{Z}, i) T^2 = T^{i-2} + 32T^{-1} + 98 + 32T + T^2.
\]

The algebra associated to the pointed match circle \(\mathcal{Z}'\) for a genus 2 surface with matching \((1 \leftrightarrow 5, 2 \leftrightarrow 6, 3 \leftrightarrow 7, 4 \leftrightarrow 8)\) has Poincaré polynomial

\[
\sum \dim_i H_i(\mathcal{A}(\mathcal{Z}', i)) T^2 = T^{2-2} + 32T^{-1} + 70 + 32T + T^2.
\]

The ranks in the genus two examples which are equal are explained by the observations that for any pointed match circle, \(\mathcal{A}(\mathcal{Z}, -k) \cong F_{2k}\). \(\mathcal{A}(\mathcal{Z}, -k + 1)\) has no differential; the dimension of \(\mathcal{A}(\mathcal{Z}, -k + 1)\) is independent of the matching; and the following:

**Remark 4:** The algebras \(\mathcal{A}(\mathcal{Z}, i)\) and \(\mathcal{A}(\mathcal{Z}, -i)\) are Koszul dual. (Here, \(\mathcal{Z}\) denotes the pointed match circle obtained by reversing the orientation on \(Z\).) Also, given a pointed matched circle \(\mathcal{Z}\) for \(F\), let \(\mathcal{Z}'\) denote the pointed matched circle corresponding to the dual handle decomposition of \(F\). Then \(\mathcal{A}(\mathcal{Z}, i)\) and \(\mathcal{A}(\mathcal{Z}', i)\) are Koszul dual. In particular, \(\mathcal{A}(\mathcal{Z}, -i)\) is quasi-isomorphic to \(\mathcal{A}(\mathcal{Z}', i)\).

**Remark 5:** In Zarev’s bordered-sutured extension of the theory (49), the strands algebra \(\mathcal{A}(n, k)\) has a topological interpretation as the algebra associated to a disk with boundary sutures.

Combinatorial Representations of Bordered 3-Manifolds

A bordered 3-manifold \(Y\) with an orientation-preserving homeomorphism \(\phi: F(\mathcal{Z}) \to \partial Y\) for some pointed matched circle \(\mathcal{Z}\). Two bordered 3-manifolds \((Y_1, \phi_1: F(\mathcal{Z}_1) \to \partial Y_1)\) and \((Y_2, \phi_2: F(\mathcal{Z}_2) \to \partial Y_2)\) are called equivalent if there is an orientation-preserving homeomorphism \(\psi: Y_1 \to Y_2\) such that \(\phi_1 = \phi_2 \circ \psi\); in particular, this implies that \(\mathcal{Z}_1 = \mathcal{Z}_2\). Bordered Floer theory associates homotopy equivalence classes of modules to equivalence classes of bordered 3-manifolds. Just as the bordered Floer algebras are associated to combinatorial representations of surfaces, not directly to surfaces, the bordered Floer modules are associated to combinatorial representations of bordered 3-manifolds.
The Closed Case. Recall that a three-dimensional handlebody is a regular neighborhood of a connected graph in $\mathbb{R}^3$. According to a classical result of Heegaard (50), every closed, orientable 3-manifold can be obtained as a union of two such handlebodies, $H_a$ and $H_b$. Such a representation is called a Heegaard splitting. A Heegaard splitting along an orientable surface $\Sigma$ of genus $g$ can be represented by a Heegaard diagram: a pair of $g$ tuples of pairwise disjoint, homologically linearly independent, embedded circles $\alpha = (\alpha_1, \ldots, \alpha_g)$ and $\beta = (\beta_1, \ldots, \beta_g)$ in $\Sigma$. These curves are chosen so that each $\alpha_i$ (respectively, $\beta_i$) bounds a disk in the handlebody $H_a$ (respectively, $H_b$). Any two Heegaard diagrams for the same manifold $Y$ are related by a sequence of moves, called Heegaard moves; see, for instance, ref. 51 or ref. 1, Sect. 2.1.

Representing 3-Manifolds with Boundary. The story extends easily to 3-manifolds with boundary, using a slight generalization of handlebodies. A compression body (with both boundaries connected) is the result of starting with a connected orientable surface $\Sigma_2$ times $[0,1]$ and then attaching thickened disks (three-dimensional 2-handles) along some number of homologically linearly independent, disjoint circles in $\Sigma_2 \times \{0\}$. A compression body has two boundary components, $\Sigma_1$ and $\Sigma_3$, with genera $g_1 \leq g_2$. Up to homeomorphism, a compression body is determined by its boundary.

A Heegaard decomposition of a 3-manifold $Y$ with nonempty, connected boundary is a decomposition $Y = H_a \cup H_b$, where $H_a$ is a compression body and $H_b$ is a handlebody. Let $\Sigma$ be the genus of $\Sigma_2$ and $k$ the genus of $\partial Y$. A Heegaard diagram for $Y$ is gotten by choosing $g$ pairwise disjoint circles $\beta_1, \ldots, \beta_k$ in $\Sigma$ and $g-k$ disjoint circles $\alpha_1', \ldots, \alpha_{g-k}'$ in $\Sigma$ so that

- The circles $\beta_1, \ldots, \beta_k$ bound disks $D_{\beta_1}, \ldots, D_{\beta_k}$ in $H_b$ such that $H_a \cup (D_{\beta_1} \cup \ldots \cup D_{\beta_k})$ is topologically a ball, and
- The circles $\alpha_1', \ldots, \alpha_{g-k}'$ bound disks $D_{\alpha_1'}, \ldots, D_{\alpha_{g-k}'}$ in $H_a$ such that $H_a \cup (D_{\alpha_1'} \cup \ldots \cup D_{\alpha_{g-k}'})$ is topologically the product of a surface and an interval.

To specify a parametrization, or bordering, of $\partial Y$, we need a little more data. A bordered Heegaard diagram for $Y$ is a tuple $\mathcal{H} = (\Sigma, \alpha_1', \ldots, \alpha_{g-k}', \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_k, z)$, where

- $\Sigma$ is an oriented surface with a single boundary component;
- $(\Sigma || D^2, \alpha', \beta)$ is a Heegaard diagram for $Y$;
- $\alpha_1', \ldots, \alpha_{g-k}'$ are pairwise disjoint, embedded arcs in $\Sigma$ with boundary on $\partial \Sigma$, and are disjoint from the $\alpha_i$;
- $\Sigma \cup (\alpha \cup \ldots \cup \alpha_{g-k}' \cup \ldots \cup \beta)$ is a disk with $2(g-k)$ holes;
- $z$ is a point in $\partial \Sigma$, disjoint from all of the $\alpha_i$.

Let $\alpha = \alpha' \cup \alpha$. A bordered Heegaard diagram $\mathcal{H}$ specifies a pointed matched circle $\mathcal{I}(\mathcal{H}) = (\mathcal{Z}, \alpha, \beta)$, two points where two points $\mathcal{a}$ are matched in $\mathcal{M}$ if they lie on the same $\alpha_i$. A bordered Heegaard diagram for $Y$ also specifies an identification $\phi: \mathcal{F}(\mathcal{Z}) \rightarrow \partial Y$, well-defined up to isotopy.

There are moves, analogous to Heegaard moves, relating any two bordered Heegaard diagrams for equivalent bordered 3-manifolds.

The Modules and Bimodules

As discussed above, there are two invariants of a 3-manifold $Y$ with boundary $F(\mathcal{Z})$. $\mathcal{C}FD(Y)$ has a straightforward module structure but a differential that counts holomorphic curves, whereas $\mathcal{C}FA(Y)$ uses holomorphic curves to define the module structure itself.

Fix a bordered Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, z)$ for $Y$. Let $\mathcal{E}(\mathcal{H})$ be the set of $g$ tuples $x = (x_i)_{i=1}^g \subset \alpha \cap \beta$ so that there is exactly one point $x_i$ on each $\beta$-circle and on each $\alpha$-circle, and there is at most one $x_i$ on each $\alpha$-arc. The invariant $\mathcal{C}FA(Y)$ is a direct sum of copies of $\mathbb{F}_2$, one for each element of $\mathcal{E}(\mathcal{H})$, whereas $\mathcal{C}FD(Y)$ is a direct sum of elementary projective $\mathcal{A}(\partial \mathcal{H})$ modules, one for each element of $\mathcal{E}(\mathcal{H})$. Let $X(\mathcal{H})$ be the $\mathbb{F}_2$-vector space generated by $\mathcal{E}(\mathcal{H})$, which is also the vector space underlying $\mathcal{C}FA(Y)$.

Each generator $x \in \mathcal{E}(\mathcal{H})$ determines a spin$^c$ structure $s(x) \in \text{spin}^c(Y)$; the construction (30) is an easy adaptation of the closed case (1, Sect. 2.6).

Before continuing to describe the bordered Floer modules, we digress to briefly discuss the moduli spaces of holomorphic curves.

Moduli Spaces of Holomorphic Curves. Fix a bordered Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, z)$. Let $\Sigma = \Sigma \setminus \partial \Sigma$. Choose a symplectic form $\omega_\Sigma$ on $\Sigma$ giving it a cylindrical end and a complex structure $j_\Sigma$ compatible with $\omega_\Sigma$, making $\Sigma$ into a punctured Riemann surface. Let $p$ denote the puncture in $\Sigma$. We choose the $\alpha_i'$ so that their intersections with $\Sigma$ (also denoted $\alpha'_j$) are cylindrical ($\mathbb{R}$-invariant) in a neighborhood of $p$.

We consider curves $u: (\Sigma, \partial \Sigma) \rightarrow (\Sigma \times [0,1] \times \mathbb{R}, \alpha \times \{1\} \times \mathbb{R} \cup \beta \times \{0\} \times \mathbb{R})$.

holomorphic with respect to an appropriate almost complex structure $J$, satisfying conditions spelled out in ref. 30. The reader may wish to simply think of a product complex structure $j_\Sigma \times j_\Sigma$, though these complex structures may not be general enough to achieve transversality.

Such holomorphic curves $u$ have asymptotics in three places:

- $\Sigma \times [0,1] \times \{-\infty\}$. We consider curves asymptotic to $g$-tuples of strips $x \times [0,1] \times \mathbb{R}$ at $-\infty$ and $y \times [0,1] \times \mathbb{R}$ at $+\infty$, where $x, y \in \mathcal{E}(\mathcal{H})$.
- $\{p\} \times [0,1] \times \mathbb{R}$, which we denote $\omega_\infty$. We consider curves asymptotic to chords $\rho_i$ in $\partial \Sigma$, a point $(1, i) \in [0,1] \times \mathbb{R}$. These are chords for the cylindrical foliation of $\partial \Sigma \times \{\omega_\infty\}$, whose leaves are the circles $\partial \Sigma \times \{(x_0, i_0)\}$.

We impose the condition that these chords $\rho_i$ not cross $z \in \partial \Sigma$.

Topological maps of this form can be grouped into homology classes. Let $\pi_1(x, y)$ denote the set of homology classes of maps asymptotic to $x \times [0,1] \at \pm \infty$ and $y \times [0,1] \at \pm \infty$. Then $\pi_1(x, y)$ is canonically isomorphic to $H_2(Y, \partial Y)$; $\pi_1(x, y)$ is nonempty if and only if $s(x) = s(y)$; and if $s(x) = s(y)$ then $\pi_1(x, y)$ is an affine copy of $H_2(Y, \partial Y)$, under concatenation by elements of $\pi_1(x, y)$ [or $\pi_1(y, x)$] (30). Again, these results are easy adaptations of the corresponding results in the closed case (1, Sect. 2.)

Note that our usage of $\pi_1(x, y)$ differs from the usage in ref. 1, where homology classes are allowed to cross $z$, but agrees with the usage in ref. 30.

Given generators $x, y \in \mathcal{E}(\mathcal{H})$, a homology class $B \in \pi_1(x, y)$, and a sequence $\rho = (\rho_1, \ldots, \rho_m)$ of sets $\rho_i \subset \{1, \ldots, m\}$ of Reeb chords, let $\mathcal{M}(x, y; \rho)$ denote the moduli space of embedded holomorphic curves $u$ in the homology class $B$, asymptotic to $x \times [0,1] \at \mp \infty$, $y \times [0,1] \at \mp \infty$, and $\rho_1 \times \{1, 2\} \at \pm \infty$, for some sequence of heights $t_1 < \cdots < t_m$. There is an action of $\mathbb{R}$ on $\mathcal{M}(x, y; \rho)$, by translation. Let $\mathcal{M}(x, y; \rho) = \mathcal{M}(x, y; \rho) / \mathbb{R}$.

The modules $\mathcal{C}FD(\mathcal{H})$ and $\mathcal{C}FA(\mathcal{H})$ will be defined using counts of zero-dimensional moduli spaces $\mathcal{M}(x, y; \rho)$. Proving that these modules satisfy $\delta^2 = 0$ and the $d_a$ relations, respectively, involves studying the ends of one-dimensional moduli spaces. These ends correspond to the following four kinds of degenerations:
1. Breaking into a two-story homologous building. That is, the $R$ coordinate of some parts of the curve go to $\pm \infty$ with respect to other parts, giving an element of $\mathbb{M}^B(x, y; \vec{p}) \times \mathbb{M}^B(x, y; \vec{p})$, where $B$ is the concatenation $B_1 \ast B_2$ and $\vec{p}$ is the concatenation $(\vec{p_1}, \vec{p_2})$.

2. Degenerations in which a boundary branch point of the projection $\pi \ast u$ approaches $e_0$, in such a way that some chord $\rho_{ij}$ splits into a pair of chords $\rho_{1i}$, $\rho_{2i}$ with $\rho_{1i} = \rho_{2i} \cup \rho_{0i}$. This degeneration results in a curve at $e_0$, a join curve, and an element of $\mathbb{M}^B(x, y; \vec{p})$, where $\vec{p}$ is obtained by replacing the chord $\rho_{ij} \in \rho_{1i} \in \rho_2$ with two chords, $\rho_{1i}$ and $\rho_{0i}$.

3. The difference in $R$ coordinates $t_{i-1} - t_i$ between two consecutive sets of chords $\rho_1$ and $\rho_{i-1}$ in $\vec{p}$ going to 0. In the process, some boundary branch points of $\rho_{i-1}$ may approach $e_0$, degenerating a split curve, along with an element of $\mathbb{M}^B(x, y; \vec{p})$, where $\vec{p} = (\rho_{1i}, \rho_{2i}, \rho_{0i}; \ldots; \rho_{0n})$ and $\rho_{0i} = (\rho_{0i}^1, \rho_{0i}^2)$ is gotten from $\rho_i \cup \rho_{i+1}$, by gluing together any pairs of chords $(\rho_{i1}, \rho_{i+1})$ where $\rho_{i1}$ ends at the starting point of $\rho_{i+1}$ (i.e., $\rho_{i1} = \rho_{i+1}$).

4. Degenerations in which a pair of boundary branch points of $\pi \ast u$ approach $e_0$, causing a pair of chords $\rho_1$ and $\rho_1$ in some $p_i$ whose endpoints $\rho_1$ and $\rho_1$ are nested, say $\rho_{11} \leq \rho_{12} \leq \rho_{13}$, to break apart and recombine into a pair of chords $\rho_1 = (\rho_{11}, \rho_{12}, \rho_{13})$ and $\rho_0 = (\rho_{12}, \rho_{13})$. This gives an odd shuffle curve at $e_0$ and an element of $\mathbb{M}^B(x, y; \vec{p})$, where $\vec{p}$ is obtained from $\vec{p}$ by replacing $\rho_{11}$ and $\rho_{12}$ in $\rho_0$ with $\rho_{01}$ and $\rho_{02}$.

See Fig. 3 for examples of the first three kinds of degenerations.

Remark 6: This analytic setup builds on the “cylindrical reformulation” of Heegaard Floer theory (52). It relates to the original formulation of Heegaard Floer theory, in terms of holomorphic disks in $\text{Sym}^*\Sigma$, by thinking of a map $D \rightarrow \text{Sym}^*\Sigma$ as a multivalued map $D \rightarrow \Sigma$ and then taking the graph. See, for instance, ref. 52, Sect. 13. Some of the results were previously proved in ref. 53.

Type D Modules. Fix a bordered Heegaard diagram $H$ and a suitable almost complex structure $I$. Let $\mathcal{L} = dH$ be the orientation reverse of the pointed matched circle given by $dH$. Given a generator $x \in \mathbb{H}(H)$, let $I_x(x)$ denote the indecomposable idempotent of $\mathcal{L}(x, 0) \subseteq \mathcal{L}(x)$ corresponding to the set of $\alpha$-arcs intersecting $x$ [opposite of $I_p(x)$], again making $X(H)$ into a module over $\mathcal{L}(x) \subseteq \mathcal{L}(x)$. Define a differential $\partial_x$ of $\mathcal{L}(x)$ on $X(H)$ by setting

$$m_{n+1}(x, \rho(\rho_1), \ldots, \rho(\rho_n)) = \sum_{y \in \mathbb{H}(H \cup x)} \sum_{\rho(\rho_i) \in \rho(\rho_{n+1})} \# \mathbb{M}^B(x, y; (\rho_1, \ldots, \rho_n)) \cdot y,$$

and extending multilinearly. As for $\mathbb{C}F^D(Y)$, to ensure finiteness of these sums, we need to assume that $H$ is provincially admissible.

Theorem 4. The operations $m_{n+1}$ satisfy the $\partial_x$ module relation.

Proof sketch: Because $\partial_x(Z)$ is a differential algebra, the $\partial_x$ relation for $\mathbb{C}F^D(Y)$ takes the form

$$0 = \sum_{i \leq j < n+2} m_{i+1}(x, a_1, \ldots, a_{i-1}, a_j) + \sum_{\rho \in \mathbb{H}(H \cup x)} \sum_{\rho(\rho_i) \in \rho(\rho_{n+1})} m_{n+1}(x, a_1, \ldots, \rho(\rho_j), \ldots, a_n),$$

$$+ \sum_{\rho \in \mathbb{H}(H \cup x)} \sum_{\rho(\rho_i) \in \rho(\rho_{n+1})} \cdot y.$$

The first term in Eq. 4 corresponds to degenerations of type 1. The second term corresponds to degenerations of types 2 and 4, depending on whether one of the strands in the crossing being resolved is horizontal (2) or not (4). The third term corresponds to degenerations of type 3. This proves the result.

Theorem 5. (30) Up to homotopy equivalence, the $\partial_x$ module $\mathbb{C}F^D(H)$ is independent of the (provincially admissible) bordered Heegaard diagram $H$ representing the bordered 3-manifold $Y$. 

See Fig. 3 for examples of the first three kinds of degenerations.

Fig. 3. Degenerations of holomorphic curves. Degenerations of types 1, 2, and 3 are shown, in that order. The dots indicate branch points, which can be thought of as the ends of cuts. (This figure is drawn from ref. 30.)
Again, the proof is similar to the closed case (1, Theorem 6.1).

**Remark 8:** Like $\widetilde{CFD}(Y)$, the module $\widetilde{CFD}(A)$ breaks up as a sum over spin structures on $Y$.

**Remark 9:** It is always possible to choose a Heegaard diagram $\mathcal{H}$ for $Y$ so that the higher products $m_n$, $n > 2$, vanish on $\widetilde{CFD}(\mathcal{H})$, so that $\widetilde{CFD}(\mathcal{H})$ is an honest differential module. One way to do so is using an analogue of nice diagrams (31).

**Bimodules.** Next, suppose $Y$ is a strongly bordered 3-manifold with two boundary components. By this we mean that we have a 3-manifold $Y$ with boundary decomposed as $\partial Y = \partial Y_1 \sqcup \partial Y_2$, homeomorphisms $\phi_2: F(\mathcal{X}_L) \to \partial Y_2$ and $\phi_2: F(\mathcal{X}_R) \to \partial Y_2$, and a framed arc $\gamma$ connecting the basepoints in $F(\mathcal{X}_L)$ and $F(\mathcal{X}_R)$ and pointing into the preferred disks of $F(\mathcal{X}_L)$ and $F(\mathcal{X}_R)$. Associated to $Y$ are bimodules $\widetilde{CFD}(Y)$, $\widetilde{CFD}(Y,\partial)$, $\widetilde{CFD}(Y,\partial,\partial)$ defined by treating, respectively, both $\partial Y_1$ and $\partial Y_2$ in type $D$ fashion; $\partial Y_1$ in type $D$ fashion, and $\partial Y_2$ in type $A$ fashion; and both $\partial Y_1$ and $\partial Y_2$ in type $A$ fashion.

An important special case of 3-manifolds with two boundary components is mapping cylinders. Given an isotopy class of maps $\phi: F(\mathcal{X}_L) \to F(\mathcal{X}_R)$ taking the distinguished disk of $F(\mathcal{X}_L)$ to the distinguished disk of $F(\mathcal{X}_R)$ and the basepoint of $F(\mathcal{X}_L)$ to the basepoint of $F(\mathcal{X}_R)$—called a strongly based mapping class —the mapping cylinder $M_\phi$ of $\phi$ is a strongly bordered 3-manifold with two boundary components. Let $\widetilde{CFD}(\phi) = \widetilde{CFD}(M_\phi)$, $\widetilde{CFD}(\phi) = \widetilde{CFD}(M_\phi)$, $\widetilde{CFD}(\phi) = \widetilde{CFD}(M_\phi)$. The set of strongly based mapping classes forms a groupoid, with objects the pointed matched circles representing genus surfaces and $\text{Hom}(\mathcal{X}_L, \mathcal{X}_R)$ the strongly based mapping classes $F(\mathcal{X}_L) \to F(\mathcal{X}_R)$. In particular, the automorphisms of a particular pointed matched circle $\mathcal{X}_L$ form the (strongly based) mapping class group.

**Remark 10:** The functors $\widetilde{CFD}(\phi) \otimes -$ give an action of the strongly based mapping class group on the (derived) category of left $\widetilde{CFD}(\mathcal{H})$ modules; this action categorifies the standard action on $\lambda^*H_1(F(\mathcal{X}_L); F_2)$. This action is faithful (55).

**Grading**

Let $\mathcal{X} = (Z, {a, M, z})$ denote the genus 1 pointed matched circle from Example 1. Consider the following elements of $\alpha(\mathcal{X}_L)$:

$x = a_1(1, 2), \{2, 3\}, (1 \mapsto 3, 2 \mapsto 2)$

$y = a_1(1, 2), \{4\}, (1 \mapsto 1, 2 \mapsto 2)$

A short computation shows that $y \cdot x = d((dx) \cdot y)$. It follows that there is no Z grading on $\alpha(\mathcal{X}_L)$ with homogeneous basic generators. A similar argument applies to $\alpha(\mathcal{X}_R, i)$ for any $\mathcal{X}_R$, as long as $\alpha(\mathcal{X}_R, i)$ involves at least two moving strands.

There is, however, a grading in a more complicated sense. Let $G$ be a group and $\lambda \in G$ a distinguished central element. A grading of a differential algebra $\mathcal{A}$ by $(G, \lambda)$ is a decomposition $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ so that $d(\mathcal{A}_g) \subseteq \mathcal{A}_{g-1}$ and $d(\mathcal{A}_{g}) \mathcal{A}_{\lambda g}$. Taking $G = \mathbb{Z}$ and $\lambda = 1$ recovers the usual notion of a Z grading of homological type. The corresponding notion for modules is a grading by a $G$ set. A grading of a left differential module $M$ by a left $G$ set $S$ is a decomposition $M = \bigoplus_{s \in S} M_s$ so that $d(M_s) \subseteq M_{s-1}$ and $d(M_s) \subseteq M_{s\lambda}$. Similarly, right modules are graded by right $G$ sets. If $M$ is graded by a $G$ set $S$ and $N$ is graded by $T$, then $M \otimes N$ is graded by $S \otimes T$, which retains an action of $\lambda$. These more general kinds of gradings have been considered by, e.g., Nastasescu and Van Oystaeyen (56).

**Deforming the Diagonal and the Pairing Theorems.**

The tensor product pairing theorems are the main motivation for the definitions of the modules and bimodules. We will sketch the proof of the archetype, Eq. 1. Fix bordered Heegaard diagrams $\mathcal{H}_1$ and $\mathcal{H}_2$ for $Y_1$ and $Y_2$, respectively, with $\partial \mathcal{H}_1 = -\partial \mathcal{H}_2$. It is easy to see that $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ is a Heegaard diagram for $Y_1 \cup_Y Y_2$.

There are two sides to the proof, one algebraic and one analytic. We start with the algebra. Typically, the $\lambda$ tensor product $M \otimes \mathcal{N}$ of $\lambda$ modules $M$ and $\mathcal{N}$ is defined using a chain complex whose underlying vector space is $M \otimes \mathcal{N} T^* \lambda \otimes \mathcal{N}$ (where $T^* \lambda$ is the tensor algebra of $\lambda$, and $k$ is the ground ring of $\lambda$—for us, the ring of idempotents). This complex is typically infinite-dimensional, and so is unlikely to align easily with $\lambda$.

However, the differential module $\mathcal{C}(\mathcal{H}_2)$ has a special form. It is given by $\mathcal{A}(\mathcal{X}_L) \otimes_{\mathcal{X}_R} X(\mathcal{X}_2)$, so the differential is encoded by a map $\partial: X(\mathcal{X}_2) \to \mathcal{A}(\mathcal{X}_L) \otimes_{\mathcal{X}_R} X(\mathcal{X}_2)$. This allows us to define a smaller model for the $\lambda$ tensor product. Let $\partial: X(\mathcal{X}_2) \to \mathcal{A}(\mathcal{X}_L) \otimes_{\mathcal{X}_R} X(\mathcal{X}_2)$ be the result of iterating $\partial$ $n$ times. For notational convenience, let $M = \mathcal{C}(\mathcal{H}_1)$ and $X = X(\mathcal{X}_2)$. Define $x(\mathcal{X}_1) \otimes \mathcal{C}(\mathcal{H}_2)$ to be the $\mathcal{X}_R$-vector space $M \otimes x(\mathcal{X}_2) X$, with differential (graphically depicted in Fig. 4)

$$\partial = \sum_{i=0}^{\infty} \{m_{i+1} \otimes x^i\} \cdot (1_{\mathcal{X}_R} \otimes \partial)^i.$$ [5]

The sum in Eq. 5 is a priori finite. To ensure that it is finite, we need to assume an additional boundedness condition on either $\mathcal{C}(\mathcal{H}_1)$ or $\mathcal{C}(\mathcal{H}_2)$. These boundedness conditions correspond to an admissibility hypothesis for $\mathcal{H}$, which in turn guarantees that $\mathcal{H}, \mathcal{H}_2$ is (weakly) admissible.

**Lemma 1.** There is a canonical homotopy equivalence

$$\mathcal{C}(\mathcal{H}_1) \otimes_{\mathcal{X}_R} \mathcal{C}(\mathcal{H}_2) \simeq \mathcal{C}(\mathcal{H}_1) \otimes_{\mathcal{X}_R} \mathcal{C}(\mathcal{H}_2).$$

The proof is straightforward. We turn to the analytic side of the argument next. Because of how the idempotents act on $\mathcal{C}(\mathcal{H}_1)$ and $\mathcal{C}(\mathcal{H}_2)$, there is an obvious identification between generators $x_r \otimes x_r$ of $\mathcal{C}(\mathcal{H}_1) \otimes \mathcal{C}(\mathcal{H}_2)$ and generators $x$ of $\mathcal{C}(\mathcal{R})$.

Let $Z = \partial \mathcal{H}_1 = \partial \mathcal{H}_2 \subset \mathcal{H}$. The differential on $\mathcal{C}(\mathcal{H})$ counts rigid $J$-holomorphic curves in $Z \times [0,1] \times \mathbb{R}$. For a sequence of almost complex structures $J$, with longer and longer necks at $Z$, such curves degenerate to pairs of curves $(u_L, u_R)$ for $\mathcal{H}_1$ and $\mathcal{H}_2$, with matching asymptotics at $\infty$. More precisely, in the limit as we stretch the neck, the moduli space degenerates to a fibered product.
Here, ev \equiv \infty. In the limit, some of the chords on the left collide, whereas some of the chords on the right become infinitely far apart. The result exactly recaptures the definitions of \( \mathcal{CF} \) and \( \mathcal{CF} \) for the differential) and the condition that 
\[ c \bigwedge \phi(\varphi_{X_{i}}) \text{ is quasi-isomorphic to } \mathcal{CF}(\mathcal{F}) \text{ and } \mathcal{CF}(\mathcal{H}) \text{ and the algebra of Eq. 5 (30).} \]

**Remark 11:** In ref. 30, we also give another proof of the pairing theorem (Eq. 1), using the technique of nice diagrams (31).

**Computing with Bordered Floer Homology**

**Computing \( \mathcal{CF} \).** Let \( Y \) be a closed 3-manifold. As discussed earlier, \( Y \) admits a Heegaard splitting into two handlebodies, glued by some homeomorphism \( \phi \) between their boundaries. Via the pairing theorems (Eqs. 2 and 3), this reduces computing \( \mathcal{HF}(L) \) to computing \( \mathcal{CFD}(H) \) for some particular bordered handlebody \( H \) of each genus \( g \) and \( \mathcal{CFDD}(\phi) \) for arbitrary \( \phi \) in the strongly based mapping class group. For an appropriate \( H \), \( \mathcal{CFD}(H) \) is easy to compute. Moreover we do not need to compute \( \mathcal{CFDD}(\phi) \) for every mapping class, just for generators for the mapping class groupoid. This groupoid has a natural set of generators: arcsides (compare refs. 57 and 58). It turns out that the type \( \mathcal{DD} \) invariants of arcsides are determined by a small amount of geometric input (essentially, the set of generators and a nondegeneracy condition for the differential) and the condition that \( \hat{\theta} = 0 \) (59).

These techniques also allow one to compute all types of the bordered invariants for any bordered 3-manifold.

**Cobordism Maps.** Next, we discuss how to compute the map \( \hat{f}_{W} : \mathcal{CF}(Y_{1}) \to \mathcal{CF}(Y_{2}) \) associated to a 4-dimensional cobordism \( W \) from \( Y_{1} \) to \( Y_{2} \). The cobordism \( W \) can be decomposed into three cobordisms \( W_{1}W_{2}W_{3} \), where \( W_{i} : Y_{i-1} \to Y_{i+1} \) consists of \( i \)-handle attachments and \( \hat{f}_{W} \) is a corresponding composition \( \hat{f}_{W_{1}} \circ \hat{f}_{W_{2}} \circ \hat{f}_{W_{3}} \).

The maps \( \hat{f}_{W_{1}} \) and \( \hat{f}_{W_{2}} \) are simple to describe: \( Y_{2} \cong Y_{1} \# (S^{2} \times S^{1}) \), whereas \( Y_{3} \cong Y_{2} \# (S^{2} \times S^{1}) \); \( \mathcal{CF}(S^{2} \times S^{1}) \) is (homotopy equivalent to) \( \mathbb{F} \oplus \mathbb{F} = H_{c}(S^{1}; \mathbb{F}) \) and the invariant \( \mathcal{HF}(L) \) satisfies a Künneth theorem for connect sums, so

\[
\mathcal{CF}(Y_{1} \# (S^{2} \times S^{1})) \cong \mathcal{CF}(Y_{1}) \otimes_{\mathbb{F}} H_{c}(T^{2}; \mathbb{F})
\]

(with respect to appropriate Heegaard diagrams), where \( T^{2} \) is the \( k \)-dimensional torus. Let \( \tilde{\eta} \) be the top-dimensional generator of \( H_{c}(T^{2}; \mathbb{F}) \) and \( \eta \) the bottom-dimensional generator of \( H_{c}(T^{2}; \mathbb{F}) \). Then \( \hat{f}_{W} \) is \( x \mapsto x \otimes \tilde{\eta} \), whereas \( \hat{f}_{W} \) takes \( x \otimes \eta \mapsto x \) and \( x \otimes \tilde{\eta} \mapsto 0 \) if \( \text{gr}(x) > \text{gr}(\eta) \).

By contrast, \( \hat{f}_{W} \) is defined by counting holomorphic triangles in a suitable Heegaard triple diagram. Two additional properties of bordered Floer theory allow us to compute \( \hat{f}_{W} \):

- The invariant \( \mathcal{CFD}(H) \) of a handlebody \( H \) is rigid, in the sense that it has no nontrivial graded automorphisms. This allows one to compute the homotopy equivalences between the results of making different choices in the computation of \( \mathcal{CF}(Y) \).
- There is a pairing theorem for holomorphic triangles.

Given these, one can compute \( \hat{f}_{W} \) as follows: Using results from the previous section, we can compute \( \mathcal{CF}(Y_{2}) \) [respectively, \( \mathcal{CF}(Y_{3}) \)] using a Heegaard decomposition making the decomposition \( Y_{2} \cong Y_{1} \# (S^{2} \times S^{1}) \) [respectively, \( Y_{1} \cong Y_{2} \# (S^{2} \times S^{1}) \)] manifest. With respect to this decomposition, the map \( \hat{f}_{W_{1}} \) (respectively, \( \hat{f}_{W_{2}} \)) is easy to read off.

To compute \( \hat{f}_{W} \), one works with Heegaard decompositions of \( Y_{2} \) and \( Y_{3} \) with respect to which the cobordism \( W \) takes a particularly simple form, replacing one of the handlebodies \( H \) of a Heegaard decomposition of \( Y_{2} \) with a differently framed handlebody \( H \). It is easy to compute the triangle map \( \mathcal{CFD}(H) \to \mathcal{CFD}(H') \). By the pairing theorem for triangles, extending this map by the identity map on the rest of the decomposition gives the map \( \hat{f}_{W} \). Finally, the rigidity result allows one to write down the isomorphisms between \( \mathcal{CF}(Y_{2}) \) [and \( \mathcal{CF}(Y_{3}) \)] computed in the two different ways. The map \( \hat{f}_{W} \) is then the composition of the three maps \( \hat{f}_{W_{1}} \) and the equivalences connecting the two different models of \( \mathcal{CF}(Y_{2}) \) and of \( \mathcal{CF}(Y_{3}) \).

The details will appear in forthcoming work.

**Polygon Maps and the Ozsváth–Szabó Spectral Sequence**. Khovanov introduced a categorification of the Jones polynomial (60). This categorification associates to an oriented link \( L \) a bigraded abelian group \( KH_{s,\ell}(L) \), the Khovanov homology of \( L \), whose graded Euler characteristic is \( q^{s} + q^{-s} \) times the Jones polynomial \( J(L) \). There is also a reduced version \( KH_{s}(L) \), whose graded Euler characteristic is simply \( J(L) \). The skein relation for \( J(L) \) is replaced by a skein exact sequence. Given a link \( L \) and a crossing \( c \) of \( L \), let \( \ell_{0} \) and \( L_{1} \) be the two resolutions of \( c \). Then there is a long exact sequence relating the (reduced) Khovanov homology groups of \( L, L_{1}, \) and \( L_{0} \).

Szabó and the second author observed that the Heegaard Floer group \( \mathcal{HF}(L) \) of the double cover of \( S^{3} \) branched over \( L \) satisfies a similar skein exact triangle to (reduced) Khovanov homology and takes the same value on an \( n \)-component unlink (with some collapse of gradings). Using these observations, they produced a spectral sequence from Khovanov homology (with \( F_{2} \) coefficients) to \( \mathcal{HF}(L) \) (26). [Because of a difference in conventions, one must take the Khovanov homology of the mirror \( r(L) \) of \( L \).] Baldwin recently showed (61) that the entire spectral sequence \( KH_{s}(r(L)) \to \mathcal{HF}(L) \) is a knot invariant.

Bordered Floer homology can be used to compute this spectral sequence (62). Write \( L \) as the plat closure of some braid \( B \), and decompose \( B \) as a product of braid generators \( s_{1} \cdots s_{n} \). The branched double cover of a braid generator \( s_{j} \) is the mapping cylinder of a Dehn twist, and the branched double covers of the plats closing \( B \) is a handlebody \( H \). So \( \mathcal{CF}(D(K)) \) is quasi-isomorphic to

\[
\mathcal{CF}(D(K))
\]
The bordered invariant of a Dehn twist $\tau$ along $\gamma \subset F(X)$ can be written as a mapping cone of a map between the identity cobordism $I = [0,1] \times F(X)$ and the manifold $Y_{\partial(I)}$ obtained by $0$-surgery on $[0,1] \times F(X)$ along $\gamma$:

$$CFA(\tau) \simeq \operatorname{Cone}(CFA(Y_{\partial(I)}) \to CFA(I)).$$

The next step is to compute the groups $CFA(\partial(I))$ for the curves $\gamma$ corresponding to the braid generators and the maps from Formulas 8 and 9, which again requires a small amount of geometry (62).

Finally, we identify this spectral sequence with the earlier spectral sequence from ref. 26. The key ingredient is another pairing theorem identifying the algebra of tensor products of mapping cones with counts of holomorphic polygons.

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