The nonlinear stability of the trivial solution to the Maxwell-Born-Infeld system

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Jared Speck

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The nonlinear stability of the trivial solution to the Maxwell-Born-Infeld system

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In this article, we use an electromagnetic gauge-free framework to establish the existence of small-data global solutions to the Maxwell-Born-Infeld (MBI) system on the Minkowski spacetime background in 1+3 dimensions. Because the nonlinearities in the system have a special null structure, we are also able to show that these solutions decay at least as fast as solutions to the linear Maxwell-Maxwell system. In addition, we show that on any Lorentzian manifold, the MBI system is hyperbolic in the interior of the field-strength regime in which its Lagrangian is real-valued.

I. INTRODUCTION

The Maxwell-Born-Infeld (MBI) system is a nonlinear model of classical electromagnetism that was introduced in the 1930s by Born and Infeld, with a precursor by Born. In this article, we study the source-free (i.e., the right-hand sides of (1a) and (1b) are 0) MBI system in the fixed spacetime \((M, g)\). By “spacetime,” we mean a four-dimensional time-oriented manifold \(M\) together with a Lorentzian metric \(g\) of signature \((- , +, +, +)\). We will assume throughout the article that \((M, g)\) is equal to \(\mathbb{R}^{1+3}\) equipped with the usual Minkowski metric, which has components \(g_{\mu \nu} = \text{diag}(-1, 1, 1, 1)\) in an inertial coordinate system \((x^0, (x^1, x^2, x^3)) \equiv (t, \mathbf{x})\). Nonetheless, much of our discussion regarding the structure of the MBI system remains valid in an arbitrary spacetime. As is explained in detail in Sec. III, the MBI equations can be expressed as

\[ d\mathcal{F} = 0, \quad (1a) \]
\[ d\mathcal{M} = 0, \quad (1b) \]

where \(d\) denotes the exterior derivative operator, the Faraday tensor \(\mathcal{F}\) (which is a two-form) is the fundamental unknown, the Maxwell tensor \(\mathcal{M}\) (which is also a two-form) is defined by

\[ \mathcal{M} = \ell^{-1}_{(MBI)} (\ast \mathcal{F} + \mathcal{i}_{(2)} \mathcal{F}), \quad (2) \]

\(\ast\) denotes the Hodge dual, \(\mathcal{i}_{(1)} \equiv \frac{1}{2} (g^{-1})^{\kappa \xi} (g^{-1})^{\mu \nu} \mathcal{F}_{\xi \eta} \mathcal{F}_{\kappa \lambda}, \quad \mathcal{i}_{(2)} \equiv \frac{1}{4} (g^{-1})^{\kappa \xi} (g^{-1})^{\mu \nu} \mathcal{F}_{\xi \eta} \ast \mathcal{F}_{\kappa \lambda}\) are the electromagnetic invariants, \((g^{-1})^{\mu \nu}\) are the components of the inverse of the spacetime metric \(g\), and \(\ell_{(MBI)} \equiv (1 + \mathcal{i}_{(1)} - \mathcal{i}_{(2)}^2)^{1/2}\). Born and Infeld’s contribution to the above system was their provision of the constitutive relation (2), while Eqs. (1a) and (1b) were postulated in the 1860s by Maxwell. We remark that Maxwell’s formulation of electromagnetism was not presented using the framework of the Faraday tensor, nor that of the familiar electric field \(E\) and magnetic induction \(B\); rather, he used the structure of quaternions to write down a system of 20 equations in 20 unknowns. The familiar “vector” formulation in terms of \(E\) and \(B\) was developed by Heaviside. We recall that in contrast to (2), Maxwell adopted the linear constitutive law \(\mathcal{M} = \ast \mathcal{F}\). Hence, we refer to the nonlinear system (1a), (1b), and (2) as the “Maxwell-Born-Infeld” equations, and the linear system (1a) and (1b), \(\mathcal{M} = \ast \mathcal{F}\) as the “Maxwell-Maxwell” equations. We summarize our main results here; they are precisely stated and proved in Secs. XII and XIII.
Main results. The trivial solution to the MBI system on the 1 + 3 dimensional Minkowski spacetime background is globally stable. More specifically, if the initial data for the MBI system are sufficiently small as measured by the weighted Sobolev norm \( \| \cdot \|_{H^N} \) defined in (326), with \( N \geq 4 \), then these data launch a unique classical solution to the MBI system existing in all of Minkowski spacetime. Moreover, these small-data solutions decay at least as fast as solutions to the linear Maxwell-Maxwell equations. In addition, the MBI system is hyperbolic in the interior of the field-strength regime in which its Lagrangian is real-valued. By “hyperbolic,” we mean that there is a local energy estimate available that can be used to prove that initial data have a non-trivial development, and that furthermore, the system has a finite speed of propagation. In particular, the system is locally well-posed in the aforementioned weighted Sobolev space.

Remark 1.0.1. Under the assumption of planar symmetry, it is known that large-data solutions to the MBI system can blow-up in finite time; see Refs. 7, 47, and 55.

Remark 1.0.2. Certain large fields can cause \( \ell_{\text{(MBI)}} \) to become complex-valued. For such fields, MBI theory is not even well defined. However, as is discussed in Remark 5.8.3, the MBI equations are well defined and hyperbolic for all finite values of the state-space variables \( (B, D) \), which are introduced in Sec. V H. Thus, the MBI system is locally well-posed for sufficiently regular initial data belonging to the interior of the region of state-space in which the equations are well defined.

Remark 1.0.3. Although we only discuss global existence to the future, our results apply just as well to the past. Our notion of “future” is determined by assumption that the vectorfield \( \partial_t \) is future-directed.

Recently, several scientific communities have expressed renewed interest in the MBI system for a variety of reasons. As an interesting example, we cite the works, \(^{30, 31}\) in which Kiessling has proposed a model of classical electrodynamics with point charges that has the promise of self-consistency; it is hoped that Kiessling’s model is mathematically well defined without truncation, regularization, or renormalization. This would remedy the infamous “electromagnetic divergence problem,” which is the following distressing fact: the motion of point charges in linear Maxwell-Maxwell theory is mathematically ill-defined under the usual Lorentz force law. Kiessling’s theory couples a first-order guiding law for the point charges to the MBI field equations. The relativistic guiding field satisfies a Hamilton-Jacobi-type PDE. In contrast to the case of the linear Maxwell-Maxwell equations, the electromagnetic potentials (an electromagnetic potential is a one-form \( A \) such that \( F = dA \)) of the solutions to the MBI system with non-accelerating point charge sources in Minkowski spacetime can be chosen to be Lipschitz continuous, even at a point charge’s location. It is expected that this continuity property should remain true even for accelerating point charges, which would then allow for a well-defined coupling to the Hamilton-Jacobi theory. As a second interesting example, we note that the MBI system has mathematical connections to string theory, for its Lagrangian [see (79)] appears in connection with the motion gauge fields (arising in the study of attached, open strings) on a D-brane (see, e.g., Ref. 25).

As is true for Kiessling, our interest in the MBI system is further motivated by results contained in Refs. 5 and 48, which show that it is the unique (up to a parameter \( \beta > 0 \) known as Born’s “aether” constant) theory of classical electromagnetism that is derivable from an action principle and that satisfies the following 5 postulates (see also the discussion in Refs. 2 and 30):

1. The field equations transform covariantly under the Poincaré group.
2. The field equations are covariant under a Weyl (gauge) group.
3. The electromagnetic energy surrounding a stationary point charge is finite.
4. The field equations reduce to the linear Maxwell-Maxwell equations in the weak field limit.
5. The solutions to the field equations are not birefringent (we will soon elaborate upon this notion).

We remark that the Maxwell-Maxwell system satisfies all of the above postulates except for (3), and that the MBI system was shown to satisfy (3) by Born in Ref. 6.
We would now like to further discuss postulate (5). Physically, it is equivalent to the statement that the “speed of light propagation” is independent of the polarization of the wave fields. Mathematically, it can be recast as a statement about the characteristic subsets of the field equations. To flesh out this notion, we need to discuss some technical details. First, we remark that Eq. (87b), which reads $H^{\mu \nu \kappa \lambda} \nabla_\mu F_{\kappa \lambda} = 0$, is equivalent to (1b) and (2) modulo equation (1a), where the tensor field $H^{\mu \nu \kappa \lambda}$ is defined in (86a). Now for each covector $\xi \in T^*_p M$, the cotangent space of $M$ at $p$, we consider the quadratic form $\chi^{\mu \nu}(\xi) \equiv H^{\mu \nu \kappa \lambda} \xi_\kappa \xi_\lambda$. Because of the properties (77a)–(77c), which are also possessed by $H^{\mu \nu \kappa \lambda}$, it follows that $\xi$ is an element of $N(\chi(\xi)) \equiv \{ \xi \in T^*_p M \mid \chi^{\mu \nu}(\xi) \xi_\mu = 0 \}$. The null space of $\chi(\xi)$ viewed as a map from $T^*_p M$ to $T^*_p M$. The characteristic subset of $T^*_p M$, which we denote by $C^*_p$, is defined to be the set of all $\xi$ such that $N(\chi(\xi))$ is strictly larger than $\text{span}(\xi)$; i.e.,

$$C^*_p \equiv \{ \xi \in T^*_p M \mid N(\chi(\xi)) \text{ span}(\xi) \neq \emptyset \}. \quad (3)$$

We remark that many authors choose to exclude $\xi = 0$ from the definition of $C^*_p$. As discussed in detail in Chap. 6 of Ref. 13, the set $C^*_p$ governs the local speeds of propagation of solutions to the MBI system. It is easy to see that $C^*_p$ is a conical set in the sense that if $\xi \in C^*_p$, then any multiple of $\xi$ is also an element of $C^*_p$. In general, this conical subset may have several different “sheets.” However, in the case of the MBI system, there is a degeneracy resulting in the presence of only a single sheet (i.e., there is only one “null cone” associated with the MBI system); this is the mathematical characterization of “no birefringence.” As we alluded to above, the Maxwell-Maxwell system also possesses this property. Moreover, in the case of the Maxwell-Maxwell system, $C^*_p$ exactly coincides with the gravitational null cone in $T^*_p M$, which is $\{ \xi \in T^*_p M \mid (g^{-1})^{\mu \nu} \xi_\mu \xi_\nu = 0 \}$. However, in a general nonlinear theory, and specifically in the case of the MBI system, $C^*_p$ does not coincide with the gravitational null cone. In the case of the MBI system, it can be shown (see Lemma 7.4.3) that $C^*_p \equiv \{ \xi \in T^*_p M \mid (b^{-1})^{\mu \nu} \xi_\mu \xi_\nu = 0 \}$, where $(b^{-1})^{\mu \nu} = (g^{-1})^{\mu \nu} - (1 + \bar{\nu}_1(\bar{\mathcal{F}}))^{-1} \mathcal{F}^{\mu \kappa} \mathcal{F}_\kappa$ is the reciprocal Maxwell-Born-Infeld metric defined in (208).

Furthermore, as an aside, we will also investigate a related issue that would be relevant if one wanted to couple the MBI system to the equations of general relativity. Namely, we prove that MBI system’s energy-momentum tensor satisfies the dominant energy condition; see Lemma 7.1.1. Physically, this means that the speeds of propagation associated with the MBI system are no larger than the speeds associated with the gravitational null cone; i.e., the “speed of MBI light is less than or equal to the speed of gravity.” Mathematically, this means that $C^*_p$ lies outside of the gravitational null cone. We remark that there is a dual picture that perhaps more intuitively corresponds to the notion of MBI light traveling “more slowly than gravity.” The dual picture refers to $C_p$, the characteristic subset of $T_p M$, where $T_p M$ is the tangent space of $M$ at $p$. The set $C_p$ is dual to $C^*_p$ in a sense defined in Ref. 13. In the dual picture, $C_p \subset T_p M$ lies inside of the gravitational null cone $\{ X \in T_p M \mid g_{\kappa \lambda} X^\kappa X^\lambda = 0 \}$. In the case of the MBI system, $C_p$ has a simple structure: $C_p = \{ X \in T_p M \mid b_{\mu \nu} X^\mu X^\nu = 0 \}$, where $b_{\mu \nu} = b(\text{MBI})(1 + \bar{\nu}_1(\mathcal{F}))(g_{\mu \nu} + \mathcal{F}_\mu \mathcal{F}_\nu)$ is the Born-Infeld metric given in (209).

A. Comparison with related work

The core of our proof is based on a blend of ideas presented in Refs. 17 and 13; in Ref. 17, Christodoulou and Klainerman used methods similar to the ones used in this article to analyze the decay properties of solutions to the linear Maxwell-Maxwell equations in Minkowski spacetime, while in Ref. 13, Christodoulou provided a framework for deriving positive “almost conserved” quantities for nonlinear, hyperbolic PDEs that are derivable from a Lagrangian, of which the MBI system is an example. In short, using the methods of Ref. 13, we are able to construct certain almost-conserved (in the small-solution regime) energies that have coercive properties nearly identical to those of the conserved quantities constructed in Ref. 17.

The aforementioned works and the present one are based on applications of a collection of geometric-analytic techniques that are applicable to a large class of hyperbolic PDEs derivable from a Lagrangian. These techniques are often collectively referred to as the vectorfield method. The term “vectorfield” refers to the fact that in typical applications, coercive quantities are constructed.
with the help of special vectorfields connected to the symmetries (or approximate symmetries) of the system. More precisely, the vectorfield method comes in two varieties, and we use both of them. The first variety is called the method of vectorfield multipliers. Roughly speaking, this method corresponds to choosing a good quantity that will be used to multiply the equations, and then integrating by parts over a well-chosen spacetime region to deduce a useful energy estimate. Over the last few decades, this fundamental method has been applied in many different contexts to a large variety of wave-like equations. This method can be viewed as an extension of Noether's theorem. In particular, we use the multiplier method framework developed in Ref. 13, a work that can be viewed as a geometric extension of Noether’s theorem to handle regularly hyperbolic (see Ref. 13 for Christodoulou’s definition of regular hyperbolicity) systems/solutions that are derivable from a Lagrangian but perhaps lack exact symmetry (and hence the energy estimates contain error terms). Specifically, to derive our main energy estimates, we make use of a Morawetz-type vectorfield $K$ [see definition (134)], which was first used in Ref. 44. Our initial construction involving the use of $K$ as a multiplier is provided in Lemma 7.4.1.

The second variety is known as the vectorfield commutator method. Originally introduced by Klainerman 32, 33 (based on some earlier ideas of John 29) in his analysis of small-data global solutions to nonlinear wave equations, this technique involves commuting the equations with well-chosen vectorfield differential operators. By “well-chosen,” we mean that a priori bounds of the commuted quantities lead to useful estimates, and also that the error terms generated by commuting have a favorable structure. We remark that the favorable structure of the error terms corresponding to the commuted quantities of interest in this article is uncovered in Proposition 7.2.1 and Lemma 9.0.5. In total, the vectorfield method has blossomed into a mature industry. As examples, we provide a non-exhaustive list of topics for which the vectorfield method has proven fruitful.

- Global nonlinear stability results for the Einstein equations 8, 18, 20, 34, 39, 40, 49, 54, 58
- Small-data global existence for nonlinear elastic waves 52
- The formation of shocks in solutions to the relativistic Euler equations 14
- Decay results for linear equations on curved backgrounds 1, 4, 21–23, 26
- The formation of trapped surfaces in vacuum solutions to the Einstein equations 16, 35
- Local existence and non-relativistic limits for the relativistic Euler equations without the use of symmetrizing variables 56, 57, 60

B. Difficulties in working with a four-potential

In various contexts during the study of linear Maxwell-Maxwell theory, authors commonly analyze the components of a four-potential $A$ and its derivatives, rather than the Faraday tensor itself (Chap. 6 of Ref. 28 is a classic reference, and Refs. 37 and 38 are examples in the context of the Einstein-Maxwell system). Recall that a four-potential is a one-form $A$ such that $F = dA$; the existence of such a one-form is guaranteed by (1a) and Poincaré’s Lemma. $A$ is not unique, for any “gauge” transformation of the form $A \rightarrow A + \nabla \gamma$, where $\gamma$ is a scalar-valued function, preserves the relation $F = dA$. A well-known method of capitalizing on this gauge freedom is to work in the Lorentz gauge, which is the added condition

$$\nabla \cdot A^c = 0. \quad (4)$$

The advantages of the Lorentz gauge are discussed below. Of course, the viability of the gauge condition (4), which can be arranged to hold initially, depends on the fact that it is preserved by the flow of an appropriate version of the Maxwell-Maxwell equations (e.g., the system (8) below).

We would now like to discuss some subtle issues concerning the difficulties that arise if one attempts to work with the Lorentz gauge when the electromagnetic equations are quasilinear. As in the remainder of the article, we assume in this section that $(M, g)$ is Minkowski spacetime, and furthermore, that we are working in an inertial coordinate system. However, these assumptions have no substantial bearing on the issues at hand, for the same issues arise in any other spacetime $(M, g)$ equipped with any coordinate system. We begin with a brief summary of the framework used for discussing an arbitrary nonlinear covariant theory of electromagnetism that is derivable from a
Lagrangian. We remark that we will slightly depart from the usual convention by referring to the Hodge dual of \( \mathcal{L} \), which we denote by \( ^\ast \mathcal{L} \), as the Lagrangian; \( \mathcal{L} \) is a four-form, while \( ^\ast \mathcal{L} \) is scalar-valued. Now if we choose to describe such a theory through the use of four-potentials \( A \), then the Lagrangian \( ^\ast \mathcal{L} = \mathcal{L}[\nabla A] \) can be written as a function of \( \nabla A \). A very detailed elaboration of this discussion can be found in Ref. 13; here, we only introduce the facts that are relevant to the issues at hand. The Euler-Lagrange equations for such a theory can be written as

\[
h^\xi_\eta \nabla_\xi \nabla_\eta A^\mu = 0, \quad (\nu = 0, 1, 2, 3),
\]

where

\[
h^\xi_\eta = \frac{\partial^2 \mathcal{L}}{\partial (\nabla_\xi A^\mu) \partial (\nabla_\eta A^\nu)}.
\]

Note that \( h \) has a symmetry property that will be important for the construction of energies; it is invariant under the following simultaneous exchange of indices:

\[
h^\eta_\xi = h^\xi_\eta.
\]

As we will see, the difficulties that arise in working with the Lorentz gauge in the quasilinear case are present because one cannot capitalize on the following remarkable simplification that occurs in linear Maxwell-Maxwell theory: Eqs. (5) can be written as a system of completely decoupled wave equations for the components of \( A \). That is, in linear Maxwell-Maxwell theory under the Lorenz gauge, the components of \( A \) are solutions to the following system:

\[
g_{\mu\nu} (g^{-1})^{\xi\eta} \nabla_\xi \nabla_\eta A^\mu = 0, \quad (\nu = 0, 1, 2, 3). \tag{8}
\]

Consequently, we have that

\[
h^\xi_\eta = g_{\mu\nu} (g^{-1})^{\xi\eta}.
\]

Because of the full decoupling of the components \( A^\nu \) at the level of the top-order (second) derivatives, we can multiply both sides of (8) by the “seemingly non-geometric” quantity \( \nabla_\eta A^\nu \) (with the index \( \nu \) downstairs!), integrate over \( \mathbb{R}^3 \), and integrate by parts to show that the following energy \( E \) is conserved for solutions to (8):

\[
E^2(t) \equiv \frac{1}{2} \sum_{\xi,\eta = 0}^3 \int_{\mathbb{R}^3} \left( \nabla_\xi A_\eta(t, x) \right)^2 \, d^3 x.
\]

We remark that these steps can alternatively be carried out using an energy current framework, similar to the energy current estimate (15) described below. In the language of Ref. 13, the special product structure of \( h^\xi_\eta \) in (9) is called separability; the existence of the conserved coercive quantity (10) is because of this additional structure, which is not typically present in the equations of a quasilinear theory.

Let us contrast this to the case of the MBI system (or any other quasilinear perturbation of linear Maxwell-Maxwell theory derivable from a covariant Lagrangian). In the case of the MBI system in Lorenz gauge, it can be shown using (4) that the MBI equations can be written in such a way that

\[
h^\xi_\eta = g_{\mu\nu} (g^{-1})^{\xi\eta} + \tilde{h}^\xi_\eta,
\]

where \( \tilde{h}^\xi_\eta \), which has the symmetry property (7), is a term that is of quadratic order in \( \nabla A \) in the small-solution regime. The corresponding system of PDEs is, therefore,

\[
g_{\mu\nu} (g^{-1})^{\xi\eta} \nabla_\xi \nabla_\eta A^\mu + \tilde{h}^\xi_\eta \nabla_\xi \nabla_\eta A^\mu = 0, \quad (\nu = 0, 1, 2, 3). \tag{12}
\]

Unfortunately, in general, it is not possible to simply multiply both sides of (12) by \( \nabla_\eta A^\nu \), integrate over \( \mathbb{R}^3 \), and integrate by parts; the tensorfield \( \tilde{h}^\xi_\eta \) in (11) is not separable in general, nor in the particular case of the MBI system. Note that this difficulty does not arise in the study of a single quasilinear wave equation; e.g., small quasilinear perturbations of the linear wave equation in Minkowski spacetime preserve hyperbolicity and the availability of a basic \( L^2 \) energy estimate.
One may attempt to resolve this difficulty by using the framework of energy currents developed in Ref. 13. A natural quantity that arises from an application of this framework is \( I(t) \), which is defined by

\[
I(t) \overset{\text{def}}{=} \int_{\mathbb{R}^3} J_{(MBI+Lorenz)}^\mu(t, \Sigma) d^3\Sigma,
\]

where the energy current \( J_{(MBI+Lorenz)}^\mu(t, \Sigma) \) is defined by

\[
J_{(MBI+Lorenz)}^\mu(t, \Sigma) \overset{\text{def}}{=} -h_{k\lambda}^\mu \left( \nabla_\nu A^\nu \right) (\nabla_\zeta A^\zeta) + \frac{1}{2} \delta_0^\mu \delta_0^\nu h_\lambda^\nu (\nabla_\zeta A^\zeta)(\nabla_\zeta A^\zeta), \quad (\mu = 0, 1, 2, 3).
\]

As explained in detail in Ref.13 and in Sec. VII, the current (14) can be constructed by contracting a certain tensor, namely, the canonical stress, against the vectorfield \( \partial_t \). The details of this construction do not concern us here. We only remark that the quantity \( I(t) \) is what one first tries to construct in an effort control solutions to the MBI system during a proof of local well posedness. On the one hand, there is an important advantage to working directly with \( \partial_t \) and as explained in Sec.VIII, we can construct suitable positive energies by working \( \partial_t \) itself. More specifically, it can be shown that

\[
\frac{d}{dt} I(t) \leq C(\|\nabla h\|_{L^\infty}) \|\nabla A\|_{L^2}^2.
\]

We remark that a quick way to see (15) is to use Eqs. (5) and the symmetry property (7) to show that \( |\nabla_\mu J_{(MBI+Lorenz)}^\mu| \leq C(\|\nabla h\|_{L^\infty}) |\nabla A|_2^2 \); (15) then follows from the divergence theorem. Alternatively, one may multiply both sides of (12) by the “geometric” quantity \( \nabla_\nu A^\nu \) (with the \( \nu \) index upstairs!) and integrate by parts with the help of the symmetry property (7), arriving at (15).

However, we quickly run into a difficulty: (11) and (14) imply that in the small-solution regime, \( J_{(MBI+Lorenz)}^\mu \) is indefinite in \( \nabla A \):

\[
J_{(MBI+Lorenz)}^0 = \frac{1}{2} \sum_{\zeta=0}^{3} g_{\zeta}\left( \nabla_\zeta A^\zeta \right) (\nabla_\zeta A^\zeta) + O(|\nabla A|_4^4).
\]

Therefore, \( I(t) \) is not a coercive quantity, and in particular, it is of no use in controlling the \( L^2 \) norms of solutions to (12).

These difficulties are not fatal in the sense that the fundamental unknown is the Faraday tensor \( F = dA \), and as explained in Sec. VIII, we can construct suitable positive energies by working directly with \( F \). More specifically, our energies control the combinations \( I_1 = \nabla_\zeta A^\zeta \), namely, that its time derivative can be bounded in terms of the \( L^2 \) norm of \( \nabla A \) itself. More specifically,

\[
\frac{d}{dt} I(t) \leq C(\|\nabla h\|_{L^\infty}) \|\nabla A\|_{L^2}^2.
\]

C. Comments on the analysis

In this section, we summarize the main ideas of our proof. We first remark that all of the discussion in this section assumes that we have fixed an inertial coordinate system \((t, \Sigma)\) on \( M \),
which is a global coordinate system in which the spacetime metric has the components $g_{\mu\nu} = \text{diag} \left( -1, 1, 1, 1 \right)$. Throughout this article, we work directly with the Faraday tensor $F$ and thus avoid the aforementioned difficulties associated with choosing a gauge for the four-potential. To analyze $F$, we use the framework of Ref. 17 and decompose it into its Minkowski null components. Before discussing the notion of null components, we first introduce the following foliations of Minkowski spacetime: the family of ingoing Minkowski null cones $C_q^-(\tau, y) = \{ (\tau, y) \mid |y| - \tau = q \}$; the family of outgoing Minkowski null cones $C_q^+(\tau, y) = \{ (\tau, y) \mid |y| + \tau = q \}$; and the spacelike hypersurfaces $\Sigma_t = \{ (\tau, y) \mid \tau = t \}$, which intersect the null cones in spheres $S_{r,t} = \{ (\tau, y) \mid \tau = t, |y| = r \}$. All of these families of surfaces will play an important role in this article.

At each non-zero spacetime point $p$, there exists a null frame $\{L, e_1, e_2\}$, where $L \equiv \partial_t - \partial_r$ is an ingoing null geodesic vectorfield tangent to the corresponding cone $C_q^-$, $L \equiv \partial_t + \partial_r$ is an outgoing null geodesic vectorfield tangent to the corresponding cone $C_q^+$ normalized by the condition $g(L, L) = -2$, and the orthonormal vectorfields $e_1, e_2$ are tangent to the corresponding sphere $S_{r,t}$ and $g$-normal to both $L, L$. In particular, the null frame forms a basis for $T_p\mathcal{M}$. Using this null frame, we can decompose $F$ into its Minkowski null components. The null components of $F$ are defined to be the following pair of one-forms $\alpha = \alpha[F], \sigma = \sigma[F]$ tangent to the $S_{r,t}$, and the following two scalar quantities $\rho = \rho[F], \sigma = \sigma[F]$

\[
\begin{align*}
\alpha_A &= F_{AL}, \\
\sigma_A &= F_{2L}, \\
\rho &= \frac{1}{2}F_{LL}, \\
\sigma &= F_{12},
\end{align*}
\]

where we have abbreviated $F_{AL} = e^a_A L^\lambda F_{\lambda A}, F_{12} = e^1_\lambda e^2_\lambda F_{\lambda \lambda},$ etc; see Sec. V for more details.

1. Linear analysis

The following decay properties, which can be expressed with the help of the null coordinates $q = r-t, s = r+t$, where $r \equiv |x|$ (note that the null coordinate $q$ is constantly equal to $q_0$ along the outgoing cone $C_{q_0}^+$ and that the null coordinate $s$ is constantly equal to $s_0$ along the ingoing cone $C_{s_0}^-$), were shown in Ref. 17 for solutions to the linear Maxwell-Maxwell system arising from data with suitable decay properties at infinity:

- The worst decaying component is $\alpha$, which decays like $(1 + s)^{-1}(1 + |q|)^{-3/2}$.
- The fastest decaying component is $\rho$, which decays like $(1 + s)^{-5/2}$.
- $\rho$ and $\sigma$ each decay at the intermediate rate $(1 + s)^{-2}(1 + |q|)^{-1/2}$.
- Any derivative tangential to the outgoing cones $C_{q_0}^+$ (i.e., $\nabla_L, \nabla_{e_\lambda}$) creates additional decay of order $(1 + s)^{-1}$, while the transversal derivative $\nabla_L$ creates additional decay of order $(1 + |q|)^{-1}$, which is weaker than $(1 + s)^{-1}$.

We remark that the finiteness of $\| (\hat{B}, \hat{D}) \|_{H^1}$ is sufficient for the above asymptotic estimates to hold, where $(\hat{B}, \hat{D})$ is the electromagnetic decomposition of the data for $F$ described in Sec. V H, and the weighted Sobolev norm $H^1_\tau$ is defined in Definition 8.1.1.

In Sec. XIII (see also Proposition 10.0.1), we show that small-data solutions to the MBI system decay at least as fast as solutions to the linear Maxwell-Maxwell system. Since the analysis of the linear theory also plays a key role in our analysis of the MBI system, we first discuss the basic strategy for establishing the aforementioned decay of solutions to the linear Maxwell-Maxwell system

\[
\begin{align*}
dF &= 0, \\
d^*F &= 0.
\end{align*}
\]
We recall that for any two-form $F$, the corresponding Maxwell-Maxwell energy momentum tensor is

$$Q^{(\text{Maxwell})}_{\mu
u} \overset{\text{def}}{=} F^\mu_{\nu} F^\nu_{\kappa} - \frac{1}{4} g^{{\mu\nu}} F_{\kappa\lambda} F^{\kappa\lambda},$$

and that if $F$ is a solution of (17), then $\nabla_\mu Q^{(\text{Maxwell})}_{\mu\nu} = 0, (\nu = 0, 1, 2, 3)$. Furthermore, using the timelike conformal Killing field $\overline{K}$, which has components $\overline{K}^0 = 1 + t^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$, $\overline{K}^j = 2t x^j$, $(j = 1, 2, 3)$, we can construct the energy current

$$J^{(\text{Maxwell})}_\mu = -Q^{\mu(\text{Maxwell})} \overline{K}^\mu.$$

Recall that a conformal Killing field is a vectorfield $X$ that satisfies $\nabla_\mu X_\nu + \nabla_\nu X_\mu = \phi X g_{\mu\nu}$ for some scalar-valued function $\phi$. Because $Q^{(\text{Maxwell})}_{\mu\nu}$ is symmetric, $(g^{-1})^{\mu\nu} Q^{(\text{Maxwell})}_{\nu\kappa} = 0$, and $\overline{K}$ is a conformal Killing field, it thus follows that for solutions to (17), we have

$$\nabla_\mu J^{\mu(\text{Maxwell})}_\nu = 0.$$  \hspace{1cm} (20)

Additionally, $Q^{(\text{Maxwell})}_{\mu\nu}$ has the following positivity property: for every pair of future-directed causal vectors $X$, $Y$, we have that $Q^{(\text{Maxwell})}_{\kappa\lambda} X^\kappa Y^\lambda \geq 0$. In particular, choosing $X^\mu \overset{\text{def}}{=} \delta^\mu_0$, $Y^\mu \overset{\text{def}}{=} \overline{K}^\mu$, it can be shown that (see Lemma 7.4.1)

$$J^0_{(\text{Maxwell})} = \frac{1}{2} \left\{ (1 + q^2) |x|^2 + (1 + s^2)|\alpha|^2 + (2 + q^2 + s^2)(\rho^2 + \sigma^2) \right\},$$

where $\alpha$, $\alpha$, $\rho$, and $\sigma$ are the null components of $F$. If we define the energy $E \geq 0$ by

$$E^2(t) \overset{\text{def}}{=} \int_{\mathbb{R}^3} J^0_{(\text{Maxwell})}(t, x) dx,$$

then it follows from (20) and the divergence theorem that $E(t)$ is constant in time if it is initially finite and if $F$ verifies (17),

$$\frac{d}{dt} (E^2(t)) = 0.$$  \hspace{1cm} (23)

The various weights in (21) are the first hint that different null components of $F$ have different $L^\infty$ decay properties. However, in a full proof of the decay, one has to commute the Maxwell equations with various conformal Killing fields and apply the global Sobolev inequality. Let us explain what we mean by this. Given any solution $F$ of (17) and any conformal Killing field $Z$, it can be shown that $L_Z F$ is also a solution to the linear Maxwell-Maxwell equations. Here, $L_Z F$ is the Lie derivative of $F$ with respect to the vectorfield $Z$. Iterating this process, we conclude that $L^I_Z F$ is a solution, where $I$ is a multi-index, and $L^I_Z$ is shorthand notation for iterated Lie derivatives with respect to vectorfields $Z \in \mathcal{Z}$. In this article, the relevant set of conformal Killing fields $\mathcal{Z}$ consists of: the 4 translations $T_\mu \overset{\text{def}}{=} \delta_\mu$, $(\mu = 0, 1, 2, 3)$; the 3 rotations $\Omega_{(i)}^{(jk)} \overset{\text{def}}{=} x_j \partial_k - x_k \partial_j, (1 \leq j < k \leq 3)$; the 3 Lorentz boosts $\Omega_{(ij)} \overset{\text{def}}{=} -t \partial_j - x_j \partial_t, (i = 1, 2, 3)$; and the scaling vectorfield $S \overset{\text{def}}{=} x^\mu \partial_\mu$.

Furthermore, as in (23), the weighted $L^2$ norms of the various null components of the $L^I_Z F$ are constant in time. Now in order to derive $L^\infty$ decay, we need to connect these weighted $L^2$ norms of $L^I_Z F$ to weighted $L^\infty$ norms of $F$. This is exactly what the global Sobolev inequality provides; see Proposition 10.0.1 for the details.

Let us also discuss the heuristic mechanism for the following previously mentioned important fact: the derivatives of $F$ in directions tangential to the $C^2_q$ (i.e., $\nabla_L, \nabla_{e_a}$) have better decay properties than the transversal derivative $\nabla_L$. As examples, we consider the outgoing null vectorfield

$L \overset{\text{def}}{=} \partial_t + \partial_r$ and the ingoing null vectorfield $L^{*} \overset{\text{def}}{=} \partial_t - \partial_r$, where $\partial_r$ denotes the radial derivative. Simple algebraic computations lead to the identities

$$L = \frac{S - \phi^2 \Omega_{(0a)}^{(0)}}{s}, \quad L^* = -\frac{S + \phi^2 \Omega_{(0a)}^{(0)}}{q}.$$  \hspace{1cm} (24)
where $\omega^i \overset{\text{def}}{=}$ $x^i/r$, and $q, s$ are the null coordinates mentioned above. Therefore, if we have achieved good control of the quantities $\nabla_\mathbf{F}$ and $\nabla_{\partial_0} \mathbf{F}$, then the first relation in (24) suggests that we can achieve even better control of the outgoing derivative $\nabla_\mathbf{F}$, because of the favorable denominator $s^{-1}$. On the other hand, the transversal derivative $\nabla_\mathcal{L} \mathbf{F}$ features a less favorable denominator $q^{-1}$.

More specifically, the $q^{-1}$ term fails to provide decay in the “wave zone” $r \approx t$, while in the entire region $\{t \geq 0\}$ we have that $s^{-1} \leq \min \{t^{-1}, r^{-1}\}$; i.e., decay in $s$ implies decay in $r$ and $t$.

2. Nonlinear analysis

We now outline the key differences between the proof of decay of solutions to the linear Maxwell-Maxwell equations, and the proof of the global existence of and decay of solutions to the MBI system in the small-data regime. To analyze solutions to the MBI system, the “working form” of which is given below in (87a) and (87b), we will use the same Minkowski null decomposition of the Faraday tensor described above. In particular, in order to derive our desired estimates, we do not need to use the characteristic geometry of the MBI system; in using the “wrong” Minkowskian geometry (which has the advantage of relative simplicity), we are deviating from the correct MBI geometry [which is governed by the reciprocal Maxwell-Born-Infeld metric $(b^{-1})\mathbf{F}$ defined in (208)] by small error terms that are controllable. Now like the Maxwell-Maxwell system, the MBI system has a corresponding divergence-free energy-momentum tensor $Q_{\mu\nu}^{MBI}$, which is given below in (182); the availability of this tensor is a well-known consequence of the fact that the MBI Lagrangian $\mathcal{L}_{MBI}$ [see (79)] depends covariantly on only the metric $g$ and the field variables $\mathbf{F}$.

This tensor can be used in conjunction with the vectorfield $\mathcal{K}$ to estimate the weighted $L^2$ norm of the solutions $\mathbf{F}$ to the MBI system. However, to estimate the weighted $L^2$ norm of $\mathcal{L}_\mathbf{F}$, we need a different tensor, which is described by Christodoulou in detail in Ref. 13: the canonical stress $\dot{Q}^{\mu\nu}[\cdot, \cdot]$ defined below in (195). We remark that in the case of the linear Maxwell-Maxwell system, the quantity $\dot{Q}^{\mu\nu}[\mathcal{L}_\mathbf{F}, \mathcal{L}_\mathbf{F}]$ coincides with the energy momentum tensor $Q_{\mu\nu}(\mathbf{F})$ constructed out of the $\mathcal{L}_\mathbf{F}$, but in a general nonlinear theory, the two tensors differ. The important point is that the $\mathcal{L}_\mathbf{F}$ are solutions to the linearized equations (186a) and (186b), which are derivable from a linearized Lagrangian $\mathcal{L}$ [see (192)] depending on the metric $g$, the linearized variables $\mathcal{F} \overset{\text{def}}{=} \mathcal{L}_\mathbf{F}$, and also the background $\mathbf{F}$. Although the dependence of $\mathcal{L}$ on the background precludes the availability of a divergence-free tensor for solutions to the linearized system, we may nevertheless use Christodoulou’s framework to construct the tensor $\dot{Q}^{\mu\nu}$. Although $\dot{Q}_{\mu\nu}$ is in general not even symmetric, nor is $\nabla_\mu (\dot{Q}^{\mu\nu}[\mathcal{L}_\mathbf{F}, \mathcal{L}_\mathbf{F}])$ non-zero, the role that $\dot{Q}^{\mu\nu}$ plays in the analysis of the linearized equations is roughly analogous to the role played by the energy momentum tensor in the original equations: $\nabla_\mu (\dot{Q}^{\mu\nu}[\mathcal{L}_\mathbf{F}, \mathcal{L}_\mathbf{F}])$, though non-zero, is lower order (i.e., for solutions to the linearized equations, $\nabla_\mu (\dot{Q}^{\mu\nu}[\mathcal{L}_\mathbf{F}, \mathcal{L}_\mathbf{F}])$ does not depend on $\nabla_\mathcal{L}_\mathbf{F}$), and furthermore, $\dot{Q}^{\mu\nu}[\cdot, \cdot]$ possesses some positivity properties under contractions against certain pairs $(\xi, X)$ consisting of a timelike covector $\xi$ and a timelike vector $X$.

Once we have $\dot{Q}^{\mu\nu}$, we can again use the vectorfield $\mathcal{K}$ construct energies $E_N[\mathcal{F}(t)]$, which are a sum over $|l| \leq N$ of the energy of $\mathcal{L}_\mathbf{F}$, that are analogous to the energies (22) defined in the Maxwell-Maxwell case. The precise definition of $E_N[\mathcal{F}(t)]$, that is constructed out of the modified canonical stress $\dot{Q}^{\mu\nu}_v$ (see Definition 7.3.3), is given in Eq. (281); the modified stress is a slightly altered version of $\dot{Q}^{\mu\nu}_v$ that we introduce for computational convenience. Now in contrast to the case of Maxwell-Maxwell solutions, for solutions to the MBI system, $E_N[\mathcal{F}(t)]$ is not constant. Additionally, for MBI solutions, the $q, s$ — weighted factors appearing in the expression $E_N[\mathcal{F}(t)]$ are not manifestly uniform, but instead depend on the solution $\mathbf{F}$ itself. For these reasons, it is convenient to introduce a norm $\| \mathcal{F}(t) \|_{L^2;N}$ whose $q, s$ — weighted factors are independent of $\mathbf{F}$; see (280). In order to compare the two quantities, we establish inequality (292b), which shows that in the small-solution regime, $E_N[\mathcal{F}(t)] \approx \| \mathcal{F}(t) \|_{L^2;N}$. The crux of the global existence proof is the following: even though $E_N[\mathcal{F}(t)]$ is not constant, we are nevertheless able to derive an a priori bound for $\| \mathcal{F}(t) \|_{L^2;N}$ which shows that it remains uniformly small on any time interval of existence for the solution. According to the continuation principle of Proposition 12.0.1, such an a priori bound for $\| \mathcal{F}(t) \|_{L^2;N}$ implies global existence when $N \geq 4$. 


Now in order to estimate \( \| F(t) \|_{\mathcal{L}^2} \), we need to handle the numerous “error” terms arising in the expression for \( \frac{d}{dt} (\mathcal{E}_N^2[F(t)]) \). The source of error terms was alluded to above, namely, that the divergence of \( \mathcal{Q}_{\mu}^a [\mathcal{L}^2_\mu \mathcal{F}, \mathcal{L}^1_\nu \mathcal{F}] \) is non-zero. The precise expression for \( \nabla_\mu (\mathcal{Q}_{\nu}^a [\mathcal{L}^2_\mu \mathcal{F}, \mathcal{L}^1_\nu \mathcal{F}]) \) is given in Lemma 7.3.1. We mention here that one of the most important terms in the expression arises from the fact that \( \mathcal{L}^2_\mu \mathcal{F} \) is a solution to the linearized equations with inhomogeneous terms; many inhomogeneous “error” terms arise from commuting the operator \( \mathcal{L}^1_\nu \mathcal{F} \) through Eq. (87b); see Proposition 7.2.1. This commuting is accomplished through the use of modified Lie derivatives \( \mathcal{L} \), which are equal to ordinary Lie derivatives plus a scalar multiple of the identity; see Definition 4.0.2 and Lemma 6.0.6. A careful analysis of the special structure of the error terms (which are discussed in Sec. IC 3), in conjunction with the global Sobolev inequality, leads to the a priori estimate (406), which is valid in the small-solution regime. We restate this inequality here for convenience:

\[
\| F(t) \|_{\mathcal{L}^2} \leq C \left\{ \| F(0) \|_{\mathcal{L}^2}^2 + \int_0^1 \frac{1}{1 + \tau^2} \| F(\tau) \|_{\mathcal{L}^2}^2 \, d\tau \right\}.
\]

(25)

Applying Gronwall’s inequality to (25), we thus conclude the desired result: \( \| F(t) \|_{\mathcal{L}^2} \) is globally bounded in time, if \( \| F(0) \|_{\mathcal{L}^2} \) is sufficiently small. In addition, we remark that the aforementioned decay properties of the solution \( F \) are a by-product of the previous analysis. More specifically, the decay properties of \( F \) follow directly from the global bound on \( \| F(t) \|_{\mathcal{L}^2} \) and the global Sobolev inequality (Proposition 10.0.1).

We have a few final comments to make concerning the smallness of the data. The initial data consist of a pair of one-forms (\( \mathcal{B}, \mathcal{D} \)) that are tangent to the Cauchy-hypersurface \( \Sigma_0 \), and that satisfy the constraints (which are familiar from linear Maxwell-Maxwell theory) \( \nabla_\mu \mathcal{B}^\mu = \nabla_\mu \mathcal{D}^\mu = 0 \). Here, \( \nabla \) denotes the Levi-Civita connection corresponding to the first fundamental form \( g \) of \( \Sigma_0 \) (see Sec. II). As is described in Sec. VI H, \( \mathcal{F}(0) \) can be constructed out of (\( \mathcal{B}, \mathcal{D} \)). However, the quantity \( \| \mathcal{F}(0) \|_{\mathcal{L}^2} \) is small, which is required to close the global existence argument, involves derivatives of \( \mathcal{F} \) that are normal to \( \Sigma_0 \) (i.e., time derivatives). To address this discrepancy, we repeatedly differentiate an appropriate version of the MBI system, which allows us to express the normal derivatives of \( \mathcal{F} \) along \( \Sigma_0 \) in terms of the tangential derivatives (i.e., spatial derivatives) of (\( \mathcal{B}, \mathcal{D} \)). Consequently, as is explained in detail in Sec. VIII A, it is possible to devise a smallness condition involving only the data (\( \mathcal{B}, \mathcal{D} \)) and their tangential derivatives, from which the smallness of \( \| \mathcal{F}(0) \|_{\mathcal{L}^2} \) necessarily follows. This allows for a “proper” formulation of the small-data global existence condition of Theorem 1 in terms of quantities inherent to the data.

3. The error terms

Let us now make a few remarks concerning the many error terms that arise in our study of \( \frac{d}{dt} (\mathcal{E}_N^2[F(t)]) \), since the study of these error terms is at the heart of our analysis. In the small-solution regime, the MBI system is a cubic quasilinear perturbation of the linear Maxwell-Maxwell system. It is well known that for linear hyperbolic PDEs whose solutions possess the decay properties of solutions to the Maxwell-Maxwell system, cubic perturbations do not destroy the existence of small-data global solutions (we are assuming that the perturbations involve only 1 or fewer derivatives, and that the perturbed system is also hyperbolic). In fact, a much shorter proof of small-data global existence could be provided by using the vectorfield \( \partial_t \) in place of the vectorfield \( \mathcal{K} \) in our construction of the energies. However, in order to show that small-data MBI solutions decay at least as fast as solutions to the linear Maxwell-Maxwell system, we make full use of the vectorfield \( \mathcal{K} \), together with an algebraic property of the MBI system: its nonlinearities have a special null structure in that they satisfy a version of the null condition. The null condition, which was first identified by Christodoulou12 and Klainerman13 in the context of nonlinear wave equations, is a collection of algebraic properties that are satisfied by special nonlinearities. Roughly speaking, when a nonlinearity satisfies the null condition, the worst kind of terms (from the point of view of decay) are not present. More specifically, in the case of the MBI system, the expression for \( \frac{d}{dt} (\mathcal{E}_N^2[F(t)]) \) involves quartic terms in \( F \) and its iterated Lie derivatives \( \mathcal{L}^2 \mathcal{F} \), multiplied by weights in \( q \) and \( s \) arising from \( \mathcal{K} \) and \( \nabla \mathcal{K} \). Because these terms are fourth order, we do not need to perform a fully detailed null decomposition in order to prove our desired estimates. That is, there is
room for imprecision; we only prove estimates that are sufficient recover the full decay properties possessed by solutions to the linear Maxwell-Maxwell system. Let us summarize the version of the null condition that we show is satisfied by the MBI system (see Sec. IX for complete details): for the products of terms in the expression for \( \frac{d}{dt} (E^\mu_\nu(\mathcal{F}(t))) \) that involve a weight of \( s \) or \( 1 + s^2 \), at most two of the four factors correspond to the worst-decaying components \( a \mathcal{L}_D \mathcal{F} \). It is also true that for the products of terms involving a weight of \( q \) or \( 1 + q^2 \), at most three of the four factors correspond to the worst decaying components \( a \mathcal{L}_D \mathcal{F} \). However, we do not make use of the availability of the one “good” factor since our estimates close without it.

4. The large-data local well-posedness of the MBI system

Finally, we would like to make a few remarks about the local well-posedness proof that is briefly sketched in Sec. XII. This result is interesting in itself because it shows the following fact, which is arguably not manifest: the MBI system’s initial value problem is locally well posed in every field-strength regime in which its Lagrangian is real-valued, i.e., in every regime in which the theory is well defined. The crucial estimate in this regard is contained in Proposition 7.4.4, which shows that it is always possible to construct an energy current for the linearized equations with positivity properties that are sufficient to deduce local well-posedness in the weighted Sobolev space of relevance for our global existence result. This fact is strongly related to the internal geometry of and the hyperbolicity of the MBI system (i.e., the structure of the characteristic subsets). That is, in large-data theory, it is important to account for the precise characteristic geometry of the equations. We account for this geometry by constructing a solution-dependent multiplier vectorfield \( X_{\text{local}} \) [see (239)] that is used to construct our local well-posedness energy current (by contracting the modified canonical stress against \( X_{\text{local}} \)). In particular, our local well-posedness current differs from the current used in our small-data global existence proof; the vectorfield \( K \) may not be a suitable multiplier for deducing large-data local well posedness because it is not adapted to the characteristic geometry of the equations.

For an alternative proof of the large-data local well-posedness of the MBI system’s initial value problem, one may consult7 (see also Ref. 51). In this work, Brenier “augments” the MBI system by taking as his 10 unknowns the non-trivial components of the electromagnetic quantities (see Sec. VH) \( B, D, P, \) and \( h \). Here, \( h \) is a scalar-valued quantity that has nothing to do with the tensorfield \( h^\mu_{\nu} \), discussed above. Along the “MBI submanifold,” \( P \) coincides with the Poynting vector \( (P = B \times D) \), and \( h \) coincides with the 00 component of the MBI energy-momentum tensor \( Q^{(MBI)}_{\mu\nu} \) [see (182)], but for general augmented MBI solutions, \( P \) and \( h \) are independent unknowns. To compensate, the additional evolution equations (1.9) and (1.10) of Ref. 7 which are redundant for solutions belonging to the MBI submanifold, were added to the MBI system (i.e., so that there are 10 equations for the 10 unknowns). From the point of view of hyperbolicity, the most important feature of this augmented system is that the function \( S(B, D, P, h) := \frac{1 + |B|^2 + |D|^2 + |P|^2}{h} \), which coincides with a constant multiple of the quantity \( h = Q^{(MBI)}_{00} \), for solutions constrained to the MBI submanifold, satisfies the properties of a smooth, strictly convex entropy function of the augmented variables \( B, D, P, h \). Thus, from the general framework of hyperbolic conservation laws (see, e.g., Ref. 19), it follows that there exists a change of state-space variables in which the augmented MBI system becomes symmetric hyperbolic. For symmetric hyperbolic systems, there exists a well-established theory of well-posedness based on energy estimates (see, e.g., Refs. 10, 19, 24, 36, 41, and 50).

D. Outline of the article

The remainder of the article is organized as follows:

- In Sec. II, we recall some basic facts from differential geometry.
- In Sec. III, we provide a detailed introduction to the MBI system.
- In Sec. IV, we discuss the collections of Minkowski conformal Killing fields that play a role in our analysis. We also introduce modified Lie derivatives, which have favorable commutation properties with the MBI equations.
In Sec. V, we introduce the Minkowski null frame and the Minkowskian null decomposition of a tensor. We then decompose the MBI system relative to a Minkowski null frame. We also introduce several electromagnetic decompositions of the Faraday and Maxwell tensors.

In Sec. VI, we provide some commutation lemmas that will be used throughout the remainder of the article, especially in Sec. X.

In Sec. VII, we discuss the energy-momentum tensor associated with the MBI system, and the canonical stress/modified canonical stress tensors associated with the equations of variation.

In Sec. VIII, we introduce the norms, seminorms, and energies that will be used in the proof of our main theorem.

In Sec. IX, we perform a partial null decomposition of the nonlinear error terms that appear in the expression for the time derivative of the energy. It is here that the special null structure is revealed.

In Sec. X, we discuss the global Sobolev inequality, which connects weighted $L^2$ bounds to weighted $L^\infty$ bounds.

In Sec. XI, we prove the $a priori$ bound (25), which is the most important inequality in the article.

In Sec. XII, we briefly discuss large-data local well-posedness for the MBI system. We also discuss the availability of a continuation principle, which provides criteria for the existence of a global classical solution.

In Sec. XIII, we combine the results of Secs. XI and XII in order to establish our main theorem.

In Appendix A, we provide some weighted Sobolev-Moser estimates that are used during our analysis of the data.

In Appendix B, we collect together much of the notation that is introduced throughout the article.

II. GEOMETRY

In this section, we recall some basic facts from differential geometry that will be used throughout the article.

A. Inertial coordinate systems, the spacetime metric, and the Riemannian metric

In Minkowski spacetime, there exists a family of global coordinate systems, which we refer to as inertial coordinate systems, in which the metric $g_{\mu\nu}$ and its inverse $(g^{-1})^{\mu\nu}$ have the following components:

$$g_{\mu\nu} = (g^{-1})^{\mu\nu} = \text{diag}(-1, 1, 1, 1).$$  \hfill (26)

It will be convenient to carry out calculations and to define various tensors relative to an inertial coordinate system. Therefore, we fix a single inertial coordinate system $\{x^\mu\}_{\mu = 0, 1, 2, 3}$ on Minkowski spacetime. For the remainder of the article, when we decompose tensors with respect to an inertial frame, it will always be relative to the frame corresponding to this fixed inertial coordinate system.

When working in this coordinate system, we often use the abbreviations

$$x^0 \overset{\text{def}}{=} t, \quad \hat{x} \overset{\text{def}}{=} (x^1, x^2, x^3), \quad \partial_t \overset{\text{def}}{=} \partial_0 = T^{(0)}.$$  \hfill (27a)

$$\partial_\mu \overset{\text{def}}{=} \frac{\partial}{\partial x^\mu}, \quad \partial_t \overset{\text{def}}{=} \partial_0 = T^{(0)}. \quad \hfill (27b)$$

We recall the following partitions of $T_pM$ and $T_p^*M$ induced by $g$.

**Definition 2.1.1 (Timelike, null, causal, and spacelike vectors):** Vectors $X \in T_pM$ are classified as timelike, null, causal, spacelike as follows, where $g(X, Y) \overset{\text{def}}{=} g_{\alpha\beta}X^\alpha Y^\beta$.

$$g(X, X) < 0 \quad \text{(timelike)}, \quad \hfill (28a)$$

$$g(X, X) = 0 \quad \text{(null)} \quad \hfill (28b)$$
\( g(X, X) \leq 0 \) (causal), \( g(X, X) \rangle 0 \) (spacelike). \hfill (28c)

Furthermore, causal vectors \( X \) are classified as future-directed or past-directed as follows:

\( g(X, T(0)) \langle 0 \) (future-directed), \hfill (29a)

\( g(X, T(0)) \rangle 0 \) (past-directed). \hfill (29b)

Covectors \( \xi_\mu \) are defined to have the same classification as their metric dual \( X^\mu \equiv (g^{-1})^{\mu \nu} \xi_\nu \). We sometimes refer to \( \xi \) as the \( g \)-dual of \( X \) in order to emphasize that this notion of duality depends on \( g \).

In order to measure the size of various tensors, it is convenient to introduce a Riemannian metric on \( \mathbb{R}^4 \). A natural choice is the Euclidean metric \( e \), which has the following components relative to an arbitrary coordinate system

\[
e_{\mu \nu} \equiv g_{\mu \nu} + 2(T(0)_\mu(T(0)_\nu).
\] \hfill (30)

In the above formula, \( T(0) \) is the “time translation” vectorfield, which is defined to coincide with \( \partial_t \) in our inertial coordinate system. Therefore, relative to this coordinate system, the metric \( e \) and its inverse \( e^{-1} \) have the following components:

\[
e_{\mu \nu} = \text{diag}(1, 1, 1, 1),
\] \hfill (31a)

\[
(e^{-1})^{\mu \nu} = \text{diag}(1, 1, 1, 1).
\] \hfill (31b)

We now define the aforementioned tensorial norm.

**Definition 2.1.2 (Euclidean norm):** If \( U \) is a tensor of type \( (n_m) \), then we define the norm \( |\cdot| \geq 0 \) of \( U \) by

\[
|U|^2 = |(e^{-1})^{\hat{k}_1 \cdots \hat{k}_m} e_{\hat{k}_1 \cdots \hat{k}_m} U_{\hat{k}_1 \cdots \hat{k}_m \cdots \hat{k}_n \cdots \hat{k}_m}|.
\] \hfill (32)

**B. Lie derivatives and covariant derivatives**

Given any pair of vectorfields \( X, Y \), we recall that relative to an arbitrary coordinate system, their *Lie bracket* \([X, Y]\) can be expressed as

\[
[X, Y]^\mu = X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu.
\] \hfill (33)

Furthermore, we have that

\[
\mathcal{L}_X Y = [X, Y],
\] \hfill (34)

where \( \mathcal{L} \) denotes the *Lie derivative operator*. Given a type \( (0)^m \) tensorfield \( U \), and vectorfields \( Y_{(1)}, \cdots, Y_{(m)} \), the Leibniz rule for \( \mathcal{L} \) implies that (34) generalizes as follows:

\[
(\mathcal{L}_X U)(Y_{(1)}, \cdots, Y_{(m)}) = X(U(Y_{(1)}, \cdots, Y_{(m)}))
\] \hfill (35)

\[ - \sum_{i=1}^m U(Y_{(1)}, \cdots, Y_{(i-1)}, [X, Y_{(i)}], Y_{(i+1)}, \cdots, Y_{(m)}). \]

**Remark 2.2.1:** The Lie derivative operator does not commute with the raising and lowering of indices via the metric \( g \). Thus, in order to avoid confusion, we use the convention that Lie
derivatives are applied to two-forms $\mathcal{F}_{\mu\nu}$ with both indices down. In particular, we use the convention
\[ \mathcal{L}_Z \mathcal{F}_{\mu\nu} \overset{\text{def}}{=} (g^{-1})^{\mu\kappa} (g^{-1})^{\nu\lambda} \mathcal{L}_Z \mathcal{F}_{\kappa\lambda}. \]

There is a unique affine connection $\nabla$, which is known as the Levi-Civita connection, that is torsion-free and compatible with the metric $g$. These properties are equivalent to the requirement that the following identities hold for all vectorfields $X, Y, Z$:
\[ \nabla_X Y - \nabla_Y X = [X, Y], \quad (36) \]
\[ \nabla_X (g(Y, Z)) = g(\nabla_X Y, Z) + g(X, \nabla_Y Z). \quad (37) \]
Furthermore, given a type $(0,1)$ tensorfield $U$, and vectorfields $Y_{(i)}$, $\cdots$, $Y_{(m)}$, the Leibniz rule implies that
\[ (\nabla_X U)(Y_{(1)}, \cdots, Y_{(m)}) = X(U(Y_{(1)}, \cdots, Y_{(m)})) - \sum_{i=1}^{m} U(Y_{(1)}, \cdots, Y_{(i-1)}, \nabla_X Y_{(i)}, Y_{(i+1)}, \cdots, Y_{(m)}). \quad (38) \]
We remark that relative to an arbitrary coordinate system, (37) is equivalent to
\[ \nabla_{\lambda} g_{\mu\nu} = 0, \quad (\lambda, \mu, \nu = 0, 1, 2, 3). \quad (39) \]
Furthermore, in our inertial coordinate system on Minkowski spacetime, if $U$ is any type $(n)$ tensorfield, then $\nabla_{\mu} U_{\mu_1,\cdots,\mu_n}^{\nu_1,\cdots,\nu_k} = \partial_{\mu} U_{\mu_1,\cdots,\mu_n}^{\nu_1,\cdots,\nu_k}$. In the above formulas and throughout the article, we use the notation
\[ \nabla_{\chi} \overset{\text{def}}{=} X^k \nabla_k. \quad (40) \]

The Riemann curvature tensor $R(\cdot, \cdot) \cdot$ is defined by the requirement that the following identities hold for all vectorfields $X, Y, Z$:
\[ \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = R(X, Y) Z + \nabla_{[X,Y]} Z. \quad (41) \]
In Minkowski spacetime, $R(\cdot, \cdot) \cdot \equiv 0$.

The following standard lemma gives a convenient formula relating Lie derivatives and covariant derivatives.

**Lemma 2.2.1 (Lie derivatives in terms of covariant derivatives):** (Ref. 62). Let $X$ be a vectorfield, and let $U$ be a tensorfield of type $(n)$. Then $\mathcal{L}_X U$ can be expressed in terms of covariant derivatives of $U$ and $X$ as follows:
\[ (\mathcal{L}_X U)_{\mu_1,\cdots,\mu_n}^{\nu_1,\cdots,\nu_k} = (\nabla_X U)_{\mu_1,\cdots,\mu_n}^{\nu_1,\cdots,\nu_k} + U_{\mu_2,\cdots,\mu_n}^{\nu_1,\cdots,\nu_k} \nabla_{\mu_1} X^\kappa + \cdots + U_{\mu_1,\cdots,\mu_n-1}^{\nu_1,\cdots,\nu_k} \nabla_{\mu_n} X^\kappa - U_{\mu_1,\cdots,\mu_n-1}^{\nu_1,\cdots,\nu_k} \nabla_{\nu_k} X^\nu_1 \cdots \nu_{k-1}. \quad (42) \]

It follows from (39) and (42) that
\[ \mathcal{L}_X g_{\mu\nu} = (X) \pi^{\mu\nu}, \quad (43) \]
where
\[ (X) \pi^{\mu\nu} \overset{\text{def}}{=} \nabla_\mu X_\nu + \nabla_\nu X_\mu \quad (44) \]
is the deformation tensor of $X$. 

\[ \]
C. Volume forms and Hodge dual

There is a canonical volume form $\epsilon_{\mu\nu\kappa\lambda}$ associated with the metric $g$. Relative to any local coordinate system, we have that

$$\epsilon_{\kappa\lambda\mu\nu} = |\det(g)|^{-1/2}[\kappa\lambda\mu\nu],$$

(45)

$$\epsilon_{\kappa\lambda\mu\nu} = -|\det(g)|^{-1/2}[\kappa\lambda\mu\nu],$$

(46)

where $[\kappa\lambda\mu\nu]$ is totally antisymmetric with normalization $[0123] = 1$. It can be checked that the covariant derivative of the volume form vanishes

$$\nabla_\beta \epsilon_{\kappa\lambda\mu\nu} = 0,$$

(47)

where $\beta, \kappa, \lambda, \mu, \nu = 0, 1, 2, 3$.

The Hodge dual operator, which we denote by $\star$, plays a fundamental role throughout our discussion.

**Definition 2.3.1 (Hodge dual):** If $F$ is any two-form, then its Hodge dual $\star F$ is defined as follows:

$$\star F_{\mu\nu} \overset{\text{def}}{=} \frac{1}{2} \epsilon^{\kappa\lambda}_{\mu\nu} F_{\kappa\lambda}.$$  

(48)

D. $\Sigma_t$, $S_{r,t}$, and the first and second fundamental forms

**Definition 2.4.1 (Important spacelike submanifolds):** The following two classes of spacelike submanifolds of Minkowski spacetime, which we define relative to the inertial coordinate system $\{x^\mu\}_{\mu=0,1,2,3}$ will play a role throughout the remainder of the article

$$\Sigma_t \overset{\text{def}}{=} \{(\tau, y) \mid \tau = t\},$$

(49)

$$S_{r,t} \overset{\text{def}}{=} \{(\tau, y) \mid \tau = t, |y| = r\},$$

(50)

where $|y| \overset{\text{def}}{=} \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}$. We refer to the $\Sigma_t$ as “time slices,” and the $S_{r,t}$ as “spheres.”

The future-directed normal to the $\Sigma_t$ is the time translation vectorfield $T_{(0)}$, while the $S_{r,t}$ have two linearly independent null normals. We denote the one pointing in the “outward” direction by $L$, and the one pointing in the “inwards” direction by $\tilde{L}$. The vectorfields $L$ and $\tilde{L}$, which are defined on $M/0$, are unique up to multiplication by a scalar function. We choose the normalization so that they have the following components relative to our inertial coordinate system:

$$L^\mu = (1, -\omega^1, -\omega^2, -\omega^3),$$

(51a)

$$\tilde{L}^\mu = (1, \omega^1, \omega^2, \omega^3),$$

(51b)

where $\omega^i = x^i/r$. With $\partial_r \overset{\text{def}}{=} \frac{1}{r} x^a \partial_a$ denoting the radial vectorfield, $L$, $\tilde{L}$ can be expressed as

$$L = \partial_t - \partial_r,$$

(52a)

$$\tilde{L} = \partial_t + \partial_r.$$  

(52b)

We remark that beginning in Sec. V.1, $L$ and $\tilde{L}$ will play a key role in the null decomposition of the MBI system.

We now recall the definitions of the first fundamental forms of $\Sigma_t$ and of $S_{r,t}$.

**Definition 2.4.2 (First fundamental forms):** The first fundamental forms of $\Sigma_t$, $S_{r,t}$, are the Riemannian metrics on $\Sigma_t$, $S_{r,t}$, respectively, induced by the spacetime metric $g$. In an arbitrary local
coordinate system, \( g, g' \) can be expressed as follows:

\[
\begin{align*}
g_{\mu \nu} & \overset{\text{def}}{=} g_{\mu \nu} + (T(0))_{\mu} (T(0))_{\nu}, \\
g_{\mu \nu} & \overset{\text{def}}{=} g_{\mu \nu} + \frac{1}{2} \left( L_{\mu} L_{\nu} - L_{\nu} L_{\mu} \right).
\end{align*}
\] (53, 54)

We remark that the tensors \( g_{\mu \nu} \overset{\text{def}}{=} \delta_{\mu}^v + (T(0))_{\mu} (T(0))^v \) and \( g'_{\mu \nu} \overset{\text{def}}{=} \delta_{\mu}^v + \frac{1}{2} \left( L_{\mu} L_{\nu} + L_{\nu} L_{\mu} \right) \) orthogonally project onto \( \Sigma_t \) and \( \Sigma' \), respectively. Furthermore, the volume forms of \( g \) and \( g' \), which we, respectively, denote by \( \epsilon \) and \( \epsilon' \), can be expressed as follows relative to an arbitrary spacetime coordinate system:

\[
\begin{align*}
\epsilon_{\nu \lambda k} & = \epsilon_{\nu \lambda k} T_{(0)}^\mu, \\
\epsilon'_{\mu \nu \lambda k} & = \frac{1}{2} \epsilon_{\nu \lambda \mu} L^\mu L^\nu.
\end{align*}
\] (55, 56)

**Definition 2.4.3 (The definition of tangent):** Let \( U \) be a type \( \left( \begin{smallmatrix} n \\ m \end{smallmatrix} \right) \) spacetime tensor. We say that \( U \) is tangent to the time slices \( \Sigma_t \) if

\[
U_{\mu_1 \ldots \mu_n} v_{1} \ldots v_{n} = g_{\mu_1} \ldots g_{\mu_n} g v_{1} \ldots g v_{n} U_{\mu_1 \ldots \mu_n} v_{1} \ldots v_{n}.
\] (57)

Equivalently, \( U \) is tangent to the \( \Sigma_t \) if and only if any contraction of \( U \) with \( T(0) \) results in 0.

Similarly, we say that \( U \) is tangent to the spheres \( S_{r,t} \) if

\[
U_{\mu_1 \ldots \mu_n} v_{1} \ldots v_{n} = g_{\mu_1} \ldots g_{\mu_n} g v_{1} \ldots g v_{n} U_{\mu_1 \ldots \mu_n} v_{1} \ldots v_{n},
\] (58)

Equivalently, \( U \) is tangent to the spheres \( S_{r,t} \) if and only if any contraction of \( U \) with either \( L \) or \( L \) results in 0.

We also recall the following relationships between the Levi-Civita connections \( \nabla \), \( \tilde{\nabla} \) corresponding to \( g, g' \) and the Levi-Civita connection \( \nabla \) corresponding to \( g \), which are valid for any tensor \( U \) of type \( \left( \begin{smallmatrix} n \\ m \end{smallmatrix} \right) \) tangent to the \( \Sigma_t, S_{r,t} \), respectively:

\[
\nabla_{\nu} U_{\mu_1 \ldots \mu_n} v_{1} \ldots v_{n} = g_{\mu_1} \ldots g_{\mu_n} g v_{1} \ldots g v_{n} \tilde{\nabla}_{\nu} U_{\mu_1 \ldots \mu_n} v_{1} \ldots v_{n}.
\] (59)

\[
\tilde{\nabla}_{\nu} U_{\mu_1 \ldots \mu_n} v_{1} \ldots v_{n} = g_{\mu_1} \ldots g_{\mu_n} g v_{1} \ldots g v_{n} \nabla_{\nu} U_{\mu_1 \ldots \mu_n} v_{1} \ldots v_{n}.
\] (60)

As in (40), throughout the article, we use the notation

\[
\nabla X \overset{\text{def}}{=} X^{\gamma} \nabla_{\gamma}, \quad (\text{if } X \text{ is tangent to } \Sigma_t),
\] (61)

\[
\tilde{\nabla} X \overset{\text{def}}{=} X^{\gamma} \tilde{\nabla}_{\gamma}, \quad (\text{if } X \text{ is tangent to } S_{r,t}).
\] (62)

We recall the definitions of the second fundamental form of the \( \Sigma_t \), and null second fundamental forms of the \( S_{r,t} \).

**Definition 2.4.4 (Second fundamental forms):** The second fundamental form of the hypersurface \( \Sigma_t \) is defined to be the tensorfield

\[
\nabla_{\mu} (T(0))_{\nu}.
\] (63)

The null second fundamental forms of the \( S_{r,t} \) are defined to be the following pair of tensorfields:

\[
\nabla_{\mu} L_{\nu}, \quad \nabla_{\mu} L_{\nu}.
\] (64)

In the next lemma, we illustrate one of the key properties of the second fundamental forms.

**Lemma 2.4.5 (Properties of the second fundamental form of \( \Sigma_t \)):** The second fundamental form \( \nabla_{\mu} (T(0))_{\nu} \) is a symmetric type \( \left( \begin{smallmatrix} 0 \\ 2 \end{smallmatrix} \right) \) tensorfield that is tangent to the time slices \( \Sigma_t \). Similarly, the null
second fundamental forms $\nabla \mu L_v, \nabla \mu L_v$ are symmetric type $(0,2)$ tensor fields that are tangent to the spheres $S_{r,t}$.

Proof: The fact that $\nabla \mu L_v, \nabla \mu L_v$ are tangent to the $S_{r,t}$ follows from contracting them with the vectors $L|_{\mu}, L|_{\nu}$, which form a basis for the $g$-orthogonal complement in $T_{p}M$ of $T_{p}S_{r,t}$, and using the properties (107a) and (107b) (which are independently derived below). For the symmetry property, it remains to investigate the case that $X, Y$ are vector fields tangent to $S_{r,t}$. Then $[X, Y]$ is also tangent to $S_{r,t}$. Therefore, using the fact that $\nabla \mu L_v$ is tangent to the $S_{r,t}$, (36), and (37), we deduce that

$$X^\mu Y^\nu \nabla \mu L_v = g(\nabla X L, Y) = \nabla_X g(L, Y) - g(L, \nabla_X Y)$$

$$= -g(L, \nabla_Y X) - g(L, [X, Y])$$

$$= -\nabla_Y g(L, X) + g(\nabla_Y L, X)$$

$$= g(\nabla_Y L, X) = Y^\mu X^\nu \nabla \mu L_v.$$ 

The proofs for $\nabla \mu L_v$ and $\nabla \mu (T_{(0)}v)$ are similar. \hfill \square

Remark 2.4.1: Lemma 6.0.13 provides very simple expressions for the null second fundamental forms.

Remark 2.4.2: By Lemma 2.4.1, we have that $g^\mu \nabla_\mu L^\nu = \nabla _\mu L^\nu$, and similarly for $L$. Therefore, we sometimes use the abbreviations $\tilde{\nabla}_\mu L^\nu \overset{\text{def}}{=} g^\mu \nabla_\mu L^\nu$ and $\bar{\nabla}_\mu L^\nu \overset{\text{def}}{=} g^\mu \nabla_\mu L^\nu$, which should cause no confusion.

To conclude this section, we recall the following basic facts concerning the metrics $g$ and $\tilde{g}$.

Lemma 2.4.2 (Covariant derivatives of $g$ and $\tilde{g}$): Let $g$ and $\tilde{g}$ be the first fundamental forms of $g$ defined in Definition 2.4.2. Let $\nabla$, $\bar{\nabla}$ be their corresponding Levi-Civita connections, as defined in (59) and (60), respectively. Then

$$\nabla_\lambda g_{\mu\nu} = 0, \quad (\lambda, \mu, \nu = 0, 1, 2, 3),$$

$$\bar{\nabla}_\lambda \tilde{g}_{\mu\nu} = 0, \quad (\lambda, \mu, \nu = 0, 1, 2, 3).$$

Proof: Lemma 2.4.2 follows from the expressions (59) and (60) and Lemma 2.4.1. \hfill \square

III. THE MAXWELL-BORN-INFELD SYSTEM

In this section, we first discuss the equations corresponding to a generic covariant theory of classical electromagnetism that is derivable from a Lagrangian. We then introduce the Maxwell-Born-Infeld model and derive several versions of its equations. The final version, namely, Eqs. (87a) and (87b), will be the one we use throughout most of the remainder of the article.

A. The Lagrangian formulation of nonlinear electromagnetism

In this section, we recall some facts from classical nonlinear electromagnetic field theory in a spacetime $(M, g)$ of signature $(-, +, +, +)$. We restrict our attention to theories of nonlinear electromagnetism derivable from a Lagrangian $\mathcal{L}$. The fundamental quantity in such a theory is the Faraday tensor $F_{\mu\nu}$, a two-form (i.e., an antisymmetric tensor field) that is postulated to be closed

$$dF = 0,$$
where $d$ denotes the exterior derivative operator. This equation, which is the first of two equations that will define a particular nonlinear theory, is known as the Faraday-Maxwell law. In local coordinates, it can be expressed in the following two ways:

$$\partial_\nu \mathcal{F}_{\mu \nu} = 0, \quad (\lambda, \mu, \nu = 0, 1, 2, 3),$$  \hspace{1cm} (69a)

$$\nabla_\nu \mathcal{F}_{\mu \nu} = 0, \quad (\lambda, \mu, \nu = 0, 1, 2, 3),$$  \hspace{1cm} (69b)

where $[\cdot \cdot \cdot]$ denotes antisymmetrization.

In any covariant theory of classical electromagnetism, $\mathcal{L}$ is a scalar-valued function of the two invariants of $\mathcal{F}$, which we denote by $\mathcal{I}_1$ and $\mathcal{I}_2$, i.e., $\mathcal{L} = \mathcal{L}(\mathcal{I}_1[\mathcal{F}], \mathcal{I}_2[\mathcal{F}])$. They can be expressed in the following ways:

$$\mathcal{I}_1 = \mathcal{I}_1[\mathcal{F}] \overset{def}{=} \frac{1}{2} (g^{-1})^{\mu \nu}(g^{-1})^{\lambda \nu} \mathcal{F}_{\lambda \mu} \mathcal{F}_{\nu \mu} = -\mathcal{F} \wedge \mathcal{F} = |B|^2 - |E|^2,$$  \hspace{1cm} (70a)

$$\mathcal{I}_2 = \mathcal{I}_2[\mathcal{F}] \overset{def}{=} \frac{1}{4} (g^{-1})^{\mu \nu}(g^{-1})^{\lambda \nu} \mathcal{F}_{\lambda \mu} \mathcal{F}_{\nu \mu} = \frac{1}{8} \mathcal{F} \wedge \mathcal{F} = E \cdot B,$$  \hspace{1cm} (70b)

where $\wedge$ denotes the wedge product, and $E$, $B$ are the electromagnetic one-forms defined in Sec. VIIH. As we will discuss in Sec. VD, the invariants $\mathcal{I}_1$ and $\mathcal{I}_2$, viewed as quadratic forms in $\mathcal{F}$, have a special algebraic structure. More specifically, we will see that from the point of view of the decay estimates of Proposition 10.0.1, the worst possible quadratic terms are absent from $\mathcal{I}_1$ and $\mathcal{I}_2$. This is the one of the fundamental reasons that small-data solutions to the MBI system decay at least as fast as solutions to the linear Maxwell-Maxwell equations.

We now introduce the Maxwell tensor $\mathcal{M}$, a two-form whose Hodge dual $\star \mathcal{M}$ is defined by

$$\star \mathcal{M}^{\mu \nu} \overset{def}{=} \frac{\partial \star \mathcal{L}}{\partial \mathcal{F}_{\mu \nu}} - \frac{\partial \mathcal{L}}{\partial \mathcal{F}^{\mu \nu}}.$$  \hspace{1cm} (71)

To complete the specification of the electromagnetic equations, we postulate that $\mathcal{M}$ is closed:

$$d \mathcal{M} = 0.$$  \hspace{1cm} (72)

Taken together, (68) and (72) are the equations for the theory arising from the Lagrangian $\mathcal{L}$. It is straightforward to verify that (68) and (72) are, respectively, equivalent to

$$\nabla_\mu \mathcal{M}^{\mu \nu} = 0, \quad (v = 0, 1, 2, 3),$$  \hspace{1cm} (73a)

$$\nabla_\mu \mathcal{M}^{\mu \nu} = 0, \quad (v = 0, 1, 2, 3).$$  \hspace{1cm} (73b)

Equations (73a) are sometimes referred to as the Bianchi identities. We furthermore remark that the solutions to (68) and (72) are exactly the stationary points (under closed variations $d \mathcal{F} = 0$ with support contained in compact subsets $\mathcal{C}$) of the action functional

$$\mathcal{A}_\mathcal{C}[\mathcal{F}] \overset{def}{=} \int_{\mathcal{C} \in \mathcal{M}} \star \mathcal{L}(\mathcal{I}_1[\mathcal{F}], \mathcal{I}_2[\mathcal{F}]) \, d\mu_g,$$  \hspace{1cm} (74)

and that (73b) are the Euler-Lagrange equations of $\star \mathcal{L}$. In the above formula, $d\mu_g \overset{def}{=} |\det(g)|^{1/2} d^4x$ is the measure associated with the spacetime volume form (45).

The Euler-Lagrange equations (73b) can be written in the following form:

$$h^{\mu \nu \lambda} \nabla_\mu \mathcal{F}_{\nu \lambda} = 0, \quad (v = 0, 1, 2, 3),$$  \hspace{1cm} (75)

where

$$h^{\mu \nu \lambda} = -\frac{1}{2} \left( \frac{\partial^2 \mathcal{L}}{\partial \mathcal{F}_{\mu \nu} \partial \mathcal{F}_{\lambda \rho}} + \frac{\partial^2 \mathcal{L}}{\partial \mathcal{F}_{\rho \nu} \partial \mathcal{F}_{\lambda \mu}} - \frac{\partial^2 \mathcal{L}}{\partial \mathcal{F}_{\mu \nu} \partial \mathcal{F}_{\lambda \rho}} - \frac{\partial^2 \mathcal{L}}{\partial \mathcal{F}_{\rho \nu} \partial \mathcal{F}_{\lambda \mu}} \right).$$  \hspace{1cm} (76)

The tensorfield $h^{\mu \nu \lambda}$, which has the properties

$$h^{\mu \nu \lambda} = -h^{\mu \nu \lambda},$$  \hspace{1cm} (77a)
\[ h^{\mu\nu\lambda\kappa} = -h_{\mu\nu\lambda\kappa}, \quad (77b) \]

\[ h^{\nu\lambda\mu\kappa} = h_{\mu\nu\lambda\kappa}, \quad (77c) \]

is of fundamental importance in this article. As is explained in Sec. VII, its algebraic and geometric properties are intimately related to the hyperbolic nature of the MBI system. In particular, the symmetry properties (77a)–(77c) are needed to ensure that the canonical stress tensor, which is defined in Sec. VII C, has a lower order divergence.

We state as a lemma the following identities, which will be used for various computations. We leave the proof as an exercise for the reader.

**Lemma 3.1.1 (Basic identities).** The following identities hold:

\[ \frac{\partial |\text{det}(g)|}{\partial g_{\mu\nu}} = |\text{det}(g)|(g^{-1})^{\mu\nu}, \quad (78a) \]

\[ \frac{\partial (g^{-1})^{\mu\lambda}}{\partial g_{\mu\nu}} = -(g^{-1})^\nu{}_{\mu}(g^{-1})^{\lambda\nu}, \quad (78b) \]

\[ \dot{\zeta}^2_{(2)} = |\text{det}(\mathcal{F})||\text{det}(g)|^{-1}, \quad (78c) \]

\[ (g^{-1})^{\nu\lambda} \mathcal{F}_{\mu\kappa} \mathcal{F}_{\nu\lambda} - (g^{-1})^{\nu\lambda} \mathcal{F}_{\mu\kappa} \star \mathcal{F}_{\nu\lambda} = \zeta_{(1)} g_{\mu\nu}, \quad (78d) \]

\[ (g^{-1})^{\nu\lambda} \mathcal{F}_{\mu\kappa} \star \mathcal{F}_{\nu\lambda} = \zeta_{(2)} g_{\mu\nu}, \quad (78e) \]

\[ \frac{\partial \zeta_{(1)}}{\partial g_{\mu\nu}} = -g_{\kappa\lambda} \mathcal{F}_{\mu\kappa} \mathcal{F}^{\nu\lambda}, \quad (78f) \]

\[ \frac{\partial \zeta_{(2)}}{\partial g_{\mu\nu}} = -\frac{1}{2} \zeta_{(2)} (g^{-1})^{\mu\nu}, \quad (78g) \]

\[ \frac{\partial \mathcal{F}^{\mu\nu}}{\partial \mathcal{F}_{\mu\nu}} = \mathcal{F}^{\mu\nu}, \quad (78h) \]

\[ \frac{\partial \mathcal{F}^{\mu\nu}}{\partial \mathcal{F}_{\kappa\lambda}} = \frac{1}{2} \epsilon^{\mu\nu\kappa\lambda}, \quad (78i) \]

\[ \frac{\partial \mathcal{F}_{\kappa\lambda}}{\partial \mathcal{F}_{\mu\nu}} = (g^{-1})^{\mu\kappa} (g^{-1})^{\nu\lambda}, \quad (78j) \]

\[ \mathcal{N}^{\mu\nu} = 2 \frac{\partial \star L}{\partial \zeta_{(1)}} \mathcal{F}^{\mu\nu} + \frac{\partial \star L}{\partial \zeta_{(2)}} \star \mathcal{F}^{\mu\nu}, \quad (78l) \]

\[ h^{\mu\nu\kappa\lambda} = -\frac{\partial \star L}{\partial \zeta_{(1)}} \left[ (g^{-1})^{\mu\kappa} (g^{-1})^{\nu\lambda} - (g^{-1})^{\mu\lambda} (g^{-1})^{\nu\kappa} \right] - \frac{1}{2} \frac{\partial \star L}{\partial \zeta_{(2)}} \epsilon^{\mu\nu\kappa\lambda} \]

\[ -2 \frac{\partial^2 \star L}{\partial \zeta_{(1)}^2} \mathcal{F}^{\mu\nu} \mathcal{F}^{\kappa\lambda} - \frac{\partial^2 \star L}{\partial \zeta_{(1)} \partial \zeta_{(2)}} \left( \mathcal{F}^{\mu\nu} \star \mathcal{F}^{\kappa\lambda} + \star \mathcal{F}^{\mu\nu} \mathcal{F}^{\kappa\lambda} \right) \]

\[ - \frac{1}{2} \frac{\partial^2 \star L}{\partial \zeta_{(2)}^2} \star \mathcal{F}^{\mu\nu} \star \mathcal{F}^{\kappa\lambda}. \quad (78m) \]
B. Derivation of the MBI equations

The Lagrangian for the MBI model is

\[ \mathcal{L}_{\text{MBI}} \overset{\text{def}}{=} \frac{1}{\beta^4} - \frac{1}{\beta^4} (1 + \beta^4 \zeta_1 |F| - \beta^8 \zeta_2 (|F|)^2)^{1/2} = \frac{1}{\beta^4} - \frac{1}{\beta^4} (\det g + \beta^2 F)^{1/2}, \]  

(79)

where \( \beta > 0 \) denotes Born’s “aether” constant. For the remainder of the article, we set \( \beta = 1 \) for simplicity; however, the analysis in the case \( \beta \neq 1 \) easily reduces to the case \( \beta = 1 \) by making change of variable \( \tilde{F} = \beta^2 F \) in the equations. For future use, we introduce the abbreviation

\[ \ell_{\text{MBI}} \overset{\text{def}}{=} (1 + \zeta_1) - \zeta_2 \]

(80)

which implies that

\[ \mathcal{L}_{\text{MBI}} = 1 - \ell_{\text{MBI}}. \]

(81)

Using (78), (80) and (81), we compute that in the MBI model, \( \mathcal{M}^{\mu \nu} \) can be expressed as follows:

\[ \mathcal{M}^{\mu \nu} = -\ell_{\text{MBI}}^{-1} \left( F^{\mu \nu} - \frac{1}{4} F_{\lambda \nu} \star F^{\lambda \mu} \right) = -\ell_{\text{MBI}}^{-1} \left( F^{\mu \nu} - \zeta_2 \star F^{\mu \nu} \right). \]

(82)

Taking the Hodge dual of (82), we have that

\[ \mathcal{M}^{\mu \nu} = \ell_{\text{MBI}}^{-1} \left( \star F^{\mu \nu} + \zeta_2 \star F^{\mu \nu} \right). \]

(83)

From (82), it follows that the Euler-Lagrange equations (73b) for the MBI model are

\[
\begin{align*}
\nabla_\mu F^{\mu \nu} - \frac{1}{4} F^{\mu \nu} \nabla_\mu (F_{\lambda \nu} \star F^{\lambda \mu}) - \zeta_2 \nabla_\mu \star F^{\mu \nu} \\
- \frac{1}{2} \ell_{\text{MBI}}^{-2} \left( F^{\mu \nu} - \frac{1}{4} F_{\lambda \nu} \star F^{\lambda \mu} \right) \nabla_\mu \left( \frac{1}{2} F_{\lambda \nu} \star F^{\lambda \mu} - \frac{1}{16} (F_{\lambda \nu} \star F^{\lambda \mu})^2 \right) = 0.
\end{align*}
\]

(84)

Furthermore, using (78), it is straightforward to compute that the tensorfield \( h^{\mu \nu \kappa \lambda} \) from (76) can be expressed as

\[
2 h^{\mu \nu \kappa \lambda} = \ell_{\text{MBI}}^{-1} \left[ (g^{-1})^{\mu \kappa} (g^{-1})^{\nu \lambda} - (g^{-1})^{\mu \lambda} (g^{-1})^{\nu \kappa} \right] - \ell_{\text{MBI}}^{-3} \left( F^{\mu \nu} \star F^{\kappa \lambda} \right)
\]

(85)

\[
+ \frac{1}{2} \ell_{\text{MBI}}^{-2} \left( F^{\mu \nu} \star F^{\kappa \lambda} + \star F^{\mu \nu} \star F^{\kappa \lambda} \right) - \left( \ell_{\text{MBI}}^{-1} + \zeta_2 \ell_{\text{MBI}}^{-3} \right) \star F^{\mu \nu} \star F^{\kappa \lambda}
\]

\[
- \ell_{\text{MBI}}^{-1} \zeta_2 \ell_{\text{MBI}}^{-3} \epsilon^{\mu \nu \kappa \lambda}.
\]

C. \( H^{\mu \nu \kappa \lambda} \) and the working version of the MBI equations

To simplify the calculations, it is convenient to perform two simple modifications of the tensorfield \( h^{\mu \nu \kappa \lambda} \) from (85), thereby obtaining a new tensorfield \( H^{\mu \nu \kappa \lambda} \); the modifications will not alter the set of solutions to the MBI system. To construct \( H^{\mu \nu \kappa \lambda} \), we first drop the \( \ell_{\text{MBI}}^{-1} \zeta_2 \ell_{\text{MBI}}^{-3} \epsilon^{\mu \nu \kappa \lambda} \) term from (85). This is permissible because its contribution to the Euler-Lagrange equations is \( 0 = \ell_{\text{MBI}}^{-1} \zeta_2 \ell_{\text{MBI}}^{-3} \epsilon^{\mu \nu \kappa \lambda} \nabla_\mu F^{\kappa \lambda} = 2 \ell_{\text{MBI}}^{-1} \zeta_2 \nabla_\mu \star F^{\mu \nu} \), on account of Eq. (73a). Next, we multiply the remaining terms in (85) by \( \ell_{\text{MBI}} \). Furthermore, it is convenient to split \( H^{\mu \nu \kappa \lambda} \) into a main term, which coincides with the tensorfield in the case of the Maxwell-Maxwell equations, and a quadratic error term, which we denote by \( H^{\Delta \mu \nu \kappa \lambda} \). The end result is

\[
H^{\mu \nu \kappa \lambda} \overset{\text{def}}{=} \ell_{\text{MBI}} \left( H^{\mu \nu \kappa \lambda} + \frac{1}{2} \ell_{\text{MBI}}^{-1} \zeta_2 \epsilon^{\mu \nu \kappa \lambda} \right)
\]

\[
= \frac{1}{2} \left[ (g^{-1})^{\mu \kappa} (g^{-1})^{\nu \lambda} - (g^{-1})^{\mu \lambda} (g^{-1})^{\nu \kappa} \right] + H^{\Delta \mu \nu \kappa \lambda},
\]

(86a)
\[ H^{\mu\nu\kappa\lambda}_{\Delta} \equiv -\frac{1}{2} \ell_{(MBI)}^{-2} \mathcal{F}^{\mu\nu} \mathcal{F}^{\kappa\lambda} + \frac{1}{2} \ell_{(MBI)}^{-2} \left( \mathcal{F}^{\mu\nu} \star \mathcal{F}^{\kappa\lambda} + \star \mathcal{F}^{\mu\nu} \mathcal{F}^{\kappa\lambda} \right) \quad (86b) \]

It follows that the system (73a), (73b), and (82) is equivalent to the following version of the MBI system:

\[ \nabla_{\lambda} \mathcal{F}^{\mu\nu} + \nabla_{\mu} \mathcal{F}^{\nu\lambda} + \nabla_{\nu} \mathcal{F}^{\lambda\mu} = 0, \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \quad (87a) \]

\[ H^{\mu\nu\kappa\lambda} \nabla_{\mu} \mathcal{F}^{\nu\kappa\lambda} = 0, \quad (\nu = 0, 1, 2, 3). \quad (87b) \]

### IV. CONFORMAL KILLING FIELDS AND MODIFIED LIE DERIVATIVES

In this section, we recall the definition of conformal Killing fields. This collection of vectorfields, which has the structure of a Lie algebra under the Lie bracket operator \([X, Y] \rightarrow [X, Y]\), generates the conformal symmetries of the spacetime \((M, g)\). We focus on the case of Minkowski spacetime, which has the maximum possible number of generators (15). In particular, we introduce several subsets of the Minkowski conformal Killing fields, each of which will play a role throughout the remainder of the article. More specifically, they appear in the definitions of the norms and energies (see Sec. VIII) that are used during our global existence argument. Finally, for a special collection of Minkowski conformal Killing fields \(Z\), we define modified Lie derivatives \( \hat{L}Z \), which are equal to ordinary Lie derivatives plus a scalar multiple of the identity. This definition is justified by Lemma 6.0.6, which shows that the operator \( \hat{L}Z \) has favorable commutation properties with the linear Maxwell-Maxwell equation \( \left[ (g^{-1})^{\mu\kappa} (g^{-1})^{\nu\lambda} - (g^{-1})^{\mu\lambda} (g^{-1})^{\nu\kappa} \right] \nabla_{\mu} \mathcal{F}^{\kappa\lambda} = 0 \).

**Definition 4.0.1 (Killing fields and conformal Killing fields):** A Killing field of the metric \(g_{\mu\nu}\) is a vectorfield \(X\) such that

\[ (X)_{\pi}^{\mu\nu} = 0, \quad (88) \]

while a conformal Killing field \(X\) satisfies

\[ (X)_{\pi}^{\mu\nu} = \phi_X g_{\mu\nu} \quad (89) \]

for some scalar-valued function \(\phi_X(t, \chi)\). In the above formulas, the deformation tensor \((X)_{\pi}^{\mu\nu}\) is defined in (44).

The conformal Killing fields of the Minkowski metric are generated by the following 15 vectorfields (see, e.g., Ref. 15):

1. the four translations \(T_{(\mu)}\), \((\mu = 0, 1, 2, 3)\),
2. the three rotations \(\Omega_{(jk)}\), \((1 \leq j < k \leq 3)\),
3. the three Lorentz boosts \(\Omega_{(ij)}\), \((j = 1, 2, 3)\),
4. the scaling vectorfield \(S\),
5. the four acceleration vectorfields \(K_{(\mu)}\), \((\mu = 0, 1, 2, 3)\).

Relative to the inertial coordinate system \(\{x^{\mu}\}\), \(\mu = 0, 1, 2, 3\), the above vectorfields can be expressed as

\[ T_{(\mu)} = \partial_{\mu}, \quad (90a) \]

\[ \Omega_{(\mu\nu)} = x_{\mu} \delta_{\nu} - x_{\nu} \delta_{\mu}, \quad (90b) \]

\[ S = x^{\kappa} \partial_{\kappa}, \quad (90c) \]

\[ K_{(\mu)} = -2x_{\mu} S + g_{\kappa\lambda} x^{\kappa} x^{\lambda} \delta_{\mu}. \quad (90d) \]
When working in our fixed inertial coordinate system, we use the notation $T_{(0)} = \partial_t$ interchangeably. In this article, we will primarily make use of the vectorfields in (1) − (4), together with $K_{(0)} + T_{(0)}$, which has the following components relative to the inertial coordinate system:

\begin{equation}
\bar{K}^0 = 1 + t^2 + (x^1)^2 + (x^2)^2 + (x^3)^2, \tag{91a}
\end{equation}

\begin{equation}
\bar{K}^j = 2tx^j, \quad (j = 1, 2, 3). \tag{91b}
\end{equation}

We remark that the translations, rotations, and boosts are Killing fields, while relative to the inertial coordinate system, we have that

\begin{equation}
\bar{c}_Z = \text{const} \text{ from (94a)}. \tag{95}
\end{equation}

It will be convenient for us to work with modified Lie derivatives $\hat{L}_Z$. We note that these are not the same modified Lie derivatives that appear in Refs. 8, 18, and 63.

**Definition 4.0.2:** For each vectorfield $Z \in \mathcal{Z}$, we define the modified Lie derivative $\hat{L}_Z$ by

\begin{equation}
\hat{L}_Z \varphi = \mathcal{L}_Z \varphi + 2c_Z, \tag{95}
\end{equation}

where $c_Z$ denotes the constant from (94a). The crucial feature of the above definition is captured by Lemma 6.0.6 below, which shows that for each $Z \in \mathcal{Z}$, the operator $\hat{L}_Z$ can be commuted through the linear Maxwell-Maxwell equation \[(g^{-1})^{\mu x}(g^{-1})^{\nu x} - (g^{-1})^{\mu x}(g^{-1})^{\nu x} \nabla_\mu F_{\nu x} = 0,\] resulting in the identity \[(g^{-1})^{\mu x}(g^{-1})^{\nu x} - (g^{-1})^{\mu x}(g^{-1})^{\nu x} \nabla_\mu \mathcal{L}_Z F_{\nu x} = 0.\] As is shown in Lemma 9.0.5 and the proof of Lemma 11.0.2, a similar result (involving nonlinear error terms) also holds for the MBI equation (87b).

We now introduce some notation that allows us to compactly express iterated derivatives.

**Definition 4.0.3 (Iterated derivatives):** If $\mathcal{A}$ is one of the sets from (93a)-(93c), then we label the vectorfields in $\mathcal{A}$ by $\mathcal{A} = \{Z^1, \ldots, Z^m\}$, where $m$ denotes the cardinality of $\mathcal{A}$. Let $l = (i_1, \ldots, i_k)$ be a multi-index of length $k$, where each $i_j \in \{1, 2, \ldots, m\}$. We define the following differential operators:

\begin{equation}
\mathcal{L}_\mathcal{A}^l \defeq \mathcal{L}_{Z^{i_1}} \circ \cdots \circ \mathcal{L}_{Z^{i_k}}, \tag{96a}
\end{equation}
\[ \mathcal{L}_Z^I(U V) = \sum_{I_1 + I_2 = I} (\mathcal{L}_Z^{I_1} U)(\mathcal{L}_Z^{I_2} V), \]  

(97)
We are particularly interested in the case that $X$ is a conformal Killing field. Under this assumption, we have the following simple corollary of the lemma.

**Corollary 4.0.3 (Corollary of Proposition 3.3 of Ref. 17):** If $X$ is a conformal Killing field and $\mathcal{F}$ is a two-form, then

$$^*(\mathcal{L}_X \mathcal{F}) = \mathcal{L}_X^* \mathcal{F}.$$  \hspace{1cm} (105)

**Proof:** Corollary 4.0.3 follows from (89) and Lemma 4.0.2.

## V. TENSORIAL DECOMPOSITIONS

In this section, we have three main goals. First, we will recall the *Minkowski null frame* decomposition of tensorfields introduced in Ref. 17. Related to this decomposition is the notion of a *null form* $\mathcal{Q}(\cdot, \cdot)$, which is a quadratic form that acts on a pair of two-forms and that has a special algebraic property: the “worst” possible quadratic combinations, from the point of view of the decay estimates of Proposition 10.0.1, are absent. Since the two invariants $\mathcal{V}_{ij}$ of the Faraday tensor are each multiples of a corresponding null form $\mathcal{Q}_{ij}(\mathcal{F}, \mathcal{F})$, the net effect is that every nonlinear term in the Euler-Lagrange equation (84) of the MBI system can be expressed as functions of null forms in $\mathcal{F}, \nabla \mathcal{F}$. Later in the article, with the help of the additional null structure present in the expression (340) for the divergence of our energy currents, together with Proposition 1, we will be able to prove sharp decay estimates for the null components of solutions $\mathcal{F}$ to the MBI system. Next in this section, we decompose the MBI system relative to the null frame. With the goal of proving the sharpest possible decay estimates, the most important equation is (143b), which shows that the “worst” derivative of the $\alpha$ null component of $\mathcal{F}$ can be expressed in terms of “good” derivatives of other null components of $\mathcal{F}$, plus an $r^{-1}$ weighted linear term, plus cubic error terms. Finally, we introduce electromagnetic decompositions of $\mathcal{F}$ and $\mathcal{M}$, where $\mathcal{M}$ is the Maxwell tensor from (83). These electromagnetic decompositions will be useful for proving various identities and inequalities concerning $\mathcal{F}$, and for expressing the smallness condition in our global existence theorem directly in terms of the data ($\dot{B}, \dot{D}$), which are one-forms inherent to the Cauchy hypersurface $\Sigma_0$.

### A. The Null frame

Before proceeding, we introduce the subsets $C^+_q$, $C^-_s$ of Minkowski spacetime, which will play a role in the sequel.

**Definition 5.1.1:** In our fixed inertial coordinate system $(t, y)$, we define the *outgoing Minkowski null cones* $C^+_q$, and *ingoing Minkowski null cones* $C^-_s$, as follows:

$$C^+_q \overset{\text{def}}{=} \{(\tau, y) \mid |y| - \tau = q\},$$  \hspace{1cm} (106a)

$$C^-_s \overset{\text{def}}{=} \{(\tau, y) \mid |y| + \tau = s\}.$$  \hspace{1cm} (106b)

In the above formulas, $y \overset{\text{def}}{=} (y^1, y^2, y^3)$, and $|y| \overset{\text{def}}{=} \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}$.

A *Minkowski null frame* is a collection of four vectorfields $\mathcal{L}, L, e_1, e_2$ that are defined on $M/0$ and that have the following properties:

- At each point $p \in M/0$, the set $\{\mathcal{L}, L, e_1, e_2\}$ spans $T_p M \cong M$.
- For each $q \in \mathbb{R}^+$, and for all nonzero $p \in C^-_s$, $|\mathcal{L}|_p$ is a future-directed, ingoing null geodesic vectorfield tangent to $C^-_s$.
- For each $q \in \mathbb{R}^+$, and for all nonzero $p \in C^+_q$, $|\mathcal{L}|_p$ is a future-directed, outgoing null geodesic vectorfield tangent to $C^+_q$.
- For all $t \in \mathbb{R}$, for all $r > 0$, $v \overset{\text{def}}{=} |x|$, and all points $p \in S_r, |\mathcal{L}|_p$ and $L_p$ are normal to $S_r$.
- $g(\mathcal{L}, L) = -2$. 


• At each point $p \in S_{r,t}$, $e_1|_p$ and $e_2|_p$ are tangent to $S_{r,t}$.

• $g(e_A, e_B) = \delta_{AB}$.

The orthonormal pair $e_1, e_2$ can be chosen to be “locally smooth” in the following sense: by $\nabla$-parallel transporting an orthonormal pair of $S_{r,t}$—tangent vectors $e_1|_p$, $e_2|_p$ defined at the point $p$, one can obtain a smooth orthonormal $S_{r,t}$—tangent vectorfield pair $e_1, e_2$ that is defined on a neighborhood of $p$. However, it is not possible to extend $e_1, e_2$ to be continuous on the full sphere $S_{r,t}$. One could cover the $S_{r,t}$ with a finite number of orthonormal vectorfield pairs that are each smooth on an open neighborhood, but these pairs would not necessarily agree on the domains of common definition. However, this fact does cause any serious difficulties in our analysis: although some of our estimates are frame-dependent (because, for example, the contraction seminorms of Definition 5.1.2 depend in a mild fashion on the frame), different choices of $e_1, e_2$ lead to equivalent estimates (see, for example, (118a)–(118c) and the statement just after). We also remark that none of our estimates will involve derivatives of the vectorfields $e_1, e_2$.

In formulas (51a) and (51b), we provided the components of $L, L$ relative to our inertial coordinate system. Relative to an arbitrary coordinate system, the aforementioned properties of $L, L, e_1, e_2$ can be expressed as follows:

\begin{equation}
\nabla L = \nabla L = 0, \tag{107a}
\end{equation}

\begin{equation}
\dot{L} = L^s = -2, \tag{107b}
\end{equation}

\begin{equation}
e^\kappa_{AB} \dot{L} = e^\kappa_{AB} L = 0, \tag{107c}
\end{equation}

\begin{equation}
g_{\kappa\lambda} e^\kappa_A e^\lambda_B = \delta_{AB}. \tag{107d}
\end{equation}

Additionally, it follows from (36) and (107b) that $L$ and $L$ commute as vectorfields:

\begin{equation}
[L, L] = 0. \tag{108}
\end{equation}

In the analysis that will follow, we will see that the decay rates of the null components (see Sec. V C) of $F$ will be distinguished according to the kinds of contractions of $F$ taken against $L, L, e_1, e_2$. With these considerations in mind, we introduce the following sets of vectorfields

\begin{equation}
\mathcal{L} \overset{\text{def}}{=} \{ L \}, \quad \mathcal{T} \overset{\text{def}}{=} \{ L, e_1, e_2 \}, \quad \mathcal{U} \overset{\text{def}}{=} \{ L, L, e_1, e_2 \}. \tag{109}
\end{equation}

We will often need to measure the size of the contractions of various tensors and their covariant derivatives against vectors belonging to the sets $\mathcal{L}, \mathcal{T}, \mathcal{U}$. This motivates the next definition.

**Definition 5.1.2 (Frame seminorms):** If $\mathcal{V}, \mathcal{W} \in \{ \mathcal{L}, \mathcal{T}, \mathcal{U} \}$ and $F$ is an arbitrary type $\binom{\gamma}{\delta}$ tensor, then we define the following pointwise seminorms:

\begin{equation}
|F|_{\mathcal{V}\mathcal{W}} \overset{\text{def}}{=} \sum_{\mathcal{V} \in \mathcal{V}, \mathcal{W} \in \mathcal{W}} |V^\kappa W^\lambda F_{\kappa\lambda}|, \tag{110a}
\end{equation}

\begin{equation}
|\nabla F|_{\mathcal{V}\mathcal{W}} \overset{\text{def}}{=} \sum_{U \in \mathcal{U}, \mathcal{V} \in \mathcal{V}, \mathcal{W} \in \mathcal{W}} |V^\kappa W^\lambda U^\gamma \nabla_\gamma F_{\kappa\lambda}|. \tag{110b}
\end{equation}

Observe that if $F$ is a two tensor, then $|F| \approx |F|_{\mathcal{U}\mathcal{U}}$, where $|F|$ is defined in (32).

**Remark 5.1.1:** A different choice of the orthonormal vectorfields $e'_1, e'_2$ would lead to equivalent seminorms. In particular, we would have, for example, that $|F|_{\mathcal{V}\mathcal{W}} \approx |F|_{\mathcal{V}'\mathcal{W}'},$ where $\mathcal{V}', \mathcal{W}'$ denote the subsets (109) corresponding to the null frame $\{ L, L, e'_1, e'_2 \}$, and the constants implicit in $\approx$ can be chosen to be independent of the choice of $e'_1, e'_2$. 
B. Null frame decomposition of a tensorfield

For an arbitrary vectorfield $X$ and frame vector $U \in \mathcal{U}$, we define

$$X_U \overset{\text{def}}{=} X_\kappa U^\kappa, \quad \text{where } X_\mu \overset{\text{def}}{=} g_{\mu \kappa} X^\kappa.$$  \hfill (111)

The components $X_U$ are known as the null components of $X$. In the sequel, we will abbreviate

$$X_A \overset{\text{def}}{=} X_{eA}, \quad \nabla_A \overset{\text{def}}{=} \nabla_{eA}, \quad \text{etc.}$$  \hfill (112)

It follows from (111) that

$$X = X^\kappa \partial_\kappa = X_L L^\kappa + X^A e_A,$$  \hfill (113a)

$$X^L = -\frac{1}{2} X_L, \quad X_L = -\frac{1}{2} X^L, \quad X^A = X_A.$$  \hfill (113b)

Furthermore, it is straightforward to verify that

$$g(X, Y) = X^\kappa Y_\kappa = -\frac{1}{2} X_L Y_L - \frac{1}{2} X_L Y_L + \delta_{AB} X_A Y_B.$$  \hfill (114)

The above null decomposition of a vectorfield generalizes in the obvious way to higher order tensorfields. In Sec. VC, we provide a detailed version of the null decomposition of two-forms $F$, since they are the fundamental unknowns in any classical theory of electromagnetism.

C. The detailed null decomposition of a two-form

Definition 5.3.1 (Null components): Let $g^\nu_\mu$ and $\epsilon^\nu_\mu$ be as defined in (54) and (56). Given any two-form $F$, we define its null components $\alpha, \alpha, \rho, \sigma$ as follows:

$$\alpha_\mu[F] = \alpha_\mu \overset{\text{def}}{=} g^\nu_\mu F_{\nu \lambda} L^\lambda,$$  \hfill (115a)

$$\alpha_\mu[F] = \alpha_\mu \overset{\text{def}}{=} g^\nu_\mu F_{\nu \lambda} L^\lambda,$$  \hfill (115b)

$$\rho[F] = \rho \overset{\text{def}}{=} \frac{1}{2} F_{k\lambda} L^k L^\lambda,$$  \hfill (115c)

$$\sigma[F] = \sigma \overset{\text{def}}{=} \frac{1}{2} \epsilon^{k\lambda} F_{k\lambda}. $$  \hfill (115d)

It is a simple exercise to check that $\alpha, \alpha$ are tangent to the spheres $S_r, r$,

$$\alpha_\kappa L^\kappa = 0, \quad \alpha_\kappa L^\kappa = 0,$$  \hfill (116a)

$$\alpha_\kappa L^\kappa = 0, \quad \alpha_\kappa L^\kappa = 0.$$  \hfill (116b)

Furthermore, relative to the null frame $\{L, L, e_1, e_2\}$, we have that

$$\alpha_A = F_{AL},$$  \hfill (117a)

$$\alpha_A = F_{AL},$$  \hfill (117b)

$$\rho = \frac{1}{2} F_{LL},$$  \hfill (117c)

$$\sigma = F_{12}.$$  \hfill (117d)

In terms of the seminorms introduced in Definition 5.1.2, it follows that

$$|F| \approx |F|_{\mathcal{U}L} \approx |\alpha| + |\alpha| + |\rho| + |\sigma|.$$  \hfill (118a)
\[ |F|_{LT} \approx |\alpha| + |\rho|, \quad (118b) \]
\[ |F|_{TT} \approx |\alpha| + |\sigma|, \quad (118c) \]

where the constants implicit in \( \approx \) can be chosen to be independent of \( e_1, e_2 \).

The null components of \( *F \) can be expressed in terms of the above null components of \( F \). Denoting the null components of \( F \) by \( \odot \alpha, \odot \rho, \odot \sigma \), we leave it as a simple exercise for the reader to check that

\[ \odot \alpha_A = -\alpha_B \not\epsilon_{BA}, \quad (119a) \]
\[ \odot \alpha_A = \alpha_B \not\epsilon_{BA}, \quad (119b) \]
\[ \odot \rho = \sigma, \quad (119c) \]
\[ \odot \sigma = -\rho. \quad (119d) \]

Above, we have used the symbol \( \odot \) in order to avoid confusion with the Hodge dual operator and its commutation properties with the null decomposition; i.e., it is not true that \( \odot(\alpha[F]) = \alpha[\odot F] \).

### D. Null forms

**Definition 5.4.1 (Null forms):** Let \( F, G \) be any pair of two-forms. We define the null forms \( Q_i(\cdot, \cdot) \) as follows:

\[ Q_i(F, G) \overset{\text{def}}{=} F^{\lambda \rho} G_{\lambda \rho}, \quad (120a) \]

\[ Q_2(F, G) \overset{\text{def}}{=} \star F^{\lambda \rho} G_{\lambda \rho}. \quad (120b) \]

It is easy to check that \( Q_i(F, G) = Q_{i}(G, F) \) for \( i = 1, 2 \).

**Remark 5.4.1:** Observe that the two invariants (70a) and (70b) of the Faraday tensor \( F \) are

\[ \mathcal{I}_i = \frac{1}{2} Q_i(F, F), \quad (121) \]

The next lemma describes the fundamental algebraic properties of null forms, i.e., the absence of the “worst possible” quadratic terms.

**Lemma 5.4.1 (Absence of the worst possible quadratic terms):** Let \( F, G \) be a pair of two-forms, let \( Q_i(\cdot, \cdot) \) be one the null forms defined in (120a) and (120b), and let \( |\cdot|_V \) be the seminorms defined in (110a). Then for \( i = 1, 2 \), the following pointwise estimate holds

\[ |Q_i(F, G)| \lesssim |F||G|_{LT} + |F|_{TT}|G|_{TT}. \quad (122) \]

**Proof:** Let \( \alpha[F], \alpha[G], \rho[F], \rho[G] \) denote the null components of \( F \) and \( G \), respectively. Then using the relations (119a)–(119d) (for the case \( i = 2 \)), we compute that

\[ F^{\lambda \rho} G_{\lambda \rho} = -\delta_{AB} \alpha_A[G] \alpha_B[F] - \delta_{AB} \alpha_B[G] \alpha_A[F] - 2 \rho[F] \rho[G] + 2 \sigma[F] \sigma[G], \quad (123) \]

\[ \star F^{\lambda \rho} G_{\lambda \rho} = \not\epsilon_{AB} \alpha_A[G] \alpha_B[F] + \not\epsilon_{AB} \alpha_B[G] \alpha_A[F] - 2 \sigma[F] \rho[G] - 2 \rho[F] \sigma[G], \quad (124) \]

from which (122) follows. \( \Box \)

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E. Intrinsic divergence and curl, and the cross product

In this section, we recall the definitions of the intrinsic divergence and curl of vector fields $U$ that are tangent to the submanifolds $\Sigma_t$ and $S_{r,t}$.

**Definition 5.5.1 (Divergence and curl):** If $U$ is a vector field tangent to $\Sigma_t$, then its intrinsic divergence and curl are defined relative to an arbitrary spacetime coordinate system as follows:

$$\text{div } U \overset{\text{def}}{=} \nabla_\kappa U^\kappa = g^\lambda_\kappa \nabla_\lambda U_\kappa,$$

$$\text{(curl } U)^\nu \overset{\text{def}}{=} \varepsilon^{\nu \kappa \lambda} \nabla_\kappa U_\lambda,$$

where $g_{\mu\nu}$ and $\varepsilon_{\nu\kappa\lambda}$ are defined in (53) and (55). Relative to the Euclidean coordinate system $\bar{x} = (x^1, x^2, x^3)$ on $\Sigma_t$, we have that

$$\text{div } U = \nabla_a U^a,$$

$$\text{(curl } U)^j \overset{\text{def}}{=} \varepsilon^{j \alpha \beta} \nabla_\alpha U_\beta,$$

If $U$ is a vector field tangent to the spheres $S_{r,t}$, then its intrinsic divergence and curl are defined to be the following scalar quantities:

$$\text{div } U \overset{\text{def}}{=} \nabla_\kappa U^\kappa = g^\lambda_\kappa \nabla_\lambda U_\kappa,$$

$$\text{(curl } U)^\nu \overset{\text{def}}{=} \varepsilon^{\nu \kappa \lambda} \nabla_\kappa U_\lambda,$$

where $\bar{g}_{\mu\nu}$ and $\bar{\varepsilon}_{\mu\nu}$ are defined in (54) and (56). Relative to a null frame, we have that

$$\text{div } U = \delta_{AB} \nabla_A U_B,$$

$$\text{(curl } U)^j \overset{\text{def}}{=} \varepsilon^{j \alpha \beta} \nabla_\alpha U_\beta.$$

Note that in the above definitions, contractions against the frame vectors $e_1, e_2$ are taken after covariant differentiation; e.g., $\nabla_\alpha U_\beta \overset{\text{def}}{=} e^\alpha_\kappa e^\kappa_\beta \nabla_\kappa U_\beta$.

**Definition 5.5.2 (Cross product):** If $U$ and $V$ are vectors tangent to $\Sigma_t$, then we define their cross product, which is also tangent to $\Sigma_t$, as follows:

$$(U \times V)^j \overset{\text{def}}{=} \varepsilon^{j \alpha \beta} U^\alpha V^\beta.$$

F. The null components of $\overline{K}$ and $\nabla \overline{K}$

In this section, we provide the null components of the vector field $\overline{K}$, which is defined in (91a), and its covariant derivative $\nabla \overline{K}$. $\overline{K}$ is central to our global existence argument because it is a fundamental ingredient in the energies we construct; see (279) and (281). We do not provide any proofs in this section, but instead leave the simple computations as an exercise for the reader.

We first note that by (52a) and (52b), $\overline{K}$ can be expressed as follows:

$$\overline{K} = \frac{1}{2} \left\{ (1 + s^2) L + (1 + q^2) L \right\}.$$

From (107c), (107d), and (134), it easily follows that

$$\overline{K}_L = -(1 + q^2),$$

$$\overline{K}_L^{' \prime} = -(1 + s^2),$$

$$\overline{K}_L^{' \prime} = -(1 + q^2).$$
Finally, we compute that

$$\nabla L K L \equiv L^\kappa L^\lambda \nabla \kappa K \lambda = -4s,$$

(136a)

$$\nabla L K L \equiv L^\kappa L^\lambda \nabla \kappa K \lambda = 4q,$$

(136b)

$$\nabla A K B \equiv e^\kappa A e^\lambda B \nabla \kappa K \lambda = 2t \delta_{AB},$$

(136c)

$$\nabla L K L \equiv L^\kappa L^\lambda \nabla \kappa K \lambda = 0,$$

(136d)

$$\nabla L K A \equiv L^\kappa e^\lambda A \nabla \kappa K \lambda = 0,$$

(136e)

$$\nabla L K L \equiv L^\kappa L^\lambda \nabla \kappa K \lambda = 0,$$

(136f)

$$\nabla L K A \equiv L^\kappa e^\lambda A \nabla \kappa K \lambda = 0,$$

(136g)

$$\nabla A K L \equiv e^\kappa A L^\lambda \nabla \kappa K \lambda = 0,$$

(136h)

$$\nabla A K L \equiv e^\kappa A L^\lambda \nabla \kappa K \lambda = 0.$$

(136i)

G. The null decomposition of the MBI system

In this section, we decompose the MBI system into equations for the null components of \( F \).

We begin by noting that straightforward computations yield that the MBI system (1a)–(2) can be expressed in the following equivalent form:

$$\nabla \lambda F_{\mu\nu} + \nabla \mu F_{\nu\lambda} + \nabla \nu F_{\lambda\mu} = 0,$$

(137a)

$$\nabla A F_{\mu\nu} + \nabla \mu^* F_{\nu A} + \nabla \nu^* F_{A\mu}$$

(137b)

$$- \frac{1}{2} \ell^{-2} (MBI) \left[ (\nabla \lambda \hat{\gamma}_1 - 2\hat{\gamma}_2 \nabla \lambda \hat{\gamma}_2) F_{\mu\nu} + (\nabla \mu \hat{\gamma}_1 - 2\hat{\gamma}_2 \nabla \mu \hat{\gamma}_2) F_{\nu\lambda} + (\nabla \nu \hat{\gamma}_1 - 2\hat{\gamma}_2 \nabla \nu \hat{\gamma}_2) F_{\lambda\mu} \right]$$

$$- \frac{1}{2} \ell^{-2} (MBI) \left[ (\nabla \lambda \hat{\gamma}_2 - 2\hat{\gamma}_1 \nabla \lambda \hat{\gamma}_1) F_{\mu\nu} + (\nabla \mu \hat{\gamma}_2 - 2\hat{\gamma}_1 \nabla \mu \hat{\gamma}_1) F_{\nu\lambda} + (\nabla \nu \hat{\gamma}_2 - 2\hat{\gamma}_1 \nabla \nu \hat{\gamma}_1) F_{\lambda\mu} \right]$$

$$+ (\nabla \lambda \hat{\gamma}_2) F_{\mu\nu} + (\nabla \mu \hat{\gamma}_2) F_{\nu\lambda} + (\nabla \nu \hat{\gamma}_2) F_{\lambda\mu} = 0.$$
Furthermore, if $U$ is a type $\left(\begin{array}{c} 0 \\ m \end{array}\right)$ tensor field, and $X_{(i)}$, $(1 \leq i \leq m)$, and $Y$ are vector fields, then by (38), we have that

$$\nabla_Y U(X_{(1)}, X_{(2)}, \ldots, X_{(m)}) = (\nabla_Y U)(X_{(1)}, X_{(2)}, \ldots, X_{(m)}) + U(\nabla_Y X_{(1)}, X_{(2)}, \ldots, X_{(m)})$$

$$+ \cdots + U(X_{(1)}, X_{(2)}, \ldots, \nabla_Y X_{(m)}). \tag{140}$$

Similarly, if $U$ is tangent to the spheres $S_{r,1}$, and $e_{B_{(1)}}, \ldots, e_{B_{(m)}} \in \{e_1, e_2\}$, then

$$\nabla_A U(e_{B_{(1)}}, e_{B_{(2)}}, \ldots, e_{B_{(m)}}) \text{ def } = \nabla_A U(e_{B_{(1)}}, e_{B_{(2)}}, \ldots, e_{B_{(m)}})$$

$$+ \cdots + U(e_{B_{(1)}}, e_{B_{(2)}}, \ldots, \nabla_A e_{B_{(m)}}). \tag{141}$$

Applying (140) and (141) to $F$, and using (119a)–(119d), (138) and (139), we compute that the identities contained in the following lemma hold.

**Lemma 5.7.1 (Relations between $\nabla$ and $\nabla_A$):** (Page 16 of Ref. 17). Let $F$ be a two-form, and let $\alpha, \alpha, \rho, \sigma$ be its null components as defined in Definition 5.3.1. Then the following identities hold:

$$\nabla_A F_{BL} = \nabla_A \alpha_B - r^{-1}(\rho \delta_{AB} + \alpha \sigma_{AB}), \tag{142a}$$

$$\nabla_A F_{BL} = \nabla_A \alpha_B - r^{-1}(\rho \delta_{AB} - \alpha \sigma_{AB}), \tag{142b}$$

$$\nabla_A^* F_{BL} = -\sigma_{CB} \nabla_A \alpha_C - r^{-1}(\sigma \delta_{AB} - \rho \sigma_{AB}), \tag{142c}$$

$$\nabla_A^* F_{BL} = \sigma_{CB} \nabla_A \alpha_C - r^{-1}(\sigma \delta_{AB} + \rho \sigma_{AB}), \tag{142d}$$

$$\frac{1}{2} \nabla_A F_{LL} = \nabla_A \rho + \frac{1}{2} r^{-1} (\alpha_A + \alpha_A), \tag{142e}$$

$$\frac{1}{2} \nabla_A^* F_{LL} = \nabla_A \sigma + \frac{1}{2} r^{-1} (-\sigma_{BA} \alpha_B + \sigma_{BA} \alpha_B). \tag{142f}$$

$$\nabla_A F_{BC} = \sigma_{BC} \left[ \nabla_A \sigma + \frac{1}{2} r^{-1} (-\sigma_{DA} \alpha_D + \sigma_{DA} \alpha_D) \right]. \tag{142g}$$

Note that in all of the above expressions, contractions are taken after differentiating: e.g., $\nabla_A F_{BL} \text{ def } = e^A_{(\underline{\underline{\mu}})} e^B_{(\underline{\underline{\nu}})} e^C_{(\underline{\underline{\lambda}})} \nabla_{\underline{\underline{\mu}}} F_{\underline{\underline{\lambda}}} \alpha_{\underline{\underline{\nu}}} \sigma_{\underline{\underline{\lambda}}}.$

We now derive the null decomposition of the MBI system. We are mainly interested in Eq. (143b), which plays a fundamental role in our proof of Lemma 10.0.5.

**Lemma 5.7.2 (Minkowskian null decomposition of the MBI system):** The MBI system can be decomposed into principal linear terms and “cubic error terms” as follows, relative to a Minkowski...
null frame:

\[
\nabla_L \alpha_A + r^{-1} \alpha_A + \nabla A \rho - \varphi_{AB} \nabla B \sigma \\
- \frac{1}{4} \epsilon_{(MBI)}^2 \left( (\nabla_L \hat{\gamma}_1) - 2 \hat{\gamma}_2 \nabla_L \hat{\gamma}_2) \alpha_A - 2 \varphi_{AB} (\nabla_B \hat{\gamma}_1) - 2 \hat{\gamma}_2 (\nabla_B \hat{\gamma}_2) \sigma \right) \\
+ (\nabla_L \hat{\gamma}_1) - 2 \hat{\gamma}_2 (\nabla_L \hat{\gamma}_2) \varphi_{AB} \alpha_B - 2 \varphi_{AB} (\nabla_B \hat{\gamma}_1) - 2 \hat{\gamma}_2 (\nabla_B \hat{\gamma}_2) \rho \\
- (\nabla_L \hat{\gamma}_1) - 2 \hat{\gamma}_2 (\nabla_L \hat{\gamma}_2) \varphi_{AB} \xi_B \\
+ \frac{1}{2} \left( (\nabla_L \hat{\gamma}_2) \varphi_{AB} \alpha_B - 2 \varphi_{AB} (\nabla_B \hat{\gamma}_2) \rho - (\nabla_L \hat{\gamma}_2) \varphi_{AB} \xi_B \right) = 0, \tag{143a}
\]

\[
\nabla_L \alpha_A - r^{-1} \alpha_A - \nabla B \rho - \varphi_{AB} \nabla B \sigma \\
- \frac{1}{4} \epsilon_{(MBI)}^2 \left( (\nabla_L \hat{\gamma}_1) - 2 \hat{\gamma}_2 \nabla_L \hat{\gamma}_2) \alpha_A - 2 \varphi_{AB} (\nabla_B \hat{\gamma}_1) - 2 \hat{\gamma}_2 (\nabla_B \hat{\gamma}_2) \sigma \right) \\
+ (\nabla_L \hat{\gamma}_1) - 2 \hat{\gamma}_2 (\nabla_L \hat{\gamma}_2) \varphi_{AB} \alpha_B - 2 \varphi_{AB} (\nabla_B \hat{\gamma}_1) - 2 \hat{\gamma}_2 (\nabla_B \hat{\gamma}_2) \rho \\
- (\nabla_L \hat{\gamma}_1) - 2 \hat{\gamma}_2 (\nabla_L \hat{\gamma}_2) \varphi_{AB} \xi_B \\
+ \frac{1}{2} \left( (\nabla_L \hat{\gamma}_2) \varphi_{AB} \alpha_B - 2 \varphi_{AB} (\nabla_B \hat{\gamma}_2) \rho - (\nabla_L \hat{\gamma}_2) \varphi_{AB} \xi_B \right) = 0, \tag{143b}
\]

\[-\text{div} \alpha - \nabla_L \rho + 2 r^{-1} \rho - \frac{1}{2} \epsilon_{(MBI)}^2 \left( (\nabla_L \hat{\gamma}_1) - 2 \hat{\gamma}_2 \nabla_L \hat{\gamma}_2) \rho - \delta_{AB} (\nabla_A \hat{\gamma}_1) - 2 \hat{\gamma}_2 (\nabla_A \hat{\gamma}_2) \alpha_B \right) \\
- \frac{1}{2} \epsilon_{(MBI)}^2 \hat{\gamma}_1 \left( (\nabla_L \hat{\gamma}_1) - 2 \hat{\gamma}_2 \nabla_L \hat{\gamma}_2) \sigma + \varphi_{AB} (\nabla_A \hat{\gamma}_1) - 2 \hat{\gamma}_2 (\nabla_A \hat{\gamma}_2) \xi_B \right) \\
+ \left( (\nabla_L \hat{\gamma}_2) \sigma + \varphi_{AB} (\nabla_A \hat{\gamma}_2) \xi_B \right) = 0, \tag{143c}
\]

\[\text{curl} \alpha + \nabla_L \sigma - 2 r^{-1} \sigma = 0, \tag{143d}\]

\[\text{div} \alpha - \nabla_L \rho + 2 r^{-1} \rho - \frac{1}{2} \epsilon_{(MBI)}^2 \left( (\nabla_L \hat{\gamma}_1) - 2 \hat{\gamma}_2 \nabla_L \hat{\gamma}_2) \rho + \delta_{AB} (\nabla_A \hat{\gamma}_1) - 2 \hat{\gamma}_2 (\nabla_A \hat{\gamma}_2) \alpha_B \right) \\
- \frac{1}{2} \epsilon_{(MBI)}^2 \hat{\gamma}_1 \left( (\nabla_L \hat{\gamma}_1) - 2 \hat{\gamma}_2 \nabla_L \hat{\gamma}_2) \sigma + \varphi_{AB} (\nabla_A \hat{\gamma}_1) - 2 \hat{\gamma}_2 (\nabla_A \hat{\gamma}_2) \xi_B \right) \\
+ \left( (\nabla_L \hat{\gamma}_2) \sigma + \varphi_{AB} (\nabla_A \hat{\gamma}_2) \xi_B \right) = 0, \tag{143e}\]

\[\text{curl} \alpha + \nabla_L \sigma + 2 r^{-1} \sigma = 0, \tag{143f}\]

where the differential operators \text{div} and \text{curl} are defined in (131) and (132).
Remark 5.7.1: If we discard the nonlinear terms, then the resulting system is the null decomposition of the Maxwell-Maxwell system.

Proof: We contract the vectors $L^\lambda L^\mu e^\nu_A$ against Eq. (137a), and then $\phi^\nu_B L^\lambda L^\mu e^\nu_B$ against Eq. (137b). Adding the resulting equations and using (142a)–(142g) gives (143a), while subtracting the resulting equations and using (142a)–(142g) gives (143b). Equations (143c) and (143d) follow from contracting $L^\lambda e^\nu_A e^\nu_B$ against (137b) and (137a), respectively, and using (142a)–(142g). Similarly, (143e) and (143f) follow from contracting $L^\lambda e^\nu_A e^\nu_B$ against (137b) and (137a), respectively, and using (142a)–(142g).

H. The electromagnetic decompositions of $\mathcal{F}$ and $\mathcal{M}$

In this section, we provide the familiar decomposition of the Faraday tensor $\mathcal{F}$ into the electric field $E_\mu$ and the magnetic induction $B_\mu$. We also introduce a related decomposition of the Maxwell tensor $\mathcal{M}$ into the electric displacement $D_\mu$ and the magnetic field $H_\mu$. For computational purposes, it will also be convenient to introduce two additional one-forms, $\mathcal{E}_\mu$ and $\mathcal{B}_\mu$, which are related to $E_\mu$ and $B_\mu$.

The decompositions require the introduction of $i$, the interior product operator, the action of which on $\mathcal{F}$ is defined by the requirement that the following relation should hold for all vectors $X, Y$:

$$i_X \mathcal{F}(Y) = \mathcal{F}(Y, X) = \mathcal{F}_{\kappa\lambda} Y^\kappa X^\lambda.$$  

(144)

In an arbitrary coordinate system, $i_X \mathcal{F}$ can be expressed as

$$(i_X \mathcal{F})_\mu \overset{\text{def}}{=} \mathcal{F}_{\mu\kappa} X^\kappa.$$  

(145)

Definition 5.8.1 (The electromagnetic components of $\mathcal{F}$): In terms of the Faraday tensor $\mathcal{F}$ and the Maxwell tensor $\mathcal{M}$ [an expression for $\mathcal{M}$ in the MBI system is given in (83)], the one-forms $E, B, D, H, \mathcal{E},$ and $\mathcal{B}$ are defined as follows:

$$E \overset{\text{def}}{=} i_{T(0)} \mathcal{F}, \quad B \overset{\text{def}}{=} -i_{T(0)} \mathcal{F}, \quad D \overset{\text{def}}{=} -i_{T(0)} \mathcal{M}, \quad H \overset{\text{def}}{=} -\iota_{T(0)} \mathcal{M},$$

(146a)

$$\mathcal{E} \overset{\text{def}}{=} i_S \mathcal{F}, \quad \mathcal{B} \overset{\text{def}}{=} i_S \mathcal{F},$$

(146b)

where $T(0)$ and $S$ are the time translation and scaling vectorfields defined in (90a) and (90c).

In components relative to the inertial coordinate system \( \{ x^\mu \}_\mu = 0, 1, 2, 3 \), we have that

$$E_\mu = \mathcal{F}_{\mu 0}, \quad B_\mu = -\mathcal{F}_{\mu 0}, \quad D_\mu = -\mathcal{M}_{\mu 0}, \quad H_\mu = -\mathcal{M}_{\mu 0}.$$  

(147a)

$$\mathcal{E}_\mu = x^\kappa \mathcal{F}_{\mu \kappa}, \quad \mathcal{B}_\mu = x^\kappa \mathcal{F}_{\mu \kappa}.$$  

(147b)

The antisymmetry of $\mathcal{F}, \mathcal{M}$ implies that $E, B, D,$ and $H$ are tangent to the hyperplane $\Sigma_t$. In the inertial coordinate system, this is equivalent to $E_0 = B_0 = D_0 = H_0 = 0$. We may therefore view these four quantities as one-forms that are intrinsic to $\Sigma_t$.

Remark 5.8.1: Our definition of $B$ coincides with the one commonly found in the physics literature, but it has the opposite sign convention of the definition given in Ref. 17.

The identity

$$g_{\kappa\lambda} X^\kappa X^\lambda \mathcal{F}_{\mu \nu} = (i_X \mathcal{F})_\mu X_\nu - (i_X \mathcal{F})_\nu X_\mu + X^\kappa (i_X \mathcal{F})^\kappa \epsilon_{\kappa\lambda\mu\nu}$$  

(148)

shows that $\mathcal{F}$ is completely determined by $X_\kappa$, $i_X \mathcal{F}$, and $i_X \mathcal{F}$ whenever $g_{\kappa\lambda} X^\kappa X^\lambda \neq 0$. In particular, $E$ and $B$ completely determine $\mathcal{F}$, and it can be checked that in the inertial coordinate system,

$$\mathcal{F}_{j0} = E_j, \quad \mathcal{F}_{jk} = \varepsilon_{jkl} B^l, \quad B_j = \frac{1}{2} F^{ab} \mathcal{F}_{ab},$$  

(149a)
and
\[ M_{j0} = -H_j, \quad M_{jk} = \epsilon_{ijk} D^i, \quad D_j = \frac{1}{2} \epsilon_j^{ab} M_{ab}. \]  

(149b)

Now in linear Maxwell-Maxwell theory, the relations \( E = D, \ B = H \) hold. In contrast, in the case of the MBI system, tedious computations lead to the following relations.

**Lemma 5.8.1 (Relations between E, D, B, and H for the MBI system):** (Refs. 2 and 30).

\[ D = \frac{E + (E_a B^a) B}{(1 + |B|^2 - |E|^2 - (E_a B^a)^2)^{1/2}}, \]  

(150a)

\[ H = \frac{B - (E_a B^a) E}{(1 + |B|^2 - |E|^2 - (E_a B^a)^2)^{1/2}}, \]  

(150b)

Furthermore, we have that

\[ E = \frac{D + B \times (D \times B)}{(1 + |B|^2 + |D|^2 + |D \times B|^2)^{1/2}}, \]  

(151a)

\[ H = \frac{B - D \times (D \times B)}{(1 + |B|^2 + |D|^2 + |D \times B|^2)^{1/2}}. \]  

(151b)

In the above formulas, \( \times \) denotes the usual intrinsic cross product on \( \Sigma_t \); see (133).

We now decompose the MBI equations into constraint and evolution equations for the one-forms \( B \) and \( D \). This decomposition will play a role in connecting the smallness of the norm \( \| F(0) \|_{L^2; N} \) to a smallness condition involving only the data \( (\hat{B}, \hat{D}) \) and their tangential derivatives.

**Proposition 5.8.2 (The MBI system in terms of B and D):** The MBI equations (87a) and (87b) are equivalent to the following constraint and evolution equations for the one-forms \( B \) and \( D \):

**Constraint equations**
\[ \text{div} \, B = 0, \]  

(152a)

\[ \text{div} \, D = 0. \]  

(152b)

**Evolution equations**
\[ \partial_t B = -\text{curl} \left\{ \frac{D + B \times (D \times B)}{(1 + |B|^2 + |D|^2 + |D \times B|^2)^{1/2}} \right\}, \]  

(153a)

\[ \partial_t D = \text{curl} \left\{ \frac{B - D \times (D \times B)}{(1 + |B|^2 + |D|^2 + |D \times B|^2)^{1/2}} \right\}. \]  

(153b)

**Remark 5.8.2:** Regarding the terminology “constraint equations” used above, we make the following remark: it follows that if (152a) and (152b) are satisfied along \( \Sigma_0 \), and if \( B, D \) are classical solutions to the evolution equations (153a) and (153b) existing on the slab \([T_-, T_+] \times \mathbb{R}^3\), then (152b) and (152a) are satisfied in the same slab; i.e., the well-known identity \( \text{div} \circ \text{curl} = 0 \) implies that \( \nabla_t \text{div} \, B = \nabla_t \text{div} \, D = 0 \).

**Remark 5.8.3:** Using definition (80), the relations (70a), (70b), and (151a), and performing some tedious calculations, we compute that

\[ \ell^2_{(MBI)} = \frac{(1 + |B|^2)^2}{1 + |B|^2 + |D|^2 + |D \times B|^2}. \]  

(154)
Consequently, as shown by Propositions 7.4.4 and 12.0.1, the regime of hyperbolicity for the MBI system has a particularly nice interpretation in terms of the state-space variables \((B, D)\): the MBI equations are well defined and hyperbolic for all finite values of \((B, D)\).

**Proof of Proposition 5.8.2:** Working in our inertial coordinate system, we set \(\nu = 0\) in Eqs. (73a) and (73b) and use (147a) to deduce (152a) and (152b).

Setting \(\lambda = 0, \mu = a, \nu = b\) in (137a), then contracting against \(\epsilon^{ab}_{\ j}\) and using (149a), we deduce that
\[
\partial_t B = -\text{curl} \ E. \tag{155a}
\]

Similarly, we use an equivalent \((\text{modulo equation (137a)})\) version of (137b), namely, \(\nabla_\lambda M_{\mu\nu} + \nabla_\mu M_{\nu\lambda} + \nabla_\nu M_{\lambda\mu} = 0\), set \(\lambda = 0, \mu = a, \nu = b\), contract against \(\epsilon^{ab}_{\ j}\), and use (149b) to deduce that
\[
\partial_t D = -\text{curl} \ H. \tag{155b}
\]

Now from (151a), (151b), (155a), and (155b), it follows that \(B\) and \(D\) satisfy (153a) and (153b).

We have thus shown that (87a) and (87b) imply (152a) and (152b) plus (153a) and (153b). The reverse implication can be shown using similar arguments.

\(\Box\)

**VI. COMMUTATION LEMMAS**

In this section, we prove some basic commutation lemmas that will be used in the following sections.

**Lemma 6.0.3 (An expression for \([\nabla_X, \nabla_Y]\):** Let \(X\) and \(Y\) be vectorfields, let \(U\) be any tensorfield, and let \(\nabla\) denote the Levi-Civita connection corresponding to the Minkowski spacetime metric. Then
\[
[\nabla_X, \nabla_Y]U = \nabla_{[X,Y]}U. \tag{156}
\]

**Proof:** Simply use (41) and the fact that in Minkowski spacetime, \(R(X, Y)Z \equiv 0\). \(\Box\)

**Lemma 6.0.4 (Estimates for the covariant derivatives of vectors in \(O\) and \(Z\)):** Let \(O\) and \(Z\) be the subsets of Minkowski conformal Killing fields defined in (93b) and (93c). Then for any \(Z \in Z\), we have that
\[
|Z| \lesssim s, \quad |\nabla Z| \lesssim 1, \quad |\nabla(2)Z| = 0. \tag{157}
\]

Furthermore, for any \(O \in O\), we have that
\[
|O| \lesssim r, \quad |\nabla O| = \text{const}, \quad |\nabla(2)O| = 0. \tag{158}
\]

**Proof:** The simple computations are easily performed in the inertial coordinate system. \(\Box\)

**Lemma 6.0.5 (An expression for \([\nabla, \hat{L}_I^Z]\):** Let \(\nabla\) denote the Levi-Civita connection corresponding to the Minkowski metric, and let \(I\) denote a multi-index for the set \(Z\) of Minkowski conformal Killing fields. Let \(\hat{L}_I^Z\) be the iterated modified Lie derivative from Definitions 4.0.2 and 4.0.3. Then
\[
[\nabla, \hat{L}_I^Z] = 0, \quad [\nabla, \hat{L}_I^Z] = 0. \tag{159}
\]

In an arbitrary coordinate system, Eqs. (159) are equivalent to the following relations, which hold for all type \(\binom{n}{m}\) tensorfields \(U\),
\[
\nabla_\mu \left( \hat{L}_I^Z U \right)_{\mu_1...\mu_n}^{\nu_1...\nu_m} = \hat{L}_I^Z \left[ \nabla_\mu U \right]_{\mu_1...\mu_n}^{\nu_1...\nu_m}, \tag{160}
\]
\[
\nabla_\mu \left( \hat{L}_I^Z U \right)_{\mu_1...\mu_n}^{\nu_1...\nu_m} = \hat{L}_I^Z \left[ \nabla_\mu U \right]_{\mu_1...\mu_n}^{\nu_1...\nu_m}.
\]

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Lemma 6.0.7 (Commutators of differential operators involving \(L, L_{\hat{Z}}, O\)): If \(O \in \mathcal{O}\), then the vectorfields, \(L, L_{\hat{Z}}, O\) mutually commute:

\[
[L, L] = 0, \quad [L, L_{\hat{Z}}] = 0, \quad [L, O] = 0, \quad [L_{\hat{Z}}, L] = 0, \quad [L_{\hat{Z}}, L_{\hat{Z}}] = 0, \quad [L_{\hat{Z}}, O] = 0, \quad [L_{\hat{Z}}, S] = 0, \quad [S, L_{\hat{Z}}] = 0.
\]

Furthermore, with \(q \overset{\text{def}}{=} r - t\), \(s \overset{\text{def}}{=} r + t\) denoting the null coordinates, and \(S \overset{\text{def}}{=} x^\kappa \partial_\kappa\) denoting the scaling vectorfield, the following differential operator commutation relations hold when applied to arbitrary tensorfields:

\[
[\nabla_L, \nabla_L] = 0, \quad [q \nabla_L, s \nabla_L] = 0, \quad [\nabla_L, \nabla_O] = 0, \quad [\nabla_L, \nabla_O] = 0, \quad [\nabla_L, L_O] = 0, \quad [\nabla_L, L_O] = 0, \quad [q \nabla_L, L_S] = 0, \quad [s \nabla_L, L_S] = 0.
\]

Proof: (164) follows from simple computations. (165a) then follows from (166) and (164), and the fact that \(\nabla_L S = \nabla_L g = 0\). (165b) also follows from (166) and (164). (165c) follows from (42), (158), and (165b). (165d) follows from (165a), the identity \(2S = S_L - g_{L_S}\), and the fact that by (42), \(L_S U = \nabla_S U + (m - n)U\) for a type \((m, n)\) tensorfield \(U\).

Lemma 6.0.8 (Vanishing derivatives): Let \(O \in \mathcal{O}\), and let \(\epsilon_{\kappa\lambda\mu\nu}\), \(\mathfrak{g}_{\mu\nu}\), and \(\mathfrak{f}_{\mu\nu}\) be as defined in (45), (54), and (56), respectively. Then

\[
\nabla_L \mathfrak{g}_{\mu\nu} = \nabla_L \mathfrak{f}_{\mu\nu} = \nabla_L \mathfrak{f}_{\mu\nu} = 0, \quad \nabla_L \mathfrak{f}_{\mu\nu} = 0, \quad \nabla_L \mathfrak{f}_{\mu\nu} = 0, \quad \nabla_L \mathfrak{f}_{\mu\nu} = 0.
\]

Proof: The relations (159) and (160) can be shown via induction in \(|l|\) using (42) and the fact that \(\nabla_{(s)} Z = 0\) [i.e., (157)].
\[ \mathcal{L}_O g_{\mu\nu} = 0, \quad (166e) \]
\[ \mathcal{L}_O \mathcal{X}_{\mu\nu} = 0. \quad (166f) \]

**Proof:** The relation (166a) follows from definition (39), (54), (107a), and (107b). Using also (47), (166b) follows similarly. (166c) follows from (47), the formula (60), and the fact that the null second fundamental forms of the spheres \( S_r \) are tangent to \( S_r \), i.e., Lemma 2.4.1. (166d) follows from (102) and the fact that \( \pi^0 \rightarrow 0 \) (i.e., \( O \) is a Killing field). (166e) and (166f) follow from definitions (45) and (54), (164), and (166d).

**Corollary 6.0.9** (\( \nabla_L, \nabla_L, \) and \( \mathcal{L}_O \) commute with the null decomposition): Let \( \mathcal{F} \) be a two-form and let \( g, \alpha, \rho, \sigma \) be its null components. Let \( O \in \mathcal{O} \). Then \( \mathcal{L}_O \alpha[\mathcal{F}] = \alpha[\mathcal{L}_O \mathcal{F}], \mathcal{L}_O \rho[\mathcal{F}] = \rho[\mathcal{L}_O \mathcal{F}], \) and \( \mathcal{L}_O \sigma[\mathcal{F}] = \sigma[\mathcal{L}_O \mathcal{F}] \). An analogous result holds the operators \( \nabla_L, \nabla_L, \) and \( \nabla_L \) commute with the null decomposition of \( \mathcal{F} \).

**Proof:** Corollary 6.0.9 follows from Definition 5.3.1, (107a), (107b), (164), (166a), (166b), (166e) and (166f).

**Lemma 6.0.10** (Commutator identities involving \( i, \mathcal{L}, \) and \( \nabla \)): Let \( X \) and \( Y \) be vector fields, and let \( \mathcal{F} \) be a two-form. Let \( i_X \mathcal{F} \) be the interior product defined by the requirement that \( i_X \mathcal{F}(Y) = \mathcal{F}(Y, X) \) holds for all pairs of vectors \( X, Y \). Then

\[ i_X \nabla_Y \mathcal{F} - \nabla_Y i_X \mathcal{F} = i_{[X,Y]} \mathcal{F}, \quad (167a) \]
\[ \nabla_X (i_Y \mathcal{F}) - i_Y (\nabla_X \mathcal{F}) = i_{[X,Y]} \mathcal{F}. \quad (167b) \]

**Proof:** The relation (167a) follows from the Leibniz rule and the fact that \(-\nabla_Y X = [X, Y]\). The relation (167b) follows from the Leibniz rule.

We leave the computations required to prove the next lemma up to the reader.

**Lemma 6.0.11** (Commutators of the Minkowski conformal Killing fields): (Page 139 of Ref. 17). The 15 generators \( \{ T_{(\mu)}, \Omega_{(\mu\nu)}, S, K_{(\mu)} \}_{0 \leq \mu, \nu \leq 3} \) of the Minkowski conformal Killing fields, which are defined in Sec. IV, satisfy the following commutation relations:

\[ [T_{(\mu)}, T_{(\nu)}] = 0, \quad (\mu, \nu = 0, 1, 2, 3), \quad (168a) \]
\[ [T_{(\kappa)}, \Omega_{(\mu\nu)}] = g_{\kappa\mu} T_{(\nu)} - g_{\kappa\nu} T_{(\mu)}, \quad (\kappa, \mu, \nu = 0, 1, 2, 3), \quad (168b) \]
\[ [T_{(\mu)}, S] = T_{(\mu)}, \quad (\mu = 0, 1, 2, 3), \quad (168c) \]
\[ [K_{(\mu)}, T_{(\nu)}] = 2g_{\mu\nu} S + 2\Omega_{(\mu\nu)}, \quad (\mu, \nu = 0, 1, 2, 3), \quad (168d) \]
\[ [\Omega_{(\kappa\lambda)}, \Omega_{(\mu\nu)}] = g_{\kappa\mu} \Omega_{(\nu\lambda)} - g_{\kappa\nu} \Omega_{(\mu\lambda)} + g_{\kappa\lambda} \Omega_{(\nu\mu)} - g_{\kappa\mu} \Omega_{(\nu\lambda)}, \quad (\kappa, \lambda, \mu, \nu = 0, 1, 2, 3), \quad (168e) \]
\[ [\Omega_{(\mu\nu)}, S] = 0, \quad (\mu, \nu = 0, 1, 2, 3), \quad (168f) \]
\[ [K_{(\kappa)}, \Omega_{(\mu\nu)}] = g_{\kappa\mu} K_{(\nu)} - g_{\kappa\nu} K_{(\mu)}, \quad (\mu, \nu = 0, 1, 2, 3), \quad (168g) \]
\[ [K_{(\mu)}, S] = K_{(\mu)}, \quad (\mu = 0, 1, 2, 3), \quad (168h) \]
\[ [K_{(\mu)}, K_{(\nu)}] = 0, \quad (\mu, \nu = 0, 1, 2, 3). \quad (168i) \]
The following simple corollary follows directly from Lemma 6.0.11.

**Corollary 6.0.12**: Let $\mathbf{T}$ and $\mathbf{Z}$ denote the Lie algebras of vectorfields generated by the sets $\mathcal{F}$ and $\mathcal{Z}$, respectively. Then for $\mu = 0, 1, 2, 3$, we have

$$[T(\mu), Z] \subset \mathbf{T},$$  \hfill (169a)

$$[S, Z] \subset \mathbf{T}. \hfill (169b)$$

\[\square\]

**Lemma 6.0.13** (Explicit expressions for the null second fundamental forms): The null second fundamental forms $\nabla_\mu L^v$ and $\nabla_\mu L^v$ (see Definition 2.4.4) can be expressed as

$$\nabla_\mu L^v = -r^{-1} g^v_\mu,$$  \hfill (170a)

$$\nabla_\mu L^v = r^{-1} g^v_\mu.$$  \hfill (170b)

Furthermore, the intrinsic covariant derivatives of $\nabla L$ and $\nabla L$ vanish:

$$\nabla_{(M)}L \overset{\text{def}}{=} \nabla_{(M-1)}\nabla L = 0, \quad (M \geq 2),$$  \hfill (171a)

$$\nabla_{(M)}L \overset{\text{def}}{=} \nabla_{(M-1)}\nabla L = 0, \quad (M \geq 2).$$  \hfill (171b)

**Proof**: (170a) and (170b) follow from simple computations. (171a) and (171b) then follow from the Leibniz rule, (67), and the fact that $\nabla L = 0$.

**Lemma 6.0.14** (Commutator of $\nabla L$, $\nabla L$, and $\nabla L$): Let $U$ be a type $L^m$ tensorfield tangent to the spheres $S_{r}$, $t$, and let $U$ be a tensorfield tangent to the spheres $S_{r}$, $t$. Then for all integers $k$, $l$, $m \geq 0$, $\nabla^k L^l U$ is a tensorfield tangent to the spheres $S_{r}$, $t$, and

$$\nabla_{(M)}L \overset{\text{def}}{=} \nabla_{(M-1)}\nabla L = 0, \quad (M \geq 2),$$  \hfill (172)

**Proof**: The fact that $\nabla^k L^l U$ is tangent to the spheres follows from contracting any of its indices with either $L$ or $L$ and using (107a) and (107b) to commute the contractions through the derivatives; the result is 0. The relation (172) follows from the Leibniz rule, (60), (166a), and Lemma 6.0.13.

**Corollary 6.0.15** (Estimates for the commutator of $\nabla L$, $\nabla L$, and $\nabla L$): Let $k$, $l$, $m \geq 0$ be integers, and let $U$ be a tensorfield tangent to the spheres $S_{r}$, $t$. Then

$$\sum_{k \leq k', l \leq 1, m \leq m} r^{k+l+m'} |\nabla_{(m')} L^k \nabla_l U| \approx \sum_{k \leq k', l \leq 1, m \leq m} r^{k+l+m'} |\nabla^k L^l U|. \hfill (173)$$

**Proof**: Inequality (173) follows inductively with the help of the Leibniz rule, Lemma 6.0.14, and the fact that $\nabla L = 0$.

The following simple lemma gives pointwise bounds for the tensorfields $L$, $L$, $g$, and their full spacetime covariant derivatives.

**Lemma 6.0.16** (Estimates for $|\nabla_{(M)} L|$, $|\nabla_{(M)} L|$, and $|\nabla_{(M)} g|$): Let $M \geq 0$ be an integer. Let $L$, $L$ be the null vectorfields defined in (51a) and (51b), and let $g$ be the first fundamental form of the $S_{r}$, $t$ defined in (54). Then

$$|\nabla_{(M)} L| \lesssim r^{-M}, \hfill (174a)$$

$$|\nabla_{(M)} L| \lesssim r^{-M}. \hfill (174b)$$
and
\[ |\nabla_M g| \lesssim r^{-M}. \tag{175} \]

**Proof:** Since in the inertial coordinate system, \( L^\mu = (-1, \omega^1, \omega^2, \omega^3), L^\mu = (1, \omega^1, \omega^2, \omega^3), \) and \( \omega^i \equiv x^i/r, \) it is easy to check directly that (174a) and (174b) hold. Inequality (175) then follows from (39), definition (54), (174a), and (174b), and the Leibniz rule. \( \square \)

**Lemma 6.0.17** (Estimates for \(|\nabla_M O|\)). Let \( O \in \mathcal{O}. \) Then
\[ |\nabla_M O| \lesssim r^{1-M}. \tag{176} \]

**Proof:** Inequality (176) follows from repeated use of (60), (158), and Lemma 6.0.16. \( \square \)

**Lemma 6.0.18** (Estimates for contractions of rotations against \( S_{r,t} - \text{tangent} \) tensorfields). Let \( U_{i_1 \cdots i_m} \) be a type \( (0, m) \) tensorfield tangent to the spheres \( S_{r,t}. \) Let \( O, t \in \{1, 2, 3\}, \) be an enumeration of the three rotational vectorfields belonging to the set \( \mathcal{O}. \) For any rotational multi-index \( I = (i_1, \cdots, i_m) \) of length \( m' \leq m, \) where \( i_1 \in \{1, 2, 3\}, \) we define the type \( (0, m') \) \( S_{r,t} - \text{tangent} \) tensorfield \( O^I U \) by \( (O^I U)_{i_1 \cdots i_m} \equiv O^i_{k_1} \cdots O^i_{k_{m'}} U_{k_1 \cdots k_{m'} i_{m'+1} \cdots i_m}. \) Then the following pointwise estimate holds
\[ \sum_{|I|=m'} |O^I U| \approx r^{m'} |U|. \tag{177} \]

**Proof:** It is straightforward to verify that at any point in \( p \in S_{r,t}, \) there exists a pair \( O_{i_1}, O_{i_2} \) of rotations that have the following two properties: (i) \( |O_{i_1}|^2, |O_{i_2}|^2 \geq r^2/3; \) (ii) \( g(O_{i_1}, O_{i_2})^2 \leq \frac{1}{2} |O_{i_1}|^2 |O_{i_2}|^2 \) (where \( g(X, Y) \equiv g_{k\ell} X^k Y^\ell \)). This latter property implies that the (smallest) angle between \( O_{i_1} \) and \( O_{i_2}, \) viewed as vectors in the two-dimensional plane \( T_p S_{r,t}, \) is at least 60°. Therefore, any covector \( \xi \in T_p S_{r,t}, \) has a corresponding \( \xi \)-dual vector \( X \in T_p S_{r,t} \) that makes an angle \( \leq 60^\circ \) with one of \( \pm O_{i_1}, \pm O_{i_2}, \) and it follows that either \( |g(O_{i_1}, X)|^2 \geq \frac{r^2}{12} |X|^2 \) or \( |g(O_{i_2}, X)|^2 \geq \frac{r^2}{12} |X|^2. \) Thus, \( \sum_{|I|=1} |O^I \xi|^2 \lesssim r^2 |\xi|^2. \)

On the other hand, the reverse inequality \( \sum_{|I|=1} |O^I \xi|^2 \gtrsim r^2 |\xi|^2 \) trivially follows from (158). We have thus shown (177) in the “base case” \( m' = m = 1. \)

To prove the general case of (177), we first choose a \( \xi \)-orthonormal basis \( B \equiv \{e_1, e_2\} \) for \( T_p S_{r,t}. \) We denote the components of \( U \) relative to this basis by \( U_{A_{i_1} \cdots A_{im}} \equiv e_{A_i}^{A_1} \cdots e_{A_m}^{A_{m'}} U_{k_1 \cdots k_{m'} i_{m'+1} \cdots i_m}, \) where \( A_i \in B \) for \( 1 \leq i \leq m. \) Then repeatedly making use of the base case, we deduce that
\[ r^{m'} |U| \approx r^{m'} \sum_{A_{i_1} \cdots A_{im} \in B} |U_{A_{i_1} \cdots A_{im}}|. \tag{178} \]
VII. THE ENERGY-MOMENTUM TENSOR AND THE CANONICAL STRESS

In this section, we discuss the building blocks of our energies. We will begin by defining the MBI system's energy-momentum tensor, which we denote by $Q^{\mu\nu}_{MBI}$, and recalling its key properties. We remark that $Q^{\mu\nu}_{MBI}$ is the usual tensor associated with energy estimates. However, in order to derive energy estimates for the derivatives of a solution, we will need a different tensor, namely, the canonical stress $\dot{Q}^{\mu\nu}$. The bulk of this section is therefore devoted to an analysis of the properties of $Q^{\mu\nu}$ and $\dot{Q}^{\mu\nu}$.

A. The energy-momentum tensor $Q$

The energy-momentum tensor corresponding to an electromagnetic Lagrangian $^*\mathcal{L}'$ is defined as follows:

$$Q^{\mu\nu} \overset{\text{def}}{=} 2 \frac{\partial^* \mathcal{L}'}{\partial g_{\mu\nu}} + (g^{-1})^{\mu\nu} \mathcal{L}', \quad (\mu, \nu = 0, 1, 2, 3).$$

(179)

It follows trivially from the definition that $Q^{\mu\nu}$ is symmetric

$$Q^{\mu\nu} = Q^{\nu\mu}. \quad \text{(180)}$$

We recall the fundamental divergence-free property: for energy-momentum tensors $Q^{\mu\nu}$ constructed out of a solution of the equations of motion (73a) and (73b) corresponding to $^*\mathcal{L}'$, we have that

$$\nabla_{\mu} Q^{\mu\nu} = 0, \quad (\nu = 0, 1, 2, 3). \quad \text{(181)}$$

In the particular case of the MBI model, (81) and Lemma 3.1.1 imply that

$$Q^{\mu\nu}_{MBI} = \ell^{-1}(MBI) (g_{\epsilon\lambda} \mathcal{F}^{\epsilon\lambda} \mathcal{F}^{\epsilon\lambda} - \frac{2}{3} \mathcal{F}^2)(g^{-1})^{\mu\nu} + (g^{-1})^{\mu\nu} (1 - \ell(MBI)). \quad \text{(182)}$$

The next lemma is not needed for any of the main results presented in this article, but it is of interest in itself. It shows that $Q^{\mu\nu}_{MBI}$ satisfies the dominant energy condition.

**Lemma 7.1.1 (The dominant energy condition for $Q_{MBI}$):** Let $\mathcal{F}$ be any two-form for which the quantity $\ell_{MBI}$ defined in (80) is positive, i.e., any $\mathcal{F}$ for which $1 + \langle\mathcal{F}\rangle - \frac{2}{3} \mathcal{F}^2 > 0$. Then the MBI energy-momentum tensor of $\mathcal{F}$ satisfies the dominant energy condition; that is, for any pair of future-directed causal vectors $X, Y$ we have that $Q_{MBI}(X, Y) \geq 0$.

**Proof:** Let $p \in M$, and let $X, Y$ be any pair of future-directed causal vectors belonging to $T_p M$. Then in the plane spanned by $X, Y$, there exists a pair of null vectors $L', L'$ such that $g(L', L') = -2$, and such that $X = aL' + bL', Y = cL' + dL'$, with $a, b, c, d \geq 0$. Let us complement $L', L'$ with a pair of orthonormal vectors $e'_1, e'_2$ belonging to the $g$–orthogonal complement of span$(L', L')$. The set $(L', L', e'_1, e'_2)$ is therefore a null frame at $p$. Let $\mathcal{F}', \alpha', \rho', \sigma'$ denote the null components of $\mathcal{F}$ with respect to this frame. By bilinearity, it suffices to check that $Q_{MBI}(L', L') \geq 0$. For $Q_{MBI}(L', L') \geq 0$, and $Q_{MBI}(L', L') \geq 0$, we leave the following two simple calculations to the reader:

$$Q_{MBI}(L', L') = \ell_{MBI}^{-1}(\mathcal{F}')^2, \quad Q_{MBI}(L', L') = \ell_{MBI}^{-1}(\alpha')^2. \quad \text{(183)}$$

For the remaining term, we first calculate that

$$\ell_{MBI} Q_{MBI}(L', L') = \alpha'^4 \alpha'^2 + 2 \rho'^2 + 2 \rho'^2 (\rho'^2 + \sigma'^2), \quad \text{(184)}$$

We now express $2 \ell_{MBI}(\ell_{MBI} - 1) = f(\mathcal{F} - \mathcal{F}^2)$, where $f(v) = 2(1 + v) - 2(1 + v)^{1/2}$, and $f(0) = 0, f'(0) = 1, f''(0) = 1/2$. Setting $v = \mathcal{F} - \mathcal{F}^2$, using the convexity of $f$, and using the identity $\mathcal{F} = -\alpha'^4 \alpha'^2 - \rho'^2 + \sigma'^2$, it thus follows that

$$\ell_{MBI} Q_{MBI}(L', L') \geq \alpha'^4 \alpha'^2 + 2 \rho'^2 + 2 \rho'^2 (\rho'^2 + \sigma'^2) \geq \rho'^2 + \sigma'^2, \quad \text{(185)}$$

This concludes our proof of Lemma 7.1.1. \qed
B. The equations of variation

The definition (281) of our energy involves integrals of weighted squares of the components of $L^l_\Sigma F$ over the spacelike hypersurfaces $\Sigma_t$. In order to prove our global existence theorem, we need to understand the evolution of the energy, which in turn requires that we investigate the evolution of the $L^l_\Sigma F$. These quantities are solutions to the equations of variation, which are the linearization of the MBI system (87a) and (87b) around the background $F$. More specifically, the equations of variation in the unknowns $\tilde F_{\mu \nu}$ are defined to be

$$\nabla_{\lambda} \tilde F_{\mu \nu} + \nabla_{\mu} \tilde F_{\nu \lambda} + \nabla_{\nu} \tilde F_{\lambda \mu} = \mathfrak{J}_{\lambda \mu \nu}, \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \quad \tag{186a}$$

$$H^{\mu \nu \kappa \lambda} \nabla_{\mu} \tilde F_{\kappa \lambda} = \mathcal{Y}_\nu, \quad (\nu = 0, 1, 2, 3), \quad \tag{186b}$$

where the tensorfield $H^{\mu \nu \kappa \lambda}$ is defined in (86a), and $\mathfrak{J}_{\lambda \mu \nu}$ and $\mathcal{Y}_\nu$ are inhomogeneous terms that need to be specified. In our applications below, the $L^l_\Sigma F$ will play the role of $F$. In order to understand the evolution of the $L^l_\Sigma F$, we of course need to understand the nature of the inhomogeneous terms appearing in the equations of variation that they satisfy. The next proposition provides detailed expressions for these inhomogeneities.

**Proposition 7.2.1 (Algebraic structure of the inhomogeneous terms):** If $F_{\mu \nu}$ is a solution to the MBI system (87a) and (87b), then $\tilde F_{\mu \nu} \stackrel{\text{def}}{=} L^l_\Sigma F_{\mu \nu}$ is a solution to the equations of variation (186a) and (186b) with inhomogeneous terms given by

$$\mathfrak{J}_{(\lambda)}^{(I)} \equiv \mathfrak{J}_{\lambda \mu \nu}, \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \quad \tag{187a}$$

$$\mathcal{Y}_{(\lambda)}^{(I)} = H^{\mu \nu \kappa \lambda} \nabla_{\mu} \left( L^l_\Sigma F_{\kappa \lambda} \right) - \hat{L}^l_\Sigma \left( H^{\mu \nu \kappa \lambda} \nabla_{\mu} F_{\kappa \lambda} \right), \quad (\nu = 0, 1, 2, 3). \quad \tag{187b}$$

In the above formula, the tensorfield $H^{\mu \nu \kappa \lambda}$, which depends quadratically on $F$, is defined in (86b), while the iterated modified Lie derivatives $\hat{L}^l_\Sigma$ are defined in Sec. IV.

**Proof:** Recall Eq. (87a), which states that $\tilde F$ is a solution to $\nabla_{\nu} \tilde F_{\mu \nu} = 0$, where $[\cdot]$ denotes antisymmetrization. Using property (159), we therefore deduce that

$$0 = L^l_\Sigma \nabla_{\nu} \tilde F_{\mu \nu} = \nabla_{\nu} L^l_\Sigma \tilde F_{\mu \nu}. \quad \tag{188}$$

This proves (187a).

To prove (187b), we first recall Eq. (87b), which states that $\tilde F$ is a solution to

$$H^{\mu \nu \kappa \lambda} \nabla_{\mu} F_{\kappa \lambda} \stackrel{\text{def}}{=} \left\{ \frac{1}{2} \left[ (g^{-1})^{\mu \kappa} (g^{-1})^{\nu \lambda} - (g^{-1})^{\mu \lambda} (g^{-1})^{\nu \kappa} \right] + H^{\mu \nu \kappa \lambda} \right\} \nabla_{\mu} F_{\kappa \lambda} = 0. \quad \tag{189}$$

Applying $\hat{L}^l_\Sigma$ to (189) and using (161), we conclude that

$$\frac{1}{2} \left[ (g^{-1})^{\mu \kappa} (g^{-1})^{\nu \lambda} - (g^{-1})^{\mu \lambda} (g^{-1})^{\nu \kappa} \right] \nabla_{\nu} L^l_\Sigma \tilde F + \hat{L}^l_\Sigma \left( H^{\mu \nu \kappa \lambda} \nabla_{\mu} F_{\kappa \lambda} \right) = 0. \quad \tag{190}$$

It therefore follows that

$$H^{\mu \nu \kappa \lambda} \nabla_{\mu} L^l_\Sigma F_{\kappa \lambda} = \left\{ \frac{1}{2} \left[ (g^{-1})^{\mu \kappa} (g^{-1})^{\nu \lambda} - (g^{-1})^{\mu \lambda} (g^{-1})^{\nu \kappa} \right] + H^{\mu \nu \kappa \lambda} \right\} \nabla_{\mu} L^l_\Sigma F_{\kappa \lambda}, \quad \tag{191}$$

which proves (187b).

C. The canonical stress $\dot{Q}$ and the modified canonical stress $\tilde{Q}$

Although the energy momentum tensor (179) is useful for estimating $F_{\mu \nu}$, it is not quite the right object for estimating its derivatives $L^l_\Sigma F$. As explained in detail in Sec. IC2, the reason is that the $L^l_\Sigma F$ are solutions not to the MBI system itself, but rather to the to the equations of variation...
(186a) and (186b), whose linearized Lagrangian, which is defined in (192) below, depends on the background $F$. Nonetheless, we will be able to construct the canonical stress $\dot{Q}_\nu^\mu$, which encodes the approximate conservation laws satisfied by solutions to the equations of variation. We remark that a general framework concerning the properties of the canonical stress was developed by Christodoulou in Ref. 13; here, we only recall its basic features. As we will see, $\dot{Q}_\nu^\mu$ has the following two key properties:

- $\nabla_\mu \dot{Q}_\nu^\mu$ is lower order (in terms of the number of derivatives).
- It has positivity properties related to, but in general distinct from those of the energy-momentum tensor $Q_\nu^\mu$.

The first property is explained in detail at the end of this section, while the second is discussed in limited fashion in Sec. VII D.

In order to explain the origin of the canonical stress, we first define the linearized Lagrangian $\dot{L}$, which is, despite its name, a quadratic form in the variations $\dot{F}$ with coefficients depending on the background $F$.

**Definition 7.3.1 (Linearized Lagrangian):** The linearized Lagrangian $\dot{L}[\dot{F}; F]$ corresponding to the Lagrangian $L[-]$ and the background $F$ is defined as follows:

$$\dot{L} = L[\dot{F}; F] \overset{\text{def}}{=} \frac{1}{2} \frac{\partial^2 \mathcal{L}[F]}{\partial F_\mu \partial F_\nu} \dot{F}_\xi \dot{F}_\eta \dot{F}_\kappa \dot{F}_\lambda = -\frac{1}{4} h^\xi_{\nu \lambda \kappa} \dot{F}_\eta \dot{F}_\kappa \dot{F}_\lambda, \tag{192}$$

where the $F-$dependent tensorfield $h^\xi_{\nu \lambda \kappa}$ is defined in (76).

The significance of $\dot{L}[\dot{F}; F]$ is that its corresponding equations of motion are the equations of variation. More specifically, if we consider $\dot{F}$ to be the unknowns, then the principal part of the Euler-Lagrange equations (assuming the stationarity of the linearized action $A_{\xi}[\dot{F}] \overset{\text{def}}{=} \int_{\mathbb{R} \otimes M} \dot{L}[\dot{F}; F] d\mu_\xi$, under closed variations of $\dot{F}$) corresponding to $\dot{L}[\dot{F}; F]$ are the linearization (around $F$) of the Euler-Lagrange equations (75) corresponding to $\mathcal{L}$.

**Definition 7.3.2 (Canonical stress):** Given a linearized Lagrangian $\dot{L} = \dot{L}[\dot{F}; F]$, we define the corresponding canonical stress $\dot{Q}_\nu^\mu \overset{\text{def}}{=} \dot{Q}_\nu^\mu[\dot{F}, F]$ as follows:

$$\dot{Q}_\nu^\mu[\dot{F}, F] \overset{\text{def}}{=} -2 \frac{\partial \dot{L}}{\partial F_\mu \nu} \dot{F}_\xi + \delta_\nu^\mu \dot{L} = h^\xi_{\nu \lambda \kappa} \dot{F}_\kappa \dot{F}_\eta \dot{F}_\lambda - \frac{1}{4} \delta_\nu^\mu h^\xi_{\nu \lambda \kappa} \dot{F}_\eta \dot{F}_\kappa \dot{F}_\lambda, \tag{193}$$

where the tensorfield $h$ is defined in (76).

**Remark 7.3.1.** In definition (193), we have suppressed the dependence of $\dot{Q}_\nu^\mu$ on $F$. Our notation $\dot{Q}_\nu^\mu[\dot{F}, F]$ emphasizes the fact that $\dot{Q}_\nu^\mu$ depends quadratically on $\dot{F}$.

In Sec. III C, we modified the tensorfield $h$ corresponding to the MBI system, obtaining a new tensorfield $H$. Therefore, for computational purposes, it is convenient to construct a modified version of the MBI canonical stress using $H$; the next definition captures this slight modification.

**Definition 7.3.3 (Modified canonical stress):** We define the modified canonical stress $\tilde{Q}_\nu^\mu$ to be the tensorfield obtained by replacing $h$ in (193) with $H$, where $H$ is defined in (86a),

$$\tilde{Q}_\nu^\mu[\dot{F}, F] \overset{\text{def}}{=} H^\xi_{\nu \lambda \kappa} \dot{F}_\kappa \dot{F}_\eta \dot{F}_\lambda - \frac{1}{4} \delta_\nu^\mu H^\xi_{\nu \lambda \kappa} \dot{F}_\eta \dot{F}_\kappa \dot{F}_\lambda. \tag{194}$$

Note that $\tilde{Q}_\nu^\mu$ depends quadratically on $\dot{F}$, and it also depends on the background $F$.

In general, $\tilde{Q}_\nu^\mu$ is not symmetric, nor is $\dot{Q}_\nu^\mu$, traceless. However, in the case of the MBI system, it can be checked that $\tilde{Q}_\nu^\mu$ is in fact traceless. More specifically, using (86a) and (86b), we compute...
that
\[
\tilde{Q}_\mu^\nu[\tilde{\mathcal{F}}, \tilde{\mathcal{F}}] = \tilde{\mathcal{F}}_\mu^\zeta \tilde{\mathcal{F}}_{\nu\zeta} - \frac{1}{4} g_{\mu\nu} \tilde{\mathcal{F}}_{\zeta\eta} \tilde{\mathcal{F}}^{\zeta\eta} + \frac{1}{2} \ell_{(\text{MBI})}^{-2} \left\{ \tilde{\mathcal{F}}_\mu^\zeta \tilde{\mathcal{F}}_{\nu\zeta} \tilde{\mathcal{F}}_{\kappa\lambda} + \frac{1}{4} g_{\mu\nu} (\tilde{\mathcal{F}}^{\zeta\lambda} \tilde{\mathcal{F}}_{\zeta\lambda})^2 \right\} \\
+ \frac{1}{2} \left( 1 + \frac{1}{2} \ell_{(\text{MBI})}^{-2} \right) \left\{ - \mathcal{F}_\mu^\zeta \tilde{\mathcal{F}}_{\nu\zeta} \tilde{\mathcal{F}}_{\kappa\lambda} + \frac{1}{4} g_{\mu\nu} (\tilde{\mathcal{F}}^{\zeta\lambda} \tilde{\mathcal{F}}_{\zeta\lambda})^2 \right\} \\
+ \frac{1}{2} \ell_{(\text{MBI})}^{-2} \mathcal{F}_\mu^\zeta \tilde{\mathcal{F}}_{\nu\zeta} \tilde{\mathcal{F}}_{\kappa\lambda} - \frac{1}{4} g_{\mu\nu} \tilde{\mathcal{F}}_{\zeta\eta} \tilde{\mathcal{F}}_{\zeta\eta} \tilde{\mathcal{F}}^{\zeta\lambda} \tilde{\mathcal{F}}_{\zeta\lambda} \tilde{\mathcal{F}}_{\kappa\lambda} \right\},
\]
from which it also follows that
\[
\tilde{Q}_\kappa^\kappa = 0.
\]

We also note that the first two terms on the right-hand side of (195) are the components of the energy-momentum tensor for the linear Maxwell-Maxwell equations (in the unknown \( \tilde{\mathcal{F}} \)).

The failure of symmetry of \( \tilde{Q}_\mu^\nu \), is captured by the following expression:
\[
\tilde{Q}_\mu^\nu - \tilde{Q}_\nu^\mu = \frac{1}{2} \ell_{(\text{MBI})}^{-2} \mathcal{F}^{\kappa\lambda} \tilde{\mathcal{F}}_{\kappa\lambda} \left\{ \mathcal{F}_\mu^\zeta \tilde{\mathcal{F}}_{\nu\zeta} - \mathcal{F}_\nu^\zeta \tilde{\mathcal{F}}_{\mu\zeta} \right\} \\
+ \frac{1}{2} \left( 1 + \frac{1}{2} \ell_{(\text{MBI})}^{-2} \right) \mathcal{F}^{\kappa\lambda} \tilde{\mathcal{F}}_{\kappa\lambda} \left\{ \mathcal{F}_\mu^\zeta \tilde{\mathcal{F}}_{\nu\zeta} - \mathcal{F}_\nu^\zeta \tilde{\mathcal{F}}_{\mu\zeta} \right\} \\
+ \frac{1}{2} \ell_{(\text{MBI})}^{-2} \mathcal{F}^{\kappa\lambda} \tilde{\mathcal{F}}_{\kappa\lambda} \left\{ \mathcal{F}_\mu^\zeta \tilde{\mathcal{F}}_{\nu\zeta} - \mathcal{F}_\nu^\zeta \tilde{\mathcal{F}}_{\mu\zeta} \right\}.
\]

We conclude this section by proving a lemma that illustrates the first key property of \( \tilde{Q}_\mu^\nu \), namely, that its divergence is lower order.

Lemma 7.3.1 (An expression for \( \nabla_\mu \tilde{Q}_\nu^\nu \)): Let \( \tilde{\mathcal{F}} \) be a solution to the equations of variation (186a) and (186b) corresponding to the background \( \mathcal{F} \), and let \( \tilde{Q}_\mu^\nu[\tilde{\mathcal{F}}, \tilde{\mathcal{F}}] \) be the MBI modified canonical stress given in (195). Let \( \tilde{\mathcal{F}}_\mu^\nu, \tilde{\mathcal{F}}^\nu, \tilde{\mathcal{F}}_\nu^\nu \) be the inhomogeneous terms on the right-hand sides of (186a) and (186b). Then
\[
\nabla_\mu \left( \tilde{Q}_\mu^\nu[\tilde{\mathcal{F}}, \tilde{\mathcal{F}}] \right) = -\frac{1}{2} H^{\kappa\nu\lambda} \tilde{\mathcal{F}}_{\zeta\eta} \tilde{\mathcal{F}}^{\kappa\lambda} \tilde{\mathcal{F}}_{\zeta\lambda} + \tilde{\mathcal{F}}_\nu^\eta \tilde{\mathcal{F}}_{\nu}^\eta + (\nabla_\mu H^{\mu\nu\kappa}) \tilde{\mathcal{F}}_{\kappa\lambda} \tilde{\mathcal{F}}^\kappa_{\nu\lambda} - \frac{1}{4} (\nabla_\mu H^{\mu\nu\kappa}) \tilde{\mathcal{F}}_{\zeta\eta} \tilde{\mathcal{F}}^{\zeta\lambda} \tilde{\mathcal{F}}_{\kappa\lambda}.
\]

Proof: Lemma 7.3.1 follows from the properties (77a)–(77c), which are satisfied by \( H^{\mu\nu\kappa} \), and from straightforward computations. \( \square \)

D. Positivity properties of the modified canonical stress

The canonical stress and the modified canonical stress have positivity properties that are analogous to, but distinct from those of the energy-momentum tensor. For a complete discussion of these properties, which are related to the geometry of the equations, readers may consult. Here, we only discuss the positivity properties that are relevant to our small-data global existence proof and our sketch of a large-data local well-posedness proof. We remark that in general, and specifically in the case of the MBI system, the geometry of the electromagnetic equations (i.e., the structure of the characteristic subsets) is distinct from the geometry corresponding to the spacetime metric \( \mathcal{G} \). However, for the Maxwell-Maxwell equations, the two geometries coincide. For purposes of constructing
an energy (see Sec. VIII) suitable for the global existence proof, the main quantity of interest will be the quadratic form \( \tilde{Q}_{\xi}^{\epsilon}(\xi^{(0)}K)[\cdot, \cdot] \), where \( \xi^{(0)} \) is the \( g \)-dual of the timelike translation Killing vectorfield \( T_{(0)} \) defined in (93a), and \( K = \frac{1}{2}\{(1 + s^2)L + (1 + q^2)L\} \) is the conformal Killing field defined in (134). As we will see in the next lemma, in the small-field regime, the resulting expression for \( \tilde{Q}_{\xi}^{\epsilon}(\xi^{(0)}K)[\cdot, \cdot] \) is a positive definite quadratic form in the null components of the variation \( \mathcal{F} \). However, different components carry different weights. Ultimately, the different weights will translate into the fact that the various null components of a solution \( \mathcal{F} \) and its derivatives have distinct rates of decay; see the global Sobolev inequality (Proposition 10.0.1).

In our proof of large-data local well-posedness (see Proposition 12.0.1), we will use a multiplier vectorfield \( X_{\text{local}} \) (see Proposition 7.4.4) in place of \( K \). The reason we use \( X_{\text{local}} \) is that the positive definiteness of \( \tilde{Q}_{\xi}^{\epsilon}(\xi^{(0)}K)[\cdot, \cdot] \) may break down in the large-data regime. In contrast, the vectorfield \( X_{\text{local}} \), which is constructed with the aid of the reciprocal Born-Infeld metric \((b^{-1})^{\mu\nu} \) [see (208) and (239)], maintains its positivity in all regimes in which the MBI equations are well-defined. Although \( X_{\text{local}} \) does not provide good \( q, s \) weights for the null components of \( \mathcal{F} \), a bound of the form \( \tilde{Q}(\xi^{(0)}, X_{\text{local}})(\mathcal{F}, \mathcal{F}) \geq C|\mathcal{F}|^2 \) for some positive constant \( C \) is sufficient to prove local existence.

We begin with a lemma that addresses the positivity properties of \( \tilde{Q}(\xi^{(0)}, K)[\cdot, \cdot] \).

**Lemma 7.4.1 (Positivity properties of \( \tilde{Q} \)):** Let \( \mathcal{F}, \mathcal{F} \) be arbitrary two-forms, and let \( \mathcal{F}, \tilde{\mathcal{F}}, \check{\mathcal{F}}, \tilde{\mathcal{F}} \) be the null components of \( \mathcal{F} \). Let \( \tilde{Q} \) be the modified canonical stress (195) associated with \( \mathcal{F}, \mathcal{F} \), and let \( T_{(0)}, K = \frac{1}{2}\{(1 + s^2)L + (1 + q^2)L\} \) be the conformal Killing fields defined in (90a) and (134), respectively. Let \( \xi^{(0)} \) be the \( g \)-dual of \( T_{(0)} \). Then there exists a constant \( \epsilon > 0 \) such that if \( |\mathcal{F}| < \epsilon \), then

\[
\tilde{Q}(\xi^{(0)}, K)[\mathcal{F}, \mathcal{F}] = \tilde{Q}_{\mu\nu} T_{(0)}^{\mu\nu} K[\mathcal{F}, \mathcal{F}] = \frac{1}{2}\left\{(1 + q^2)|\tilde{\mathcal{F}}|^2 + (1 + s^2)|\tilde{\mathcal{F}}|^2 + (2 + q^2 + s^2)(\rho^2 + \sigma^2)\right\} \tag{199}
\]

\[+ |\tilde{\mathcal{F}}|^2 O((1 + s^2)|\Sigma_{\mathcal{F}}^2|) + (1 + s^2)(|\tilde{\mathcal{F}}|^2 + \rho^2 + \sigma^2)O(|\mathcal{F}|^2) \]

\[+ (1 + |q|)^2(|\tilde{\mathcal{F}}|^2 + |\tilde{\mathcal{F}}|^2 + \rho^2 + \sigma^2)O(|\mathcal{F}|^2).\]

In (199), \( O(U) \) denotes a quantity which is \( \leq CU \) in magnitude for some positive constant \( C \).

**Remark 7.4.1:** As we shall see, in the small-solution regime, the dominant term on the right-hand side of (199) is \( \frac{1}{2}\{(1 + q^2)|\tilde{\mathcal{F}}|^2 + (1 + s^2)|\tilde{\mathcal{F}}|^2 + (2 + q^2 + s^2)(\rho^2 + \sigma^2)\} \). This motivates our definition of the weighted pointwise norm (277) below. Furthermore, at first sight, one might worry that the \( (1 + s^2) \) factor could cause the “error terms” \( O(\cdots) \) to become large. This worry is alleviated by Corollary 10.0.2, which shows that under suitable smallness assumptions on \( \mathcal{F} \), the term \( |\tilde{\mathcal{F}}|^2 O((1 + s^2)|\Sigma_{\mathcal{F}}^2|) \) is in fact small compared to the \( (1 + q^2)|\tilde{\mathcal{F}}|^2 \) term in the first line on the right-hand side of (199).

**Proof:** Using (195), we decompose \( \tilde{Q} \) into its linear “Maxwell” part \( \tilde{Q}^{(\text{Maxwell})} \), and the remaining “error terms”:

\[
\tilde{Q}_{\mu\nu} = \tilde{Q}^{(\text{Maxwell})}_{\mu\nu} + \frac{1}{2} \tilde{\Sigma}^{(\text{MBI})}_{(2)} \left\{- \mathcal{F}_{\mu}^{\epsilon} \mathcal{F}_{\nu}^{\epsilon} \mathcal{F}_{\kappa}^{\epsilon} \mathcal{F}_{\lambda}^{\epsilon} + \frac{1}{4} g_{\mu\nu}(\mathcal{F}_{\kappa}^{\epsilon} \mathcal{F}_{\lambda}^{\epsilon})^2 \right\}
\]

\[+ \frac{1}{2} \tilde{\Sigma}^{(\text{MBI})}_{(2)} \left\{- \mathcal{F}_{\mu}^{\epsilon} \mathcal{F}_{\nu}^{\epsilon} \mathcal{F}_{\kappa}^{\epsilon} \mathcal{F}_{\lambda}^{\epsilon} - \frac{1}{4} g_{\mu\nu}(\mathcal{F}_{\kappa}^{\epsilon} \mathcal{F}_{\lambda}^{\epsilon})^2 \right\}
\]

\[+ \frac{1}{2} \tilde{\Sigma}^{(\text{MBI})}_{(2)} \left\{ \mathcal{F}_{\mu}^{\epsilon} \mathcal{F}_{\nu}^{\epsilon} \mathcal{F}_{\kappa}^{\epsilon} \mathcal{F}_{\lambda}^{\epsilon} - \frac{1}{4} g_{\mu\nu}(\mathcal{F}_{\kappa}^{\epsilon} \mathcal{F}_{\lambda}^{\epsilon})^2 \right\}, \tag{200}
\]
As in the proof of Lemma 7.1.1, we leave it as a simple exercise for the reader to check that

$$\tilde{Q}_{\mu\nu}^{(\text{Maxwell})}(L, L) = |\dot{\mathbf{a}}|^2, \quad \tilde{Q}_{\mu\nu}^{(\text{Maxwell})}(L, \ell) = |\dot{\mathbf{a}}|^2, \quad \tilde{Q}_{\mu\nu}^{(\text{Maxwell})}(\ell, L) = (\rho^2 + \sigma^2).$$

(202)

Since $$\tilde{Q}_{\mu\nu}^{(\text{Maxwell})}(T(0), \tilde{K}) = \tilde{Q}_{\mu\nu}^{(\text{Maxwell})}(\frac{1}{2}(L + L), (1 + s^2)L + (1 + q^2)L),$$ it easily follows from (202) that

$$\tilde{Q}_{\mu\nu}^{(\text{Maxwell})}(T(0), \tilde{K}) = \frac{1}{2} \left\{ (1 + q^2)|\dot{\mathbf{a}}|^2 + (1 + s^2)|\dot{\mathbf{a}}|^2 + (2 + q^2 + s^2)(\rho^2 + \sigma^2) \right\},$$

(203)

which gives the principal term (i.e., the term in braces) on the right-hand side of (199). Note that in the above formulas, we have abused notation by identifying covectors with their $g -$ dual vectors.

To bound the error terms, we first note that $\tilde{Q}_{\mu\nu}^{(\text{MBI})}$ is an order 1 factor when $|\mathcal{F}|$ is sufficiently small, so that we may ignore it. With this fact in mind, we then evenly divide the 8 error terms on the right-hand side of (200) into two classes: those that contain the $g_{\mu\nu}$ factor, and those that do not. For the first class, we note that $|g_{\mu\nu}T_{\mu}^{\nu}\tilde{K}^\epsilon| = \frac{1}{2}|2 + q^2 + s^2| \lesssim (1 + s^2)$. Therefore, from Lemma 5.4.1, it follows that

$$|g_{\mu\nu}T_{\mu}^{\nu}\tilde{K}^\epsilon(\mathcal{F}^{\epsilon \mathbf{k}}\mathcal{F}_{\epsilon \mathbf{k}})^2| \lesssim (1 + q^2)|\mathcal{F}|^2|\mathcal{F}|^2 + (1 + s^2)|\mathcal{F}|^2|\mathcal{F}|^2,$$

(204)

and similarly for the other 3 terms of this type.

For the first term belonging to the second class, we use the null decomposition (135a)-(135c) of $\tilde{K}$ to conclude that

$$|T_{\mu}^{\nu}\tilde{K}^\epsilon\mathcal{F}_{\mu}^{\xi}\mathcal{F}_{\nu}^{\xi}| \lesssim (1 + q^2)|\mathcal{F}| + (1 + s^2)|\mathcal{F}|,$$

(205)

Therefore, by a second application of Lemma 5.4.1 and the inequality $|ab| \lesssim a^2 + b^2$, it follows that

$$|T_{\mu}^{\nu}\tilde{K}^\epsilon\mathcal{F}_{\mu}^{\xi}\mathcal{F}_{\nu}^{\xi}\mathcal{F}_{\epsilon \mathbf{k}}| \lesssim (1 + q^2)|\mathcal{F}|^2|\mathcal{F}|^2 + (1 + s^2)|\mathcal{F}|^2|\mathcal{F}|^2$$

(206)

$$+ (1 + s^2)|\mathcal{F}|^2|\mathcal{F}|^2 + (1 + s^2)|\mathcal{F}|^2|\mathcal{F}|^2$$

$$\lesssim (1 + q^2)|\mathcal{F}|^2(|\dot{\mathbf{a}}|^2 + |\mathbf{a}|^2 + \rho^2 + \sigma^2)$$

$$+ (1 + s^2)|\mathcal{F}|^2|\dot{\mathbf{a}}|^2 + (1 + s^2)|\mathcal{F}|^2(|\dot{\mathbf{a}}|^2 + \rho^2 + \sigma^2).$$

The remaining 3 terms of this type can be estimated similarly. Inequality (199) thus follows.

The next definition characterizes the region of state-space in which the MBI equations are well defined. As we will see, this region coincides with the region in which the system is hyperbolic.

**Definition 7.4.1 (The regime of hyperbolicity $\mathcal{H}$):** We define the regime of hyperbolicity to be the subset

$$\mathcal{H} \overset{\text{def}}{=} \{ \mathcal{F} \mid \ell_{(\text{MBI})}[\mathcal{F}] > 0 \},$$

(207)

where $\ell_{(\text{MBI})}[\mathcal{F}] = (1 + \zeta_{(1)}[\mathcal{F}] - \zeta_{(2)}^2[\mathcal{F}])^{1/2}$ is as in (80), and $\zeta_{(1)}[\mathcal{F}]$ and $\zeta_{(2)}[\mathcal{F}]$ are defined in (70a) and (70b).
We now introduce the reciprocal Maxwell-Born-Infeld metric \((b^{-1})^{\mu\nu}\), which is a fundamental ingredient in our construction of the positive energy densities of Proposition 7.4.4. These energy densities play a central role in our proof of large-data local well-posedness (see Proposition 12.0.1).

**Definition 7.4.2 (The reciprocal Maxwell-Born-Infeld metric):** Let \(\mathcal{F}\) be a two-form. We define \((b^{-1})^{\mu\nu}\), the reciprocal Maxwell-Born-Infeld metric, to be the following \(\mathcal{F}\)-dependent tensor field:

\[
(b^{-1})^{\mu\nu} \overset{\text{def}}{=} (g^{-1})^{\mu\nu} - (1 + \zeta_{(1)}[\mathcal{F}])^{-1} \mathcal{F}^{\mu\kappa} \mathcal{F}^{\nu}_{\kappa}
\]  

(208)

where the final equality follows from (78d).

It is straightforward to verify with the help of (78d) and (78e) that whenever \(\mathcal{F} \in \mathcal{H}\), the inverse of \((b^{-1})^{\mu\nu}\), which we denote by \(b^{\mu\nu}\), exists and can be expressed as

\[
b^{\mu\nu} = \ell^{-2}_{(MBI)} (1 + \zeta_{(1)}[\mathcal{F}]) \left( \delta_{\mu\nu} + \mathcal{F}^{\mu}_{\kappa} \mathcal{F}^{\nu}_{\kappa} \right).
\]  

(209)

The next lemma describes some basic properties of \((b^{-1})^{\mu\nu}\).

**Lemma 7.4.2 (Basic properties of \((b^{-1})^{\mu\nu}\):** Suppose that \(\mathcal{F} \in \mathcal{H}\), where \(\mathcal{H}\) is defined in (207). Then \((b^{-1})^{\mu\nu}\) is a Lorentzian metric of signature \((- , +, +, +)\). Furthermore, if \((g^{-1})^{\kappa\lambda} \xi^\kappa \xi^\lambda < 0\), then \((b^{-1})^{\kappa\lambda} \xi^\kappa \xi^\lambda < 0\).

**Proof:** We first prove the second statement. To this end, we note that by the antisymmetry of \(\mathcal{F}\), we have \(\mathcal{F}^{\kappa\lambda} \xi^\kappa \xi^\lambda = 0\). Equivalently, the vector \(V^\mu \overset{\text{def}}{=} \mathcal{F}^{\mu\kappa} \xi^\kappa\), verifies \(V^\mu = 0\). Thus, if \((g^{-1})^{\kappa\lambda} \xi^\kappa \xi^\lambda < 0\) (i.e., if \(\xi^\kappa\xi^\lambda\) is \(g\)-timelike), it follows that \(V^\mu\) is \(g\)-spacelike. Therefore, \(0 \leq V^\mu V^\mu = \mathcal{F}^{\mu\kappa} \mathcal{F}^{\nu}_{\kappa} \xi^\kappa \xi^\nu\). Since \(1 + \zeta_{(1)} > 0 \) whenever \(\mathcal{F} \in \mathcal{H}\), the expression (208) shows that \((b^{-1})^{\kappa\lambda} \xi^\kappa \xi^\lambda < 0\) whenever \((g^{-1})^{\kappa\lambda} \xi^\kappa \xi^\lambda < 0\), as desired.

To prove the first statement, we will use the electromagnetic decompositions of Sec. V.H. We first Euclidean-rotate the rectangular coordinates so that \(E_{\mu} = (0, E_1, 0, 0)\) and \(B_{\mu} = (0, B_1, B_2, 0)\). Using (147a) and (209), we then calculate that relative to this frame, we have

\[
\ell^{-2}_{(MBI)} (1 + \zeta_{(1)})^{-1} b^{\mu\nu} = 
\begin{pmatrix}
-1 + (E_1)^2 & 0 & 0 & -E_1 B_2 \\
0 & 1 + (B_2)^2 - (E_1)^2 & -B_1 B_2 & 0 \\
0 & -B_1 B_2 & 1 + (B_1)^2 & 0 \\
-E_1 B_2 & 0 & 0 & 1 + (B_1)^2 + (B_2)^2
\end{pmatrix}.
\]  

(210)

Let \(M_i\) denote the \(i \times i\) diagonal sub-block matrix whose lower right-hand entry coincides with the lower right-hand corner of (210) (so that \(M_1 = 1 + (B_1)^2 + (B_2)^2\), \(M_2 = 1 + (B_1)^2 + (B_2)^2\), etc). Using (70a), (70b), and (80), we calculate that

\[
\det(M_1) = 1 + (B_1)^2 + (B_2)^2,
\]  

(211)

\[
\det(M_2) = [1 + (B_1)^2][1 + (B_1)^2 + (B_2)^2],
\]  

(212)

\[
\det(M_3) = \ell^{-2}_{(MBI)} [1 + (B_1)^2 + (B_2)^2],
\]  

(213)

\[
\det(M_4) = \det(b) = -\ell^{-2}_{(MBI)} [1 + (B_1)^2 + (B_2)^2 + (E_1)^2 + (E_1)^2 (B_1)^2 + 2(E_1)^2 (B_2)^2] .
\]  

(214)

Recall that \(\ell_{(MBI)} > 0\) whenever \(\mathcal{F} \in \mathcal{H}\). Thus, by Sylvester’s criterion, \(M_3\) is positive definite. Furthermore, \(b_{\mu\nu}\) is not positive definite since its determinant is strictly negative. In particular, the
maximal dimension of all subspaces of \( \mathbb{R}^4 \) on which \( b_{\mu \nu} \) is positive definite is 3. It follows that \( b_{\mu \nu} \) is a Lorentzian matrix of signature \((- , + , + , +)\).

\[ \square \]

The next lemma shows that the characteristic subset of the cotangent space for the MBI system is precisely the collection of covectors that are \((b^{-1}) = 0\).

**Lemma 7.4.3 (The characteristic subset of \( T^*_p M \) for the MBI system):** Suppose that \( F \in \mathscr{F} \). Then the characteristic subset \( C^*_p \) of the cotangent space \( T^*_p M \) for the MBI system [see Eq. (3)] is precisely the set of covectors \( \xi \in T^*_p M \) such that

\[ (b^{-1})^{\mu \nu} \xi_\mu \xi_\nu = 0, \tag{215} \]

where \((b^{-1})^{\mu \nu}\) is defined in (208).

**Proof:** Given \( \xi \in T^*_p M \), we define the following two vectors for computational convenience:

\[ V^\mu = F^{\mu \kappa} \xi_\kappa, \tag{216} \]
\[ W^\mu = \ast F^{\mu \kappa} \xi_\kappa. \tag{217} \]

Note that

\[ \xi_\kappa V^\kappa = 0, \tag{218} \]
\[ \xi_\kappa W^\kappa = 0. \tag{219} \]

Furthermore, from (78d) and (78e), it follows that

\[ W_\kappa V^\kappa = \iota_{(2)} \xi_\kappa \xi^\kappa, \tag{220} \]
\[ V_\kappa V^\kappa = W_\kappa W^\kappa + \iota_{(1)} \xi_\kappa \xi^\kappa. \tag{221} \]

Recall that \( \chi^{\mu \nu}(\xi) \overset{\text{def}}{=} H^{\mu \nu \kappa \lambda} \xi_\kappa \xi_\lambda \), where \( H^{\mu \nu \kappa \lambda} \) is defined in (86a), and that \( C^*_p = \{ \xi \in T^*_p M \mid N(\chi(\xi)) \setminus \text{span(\xi)} \neq \emptyset \} \). Using (86a), we compute that the following relation holds for any \( \xi \in T^*_p M \),

\[ 2 \mathcal{L}_{(MBI)}^{\mu \nu}(\xi) \xi_\kappa = e_2^{(MBI)} H^{\mu \nu \kappa \lambda} \xi_\kappa \xi_\lambda - \mathcal{L}_{(MBI)}^{\mu \nu} \xi_\kappa \xi_\lambda - V_\kappa V^\kappa \]
\[ \quad + \iota_{(2)} V^\mu \xi_\mu W^\kappa + \iota_{(2)} W^\mu \xi_\mu V^\kappa - (1 + \iota_{(1)}) W^\mu \xi_\mu W^\kappa. \tag{222} \]

By the definition of \( C^*_p \), we are seeking all covectors \( \xi \in T^*_p M \) such that there exists a \( \zeta \in N(\chi(\xi)) \setminus \text{span(\xi)} \) with \( \chi^{\mu \nu}(\xi) \zeta_\nu = 0 \) for \( \mu = 0, 1, 2, 3 \). Let us first consider the case that \( V = W = 0 \). Then by (208), we have that \((b^{-1})^{\mu \nu} \xi_\mu \xi_\nu = 0\) if and only if \( \xi_\kappa \xi^\kappa = 0 \). Thus, we must show that such a \( \zeta \) exists if and only if \( \xi_\kappa \xi^\kappa = 0 \). If \( \xi_\kappa \xi^\kappa = 0 \), then it follows from (222) that for any \( \zeta \) with \( \xi_\kappa \xi^\kappa = 0 \), we have \( \chi^{\mu \nu}(\xi) \zeta_\nu = 0 \). Conversely, if \( \xi_\kappa \xi^\kappa \neq 0 \), then it is impossible to find such a \( \zeta \). This completes the proof in this case. We will therefore assume for the remainder of the proof that \( V \) and \( W \) are not simultaneously 0.

In the next case we consider, we also assume that \( \xi_\kappa \xi^\kappa \neq 0 \). In the remainder of the proof, we will slightly abuse notation by identifying vectors with their \( g \) – dual covectors. Under the present assumptions, Eq. (222) shows that any viable \( \zeta \) must satisfy \( \zeta \in \text{span}(\xi, V, W) \). Since \( \chi^{\mu \nu}(\xi) \zeta_\nu \equiv 0 \), we can assume without loss of generality that \( \zeta \in \text{span}(V, W) \). That is, for viable \( \zeta \), there exist \( (c_1, c_2) \neq (0, 0) \) such that

\[ \zeta_\mu = c_1 V_\mu + c_2 W_\mu. \tag{223} \]

From (218)–(219) and (223), it follows that

\[ \zeta_\kappa \xi^\kappa = 0, \tag{224} \]
\[ \zeta_{\varepsilon} V^\varepsilon = c_1 V_e V^\varepsilon + c_2 \dot{\gamma}(1) \xi \xi^\varepsilon, \]  
(225)

\[ \zeta_{\varepsilon} W^\varepsilon = c_1 \dot{\gamma}(2) \xi \xi^\varepsilon + c_2 V_e V^\varepsilon - c_2 \dot{\gamma}(1) \xi \xi^\varepsilon. \]  
(226)

Inserting (223) and (224)–(226) into (222), we compute that \( \chi^{\mu\nu}(\xi)\zeta_\nu = 0 \) if and only if

\[ V^\mu \left[ (1 + \dot{\gamma}(1)) \xi \xi^\varepsilon - V_e V^\varepsilon \right] \{c_1 - c_2 \dot{\gamma}(2)\} \]

\[ + W^\mu \left[ (1 + \dot{\gamma}(1)) \xi \xi^\varepsilon - V_e V^\varepsilon \right] \{-c_1 \dot{\gamma}(2) + c_2 (1 + \dot{\gamma}(1))\} = 0. \]  
(227)

With the help of (220) and (221), it is straightforward to verify that there exists a solution \((c_1, c_2)\) to (227) that corresponds to \( \zeta \neq 0 \) if and only if \((1 + \dot{\gamma}(1)) \xi \xi^\varepsilon - V_e V^\varepsilon = 0 \). Equivalently, there exists a \( \xi \in N(\chi(\xi)) \setminus \text{span}(\xi) \) if and only if

\[ (b^{-1})^{\mu\nu} \xi_\mu \xi_\nu = 0, \]  
(228)

where \((b^{-1})^{\mu\nu}\) is defined in (208). We have thus reached the desired conclusion in this case.

We now address the case \( \xi_{\varepsilon} \xi^\varepsilon = 0 \). By (208), in order to reach the desired conclusion, we must show that a viable \( \zeta \) exists if and only if \( V_e V^\varepsilon = 0 \). We will use the notation \( \xi^\perp \overset{\text{def}}{=} \{X \mid \chi_e \xi^\varepsilon = 0\} \).

We will first show that \( V_e V^\varepsilon = 0 \) implies that \( \xi \in C^*_{\text{p}} \). We begin by observing that if \( V_e V^\varepsilon = 0 \), then since \( \xi \) is \( g \)-null and \( V \in \xi^\perp \), it necessarily follows that \( V \in \text{span}(\xi) \). From (219) and (221), it also follows that \( W_e W^\varepsilon = 0 \) and \( W \in \text{span}(\xi) \). Hence, the subspace \( \xi^\perp \cap V^\perp \cap W^\perp \) is three-dimensional. Thus, there necessarily exists a covector \( \zeta \in (\xi^\perp \cap V^\perp \cap W^\perp) \setminus \text{span}(\xi) \). The relation (222) implies that any such \( \zeta \) verifies \( \chi^{\mu\nu}(\xi)\zeta_\nu = 0 \), and we conclude that \( \zeta \in C^*_{\text{p}} \).

Conversely, if \( V_e V^\varepsilon \neq 0 \), then (218)–(221) imply that \( V, W \) are a pair of non-zero \( g \)-orthogonal covectors that belong to \( \xi^\perp \). In particular, \( \xi, V, W \) are linearly independent, and

\[ \xi^\perp = \text{span}(\xi, V, W). \]  
(229)

It therefore follows from (222) that \( \chi^{\mu\nu}(\xi)\zeta_\nu = 0 \) if and only if

\[ -\ell^2_{(\text{MBI})} \xi \xi^\varepsilon = 0, \]  
(230)

\[ -\zeta_{\varepsilon} V^\varepsilon + \dot{\gamma}(2) \xi W^\varepsilon = 0, \]  
(231)

\[ \dot{\gamma}(2) \xi W^\varepsilon - (1 + \dot{\gamma}(1)) \xi_e W^\varepsilon = 0. \]  
(232)

From (231) and (232), we deduce that

\[ 0 = (1 + \dot{\gamma}(1) - \dot{\gamma}(2))^2 \xi_e W^\varepsilon = \ell^2_{(\text{MBI})} \xi_e W^\varepsilon. \]  
(233)

Since \( \ell^2_{(\text{MBI})} > 0 \) whenever \( F \in \mathcal{H} \), it follows from (233) that \( \zeta_{\varepsilon} W^\varepsilon = 0 \). Then (231) implies that \( \zeta_{\varepsilon} W^\varepsilon = 0 \). Furthermore, (230) implies that \( \zeta_e \xi^\varepsilon = 0 \). With the help of of these three relations and (229), we deduce that \( \chi^{\mu\nu}(\xi)\zeta_\nu = 0 \) if and only if \( \zeta \in \text{span}(\xi) \). We therefore conclude that \( \xi \notin C^*_{\text{p}} \) as desired.

The next proposition provides the fundamental estimate that is needed to deduce the “large-data” local existence result of Proposition 12.0.1.

**Proposition 7.4.4 (Positive definite energy densities within the regime of hyperbolicity: \( \mathcal{H} \)):**

Let \( \mathcal{R} \) be a compact subset of the regime of hyperbolicity \( \mathcal{H} \), which is defined in Definition 7.4.1. Let \( F \in \mathcal{R} \), let \( \mathcal{F} \) be any two-form, and let \( \hat{Q}^{\mu\nu} [F, F] \) be the modified canonical stress tensor (195) for the MBI equations of variation corresponding to the background \( F \) and the variation \( \mathcal{F} \). Let \( X_{\text{local}} \) be the \( F \)-dependent vectorfield defined below in (239). Then there exists a constant \( C_\mathcal{R} > 0 \), depending only on \( \mathcal{R} \), such that the quadratic form \( \hat{Q}(\xi(0), X_{\text{local}})(\cdot, \cdot) \) is positive definite and such that the following estimate holds:

\[ \hat{Q}^{\mu\nu}(\xi(0), X_{\text{local}})(\mathcal{F}, \mathcal{F}) \geq C_\mathcal{R} |\mathcal{F}|^2. \]  
(234)
In the above expression, the covector $\xi^{(0)}$ is the $g$–dual of the time translation Killing field $T_{(0)}$, which is defined in (90a).

Remark 7.4.2: An analogous version of Proposition 7.4.4 holds in any spacetime $(M, g)$. We remark that in a general curved spacetime, the modified canonical stress tensor depends on the metric $g$ and the background two-form $F$, but not on any connection coefficients or curvature components of $g$ (i.e., not on any derivatives of $g$). This fact follows from the definition (193) of the canonical stress, which remains valid in a general spacetime.

Proof: Let $(E, B)$ and $(\mathcal{E}, \mathcal{B})$ be the electromagnetic decompositions of $\mathcal{F}$ and $\mathcal{F}$ described in Sec. V H. Then simple calculations imply that the following identities hold in the inertial coordinate system:

\[ \mathcal{F}^*_0 \mathcal{F}_{0\kappa} = E_\kappa \dot{E}^\kappa, \]
\[ \star \mathcal{F}^*_0 \mathcal{F}_{0\kappa} = -B_\kappa \dot{E}^\kappa, \]
\[ \mathcal{F}^{\kappa\lambda} \mathcal{F}_{\kappa\lambda} = 2B_\kappa B^\kappa - 2E_\kappa \dot{E}^\kappa, \]
\[ \star \mathcal{F}^{\kappa\lambda} \mathcal{F}_{\kappa\lambda} = 2B_\kappa \dot{E}^\kappa + 2E_\kappa B^\kappa. \]

We now discuss how to construct a suitable (for deducing local well-posedness) multiplier vector field $X^\mu$ from a $g$–timelike covector $\zeta_\mu$. The discussion in this paragraph is based on the results of Ref. 13. Now as shown in Ref. 13, in order for $\int_{\mathbb{R}} Q^\mu_\nu \xi^{(0)} X^\nu [\mathcal{F}, \mathcal{F}] d^3x$ to be bounded from below by a multiple of $\int_{\mathbb{R}} |\mathcal{F}|^2 d^3x$, it is sufficient that $\xi^{(0)}$ be a covector lying strictly inside the boundary of the characteristic subset of $T_p M$, and that $X$ be a vector lying strictly inside the boundary of the characteristic subset of $T_p M$ satisfying $\xi^{(0)}(X) \equiv \xi^{(0)} X^\kappa < 0$ (we remark that many of our definitions differ from those in Ref. 13 by a minus sign). More precisely, as discussed in Ref. 13, in general, $\xi^{(0)}$ must be an element of the inner core of the characteristic subset of $T^*_p M$, and $X$ an element of the inner core of the characteristic subset of $T_p M$. In the case of the MBI system, the inner core of the characteristic subset of $T^*_p M$ coincides with the interior of the boundary of the characteristic subset of $T^*_p M$, and similarly for the inner core of the characteristic subset of $T_p M$. This is because Lemma 7.4.3 implies that the characteristic subset of $T^*_p M$ in the case of the MBI system has a particularly simple structure: it is the cone $\{ \xi \in T^*_p M \mid (b^{-1})^{\kappa\lambda} \xi_\kappa \xi_\lambda = 0 \}$. Furthermore, the results of Ref. 13 imply that the characteristic subset of $T_p M$ is the cone $\{ X \in T_p M \mid b_{\alpha\beta} X^\alpha X^\beta = 0 \}$, where $b_{\mu\nu}$ is defined in (209). Thus, the results of Ref. 13 imply that the inner cores of the two characteristic subsets are, respectively, $\{ \xi \in T^*_p M \mid (b^{-1})^{\kappa\lambda} \xi_\kappa \xi_\lambda < 0 \}$ and $\{ X \in T_p M \mid b_{\alpha\beta} X^\alpha X^\beta < 0 \}$. Hence, in the case of the MBI system, Lemma 7.4.2 implies that $\zeta^{(0)}$ in fact belongs to the inner core of the characteristic subset of $T^*_p M$. Furthermore, we can construct a suitable multiplier vector field $X^\mu$ by choosing $\zeta_\mu$ to be any $g$–timelike covector and then setting $X^\mu \equiv (b^{-1})^{\mu\kappa} \zeta_\kappa$.

Rather than relying on the above abstract framework, we will instead directly show the positive definiteness of the quadratic form $Q^\mu_\nu \xi^{(0)} X^\nu [\cdot, \cdot]$ for a particular choice of $X^\mu \equiv (b^{-1})^{\mu\kappa} \zeta_\kappa$: this pointwise positivity is stronger than the integrated positivity that follows from the general framework. Specifically, we make the simple choice $\zeta = \xi^{(0)}$. Furthermore, motivated by the above discussion, we define the multiplier vector field $X_{local}$ by

\[ X^\mu_{local} \equiv 2\ell^2_{(MBI)} (1 + \dot{\gamma}_1 [\mathcal{F}])(b^{-1})^{\mu\kappa} \xi^{(0)} = 2\ell^2_{(MBI)} (1 + \dot{\gamma}_1 [\mathcal{F}]) T^\mu_{(0)} - 2\ell^2_{(MBI)} T^0_{(0)} \mathcal{F}^\mu_{\kappa} \mathcal{F}_{\kappa\kappa}, \]

where the last equality is valid in the inertial coordinate system. The $2\ell^2_{(MBI)} (1 + \dot{\gamma}_1 [\mathcal{F}])$ factor on the left-hand side of (239) is a normalization factor that was chosen out of computational
convenience. Using the identities (235)–(238), it can be checked that

\[ \widetilde{Q}^{\mu} v^{(0)} X_{\text{local}}[F, \tilde{F}] \overset{\text{def}}{=} \widetilde{Q}^{\mu} v^{(0)} X_{\text{local}} = 2(1 + |B|^{2})\ell_{(MBI)}^{2} \widetilde{Q}_{00} - 2\ell_{(MBI)}^{2} \widetilde{Q}_{0a} F^{a} F_{0a}. \]  

Our goal is to show that the right-hand side of (240) is a uniformly positive definite quadratic form in \( \tilde{F} \) whenever \( F \in \mathcal{R} \). Using (70a), (70b), (195), and (235)–(238), and defining (for notational convenience)

\[ (U, V) \overset{\text{def}}{=} U_{a} V^{a} \]  

for any pair of vectors \( U, V \in \Sigma_{\ell} \), we compute that

\[ 2\ell_{(MBI)}^{2} \widetilde{Q}_{00} = \ell_{(MBI)}^{2} \left[ |\tilde{E}|^{2} + |\tilde{B}|^{2} \right] + \left\{ \langle E, \tilde{E} \rangle^{2} + (1 + |B|^{2}) - |E|^{2} \right\} \langle B, \tilde{E} \rangle^{2} + 2\langle B, \tilde{E} \rangle \langle E, \tilde{E} \rangle \langle B, \tilde{E} \rangle \right\} \]  

and

\[ \ell_{(MBI)}^{2} \widetilde{Q}_{0a} F^{a} F_{0a} = -\langle B, \tilde{B} \rangle^{2} \left[ |E|^{2} + |E, B|^{2} \right] + \langle B, \tilde{B} \rangle \langle E, \tilde{E} \rangle (1 + |B|^{2}) \langle E, B \rangle + \langle B, \tilde{B} \rangle \langle E, \tilde{E} \rangle (1 + |B|^{2}) \]  

\[ -\langle E, \tilde{B} \rangle^{2} \left[ (1 + |B|^{2}) - |E|^{2} \right] B|^{2} + \langle E, B \rangle^{2} \right\} \]  

\[ -\langle E, \tilde{B} \rangle \langle E, \tilde{E} \rangle (1 + |B|^{2}) \]  

\[ -\langle E, \tilde{B} \rangle \langle E, \tilde{E} \rangle (1 + |B|^{2}) \]  

To simplify the subsequent analysis, we choose a frame of \( g \)-orthonormal vectors \( \{ e_{\parallel}, e_{\perp}, e_{\times} \} \subset \Sigma_{\ell} \) such that \( E \in \text{span}(e_{\parallel}) \), and such that \( B \in \text{span}(e_{\perp}, e_{\times}) \). We can therefore express (in a slight abuse of notation) the electric field as \( E_{\ell} \) and the magnetic induction as \( B_{\parallel} e_{\parallel} + B_{\perp} e_{\perp} \). Furthermore, we can decompose \( B \) and \( \tilde{E} \) relative to this frame as

\[ B = B_{\parallel} e_{\parallel} + B_{\perp} e_{\perp} + B_{\times} e_{\times}, \]  

\[ \tilde{E} = \tilde{E}_{\parallel} e_{\parallel} + \tilde{E}_{\perp} e_{\perp} + \tilde{E}_{\times} e_{\times}, \]  

where

\[ \langle B, e_{\parallel} \rangle \overset{\text{def}}{=} B_{\parallel}, \]  

\[ \langle B, e_{\perp} \rangle \overset{\text{def}}{=} B_{\perp}, \]  

\[ \langle E, e_{\parallel} \rangle \overset{\text{def}}{=} E_{\parallel}, \]  

\[ \langle E, e_{\perp} \rangle \overset{\text{def}}{=} E_{\perp}, \]  

and

\[ |B|^{2} = B_{\parallel}^{2} + B_{\perp}^{2} + B_{\times}^{2}, \]  

\[ |\tilde{E}|^{2} = \tilde{E}_{\parallel}^{2} + \tilde{E}_{\perp}^{2} + \tilde{E}_{\times}^{2}. \]  

We now decompose \( \Sigma_{\ell} \) as a \( \mathbb{R} \)-orthogonal direct sum

\[ \Sigma_{\ell} = \Sigma_{\times} \oplus \Sigma_{\parallel,\perp}. \]
\[ \Sigma_\times = \text{span}(e_\times), \]  

\[ \Sigma_{1,\perp} = \text{span}(e_\parallel, e_\perp). \]  

Then it is easy to see that the quadratic form \( \tilde{Q}_\nu^\mu \xi_\nu^{(0)} X_{\text{local}}^{\nu} [\cdot, \cdot] \) can be decomposed as

\[ \tilde{Q}_\nu^\mu \xi_\nu^{(0)} X_{\text{local}}^{\nu} [\mathcal{F}, \mathcal{F}] = \tilde{Q}_\nu^\mu \xi_\nu^{(0)} X_{\text{local}}^{\nu} [\Sigma_\times, \mathcal{F}, \mathcal{F}] + \tilde{Q}_\nu^\mu \xi_\nu^{(0)} X_{\text{local}}^{\nu} [\Sigma_{1,\perp}, \mathcal{F}, \mathcal{F}], \]  

where \( \tilde{Q}_\nu^\mu \xi_\nu^{(0)} X_{\text{local}}^{\nu} [\cdot, \cdot] \) denotes the restriction of \( \tilde{Q}_\nu^\mu \xi_\nu^{(0)} X_{\text{local}}^{\nu} [\cdot, \cdot] \) to \( \Sigma_\times \) and \( \tilde{Q}_\nu^\mu \xi_\nu^{(0)} X_{\text{local}}^{\nu} [\underline{\cdot}, \cdot] \) denotes the restriction of \( \tilde{Q}_\nu^\mu \xi_\nu^{(0)} X_{\text{local}}^{\nu} [\cdot, \cdot] \) to \( \Sigma_{1,\perp} \). In fact, the only term on the right-hand side of (240) that involves the components \( \hat{E}_\times \) and \( \hat{B}_\times \) comes from the term \( \ell_{(MBI)}^2 \left[ |\hat{E}|^2 + |\hat{B}|^2 \right] \) on the right-hand side of (242), which leads to the trivial inequality

\[ \tilde{Q}_\nu^\mu \xi_\nu^{(0)} |\Sigma_\times, \mathcal{F}, \mathcal{F}] \geq \ell_{(MBI)}^2 (1 + |B|^2)(\hat{E}_\times^2 + \hat{B}_\times^2). \]  

We note that the uniform positivity of the right-hand side of (256) for \( \mathcal{F} \in \mathcal{K} \) is manifest, since \( \ell_{(MBI)}^2 |\mathcal{F}| \) is uniformly positive whenever \( \mathcal{F} \in \mathcal{K} \). We therefore conclude from (256) that

\[ \tilde{Q}_\nu^\mu \xi_\nu^{(0)} |\Sigma_\times, \mathcal{F}, \mathcal{F}] \geq C_\mathcal{K} (\hat{E}_\times^2 + \hat{B}_\times^2). \]  

We now address the components \((\hat{B}_1, \hat{B}_\perp, \hat{E}_1, \hat{E}_\perp)\). After many tedious but straightforward calculations, it follows that \( \tilde{Q}_\nu^\mu \xi_\nu^{(0)} X_{\text{local}}^{\nu} [\underline{\cdot}, \cdot] \) can be viewed as a quadratic form in the components \((\hat{B}_\parallel, \hat{B}_\perp, \hat{E}_1, \hat{E}_\perp)\), whose corresponding symmetric matrix \( \{A_{ij}\}_{1 \leq i, j \leq 4} \) has the following entries:

\[ A_{11} = (\ell_{(MBI)}^2 + E^2 B_\perp^2)(1 + B_\parallel^2 - E^2), \]

\[ A_{12} = -(\ell_{(MBI)}^2 + E^2 B_\parallel^2)B_\parallel B_\perp, \]

\[ A_{13} = (1 + |B|^2)EB_\parallel B_\perp, \]

\[ A_{14} = (1 + |B|^2)(1 + B_\parallel^2 - E^2)EB_\perp, \]

\[ A_{22} = (\ell_{(MBI)}^2 + E^2 B_\parallel^2)(1 + B_\perp^2), \]

\[ A_{23} = -(1 + |B|^2)(1 + B_\perp^2)EB_\parallel, \]

\[ A_{24} = -(1 + |B|^2)EB_\perp B_\perp, \]

\[ A_{33} = (1 + |B|^2)^2(1 + B_\perp^2), \]

\[ A_{34} = (1 + |B|^2)^2B_\perp B_\perp, \]

\[ A_{44} = (1 + |B|^2)^2(1 + B_\perp^2 - E^2). \]

By Sylvester’s criterion, the positive definiteness of \( \{A_{ij}\}_{1 \leq i, j \leq 4} \) follows from the positivity of the following four quantities, which are the determinants of an increasing sequence of sub-blocks along the diagonal:

\[ A_{11} = (\ell_{(MBI)}^2 + E^2 B_\perp^2)(1 + B_\parallel^2 - E^2), \]

\[ \text{det} \left( \{A_{ij}\}_{1 \leq i, j \leq 2} \right) = (\ell_{(MBI)}^2 + E^2 B_\perp^2)^2 \ell_{(MBI)}^2, \]  

\[ \text{det} \left( \{A_{ij}\}_{1 \leq i, j \leq 3} \right) = (\ell_{(MBI)}^2 + E^2 B_\perp^2)^3 \ell_{(MBI)}^2, \]  

\[ \text{det} \left( \{A_{ij}\}_{1 \leq i, j \leq 4} \right) = (\ell_{(MBI)}^2 + E^2 B_\perp^2)^4 \ell_{(MBI)}^2. \]
We remark that we calculated the right-hand sides of (268) and (269) by hand, while we used version 11 of MAPLE to compute the right-hand sides of (270) and (271).

As before, the uniform positivity of the right-hand sides of (269)–(271) for \( \mathcal{F} \in \mathcal{R} \) is manifest. The uniform positivity of the right-hand side of (268) follows from the fact that

\[
\ell_\text{(MBI)}^2[\mathcal{F}] = \delta \iff E^2 = 1 + \frac{B_\perp^2}{1 + B_\parallel^2} - \frac{\delta}{1 + B_\parallel^2}.
\]

We have thus shown that

\[
\tilde{\mathcal{Q}}^\mu_{\nu} \xi^{(0)}_{X_{\text{local}}}[\mathcal{F}, \dot{\mathcal{F}}] \geq C_K(B_\parallel^2 + B_\perp^2 + \dot{E}_1^2 + \dot{E}_2^2).
\]

Combining, (255), (257), and (273), and using the relation (290a) proved independently below, we deduce that

\[
\tilde{\mathcal{Q}}^\mu_{\nu} \xi^{(0)}_{X_{\text{local}}}[\mathcal{F}, \dot{\mathcal{F}}] \geq C_K(B_\parallel^2 + B_\perp^2 + B_\perp^2 + \dot{E}_\perp^2 + \dot{E}_\parallel^2) = C_K(|\dot{\mathcal{E}}_N|^2 + |B|^2)
\]

\[
\geq C_K|\mathcal{F}|^2.
\]

We have thus shown the desired estimate (234).

\[\square\]

**VIII. NORMS, SEMINORMS, ENERGIES, AND COMPARISON LEMMAS**

In this section, we introduce a collection of norms and seminorms that will be used in the remaining sections to estimate solutions to the MBI system. We also introduce the energy \( \mathcal{E}_N = \mathcal{E}_N[\mathcal{F}(t)] \), a related positive integral quantity that is constructed via the modified canonical stress and the “multiplier” vectorfield \( \tilde{K} \). In Sec. XI, we will study the time derivative of \( \mathcal{E}_N \), and in order to close the estimates, we need to prove inequalities that relate the norms to the energy; we provide these inequalities in Proposition 4. We also introduce another norm \( \| \cdot \|_H \) on the electromagnetic initial data \((\ddot{B}, \dot{D})\), and for small-data, we prove that this norm is equivalent to \( \mathcal{E}_N[\mathcal{F}(0)] \). This allows us to express the global existence smallness condition of Theorem 1 in terms of the data \((\ddot{B}, \dot{D})\), which are inherent to the Cauchy hypersurface \( \Sigma_0 \). In particular, the norm on \((B, D)\) does not involve normal (i.e., time) derivatives. Finally, in (340), we provide a preliminary expression that will be needed in our estimates of \( \frac{d}{dt} (\mathcal{E}_N[\mathcal{F}(t)]) \). The expression (340) motivates Sec. IX, in which we provide null structure estimates for the terms appearing in it.

**Definition 8.0.3 (Pointwise norms):** Let \( N \geq 0 \) be an integer. If \( \mathcal{A} \) is one of the sets \( \mathcal{F}, \mathcal{O}, \) or \( \mathcal{Z} \) defined in (93a)–(93c), and \( U \) is any tensorfield, then we define the following pointwise norms of \( U \),

\[
|U|_{L_\mathcal{A}; N} \overset{\text{def}}{=} \sum_{|I| \leq N} |\mathcal{L}_\mathcal{A}^I U|,
\]

\[
|U|_{\nabla_\mathcal{A}; N}^2 \overset{\text{def}}{=} \sum_{|I| \leq N} |\nabla_\mathcal{A}^I U|^2.
\]

Furthermore, if \( U \) is tangent to the spheres \( S_{\gamma, t} \), and \( \nabla \) denotes the Levi-Civita connection corresponding to \( g \), then we define

\[
|U|_{\nabla_\mathcal{O}; N} \overset{\text{def}}{=} \sum_{|I| \leq N} |\nabla_{\mathcal{O}}^I U|^2.
\]

In the above formulas, the pointwise norm \( | \cdot | \) is defined in (32), while the iterated derivatives \( \nabla_\mathcal{A}^I \), etc., are defined in Definition 4.0.3.
Definition 8.0.4 (Weighted pointwise norm): Let \( \widetilde{Q}^{(\text{Maxwell})} \) be the Maxwellian canonical stress corresponding to the two-form \( \mathcal{F} \) defined in (201), and let \( \zeta, \bar{\zeta}, \rho, \bar{\rho} \) be the null components of \( \mathcal{F} \). Let \( \mathcal{A} \) be one of the sets \( \mathcal{S}, \mathcal{O}, \) or \( \mathcal{E} \) defined in (93a)–(93c), and let \( \mathcal{K} \) be the conformal Killing field defined in (91a). Then for each integer \( N \geq 0 \), we define the following pointwise norms of \( \mathcal{F} \):

\[ \| \mathcal{F} \|_{L^2_{\xi,N}} = \| Q^{(\text{Maxwell})}(\xi^{(0)}, \mathcal{K})[\mathcal{F}, \mathcal{F}] \| = \frac{1}{2} \left\{ (1 + q)^2 |\zeta|^2 + (1 + s^2)|\bar{\zeta}|^2 + (2 + q^2 + s^2)(\rho^2 + \bar{\rho}^2) \right\} , \]

(277)

where \( \xi^{(0)} \) is the \( g \)–dual of the time translation vectorfield \( T^{(0)} \).

Notice that the different null components of \( \mathcal{F} \) in (277) carry different weights. Additionally, note that \( (1 + |q|^2)|\zeta|^2 \leq 2 \| \mathcal{F} \|_{L^2}^2 \).

We now define the key family of vectorfields \( \mathcal{J}^\mu \) that will be used in our proof of small-data global existence. The \( \mathcal{J}^\mu \) will be used in the divergence theorem to derive energy estimates for solutions.

Definition 8.0.5 (Energy current): Let \( \widetilde{Q}[\mathcal{F}, \mathcal{F}] \) be the MBI modified canonical stress (195) corresponding to the “background” \( \mathcal{F} \) and the variation \( \mathcal{F} \), and let \( \mathcal{K} \) be the conformal Killing field defined in (91a). We define the energy current \( \mathcal{J}^\mu[\mathcal{F}] \) corresponding to \( \mathcal{F} \) to be the following vectorfield:

\[ \mathcal{J}^\mu[\mathcal{F}] \triangleq -\frac{\widetilde{Q}^\mu}{\nu} \mathcal{K}^\nu[\mathcal{F}, \mathcal{F}] . \]

(279)

We note that \( \mathcal{J}^\mu[\mathcal{F}] \) depends on the background \( \mathcal{F} \) through \( \widetilde{Q}^\mu \), that it depends quadratically on \( \mathcal{F} \), and that \( \mathcal{J}^\mu[\mathcal{F}] = \widetilde{Q}(\xi^{(0)}, \mathcal{K})[\mathcal{F}, \mathcal{F}] \).

Our proof of small-data global existence will make use of the following two weighted norms and energies.

Definition 8.0.6 (Weighted \( L^2 \) norm and energy): Let \( N \geq 0 \) be an integer, and let \( \mathcal{F} \) and \( \mathcal{F} \) be a pair of two-forms. We define the weighted integral norm \( \| \mathcal{F}(t) \|_{L^2_{\xi,N}} \) of \( \mathcal{F} \) as follows:

\[ \| \mathcal{F}(t) \|_{L^2_{\xi,N}} \triangleq \left( \int_{\mathbb{R}^3} \| \mathcal{F}(t, \chi) \|_{L^2_{\xi,N}}^2 \, d^3 \chi \right)^{1/2} . \]

(280)

Furthermore, we define the energy \( \mathcal{E}_N[\mathcal{F}(t)] \) of \( \mathcal{F} \) as follows:

\[ \mathcal{E}_N[\mathcal{F}(t)] \triangleq \left( \sum_{|\ell| \leq N} \int_{\mathbb{R}^3} \mathcal{J}^\mu[\mathcal{F}(t, \chi)] \, d^3 \chi \right)^{1/2} , \]

(281)

where the component \( \mathcal{J}^\mu[\mathcal{F}(t, \chi)] \) is defined in (279).

Remark 8.0.3: The \( q, s \) weights under the integral in definition (280) are exactly the ones needed in the global Sobolev inequality (Proposition 10.0.1).

In the next lemma and corollary, we provide some comparison estimates that will play an important role in the analysis of Sec. X.

Lemma 8.0.5 (Equivalence of various pointwise norms involving angular derivatives): Let \( N \geq 0 \) be an integer. Then for any tensorfield \( U \), we have that

\[ |U|_{L^2_{\xi,N}} \approx |U|_{V^2_{\xi,N}} . \]

(282)
Furthermore, if $U$ is type $\binom{0}{m}$ and tangent to the spheres $S_{r,1}$, then we have the following estimates:

\begin{align*}
|U|_{\mathcal{F}_c;N} &\approx \sum_{n=0}^{N} r^n |\nabla (\alpha) U|, & (283a) \\
|U|_{\mathcal{F}_c;N} &\approx |U|_{\mathcal{L}_c;N}, & (283b) \\
|U|_{\mathcal{L}_c;N} &\approx \sum_{n=0}^{N} r^n |\nabla (\alpha) U|. & (283c)
\end{align*}

**Proof:** Inequality (282) can be proved inductively using the relation (42), together with the inequalities (158).

To prove (283a), we use the notation defined in Lemma 6.0.18. For notational simplicity, we suppress the indices of $U = U_{\mu_1, \ldots, \mu_m}$. We first note that Lemma 6.0.18 implies that for each integer $N \geq 0$, the inequalities in (283a) are equivalent to the following inequalities:

\begin{equation}
|U|_{\mathcal{F}_c;N} \overset{\text{def}}{=} \sum_{|I| \leq N} |\nabla^I U| \approx \sum_{|I| \leq N} |O^I \nabla^{|I|} U|. 
\end{equation}

To prove (284), we argue by induction, the base case $N = 0$ being trivial. For the induction step, we assume that the inequality is true in the case $N$. Let $I = (i_1, \ldots, i_k)$ be a rotational multi-index with $|I| = k \leq N + 1$. Then by the Leibniz rule and (176), we have that

\begin{equation}
\left| \nabla^I U \right| \leq \sum_{p=0}^{k-1} r^p |\nabla^I U| \lesssim |U|_{\mathcal{F}_c;k-1},
\end{equation}

where in the last step, we have used (283a) under the induction hypothesis. Summing over all $|I| = k \leq N + 1$, we conclude that

\begin{equation}
|U|_{\mathcal{F}_c;N+1} \lesssim |U|_{\mathcal{F}_c;N} \lesssim \sum_{|I| \leq N} |O^I \nabla^{|I|} U|,
\end{equation}

where in the last step, we have used (284) under the induction hypothesis. From (286), the induction step for (284) easily follows.

To prove (283b), we also argue by induction using the identity (42) (which is valid if $\nabla$ is replaced with $\mathcal{F}$), the base case being trivial. We assume that the inequalities hold in the case $N$. Let $I$ be any rotational multi-index with $|I| = k \leq N + 1$. Repeatedly applying the Leibniz rule and identity (42), and using (176) plus (283a), we deduce the following inequalities:

\begin{equation}
\left| \mathcal{L}_c^I U \right| \leq \left| \mathcal{L}_c^I U - \nabla^I U \right| \lesssim \sum_{p=0}^{k-1} r^p |\nabla^I U| \lesssim |U|_{\mathcal{F}_c;k-1}.
\end{equation}
Summing over all \(|I| = k \leq N + 1\), we conclude that

\[
|U|_{L^2; N+1} - |U|_{L^2; N} \lesssim |U|_{L^2; N} \lesssim |U|_{L^2; N},
\]

(287)

where in the last step, we have used the induction hypothesis. The induction step easily follows from (287), which completes the proof of (283b).

The estimate (283c) then follows trivially from (283a) and (283b).

\[\square\]

**Corollary 8.0.6 (Comparison between \(|\nabla^k L^j U|_{L^2; M}\) and \(|\nabla^k L^j U|_{L^2; M}\))**: If \(k, l, M \geq 0\) are integers, and \(U\) is any tensorfield tangent to the spheres \(S_{t, r}\), then we have the following estimate:

\[
\sum_{k' \leq k, l' \leq l} r^{k'+l'}|\nabla^k L^j U|_{L^2; M} \approx \sum_{k' \leq k, l' \leq l} r^{k+l'}|\nabla^k L^j U|_{L^2; M}.
\]

(288)

**Proof**: Corollary 8.0.6 follows from Corollary 6.0.15 and Lemma 8.0.5.

\[\square\]

In the next proposition, we collect together a large number of comparison estimates that will be used throughout the remainder of the article.

**Proposition 8.0.7 (Comparison estimates)**: Let \(M \geq 0\) be an integer, and let \(U\) be any tensorfield. Then the following pointwise estimates hold:

\[
|U|_{L^2; M} \approx |U|_{L^2; M},
\]

(289a)

\[
|\nabla(M) U| \approx \sum_{|I|=M} |\nabla|U| = \sum_{|I|=M} |L^j U|,
\]

(289b)

\[
\sum_{n=0}^{M}(1 + |q|)|\nabla U| \lesssim |U|_{L^2; M},
\]

(289c)

\[
|U|_{L^2; M} \lesssim \sum_{n=0}^{M}(1 + s)|\nabla(n) U|,
\]

(289d)

\[
|U|_{L^2; M} \approx \sum_{n=0}^{M}(1 + r^2)|\nabla(n) U|^2_{|\Sigma_0|}.
\]

(289e)

Although \(|\Sigma_0|\) denotes evaluation along \(\Sigma_0\) in the above formulas, we emphasize that \(\nabla(n) U\) denotes the full spacetime covariant derivative operator of order \(n\).

Let \(F\) be an arbitrary two-form, let \(\alpha, \beta, \rho, \sigma\) be its null components as defined in Sec. VC, and let \(E, B, \xi, \Phi\) be its electromagnetic decompositions as defined in Sec. VH. Let \(A\) be one of the three sets \(\mathcal{F}, \mathcal{O}, \mathcal{Z}\) defined in (93a)–(93c). Let \(q \equiv r - t, s \equiv r + t\) denote the null coordinates. Then the following pointwise estimates hold:

\[
|F|^2 = 2|E|^2 + |B|^2,
\]

(290a)

\[
|F|^2 = |\alpha|^2 + |\beta|^2 + 2(\rho^2 + \sigma^2),
\]

(290b)

\[
|\star F|^2 = |E|^2 + |B|^2 + |\xi|^2 + |\Phi|^2,
\]

(290c)

\[
\quad = \frac{1}{2} \left\{(1 + q^2)|\alpha|^2 + (1 + s^2)|\xi|^2 + (2 + q^2 + s^2)(\rho^2 + \sigma^2)\right\},
\]

\[
|\star F|^2_{L^2; M} = \sum_{|I| \leq M} \left| i_{T^n} L^j A F \right|^2 + \left| i_{T^n} L^j \star A F \right|^2 + \left| i_{S} L^j A F \right|^2 + \left| i_{S} L^j \star A F \right|^2,
\]

(290d)

\[
|\star F|^2_{L^2; M} = \sum_{|I| \leq M} \left| i_{T^n} L^j A F \right|^2 + \left| i_{T^n} L^j \star A F \right|^2 + \left| i_{S} L^j A F \right|^2 + \left| i_{S} L^j \star A F \right|^2,
\]

(290e)
\[ |\nabla (n) F|^2 \approx |\nabla (n) E|^2 + |\nabla (n) B|^2, \quad (290f) \]

\[ \| F \|^2_{L^2;M} \approx \| E \|^2_{L^2;M} + \| B \|^2_{L^2;M} + \| \mathfrak{E} \|^2_{L^2;M} + \| \mathfrak{B} \|^2_{L^2;M}. \quad (290g) \]

\[ |E|_{L^2;M}^2 + |B|_{L^2;M}^2 + |\mathfrak{E}|_{L^2;M}^2 + |\mathfrak{B}|_{L^2;M}^2 \lesssim \sum_{n=0}^M (1 + \rho)^{2(n+1)}(|\nabla (n) E|^2 + |\nabla (n) B|^2), \quad (290h) \]

\[ |\nabla F|_{\mathcal{W}^*} \approx \sum_{|I| = 1} |L^I_F F|_{\mathcal{W}^*}, \quad (290i) \]

\[ \sum_{|I| \leq M} \left( (1 + |q|)^{|I|} |\nabla (n) F| \right) \lesssim \| F \|_{L^2;M}, \quad (290j) \]

\[ \sum_{n=0}^M \left( (1 + |q|)^{n+1} |\nabla (n) F| \right) \lesssim \| F \|_{L^2;M}, \quad (290k) \]

\[ \| F \|^2_{L^2;M} |_{\Sigma_0} \approx \sum_{n=0}^M (1 + \rho)^{n+1} \left( |\nabla (n) E|^2 |_{\Sigma_0} + |\nabla (n) B|^2 |_{\Sigma_0} \right). \quad (290l) \]

Although \(|\nabla (n) F|_{\Sigma_0}\) denotes evaluation along \(\Sigma_0\) in the above formulas, we emphasize that \(\nabla (n)\) denotes the full spacetime covariant derivative operator of order \(n\). Furthermore, in (290i) and (290j), \(I\) is a translational multi-index.

Let \(F\) be any two-form. Then for any \(r > 0\), \(r \overset{\text{def}}{=} |x|\), we have that

\[ \sum_{k + |I| = 0}^M (1 + |q|)^{|I|} |\nabla (n) F|_{L^2;M-k-|I|} \lesssim \| F \|_{L^2;M}, \quad (291) \]

Let \(\tilde{F}\) be another arbitrary two-form, and let \(\tilde{J}^0_F [\tilde{F}]\) be the energy current vectorfield defined in (279). Then there exists a constant \(\epsilon > 0\) such that if \(|F| \leq \epsilon\), then the following comparison estimates hold for the pointwise norms \(\tilde{J}^0_F [\cdot]\) and \(|\cdot|\) and the integral norms \(E_M [\cdot]\) and \(||\cdot||_{L^2;M}\),

\[ \tilde{J}^0_F [\tilde{F}] \approx |\tilde{F}|^2, \quad (292a) \]

\[ E_M [\tilde{F}] \approx ||\tilde{F}||_{L^2;M}. \quad (292b) \]

**Remark 8.0.4:** We remark that the estimates (289a)-(291) either were proved directly by Christodoulou and Klainerman in Ref. 17, or follow easily from the estimates of Ref. 17; we nevertheless provide proofs here for convenience.

**Proof of Proposition 8.0.7**

**Proof of (289a):** The estimate (289a) can be proved inductively using (42) and Lemma 6.0.4.

**Proof of (289b)-(289e):** The estimates in (289) are easily deduced in the inertial coordinate system \(\{x^i\}_{i=0,1,2,3}\).

To prove (289c), we first note that in the inertial coordinate system, the following identity holds for \(\mu = 0, 1, 2, 3\):

\[ T(\mu) = \frac{x^\mu \Omega_{(k)\mu} + x_\mu S}{qs}. \quad (293) \]
Therefore, we have that
\[ |\nabla_{T_{\mu}} u| = |(q s)^{-1}|x^s \nabla_{\Omega_{\mu}} u + x^s \nabla_s u| \lesssim |q^{-1}|u|_{\nabla, 1}. \tag{294} \]
Since we also have the trivial identity |\nabla_{T_{\mu}} u| \leq |u|_{\nabla, 1}, it follows that
\[ |\nabla_{T_{\mu}} u| \lesssim (1 + |q|)^{-1}|u|_{\nabla, 1}. \tag{295} \]

Similarly, we use Lemma 6.0.3, Corollary 6.0.12, and (293) to inductively derive the following inequality:
\[ |\nabla_{T_{\mu_1}} \cdots \nabla_{T_{\mu_n}} u| \lesssim (1 + |q|)^{-n}|u|_{\nabla, n}. \tag{296} \]
Combining (296) and (289b), we deduce (289c).

Inequality (289d) follows trivially from (157) and the Leibniz rule. Inequality (289e) then follows from (289c) and (289d) and the fact that q = s = r along Σ_0.

**Proof of (290a)–(290f):** The relation (290a) can easily be verified in the inertial coordinate system using (149a) and the definition of the Euclidean norm |·|.

To prove the remaining relations, we first note that if X ∈ span(L, L), then it is straightforward to verify that
\[ |i_X F|^2 = \frac{1}{4} (X_L)^2 |\alpha|^2 + \frac{1}{4} (X_L)^2 |\alpha|^2 + \frac{1}{2} X_L X_L g(\alpha, \alpha) + \frac{1}{2} \left\{ (X_L)^2 + (X_L)^2 \right\} \rho^2, \tag{297} \]
\[ |i_X \star F|^2 = \frac{1}{4} (X_L)^2 |\alpha|^2 + \frac{1}{4} (X_L)^2 |\alpha|^2 - \frac{1}{2} X_L X_L g(\alpha, \alpha) + \frac{1}{2} \left\{ (X_L)^2 + (X_L)^2 \right\} \sigma^2, \tag{298} \]
where g(\alpha, \alpha) = \xi^\kappa \alpha_\kappa \alpha_\kappa = (q^{-1})^{\kappa\lambda} \alpha_\kappa \alpha_\kappa. Therefore,
\[ |i_X F|^2 + |i_X \star F|^2 = \frac{1}{2} \left\{ (X_L)^2 |\alpha|^2 + (X_L)^2 |\alpha|^2 + \left\{ (X_L)^2 + (X_L)^2 \right\} (\rho^2 + \sigma^2) \right\}. \tag{299} \]

Now taking first X = S, and then X = T_{\mu}, using S_L = q, S_L = -s, T_{\mu} = T_{\mu} = -1, and recalling that E = i_S F, B = i_S \star F, E = i_{T_{\mu}} F, B = -i_{T_{\mu}} \star F, we find that
\[ |E|^2 + |B|^2 = \frac{1}{2} \left\{ q^2 |\alpha|^2 + \sigma^2 |\alpha|^2 + (q^2 + \sigma^2)(\rho^2 + \sigma^2) \right\}, \tag{300} \]
\[ |E|^2 + |B|^2 = \frac{1}{2} \left\{ |\alpha|^2 + |\alpha|^2 + 2(\rho^2 + \sigma^2) \right\}. \tag{301} \]
Adding (300) and (301), and comparing with (277), we conclude the following:
\[ |\mathcal{E}|^2 + |\mathcal{B}|^2 = |E|^2 + |B|^2 = |\mathcal{E}|^2 + |\mathcal{B}|^2. \tag{302} \]
We have thus proved (290a)–(290d). With the help of Corollary 4.0.3, (290e) follows similarly.

Inequality (290f) follows from (289b), (290a), and the fact that the electric field and magnetic induction one-forms associated with \nabla_{L} F are, respectively, \nabla_{L} E and \nabla_{L} B.

**Proof of (290g):** From the commutation identities (167a), (168a), (169a) and (169b), it follows that for any \mathcal{S} – multi-index \mathcal{I}, we have
\[ |i_{T_{\mu}} \mathcal{L}_2 \mathcal{F} - \mathcal{L}_2 i_{T_{\mu}} \mathcal{F}| \lesssim \sum_{\mu = 0}^{3} \sum_{|\mathcal{J}| \leq |\mathcal{I}| - 1} |i_{T_{\mu}} \mathcal{L}_2 \mathcal{F}|, \tag{303} \]
\[ |i_{T_{\mu}} \mathcal{L}_2 \star \mathcal{F} - \mathcal{L}_2 i_{T_{\mu}} \star \mathcal{F}| \lesssim \sum_{\mu = 0}^{3} \sum_{|\mathcal{J}| \leq |\mathcal{I}| - 1} |i_{T_{\mu}} \mathcal{L}_2 \star \mathcal{F}|, \tag{304} \]
\[ |i_{S} \mathcal{L}_2 \mathcal{F} - \mathcal{L}_2 i_{S} \mathcal{F}| \lesssim \sum_{\mu = 0}^{3} \sum_{|\mathcal{J}| \leq |\mathcal{I}| - 1} |i_{T_{\mu}} \mathcal{L}_2 \mathcal{F}|. \tag{305} \]
Furthermore, it follows from (290b) and (299) that for any two-form $\mathcal{F}$ and each translational Killing field $T_\mu$, $(\mu = 0, 1, 2, 3)$, we have that
\[
|i_{T_\mu}\mathcal{F}|^2 + |i_{T_\mu}^*\mathcal{F}|^2 \lesssim |\mathcal{F}|^2 \lesssim |\mathcal{F}|^2.
\]  
(307)

Combining (303)–(306) with (307), we deduce that
\[
|i_{T_\mu}\mathcal{L}^I_{\mathcal{Z}}\mathcal{F} - i_{T_\mu}^I\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}| \lesssim |\mathcal{F}|_{\mathcal{L}:|I|\rightarrow |I| - 1}.
\]  
(308)

We will now prove (290g) by induction. The base case was established in (290c). We thus inductively assume that (290g) holds in the case $M - 1$. From (290c), (308) and (309), it follows that
\[
\mathcal{F}|_{\mathcal{L}:M}^2 = |\mathcal{F}|_{\mathcal{L}:M-1}^2 + \sum_{|I|=M} \left( (i_{T_\mu}\mathcal{L}^I_{\mathcal{Z}}\mathcal{F})^2 + (i_{T_\mu}^I\mathcal{L}^I_{\mathcal{Z}}\mathcal{F})^2 + (i_{S}\mathcal{L}^I_{\mathcal{Z}}\mathcal{F})^2 + (i_{S}^I\mathcal{L}^I_{\mathcal{Z}}\mathcal{F})^2 \right)
\]  
(310)

where the induction hypothesis was used to deduce the next to last line.

For the opposite inequality, we again use (290c), (308), and (309) and the induction hypothesis to conclude that
\[
|E|^2_{\mathcal{L}:M} + |B|^2_{\mathcal{L}:M} + |\mathcal{E}|^2_{\mathcal{L}:M} + |\mathcal{B}|^2_{\mathcal{L}:M}
\]  
(311)

Proof of (290j): From the facts that $\nabla_{T_\mu} S = T_\mu$ and $\nabla_{T_\mu} T_\mu = 0$ for $\mu = 0, 1, 2, 3$, together with the commutation identity (167b), it follows that
\[
|\nabla_{\mathcal{L}}^I i_{T_\mu}^I\mathcal{F} - i_{T_\mu}^I \nabla_{\mathcal{L}}^I \mathcal{F}| = 0.
\]  
(312)

\[
|\nabla_{\mathcal{L}}^I i_{T_\mu}^I\mathcal{F} - i_{T_\mu}^I \nabla_{\mathcal{L}}^I \mathcal{F}| = 0,
\]  
(313)

\[
|\nabla_{\mathcal{L}}^I i_S \mathcal{F} - i_S \nabla_{\mathcal{L}}^I \mathcal{F}| \lesssim \sum_{|I|=0}^{3} \sum_{|J|=|I| - 1} |i_{T_\mu}^I \nabla_{\mathcal{L}}^I \mathcal{F}|.
\]  
(314)
Furthermore, from definition (277) and (290a), it follows that any two-form $\mathcal{F}$ satisfies the following inequality:

$$\|\nabla^T F - i s\nabla^T F\| \lesssim \sum_{\mu=0}^3 \sum_{|I| \leq |J|-1} |i_{\mathcal{F}_{\mu}} \nabla^T F|.$$  (315)

It thus follows from (312)–(315) and (316) that

$$|i_{\mathcal{F}_{\mu}} \mathcal{F}|^2 + |i_{\mathcal{F}_{\mu}} \mathcal{F}|^2 \lesssim (1 + |q|)^{-2} \| \mathcal{F} \|^2.$$  (316)

We remark that we have used the induction hypothesis to arrive at the last inequality.

From (290d), (312), (313), (317) and (318), it follows that

$$|\nabla^T F - i s\nabla^T F|^2 \lesssim (1 + |q|)^{-2} \| \mathcal{F} \|^2_{|T_{\mu}|}.  \quad (317)$$

$$|\nabla^T F - i s\nabla^T F|^2 \lesssim (1 + |q|)^{-2} \| \mathcal{F} \|^2_{|T_{\mu}|}.  \quad (318)$$

We now prove (290j) by induction, the base case being the trivial inequality $\| \mathcal{F} \|^2 \lesssim \| \mathcal{F} \|^2$. We assume that the inequality holds in the case $M = 1$, and we let $I$ be any multi-index $I$ with $|I| = M$. From (290d), (312), (313), (317) and (318), it follows that

$$\| \nabla^T F \|^2 = |i_{\mathcal{F}_{\mu}} \nabla^T F|^2 + |i_{\mathcal{F}_{\mu}} \nabla^T F|^2 + |i_{\mathcal{F}_{\mu}} \nabla^T F|^2 + |i_{\mathcal{F}_{\mu}} \nabla^T F|^2$$

$$\lesssim |\nabla^T F|^2 + |\nabla^T F|^2 + |\nabla^T F|^2 + (1 + |q|)^{-2} \| \mathcal{F} \|^2_{|T_{\mu}|, M}$$

$$\lesssim |\nabla^T F|^2 + |\nabla^T B|^2 + |\nabla^T F|^2 + |\nabla^T F|^2 + (1 + |q|)^{-2} \| \mathcal{F} \|^2_{|T_{\mu}|, M}.$$  (319)

We remark that we have used the induction hypothesis to arrive at the last inequality.

To estimate the $|\nabla^T F|^2 + |\nabla^T B|^2 + |\nabla^T F|^2 + |\nabla^T B|^2$ term on the right-hand side of (319), we use (289a), (289c), and (290g) to deduce that

$$|\nabla^T F|^2 + |\nabla^T B|^2 + |\nabla^T F|^2 + |\nabla^T B|^2 \lesssim (1 + |q|)^{-2} \bigg( E_{|T_{\mu}|, M}^2 + B_{|T_{\mu}|, M}^2 + E_{|T_{\mu}|, M}^2 + B_{|T_{\mu}|, M}^2 \bigg)$$

$$\approx (1 + |q|)^{-2} \| \mathcal{F} \|^2_{|T_{\mu}|, M}.  \quad (320)$$

Combining (319) and (320), we deduce the induction step. This completes the proof of (290j).

**Proof of (290h), (290i), (290k), and (290):** In the inertial coordinate system $\{x^\mu\}_{\mu = 0, 1, 2, 3}$, inequality (290h) follows from the definition (147b) of $\mathcal{E}$ and $\mathcal{B}$, (290d) and (290f), the Leibniz rule, and the fact that the coordinate functions $x^\mu$ satisfy $|x^\mu| \leq s$, $|\nabla x^\mu| \leq 1$, and $|\nabla x^\mu|^2 = 0$.

Inequalities (290i) follow from the fact that in the inertial coordinate system $\{x^\mu\}_{\mu = 0, 1, 2, 3}$, we have $|\nabla F|=\sum_{\mu=0}^3 |\nabla_{\mu} F|_{|\nabla|} = \sum_{\mu=0}^3 |\mathcal{L}_{\mu, F}|_{|\nabla|}$.

Inequality (290k) follows from (289b), (290j), and the fact that $(1 + |q|)\|F\| \lesssim \|F\|$ holds for any two-form $\mathcal{F}$.

Inequality (290l) then follows from (289a), (290f), (290g), (290h), (290k), and that fact that $q = s = r$ along $\Sigma_0$.

**Proof of (291):** Let $k \geq 0$ be an integer. Then since $L = T(0) - \omega^\alpha T_{\alpha \beta}$, $\omega^\alpha \overset{\text{def}}{=} x^\alpha/r$, we use (290j) and the fact that $T(0)\omega^\beta = \omega^\beta T_{\alpha \beta} \omega^\alpha = 0$, $(j = 1, 2, 3)$, to conclude that the following inequality holds for $r > 0$:

$$\| \nabla^L F \| \lesssim (1 + |q|)^{-k} \| \mathcal{F} \|^2_{|L_{\mu}|, k}.  \quad (321)$$

Next, we observe the following identity, which holds for any two-form $\mathcal{F}$:

$$s \nabla_L \mathcal{F} = 2L_{\mu} \mathcal{F} + q \nabla_L \mathcal{F} - 4 \mathcal{F}.  \quad (322)$$
Now using the commutation properties of Lemma 6.0.7, we may iterate (322) to conclude that there exist constants $C_{abl}$ such that

$$(s \nabla_L)^l F = \sum_{b=0}^{l} \sum_{a=0}^{b} C_{abl}(q \nabla_L)^a L_{S}^{b-a} F. \tag{323}$$

From the facts that $-\nabla_L q = \nabla_L s = 2$, it follows from (323) that there exist constants $\tilde{C}_{abl}$ such that

$$s^l \nabla_L F = \sum_{b=0}^{l} \sum_{a=0}^{b} \tilde{C}_{abl} q^a \nabla_L^{b-a} F. \tag{324}$$

Furthermore, if $I$ is any rotational multi-index, then from Lemma 6.0.7, the relations $\nabla_L s = \nabla_L q = L_{O}^{0}s = L_{O}^{0}q = 0$ and $\nabla_L q = -2$, and (321), it follows that

$$s^l \left| L_{O}^{0} \nabla_L \nabla_k F \right| \lesssim \sum_{b=0}^{l} \sum_{a=0}^{b} \left| q \right|^a \left| \nabla_L^{a+k} L_{O}^{0} L_{S}^{b-a} F \right| \lesssim \sum_{b=0}^{l} \sum_{a=0}^{b} \left| q \right|^a (1 + |q|)^{(a+k)} \left| F \right|_{L_{E}^{k} k+b+l}} \lesssim (1 + |q|)^{-k} \left| F \right|_{L_{E}^{k} k+b+l}}. \tag{325}$$

Multiplying each side of (325) by $(1 + |q|)^k$ and summing over all rotational multi-indices $|I| \leq M - k - l$, we arrive at (291).

**Proof of (292a) and (292b):** Since $J_p^q[\mathcal{F}] = \tilde{Q}(\xi^{(0)}, \mathcal{K})[\mathcal{F}, \mathcal{F}]$, (292a) is a simple consequence of Lemma 7.4.1. Inequality (292b) then follows from integrating inequality (292a) over $\Sigma_i$. \hfill \Box

### A. Norms for $B$ and $D$

In this section, we introduce a weighted Sobolev norm that will be used to express our global existence smallness condition in terms of the restriction of $(B, D)$ to $\Sigma_0$, which we denote by $(\tilde{B}, \tilde{D})$.

This norm is computed using only quantities that are inherent to $\Sigma_0$. However, during the course of our global existence argument, we analyze the norm $\| F(t) \|_{L_{E}^{N} \mathcal{N}}$, and in particular that we need the smallness of $\| F(0) \|_{L_{E}^{N} \mathcal{N}}$ to close our estimates. Since this latter quantity involves derivatives that are normal to $\Sigma_0$, we need to use the MBI equations to relate the size of $\| F(0) \|_{L_{E}^{N} \mathcal{N}}$ to the size of $(\tilde{B}, \tilde{D})$ as measured by the $\Sigma_0$–inherent norm. This is accomplished in Lemma 8.1.1.

We now define the aforementioned weighted Sobolev norm.

**Definition 8.1.1 (Weighted Sobolev norm):** Let $U$ be a tensorfield tangent to the Cauchy hypersurface $\Sigma_0$. Let $d(\chi)$ denote the Riemannian distance from the origin in $\Sigma_0$ to the point $\chi \in \Sigma_0$; i.e., in a Euclidean coordinate system $\{ v_i \}_{i=1,2,3}$ on $\Sigma$, $d(\chi) = |\chi| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. Then for any integer $N \geq 0$, and any real number $\delta$, we define the $H_{\delta}^N$ norm of $U$ by

$$\| U \|_{H_{\delta}^N}^2 \overset{\text{def}}{=} \sum_{n=0}^{N} \int_{\Sigma_0} (1 + d^2(\chi))^{(d+n)} |\nabla_m U(\chi)|^2 \, d^3 \chi. \tag{326}$$

As discussed in detail in Ref. 9, the following norm, which is used in our proof of Lemma 8.1.2, can be bounded by a suitable choice of one of the above weighted Sobolev norms; see Lemma A-2.

**Definition 8.1.2 (Weighted $C^N$ norm):** Let $U$ be a tensorfield tangent to the Cauchy hypersurface $\Sigma_0$. Let $d(\chi)$ be as in Definition 8.1.1. Then for any integer $N \geq 0$, and any real number $\delta$, we define
the $C^N$ norm of $U$ by
\[ \|U\|_{C^N}^2 \overset{\text{def}}{=} \sum_{n=0}^{N} \sup_{\Sigma_0} (1 + d^2(x))^{(d+n)} |\nabla_{(n)} U(x)|^2. \]  
\[ \tag{327} \]

The next lemma is used in the proof of the subsequent lemma, which is the main result of this section.

**Lemma 8.1.1 (An integral comparison estimate):** Let $N \geq 0$ be an integer. Let $\mathcal{F}(t, x)$ be a two-form, and let $(E(t, x), B(t, x))$ be its electromagnetic decomposition as defined in Sec. V H. Then
\[ \| \mathcal{F}(0) \|_{L^2; N} \approx \sum_{n=0}^{N} \int_{\mathbb{R}^d} (1 + |y|^{d+1}) (|\nabla_{(n)} E(0, y)|^2 + |\nabla_{(n)} B(0, y)|^2) d^d y. \]  
\[ \tag{328} \]

**Remark 8.1.1:** Note that on the right-hand side of (328), $\nabla_{(n)}$ denotes the full spacetime covariant derivative operator of order $n$. In particular, $\nabla_{(n)} E$ and $\nabla_{(n)} B$ involve both tangential and normal derivatives of $E$ and $B$.

**Proof:** Inequality (328) follows directly from integrating (290) over $\Sigma_0$. \qed

We now state and prove the main result of this section.

**Lemma 8.1.2 (A comparison estimate for $\| \mathcal{F}(0) \|_{L^2; N}$ and $\| (\hat{B}, \hat{D}) \|_{H^N}$):** Let $N \geq 2$ be an integer: Let $\mathcal{F}$ be a solution to the MBI system, and let $(\hat{B}, \hat{D})$ be its electromagnetic decomposition along the Cauchy hypersurface $\Sigma_0$ as defined in Sec. V H. Let $\| \cdot \|_{L^2; N}$ and $\| \cdot \|_{H^N}$ be the weighted integral norms defined in (280) and (326), respectively. Then there exists a constant $\epsilon > 0$ such that if $\| (\hat{B}, \hat{D}) \|_{H^N} < \epsilon$, then
\[ \| \mathcal{F}(0) \|_{L^2; N} \approx \| (\hat{B}, \hat{D}) \|_{H^N}. \]  
\[ \tag{329} \]

**Proof:** We first recall that by (151a), we have that
\[ \hat{E} = \frac{\hat{D} + \hat{B} \times (\hat{D} \times \hat{B})}{(1 + |\hat{B}|^2 + |\hat{D}|^2 + |\hat{D} \times \hat{B}|^2)^{1/2}}. \]  
\[ \tag{330} \]

With the help of Corollary A-5, it follows from (330) that if $\epsilon$ is sufficiently small, then
\[ \| (\hat{B}, \hat{D}) \|_{H^N} \approx \| (\hat{E}, \hat{B}) \|_{H^N}. \]  
\[ \tag{331} \]

We now introduce the abbreviations $V \overset{\text{def}}{=} (B, D), \hat{V} \overset{\text{def}}{=} (\hat{B}, \hat{D})$ and rewrite the MBI evolution equations (153a) and (153b) as
\[ \nabla_{T_\epsilon} V = \mathcal{L}(\nabla V) + \mathfrak{F}(V, \nabla V), \]  
\[ \tag{332} \]

where $\mathcal{L}(\cdot)$ is a constant linear map (i.e., a constant-coefficient matrix), and $\mathfrak{F}(V, \nabla V)$, which consists of “cubic-order” error terms, is a smooth function $V, \nabla V$. We will make use of the following consequence of Lemma A-2:
\[ \| V \|_{C^{N-2}} \lesssim \| V \|_{H^N}. \]  
\[ \tag{333} \]

We now repeatedly differentiate (332) with $\nabla_{T_\epsilon}$ and $\nabla$, use the resulting equations to replace $\nabla_{T_\epsilon}$ derivatives with $\Sigma_0$ — tangential derivatives, and use (333) to inductively conclude that if $\epsilon$ is sufficiently small and $0 \leq n \leq N$, then
\[ |\nabla_{(n)} V|_{n=0} \lesssim |\nabla_{(n)} \hat{V}| + \sum_{i=1}^{n} \prod_{j=1}^{m_i} |\nabla_{(m_j)} \hat{V}|, \]  
\[ \tag{334} \]

where the sum is taken over all non-negative integers $m_1, m_2, \ldots, m_n$ such that $\sum_{i=1}^{n} m_i = n$. We remark that the implicit constant in (334) depends on $\mathfrak{F}$ and its first $n$ derivatives with respect to
V, ∇V, and that we have used (333) to bound non-differentiated factors of ˙V in L∞ from above by Cε.

Multiplying each side of (334) by (1 + |x|²)ⁿ⁺¹/² = (1 + |x|²)(1 + ∑ₘ=₁ᵐₗₘ)/², squaring, integrating, and using Corollary A-4, we deduce that

\[
\int_{\mathbb{R}^3} (1 + |y|²)^{n+1} |\nabla_n V(0, y)|² d³y \lesssim \sum_{a=0}^N \int_{\mathbb{R}^3} (1 + |y|²)^{n+1} |\nabla_n \tilde{V}(y)|² d³y \lesssim \| (\tilde{B}, \tilde{D}) \|^²_{H^n_1}.
\]

We have thus shown that for sufficiently small ε, we have

\[
\sum_{n=0}^N \int_{\mathbb{R}^3} (1 + |y|²)^{n+1} (|\nabla_n B(0, y)|² + |\nabla_n D(0, y)|²) d³y \approx \| (\tilde{B}, \tilde{D}) \|^²_{H^n_1},
\]

the ≳ direction inequality being trivial. We remark that the left-hand side of (336) involves normal derivatives, while the right-hand side involves only Σ₀⁻ tangential derivatives.

Similarly, with the help of the relation (151a), it is straightforward to prove the following inequalities, which are valid for all sufficiently small ε:

\[
\int_{\mathbb{R}^3} (1 + |y|²)^{n+1} (|\nabla_n E(0, y)|² + |\nabla_n B(0, y)|²) d³y \approx \| (\tilde{E}, \tilde{B}) \|^²_{H^n_1}.
\]

From (331), (336) and (337), it follows that

\[
\sum_{n=0}^N \int_{\mathbb{R}^3} (1 + |y|²)^{n+1} (|\nabla_n E(0, y)|² + |\nabla_n B(0, y)|²) d³y \approx \| (\tilde{B}, \tilde{D}) \|^²_{H^n_1}.
\]

The desired inequalities (329) now follow from (328) and (338).

\[\Box\]

**B. An expression for \( \nabla_\mu (J_\mu^\nu [\tilde{\mathcal{F}}]) \)**

In Sec. XI, we will bound (from above) the time derivative of the energy \( E_\mathcal{N}[\mathcal{F}] \) defined in (281). By the divergence theorem, the analysis amounts to estimating the \( L^1 \) norms of the quantities \( \nabla_\mu (\tilde{J}_\mu^\nu [\tilde{\mathcal{F}}]) \), where \( \tilde{\mathcal{F}} \) is a solution to the equations of variation (186a) and (186b).

To begin, we compute that for any vectorfield \( X \),

\[
\nabla_\mu (\tilde{Q}_\mu^\nu X^\nu) = (\nabla_\mu \tilde{Q}_\mu^\nu) X^\nu + \frac{1}{2} \tilde{Q}_\mu^{\nu\sigma} \pi_{\mu\sigma} + \frac{1}{2} (\tilde{Q}_\mu^{\mu\nu} - \tilde{Q}_\nu^{\mu\mu}) \nabla_\mu X^\nu,
\]

where \( (X) \pi_{\mu\nu} \) is defined in (44). Now according to (92c), we have that \( (X) \pi_{\mu\nu} = 4t g_{\mu\nu} \). Therefore, by (196), the \( \tilde{Q}_\mu^{\mu\nu} (X) \pi_{\mu\nu} \) term on the right-hand side of (339) vanishes. If we also make use of equations (197) and (198), then we easily arrive at the following lemma.

**Lemma 8.2.1 (An expression for \( \nabla_\mu (J_\mu^\nu [\tilde{\mathcal{F}}]) \))** Let the two-form \( \tilde{\mathcal{F}} \) be a solution to the equations of variation (186a) and (186b) associated with the background two-form \( \mathcal{F} \): Let
$J^\mu [\mathcal{F}] = -\tilde{Q}_\mu \mathcal{K}^\nu [\mathcal{F}, \mathcal{F}]$ be the energy current vectorfield defined in (279). Then

$$\nabla_\mu (J^\mu [\mathcal{F}]) = \frac{1}{2} \mathcal{H}^{\mu \nu \kappa \lambda} \mathcal{F}_{\nu \kappa \lambda} \mathcal{K}^\mu - \mathcal{K}^\mu \mathcal{F}_{\nu \eta} \mathcal{Y}^\eta - (\nabla_\mu \mathcal{H}^{\mu \nu \kappa \lambda}) \mathcal{F}_{\nu \kappa \lambda} \mathcal{F}_{\nu \kappa \lambda} \mathcal{K}^\mu + \frac{1}{4} (\mathcal{K}^{\nu} \nabla_\nu \mathcal{H}^{\mu \nu \kappa \lambda}) \mathcal{F}_{\nu \kappa \lambda} \mathcal{F}_{\nu \kappa \lambda} \mathcal{K}^\mu$$

$$- \frac{1}{4} \left[ (1 + \frac{1}{2} \mathcal{C}_{\text{MBI}} \mathcal{H}) \mathcal{F}^{\nu \kappa \lambda} \mathcal{F}_{\nu \kappa \lambda} \mathcal{F}^{\mu \kappa \lambda} \mathcal{F}_{\mu \kappa \lambda} \mathcal{K}^\mu - \mathcal{F}^{\nu \kappa \lambda} \mathcal{F}_{\nu \kappa \lambda} \mathcal{F}^{\mu \kappa \lambda} \mathcal{F}_{\mu \kappa \lambda} \mathcal{K}^\mu \right] \nabla_\mu \mathcal{K}^\nu. \quad (340)$$

Remark 8.2.1: Because of (187a), the $\frac{1}{2} \mathcal{H}^{\mu \nu \kappa \lambda} \mathcal{F}_{\nu \kappa \lambda} \mathcal{K}^\mu$ term on the right-hand side of (340) is 0 for all of the variations of interest in this article.

IX. THE SPECIAL NULL STRUCTURE AND GEOMETRIC/ALGEBRAIC ESTIMATES OF THE NONLINEARITIES

In this section, we provide a partial null decomposition of the terms appearing on the right-hand side of Eq. (340), where $\mathcal{F}$ is equal to one of the iterated Lie derivatives $L^j_\mathcal{F}$ of a solution $\mathcal{F}$ to the MBI system. By “partial,” we mean that we can prove our desired estimates without performing a fully detailed null decomposition. These geometric/algebraic estimates form the backbone of the proof of Proposition 11.0.1, which is our main energy estimate. It is in this section that the special null structure is revealed; as we will see, and as is described at the end of Sec. 1C, the worst possible combinations of terms are absent from the right-hand side of (340).

We begin with the following simple lemma, which shows that covariant and Lie derivatives of null form expressions can also be expressed in terms of null forms.

Lemma 9.0.2 (Leibniz rule for null forms): Let $m \geq 0$ be an integer, and let $Q_{i,j} (\cdot, \cdot)$, $i = 1, 2$, denote the null forms defined in (120a) and (120b). Then for any vectorfield $X$ and any pair of two-forms $\mathcal{F}$, $\mathcal{G}$, and $i = 1, 2$, we have that

$$\nabla_X^m (Q_{i,j} (\mathcal{F}, \mathcal{G})) = \sum_{a+b=m} \binom{m}{a} Q_{i,j} (\nabla_X^a \mathcal{F}, \nabla_X^b \mathcal{G}). \quad (341)$$

In the above expression, $\nabla_X^m \overset{\text{def}}{=} \nabla_X \circ \nabla_X \circ \cdots \circ \nabla_X$.

Furthermore, under the convention of Remark 2.2.1, for any $Z$-multi-index $I$, there exist constants $C_{I_1, I_2}$, such that

$$L^I_Z (Q_{i,j} (\mathcal{F}, \mathcal{G})) = \sum_{|I_1| + |I_2| \leq |I|} C_{I_1, I_2} Q_{i,j} (L^I_{Z_1} \mathcal{F}, L^I_{Z_2} \mathcal{G}). \quad (342)$$

Proof: To prove (341) in the case $i = 1$, we use the Leibniz rule and the fact that $\nabla_X g = 0$. In the case $i = 2$, we use similar reasoning, plus Lemma 4.0.1.

To prove (342) in the case $i = 1$, we use the Leibniz rule and the fact that (94b) holds for any $Z \in Z$. In the case $i = 2$, we use similar reasoning, plus Corollary 4.0.3.
We now state a lemma concerning the null structure of some of the factors appearing in the terms in braces on the right-hand side of (340). This lemma is in the spirit of Lemma 5.4.1.

**Lemma 9.0.3 (Null structure estimate I):** If $\mathcal{F}$, $\mathcal{G}$ are a pair of two-forms, then
\[
|\mathcal{F}^\xi \mathcal{G}_{\mu\nu} \nabla^\mu H_{\Delta}^{\mu\nu\xi}| \lesssim s(|\mathcal{F}|_{\mathcal{L}^2} |\mathcal{G}| + |\mathcal{F}| |\mathcal{G}|_{\mathcal{L}^4} + |\mathcal{F}|_{\mathcal{T}^2} |\mathcal{G}|_{\mathcal{T}^2}).
\]

**Proof:** Lemma 9.0.3 follows from (114) together with the null decomposition of $\nabla \mathcal{K}$ given in (136a)–(136i).

The next lemma is a technical precursor to the subsequent one.

**Lemma 9.0.4 (Null structure estimate II):** Let $N \geq 4$ be an integer. Let $\mathcal{F}$ be a two-form, and let $H_{\Delta}^{\mu\nu\xi\lambda}$ be the ($\mathcal{F}$–dependent) tensor field defined in (86b). Suppose that $J$, $J'$ are $\mathbb{Z}$–multi-indices, and that $|J| \leq N$. Then there exists a constant $\epsilon > 0$ such that if $|\mathcal{F}|_{\mathcal{L}^2;[N/2]} \leq \epsilon$, then the following pointwise estimate holds for $\xi = 0, 1, 2, 3$ relative to the inertial coordinate system:
\[
\left|(\mathcal{L}_2^J H_{\Delta}^{\mu\nu\xi\lambda}) \nabla_\mu \mathcal{L}_2^{J'} \mathcal{F}_{\xi\lambda}\right| \lesssim \sum_{|J_1| + |J_2| \leq |J|} |\mathcal{L}_{22}^J \mathcal{F}| |\mathcal{L}_2^{J',\mathcal{L}_2^J \mathcal{F}} + |\mathcal{L}_2^J \mathcal{F}|_{\mathcal{L}^2} |\mathcal{L}_2^J \mathcal{F}||\mathcal{L}_2^{J'} \mathcal{F}||\mathcal{T}_2^{J'} \mathcal{F}||\mathcal{T}_2^J \mathcal{F}||_{\mathcal{T}^2}.
\]

**Remark 9.0.2:** In the above inequality, $[N/2]$ denotes the largest integer less than or equal to $N/2$, and the seminorms $| \cdot |_{\mathcal{L}^2; \mathcal{L}^4; \mathcal{T}^2}$, etc., are defined in Definition 5.1.2.

**Proof:** We begin by decomposing $H_{\Delta}^{\mu\nu\xi\lambda} = (i)\mu\nu\xi\lambda + (ii)\mu\nu\xi\lambda + (iii)\mu\nu\xi\lambda + (iv)\mu\nu\xi\lambda$, where
\[
(i)\mu\nu\xi\lambda \overset{\text{def}}{=} -\frac{1}{2} (g^{-1})^\mu\nu \tilde{g}^\xi \tilde{g}^\lambda (g^{-1})^\xi\lambda \mathcal{F}^\mu_{\mathcal{L}^2} \mathcal{F}^\nu_{\mathcal{L}^2},
\]
\[
(ii)\mu\nu\xi\lambda \overset{\text{def}}{=} -\frac{1}{2} (g^{-1})^\mu\nu \tilde{g}^\xi \tilde{g}^\lambda (g^{-1})^\xi\lambda \mathcal{F}^\mu_{(\mathcal{L}^2)^2} \mathcal{F}^\nu_{(\mathcal{L}^2)^2} \mathcal{F}^\xi\lambda,
\]
\[
(iii)\mu\nu\xi\lambda \overset{\text{def}}{=} -\frac{1}{2} (g^{-1})^\mu\nu \tilde{g}^\xi \tilde{g}^\lambda (g^{-1})^\xi\lambda \mathcal{F}^\mu_{(\mathcal{L}^2)^2} \mathcal{F}^\nu_{(\mathcal{L}^2)^2} \mathcal{F}^\xi\lambda,
\]
\[
(iv)\mu\nu\xi\lambda \overset{\text{def}}{=} -\frac{1}{2} (g^{-1})^\mu\nu \tilde{g}^\xi \tilde{g}^\lambda (g^{-1})^\xi\lambda \mathcal{F}^\mu_{(\mathcal{L}^2)^2} \mathcal{F}^\nu_{(\mathcal{L}^2)^2} \mathcal{F}^\xi\lambda \left(1 + \mathcal{L}^2_{\mathcal{L}^2} \mathcal{F}^\mu_{\mathcal{L}^2} \mathcal{F}^\nu_{\mathcal{L}^2} \mathcal{F}^\xi\lambda \right).
\]

Note that to avoid the possible confusion described in Remark 2.2.1, we have lowered all of the indices on $\mathcal{F}$ in preparation for Lie differentiation. We now claim that if $|\mathcal{F}|_{\mathcal{L}^2;[N/2]}$ is sufficiently small, then
\[
\left|(\mathcal{L}_2^J H_{\Delta}^{\mu\nu\xi\lambda}) \nabla_\mu \mathcal{L}_2^{J'} \mathcal{F}_{\xi\lambda}\right| \lesssim \sum_{i=1}^2 \sum_{|I_1| + |I_2| \leq |I|} |\mathcal{L}_2^{I_1} \mathcal{F}| |\mathcal{L}_2^{I_2} \mathcal{F}, \mathcal{L}_2^{J'} \mathcal{F}||\mathcal{L}_2^{J'} \mathcal{F}||\mathcal{T}_2^{J'} \mathcal{F}||\mathcal{T}_2^J \mathcal{F}||_{\mathcal{T}^2}.
\]

Inequality (344) then follows from (350) and Lemma 5.4.1.

Since the analysis is the roughly the same for each piece, we will focus only on the term $(iv)$. Differentiating the term $(iv)$ with $\mathcal{L}_2^J$, contracting with $\nabla_\mu \mathcal{L}_2^{J'} \mathcal{F}_{\xi\lambda}$, and using (94b) plus Corollary 4.0.3, we see that a typical term that arises after expanding via the Leibniz rule is of the form
\[
(g^{-1})^\mu\nu \tilde{g}^\xi \tilde{g}^\lambda (g^{-1})^\xi\lambda \left[\mathcal{L}_2^J \left[1 + \mathcal{L}^2_{\mathcal{L}^2} \mathcal{F}^\mu_{\mathcal{L}^2} \mathcal{F}^\nu_{\mathcal{L}^2} \mathcal{F}^\xi\lambda \right] \nabla_\mu \mathcal{L}_2^{J'} \mathcal{F}_{\xi\lambda}\right],
\]
where $|I_1| + |I_2| \leq |I|$, Note that the factor $(g^{-1})^\mu\nu \tilde{g}^\xi \tilde{g}^\lambda \mathcal{L}_2^{J'} \mathcal{F}_{\xi\lambda}$ is equal to $\mathcal{Q}_{(\mathcal{L}_2^J \mathcal{F}, \mathcal{L}_2^{J'} \mathcal{F})}$. Now among the remaining factors...
that if \( f \) is bounded in magnitude from above by the right-hand side of (350). This completes the proof for (9.0.5), it follows that

\[
\text{Corollary 4.0.3, if } H_{\mu}^\mathfrak{c} \text{, then (9.0.6), }
\]

\[
\text{Lemma 9.0.5 (Null structure estimate III): Let } N \geq 4 \text{ be an integer, let } F \text{ be a two-form, and let } H_{\mu}^\mathfrak{c} \text{ be the } (F-\text{dependent) tensorfield defined in (86b). Then there exists a constant } \epsilon > 0 \text{ such that if } |F|_{Z;|N/2|} \leq \epsilon, \text{ and } I \text{ is any } Z-\text{multi-index satisfying } |I| \leq N, \text{ then the following pointwise estimate holds:}
\]

\[
\left| H_{\Delta}^{\mu \mathfrak{c}} \nabla_{\mu} \left( L_{Z}^{I} F_{\kappa \lambda} \right) - \hat{L}_{Z}^{I} \left( H_{\Delta}^{\mu \mathfrak{c}} \nabla_{\mu} F_{\kappa \lambda} \right) \right| (L_{Z}^{I} F_{\nu \eta}) \mathcal{K}^{\nu}
\]

\[
\lesssim \left\{ 1 + s \right\}^{2} |F|_{\ell 2} + (1 + |q|)^{2} |L_{Z}^{I} F| \sum_{|J|+|I| \leq |I| \leq N/2} \left| \left( L_{Z}^{I} H_{\Delta}^{\mu \mathfrak{c}} \nabla_{\mu} L_{Z}^{I} F_{\kappa \lambda} \right) \right|
\]

\[
+ (1 + |q|)^{2} |L_{Z}^{I} F| \sum_{|J| \leq |I| \leq N/2} \left| \left( L_{Z}^{I} H_{\Delta}^{\mu \mathfrak{c}} \nabla_{\mu} L_{Z}^{I} F_{\kappa \lambda} \right) \right|.
\]

\[
\text{Proof: From the definition (95) of } \hat{L}_{Z}^{I}, \text{ the null decomposition (135a)–(135c) for } \mathcal{K}, \text{ and Lemma 6.0.5, it follows that}
\]

\[
\left| H_{\Delta}^{\mu \mathfrak{c}} \nabla_{\mu} \left( L_{Z}^{I} F_{\kappa \lambda} \right) - \hat{L}_{Z}^{I} \left( H_{\Delta}^{\mu \mathfrak{c}} \nabla_{\mu} F_{\kappa \lambda} \right) \right| (L_{Z}^{I} F_{\nu \eta} \mathcal{K}^{\nu})
\]

\[
\lesssim \left\{ 1 + s \right\}^{2} |F|_{\ell 2} \sum_{|J|+|I| \leq |I| \leq N/2} \left| \left( L_{Z}^{I} H_{\Delta}^{\mu \mathfrak{c}} \nabla_{\mu} L_{Z}^{I} F_{\kappa \lambda} \right) \right|
\]

\[
+ (1 + |q|)^{2} |L_{Z}^{I} F| \sum_{|J| \leq |I| \leq N/2} \left| \left( L_{Z}^{I} H_{\Delta}^{\mu \mathfrak{c}} \nabla_{\mu} L_{Z}^{I} F_{\kappa \lambda} \right) \right|.
\]

Inequality (352) now follows from applying inequality (344) to the terms on the right-hand side of (353).

\[
\text{Lemma 9.0.6: Under the assumptions of Lemma 9.0.5, we have that}
\]

\[
\sum_{|I| \leq N} \left| H_{\Delta}^{\mu \mathfrak{c}} \nabla_{\mu} \left( L_{Z}^{I} F_{\kappa \lambda} \right) - \hat{L}_{Z}^{I} \left( H_{\Delta}^{\mu \mathfrak{c}} \nabla_{\mu} F_{\kappa \lambda} \right) \right| (L_{Z}^{I} F_{\nu \eta} \mathcal{K}^{\nu})
\]

\[
\lesssim \sum_{|I| \leq N/2} \left( 1 + s \right) \left| \left( L_{Z}^{I} F_{\nu \eta} \mathcal{K}^{\nu} \right) \right|
\]

\[
+ \left( 1 + |q| \right) \left| \left( L_{Z}^{I} F_{\nu \eta} \mathcal{K}^{\nu} \right) \right|
\]

\[
+ \sum_{|J| \leq N/2} \left( 1 + |q| \right) \left| \left( L_{Z}^{I} F_{\nu \eta} \mathcal{K}^{\nu} \right) \right|.
\]
Proof: Inequality (354) follows from (352) and from simple algebraic estimates of the form \( ab \lesssim a^2 + b^2 \).

The following lemma will be used to control the terms \((\nabla_{\mu} H^{\mu_{\xi\lambda}})_{\xi\lambda,\xi} \hat{F}_{\xi} \hat{F}_{\xi} \hat{K}^{-}\) appearing on the right-hand side of (340).

Lemma 9.0.7 (Null structure estimate IV): Let \( \hat{F}, \hat{F} \) be a pair of two-forms, and let \( H^{\mu_{\xi\lambda}}_{\Delta} \) be the \((\hat{F}-\text{dependent})\) tensor field defined in (86b). Then there exists a constant \( \epsilon > 0 \) such that if \( |\hat{F}|, |\nabla \hat{F}| \leq \epsilon \), then the following pointwise estimates hold:

\[
|\nabla_{\mu} H^{\mu_{\xi\lambda}}_{\Delta} \hat{F}_{\xi}, \hat{F}_{\xi} \hat{F}_{\xi} \hat{K}^{-}| \lesssim (1 + s)^2 \sum_{|l| \leq 1} \left| L_{L}^{2} \hat{F}_{2}^{2} \hat{F}^{2} \right| + \left| L_{T}^{2} \hat{F}_{2}^{2} \hat{F}^{2} \right| + \left| L_{T}^{2} \hat{F}_{2}^{2} \hat{F}^{2} \right|
\]

\[
+ (1 + |q|)^{2} \sum_{|l| \leq 1} \left| L_{2}^{2} \hat{F}^{2} \hat{F}^{2} \right|,
\]

\[
|\hat{F}^{v} \nabla_{\mu} H^{\mu_{\xi\lambda}}_{\Delta} \hat{F}_{\xi}, \hat{F}_{\xi} \hat{F}_{\xi} | \lesssim (1 + s)^2 \sum_{|l| \leq 1} \left| L_{L}^{2} \hat{F}_{2}^{2} \hat{F}^{2} \right| + \left| L_{T}^{2} \hat{F}_{2}^{2} \hat{F}^{2} \right| + \left| L_{T}^{2} \hat{F}_{2}^{2} \hat{F}^{2} \right|
\]

Proof: Consider the decomposition

\[
\nabla_{\mu} H^{\mu_{\xi\lambda}}_{\Delta} = \nabla_{\mu}(i^{\xi\lambda}) + \nabla_{\mu}(ii^{\xi\lambda}) + \nabla_{\mu}(iii^{\xi\lambda}) + \nabla_{\mu}(iv^{\xi\lambda})
\]

implied by (346)–(349). We will focus only on the case of term \((i)\); terms \((ii) - (iv)\) can be handled similarly. We now further decompose

\[
\nabla_{\mu}(i^{\xi\lambda}) = (i'')^{\xi\lambda} + (i''')^{\xi\lambda},
\]

where

\[
(i'')^{\xi\lambda} \overset{\text{def}}{=} \ell^{-2} (\nabla_{\mu} \hat{F}^{\xi\lambda}) \hat{F}^{\xi\lambda},
\]

\[
(i''')^{\xi\lambda} \overset{\text{def}}{=} (\nabla_{\mu} \ell^{-2} (\nabla_{\mu} \hat{F}^{\xi\lambda})) \hat{F}^{\xi\lambda}.
\]

From (114), the null decomposition (135a)–(135c) for \( \hat{K} \), (289b) with \( M = 1 \), and the fact that \( \nabla_{\mu} \hat{F}^{\xi\lambda} = \hat{L}_{\mu\nu} \hat{F}^{\xi\lambda} \) in the inertial coordinate system, it follows that if \( \epsilon \) is sufficiently small, then

\[
\left| (i'')^{\xi\lambda} \hat{F}_{\xi}, \hat{F}_{\xi} \hat{K}^{-} \hat{F}_{\xi} \right| \lesssim \sum_{\mu=0}^{3} (1 + s)^{2} \left| \hat{F}_{2}^{2} \hat{F}^{2} \right| + (1 + |q|)^{2} \left| \hat{F}_{2}^{2} \hat{F}^{2} \right| \left| Q_{(1)}(\hat{F}, \hat{F}) \right|,
\]

\[
\left| (i'')^{\xi\lambda} \hat{F}_{\xi}, \hat{F}_{\xi} \hat{F}_{\xi} \hat{K}^{-} \hat{F}_{\xi} \right| \lesssim \sum_{\mu=0}^{3} (1 + s)^{2} \left| \hat{F}_{2}^{2} \hat{F}^{2} \right| + (1 + |q|)^{2} \left| \hat{F}_{2}^{2} \hat{F}^{2} \right| \left| Q_{(1)}(\hat{F}, \hat{F}) \right|,
\]

where the \( Q_{(1)}(\cdot, \cdot) \) terms arise from the \( \kappa, \lambda \) indices. Also applying the null structure estimates of Lemma 9.0.2 to the \( Q_{(1)}(\cdot, \cdot) \) terms and using simple algebraic estimates of the form \(|ab| \lesssim a^2 + b^2|\), we conclude that each of the right-hand sides of (362)–(364) are \( \lesssim \) the right-hand side of (355). Consequently, the same is true of \( \left| \nabla_{\mu}(i^{\xi\lambda}) \hat{F}_{\xi}, \hat{F}_{\xi} \hat{K}^{-} \hat{F}_{\xi} \right| \). Note in particular that our estimates for the terms in (362)–(364) involving the \((1 + |q|)^2\) factor are not optimal; we have simply bounded them by \((1 + |q|)^2 \sum_{|l| \leq 1} L_{2}^{2} \hat{F}^{2} \hat{F}^{2} \). The cases \((ii) - (iv)\) can be handled similarly. Also note that because of the \( \t^{2}_{1} \) and \( \t^{2}_{2} \) terms in (347)–(349), these cases involve terms which are order-6 and order-8 in \( \hat{F} \). Because of our smallness assumption on \( \hat{F} \), the “additional” powers of \( \hat{F} \) afforded
by $\zeta(2)$ and $\zeta^2(2)$, can be bounded by small constants, and the analysis can proceed as in case (i). This completes our proof of (355).

To prove (356), we first consider the decomposition
\[
(K^\nu \nabla_\Lambda H_\Lambda^{\zeta\eta\kappa\lambda})\mathcal{F}_{\zeta\eta\kappa\lambda} = [K^\nu \nabla_\Lambda (i)^{\zeta\eta\kappa\lambda}]\mathcal{F}_{\zeta\eta\kappa\lambda} + [K^\nu \nabla_\Lambda (ii)^{\zeta\eta\kappa\lambda}]\mathcal{F}_{\zeta\eta\kappa\lambda} + [K^\nu \nabla_\Lambda (iii)^{\zeta\eta\kappa\lambda}]\mathcal{F}_{\zeta\eta\kappa\lambda} + [K^\nu \nabla_\Lambda (iv)^{\zeta\eta\kappa\lambda}]\mathcal{F}_{\zeta\eta\kappa\lambda}
\] (365)
implied by (346)–(349). We will focus only on the case of term (i); terms (ii) – (iv) can be handled similarly. We now further decompose
\[
[K^\nu \nabla_\Lambda (i)^{\zeta\eta\kappa\lambda}] = (I')^{\zeta\eta\kappa\lambda} + (I'')^{\zeta\eta\kappa\lambda} + (I''')^{\zeta\eta\kappa\lambda},
\] (366)
where
\[
(I')^{\zeta\eta\kappa\lambda} \overset{\text{def}}{=} \ell_{(MBI)}^{-2} (K^\nu \nabla_\Lambda F^{\zeta\eta\kappa\lambda}),
\] (367)
\[
(I'')^{\zeta\eta\kappa\lambda} \overset{\text{def}}{=} \ell_{(MBI)}^{-2} F^{\zeta\eta\kappa\lambda} K^\nu \nabla_\Lambda F^{\zeta\eta\kappa\lambda},
\] (368)
\[
(I''')^{\zeta\eta\kappa\lambda} \overset{\text{def}}{=} (K^\nu \nabla_\Lambda \ell_{(MBI)}^{-2}) F^{\zeta\eta\kappa\lambda}.
\] (369)
From the null decomposition $K^\nu \nabla_\nu = \frac{1}{2} \{(1 + s^2) \nabla_L + (1 + q^2) \nabla_{L L}\}$, it follows that if $\epsilon$ is sufficiently small, then
\[
||(I')^{\zeta\eta\kappa\lambda} \mathcal{F}_{\zeta\eta\kappa\lambda}|| \lesssim \left((1 + s^2) Q_{(1)}(\nabla_L \mathcal{F}, \mathcal{F}) + (1 + |q|)^2 Q_{(1)}(\nabla_{LL} \mathcal{F}, \mathcal{F}) \right) ||Q_{(1)}(\mathcal{F}, \mathcal{F})||,
\] (370)
\[
||(I'')^{\zeta\eta\kappa\lambda} \mathcal{F}_{\zeta\eta\kappa\lambda}|| \lesssim \left((1 + s^2) Q_{(1)}(\nabla_L \mathcal{F}, \mathcal{F}) + (1 + |q|)^2 Q_{(1)}(\nabla_{LL} \mathcal{F}, \mathcal{F}) \right) ||Q_{(1)}(\mathcal{F}, \mathcal{F})||,
\] (371)
\[
||(I'''')^{\zeta\eta\kappa\lambda} \mathcal{F}_{\zeta\eta\kappa\lambda}|| \lesssim \sum_{i=1}^{2} \left((1 + s^2) Q_{(i)}(\nabla_L \mathcal{F}, \mathcal{F}) + (1 + |q|)^2 Q_{(i)}(\nabla_{LL} \mathcal{F}, \mathcal{F}) \right) ||Q_{(1)}(\mathcal{F}, \mathcal{F})||^2,
\] (372)
where the $Q_{(1)}(\nabla_L \mathcal{F}, \mathcal{F}), Q_{(1)}(\nabla_{LL} \mathcal{F}, \mathcal{F})$ terms in (370) arise from the $\eta, \zeta$ indices, the $Q_{(1)}(\mathcal{F}, \mathcal{F})$ terms in (370) arise from the $\kappa, \lambda$ indices, the estimate (371) follows from (370) by interchanging the roles of $\eta, \zeta$ and $\kappa, \lambda$, the $|Q_{(1)}(\mathcal{F}, \mathcal{F})|^2$ term in (372) arises from both the $\eta$ and $\kappa$ indices, and the $Q_{(i)}(\nabla_L \mathcal{F}, \mathcal{F}), Q_{(i)}(\nabla_{LL} \mathcal{F}, \mathcal{F})$ terms in (372) arise from the fact that $\ell_{(MBI)}$ can be expressed as a function of null forms; see (80) and Remark 5.4.1. We remark that in deriving (370)–(372), we have again used the smallness assumption on $\mathcal{F}$, which in particular implies that $\ell_{(MBI)} \approx 1$ and that order-4 (in $\mathcal{F}$) terms can be bounded by a small constant times order-2 terms. Also applying the null structure estimates of Lemma 5.4.1 to the $Q_{(i)}(\cdot, \cdot)$ terms, using (290), and using simple algebraic estimates of the form $|ab| \lesssim a^2 + b^2$, we conclude that each of the right-hand sides of (370)–(372) are $\lesssim$ the right-hand side of (356) if $\epsilon$ is sufficiently small. Therefore, the same is true of $[K^\nu \nabla_\Lambda (i)^{\zeta\eta\kappa\lambda}]$. We remark that we do not need to uncover the full null structure of the right-hand sides of (370)–(372) in order to conclude the desired estimates; rather, we only need the fact that the right-hand sides of (370)–(372) are at least quadratic in the $Q_{(i)}(\cdot, \cdot)$. The cases (ii) – (iv) can be handled similarly. 

Finally, the last lemma in this section will be used to control the terms inside braces on the right-hand side of (340).

Lemma 9.0.8 (Null structure estimate V): There exists a constant \( \epsilon > 0 \) so that if \( |F|, |\nabla F| \leq \epsilon \), then the following pointwise estimate holds:

\[
\left| \ell^{2}_{(M_{BI})} F_{\alpha \lambda} \frac{F_{\nu \xi}}{F_{\nu \xi}^{\mu}} \left( \frac{\nabla^{\nu} x}{\nabla^{\nu} x} - \frac{\nabla^{\nu} x}{\nabla^{\nu} x} \right)^{\mu} \right| 
+ \left( 1 + \ell^{2}_{(M_{BI})} \right) F_{\alpha \lambda} \frac{F_{\nu \xi}}{F_{\nu \xi}^{\mu}} \left( \frac{\nabla^{\nu} x}{\nabla^{\nu} x} - \frac{\nabla^{\nu} x}{\nabla^{\nu} x} \right)^{\mu} 
\]

\[
\leq s \left( |F|_{L^{2}}^{2} |F|^{2} |F|_{L^{2}}^{2} + |F|_{L^{2}}^{4} |F|_{L^{2}}^{2} \right) 
\]

\[ \text{Proof:} \text{ Inequality (373) follows from (119a)–(119d), Lemma 5.4.1, and Lemma 9.0.3.} \]

\[ \Box \]

X. THE GLOBAL SOBOLEV INEQUALITY

In this section, we recall a version of the global Sobolev inequality that was proved in Ref. 17. This fundamental inequality allows us to deduce weighted \( L^{\infty} \) bounds for a two-form \( F \) from weighted \( L^{2} \) bounds of the quantities \( L^{\infty} F \). It provides the mechanism for deducing the \( \frac{1}{1+T} \) factor in our estimate (403). Although many of the estimates in this section were proved in Ref. 17, we reprove some of them for convenience. However, in order to derive the improved decay estimates (376a) and (376b) for the \( \alpha \) component of a solution \( F \) to the MBI system, we will have to make use of the null decomposition equation (143a). The fact that inequalities (376a) and (376b) hold is another manifestation that the nonlinearities in the MBI system have a special null structure.

Proposition 10.0.1 (Global Sobolev inequality): Let \( F \) be a two-form, and let \( \alpha, \beta, \rho, \sigma \) be its null components as defined in (115a)–(115d). Let \( M \geq 2 \) be an integer, and let \( k, l, m \) be non-negative integers verifying the inequalities stated below. Then in the interior region \( \{ (t, x) \mid |x| \leq 1 + t/2 \} \), we have that

\[
|\nabla_{m} F(t, x)| \lesssim (1 + s)^{-5/2-m} \| F \|_{L^{2}; M}, \quad m = 0, \cdots, M - 2.
\]

In the exterior region \( \{ (t, x) \mid |x| \geq 1 + t/2 \} \), we have the following estimates:

\[
|\nabla_{L}^{k} \nabla_{L}^{l} \nabla_{m} \nabla(t, x)| \lesssim (1 + s)^{-l-m} (1 + |q|)^{-3/2-k} \| F \|_{L^{2}; M}, \quad 0 \leq k + l + m \leq M - 2,
\]

\[ (375a) \]

\[
|\nabla_{L}^{k} \nabla_{L}^{l} \nabla_{m} (\alpha(t, x), \rho(t, x), \sigma(t, x))| \lesssim (1 + s)^{-l-m} (1 + |q|)^{-1/2-k} \| F \|_{L^{2}; M},
\]

\[ 0 \leq k + l + m \leq M - 2. \]

\[ (375b) \]

Furthermore, if \( F \) is a solution to the MBI system (87a) and (87b), then in the exterior region, we have the following improved estimates for \( \alpha \):

\[
|\nabla_{L}^{l} \nabla_{m} \alpha(t, x)| \lesssim (1 + s)^{-5/2-l-m} \| F \|_{L^{2}; M}, \quad 0 \leq l + m \leq M - 2,
\]

\[ (376a) \]

\[
|\nabla_{L}^{k+l} \nabla_{L}^{l} \nabla_{m} \alpha(t, x)| \lesssim (1 + s)^{-3-l-m} (1 + |q|)^{-1/2-k} \| F \|_{L^{2}; M},
\]

\[ 0 \leq k + l + m \leq M - 3 \quad (\text{if } M \geq 3). \]

\[ (376b) \]
The following corollary follows easily by using \( L^r_2 \mathcal{F} \) in place of \( \mathcal{F} \) in Proposition 10.0.1. It plays a fundamental role in our proof of Proposition 11.0.1.

**Corollary 10.0.2:** Let \( \mathcal{F} \) be any two-form, and let \( I \) be any \( \mathcal{Z} \)-multi-index such that \( |I| \leq M - 2 \). Let the pointwise seminorms \( | \cdot |_{\mathcal{W}} \) be as in Definition 5.1.2. Then with \( r \overset{\text{def}}{=} |x|, \ q \overset{\text{def}}{=} r - t, \ s \overset{\text{def}}{=} r + t, \) we have that

\[
|L^r_2 \mathcal{F}| \lesssim (1 + s)^{-1}(1 + |q|)^{-3/2} \| \mathcal{F} \|_{L^2_2}.
\]

\[
|L^r_2 \mathcal{F}|_{LU} \lesssim (1 + s)^{-1}(1 + |q|)^{-1/2} \| \mathcal{F} \|_{L^2_2}.
\]

\[
|L^r_2 \mathcal{F}|_{TT} \lesssim (1 + s)^{-2}(1 + |q|)^{-1/2} \| \mathcal{F} \|_{L^2_2}.
\]

The proof of Proposition 10.0.1 is heavily based on the next lemma, which was proved in Ref. 17.

**Lemma 10.0.3 (Weighted Sobolev embedding):** (Lemmas 2.2 and 2.3 of Ref. 17). Let \( U(x) \) be a tensorfield defined on Euclidean space \( \mathbb{R}^3 \). Then for any real number \( t \geq 1 \), we have that

\[
\sup_{|y| \geq 1 + t/2} |U(x)| \lesssim (1 + t)^{-3/2} \left( \sum_{m=0}^{2m} t^{2m} \int_{|y| \leq 1 + 3m/4} |\nabla_{(m)} U(y)|^2 d^3 y \right)^{1/2}.
\]

For all \( x \in \mathbb{R}^3 \) such that \( |x| \overset{\text{def}}{=} |r| \geq 1 \), we have that

\[
|U(x)| \lesssim r^{-3/2} \left( \int_{|y| \geq r} |U(y)|_{L^2_2}^2 + |y|^2 |\nabla_{\tilde{r}} U(y)|_{L^2_{\tilde{r},1}}^2 d^3 y \right)^{1/2},
\]

where \( \tilde{r} \overset{\text{def}}{=} \partial_r \) is the radial derivative.

For all real numbers \( t \geq 0 \) and all \( x \in \mathbb{R}^3 \) such that \( |x| \overset{\text{def}}{=} r \geq 1 + t/2 \), we have that

\[
|U(x)| \lesssim (1 + s)^{-1}(1 + |q|)^{-1/2} \left( \int_{|y| \geq 1 + t/2 + 1} |U(y)|_{L^2_2}^2 + |y|^2 |\nabla_{\tilde{r}} U(y)|_{L^2_{\tilde{r},1}}^2 d^3 y \right)^{1/2}.
\]

Before proving Proposition 10.0.1, we prove two additional technical lemmas.

**Lemma 10.0.4 (Corollary of Lemma 3.3 of Ref. 17):** Let \( M \geq 0 \) be an integer. Let \( \mathcal{F} \) be a two-form, and let \( \alpha, \ \rho, \ \sigma \) be its null components as defined in (115a)–(115d). Then with \( r \overset{\text{def}}{=} |x|, \ s \overset{\text{def}}{=} r + t, \ q \overset{\text{def}}{=} r - t, \) the following pointwise inequality holds:

\[
\sum_{k+l=0}^{M} (1 + |q|)^{s^l} \left\{ (1 + |q|)|\nabla_L^k \nabla_L^l \alpha|_{L^2_2} + s|\nabla_L^{k+1} \alpha|_{L^2_2} + s|\nabla_L^{k+1} \rho|_{L^2_2} \right\} \lesssim \| \mathcal{F} \|_{L^2_2}.
\]

**Proof:** Inequality (381) follows from Corollary 6.0.9, (290c), and (291).

**Lemma 10.0.5 (Improved pointwise estimates for \( \alpha \) and \( \nabla_L \alpha \)):** Let \( M \geq 1 \) be an integer. Let \( \mathcal{F} \) be a solution to the MBI system, and let \( \alpha, \ \rho, \ \sigma \) be its null components. Then there exists an
\[ \epsilon > 0 \text{ such that if } \| F \|_{L^2(M/2)} \leq \epsilon, \text{ then the following pointwise inequality holds in the exterior region } \{(t, x) \mid |x| \geq 1 + t/2\} : \]

\[ r^M \sum_{l=0}^{M-1} r^l |\nabla^l_L \alpha|_{L^2(M-l)} + r^2 \sum_{l=0}^{M-1} r^l |\nabla^l_L \nabla^l_L \alpha|_{L^2(M-l-1)} \lesssim \| F \|_{L^2(M).} \quad (382) \]

**Proof:** To deduce that the first sum on the left-hand side of (382) is \( \lesssim \| F \|_{L^2(M)} \), we simply apply Lemma 10.0.4. To estimate the second sum on the left-hand side of (382), we first recall Eq. (143b) satisfied by the null components of \( F \); we write the equation relative to an arbitrary coordinate system, instead of a null frame:

\[ \nabla^2_L \alpha^\mu - r^{-1} \alpha^\mu - \Psi^\mu_\rho \nabla^\nu_\sigma - \phi^\nu_\mu \nabla^\nu_\sigma \]

\[ - \frac{1}{4} \ell_{(MB)}^2 \left( (\nabla^2_L \tilde{\alpha}^i_1 - 2 \tilde{\alpha}^i_2 \nabla^i_2 \tilde{\alpha}^i_2) \alpha^\mu - 2(\phi^\nu_\mu \nabla^\nu_\sigma \tilde{\alpha}^i_2 - \phi^\nu_\nu \nabla^\nu_\sigma \tilde{\alpha}^i_2) \right) \]

\[ + (\nabla^2_L \tilde{\alpha}^i_1 - 2 \tilde{\alpha}^i_2 \nabla^i_2 \tilde{\alpha}^i_2) \Phi^\mu_\rho \]

\[ - \frac{1}{4} \ell_{(MB)}^2 \left( (\nabla^2_L \tilde{\alpha}^i_1 - 2 \tilde{\alpha}^i_2 \nabla^i_2 \tilde{\alpha}^i_2) \Phi^\nu_\nu \nabla^\nu_\nu \tilde{\alpha}^i_2 - \phi^\nu_\nu \nabla^\nu_\nu \tilde{\alpha}^i_2 \right) \rho \]

\[ - (\nabla^2_L \tilde{\alpha}^i_1 - 2 \tilde{\alpha}^i_2 \nabla^i_2 \tilde{\alpha}^i_2) \Phi^\nu_\mu \nabla^\nu_\mu \tilde{\alpha}^i_2 \]

\[ + \frac{1}{2} \left( (\nabla^2_L \tilde{\alpha}^i_2) \Phi^\mu_\nu \nabla^\nu_\mu \tilde{\alpha}^i_2 - 2(\phi^\nu_\mu \nabla^\nu_\nu \tilde{\alpha}^i_2) \right) \rho - (\nabla^2_L \tilde{\alpha}^i_2) \Phi^\mu_\mu \tilde{\alpha}^i_2 \cdot 0. \quad (383) \]

We now differentiate each side of (383) with \( \nabla^k_L \nabla^l_L \), and \( L^k_\alpha \), where \( l \) is a rotational multi-index satisfying \(|l| \leq M - k - l - 1\), multiply by \( r^2(1 + |q|)^{k} \), and use Lemmas 6.0.5, 6.0.7, 6.0.8, Corollaries VI.0.9 and VIII.0.6 (to bound the \( \Psi \) derivatives by \( r^{-1} \) weighted rotational Lie derivatives), Lemma 9.0.2, the fact that \( \nabla^r_L \rho = -\nabla^r_L \tilde{\rho} = 1 \), the fact that \(|1 + |q|| \lesssim r \) in the exterior region, and the assumption that \(|F|_{L^2(M/2)} \leq \epsilon \) is sufficiently small to derive the following inequality:

\[ r^2 r^l(1 + |q|)^{k} |\nabla^{k+l}_L \alpha|_{L^2(M-k-l)} \leq r \sum_{0 \leq k \leq l, \ 0 \leq f \leq l} r^l(1 + |q|)^{k} |\nabla^k_L \nabla^f_L \alpha|_{L^2(M-k-l)} \]

\[ + r^2 r^l(1 + |q|)^{k} \sum \text{nonlinear terms.} \quad (384) \]

After fully expanding via the Leibniz rule and using the smallness of \(|F|_{L^2(M/2)} \), we deduce that up to order 1 factors, each nonlinear term on the right-hand side of (384) can be bounded by one of the three following types of terms:

\[ (i) = \left| Q_{(i)}(\nabla^k_L \nabla^f_L F, \nabla^k_L \nabla^f_L L^k_\alpha) \right| |\nabla^k_L \nabla^f_L L^k_\alpha| \prod_{4 \leq k, b, c} |\nabla^k_L \nabla^f_L L^k_\alpha|, \quad (385) \]

\[ (ii) = \left| Q_{(i)}(\nabla^k_L \Psi^f_L, \Phi^f_L \alpha, \rho, \sigma) \right| \prod_{4 \leq k, b, c} |\nabla^k_L \nabla^f_L L^k_\alpha|, \quad (386) \]

\[ (iii) = \left| Q_{(i)}(\nabla^k_L \nabla^{k+1}_L F, \nabla^k_L \nabla^{k+1}_L L^k_\alpha) \right| |\nabla^k_L \nabla^{k+1}_L L^k_\alpha| \prod_{4 \leq k, b, c} |\nabla^k_L \nabla^{k+1}_L L^k_\alpha|, \quad (387) \]

where the \( k_a \) are non-negative integers such that \( \sum_a k_a = k \), the \( l_b \) are non-negative integers such that \( \sum_b l_b = l \), the \( I_c \) are rotational multi-indices such that \( \sum_c |I_c| \leq M - k - l - 1 \), and the
$Q_{(j)}(\cdot, \cdot)$ are null forms arising from the $\hat{\zeta}(\cdot)$. We remark that the type (i) terms arise from, e.g., the term $\ell^{2}_{MBI}(\nabla_{\perp}^{2} \hat{u}_{1})_{\alpha,\mu}$ on the right-hand side of (383), the type (ii) terms arise from, e.g., the term $\ell^{2}_{MBI}(\nabla_{\perp}^{2} \hat{v}_{1})_{\alpha}$, and the type (iii) terms arise from, e.g., the term $\ell^{2}_{MBI}(\nabla_{\perp}^{2} \hat{v}_{1})_{\alpha,\mu}$.

The first term on the right-hand side of (384) is manifestly bounded by the left-hand side of (381). Therefore, in order to prove (382), what remains to be shown is that the nonlinear terms of type (i) – (iii) are each $\lesssim r^{-2}r^{-1}(1 + |q|)^{-k} F \mathbf{1}_{L^{2};M}$. For the type (i) terms, we use Corollary 6.0.9, (381) and the smallness of $|F| \mathbf{1}_{L^{2};M}$ to deduce the desired estimate

$$|\langle i \rangle| \lesssim r^{-2}r^{-1}(1 + |q|)^{-k} F \mathbf{1}_{L^{2};M} \quad \text{(388)}$$

where the $r^{-2}$ factor in (388) arises from the fact that at least 2 of the factors on the right-hand side of (385) involve derivatives of the more rapidly decaying terms of $\nabla_{\perp}$. In fact, we note that they have even better decay than the $\nabla_{\perp}$ derivatives. In fact, we note that they have even better decay since they have an angular derivative $\nabla_{\perp}^{\alpha,\alpha,\alpha}$ in place of one the $\nabla_{\perp}$; we do not make use of this fact.

To bound the type (iii) terms, we first note that only one factor on the right-hand side of (386) involves the fast-decaying terms $\alpha, \rho, \sigma$; it arises from the $Q_{(j)}$. However, there is an additional power of $\nabla_{\perp}$ available to compensate. Therefore, we again use (381) to deduce that

$$|\langle iii \rangle| \lesssim r^{-2}r^{-1}(1 + |q|)^{-k} F \mathbf{1}_{L^{2};M} \quad \text{(389)}$$

We now set $k = 0$ and combine the estimates for the linear terms and the nonlinear terms, thus arriving at the estimate (382). We remark that we will use the expressions (385)–(387) in the case $k \geq 1$ later in the article, during our proof of (376b).

Armed with the previous estimates, we are now ready for the proof of the proposition.

**A. Proof of Proposition 10.0.1 (global Sobolev inequalities)**

Most of these estimates were proved as Theorems 3.1 and 3.2 of Ref. 17. In particular, we do not repeat the proof of (374). However, our proofs of (376a) and (376b) involve modifications of the arguments that take into account the special nonlinear structure of the MBI system. Therefore, we prove (375a) and (376b) in complete detail, and supply some additional details not contained in Ref. 17.

The arguments we give concern the exterior region $\{(t, x) \mid |x| \geq 1 + t/2\}$. To begin, we square inequality (381) and integrate over the exterior region, thereby obtaining the following inequality:

$$\sum_{k+l=0}^{M} \int_{|y| \geq 1+t/2} \left(1 + |y| - t \right)^{k} |y|^{l} \left(1 + |y| - t \right)^{k} \nabla_{\perp}^{k} \nabla_{\perp}^{l} \hat{u}(t, y) \right|_{L^{2};M}^{2} \quad \text{(390)}$$

For each $k' + l' + m = 0, \ldots, M - 2$ and each rotational multi-index $|l| \leq m$, we define $U_{k',l',l}(t, x) \equiv r^{l'}(\sqrt{1 + q^{2}})^{k'+l'+l} \nabla_{\perp}^{l'} \hat{u}^{l'}_{\perp} \alpha$ or $U_{k',l',l}(t, x) \equiv r^{l'}(\sqrt{1 + q^{2}})^{k'+l'+l} \nabla_{\perp}^{l'} \hat{u}^{l'}_{\perp} \sigma_{\perp}$. If $\hat{N}$ denotes the outward normal to the sphere $S_{r,\perp}$, then the fact that $\nabla_{\perp} \hat{N} = \partial_{r}$ implies that $\nabla_{\perp} \left(1 + q^{2}\right)^{1/2} \lesssim 1$. Using this estimate, (165c), (390), and the fact that $\nabla_{\perp} = \frac{1}{2} (\nabla_{\perp} - \nabla_{\perp})$, we arrive at the following inequality:

$$\int_{|y| \geq 1+t/2} \left|U_{k',l',l}(t, y)\right|_{L^{2};M}^{2} + \left(1 + |y| - t \right)^{k} \nabla_{\perp} \nabla_{\perp}^{l'} \hat{u}(t, y) \right|_{L^{2};M}^{2} \quad \text{(391)}$$
From (380) and (391), we conclude that in the exterior region, we have that
\[ |U_{k,t}(t, x)| \lesssim (1 + s)^{-3/2} |F(t)|_{L^2; M}. \] (392)
Summing over all \(|l| \leq m\) and using (165c) plus (392), we deduce that
\[ |\nabla^k L^l g(t, x)|_{L^2; M} \lesssim (1 + s)^{-1} (1 + |q|)^{-3/2-k} |F(t)|_{L^2; M}, \] (393)
and
\[ |\nabla^k L^l (\alpha(t, x), \rho(t, x), \sigma(t, x))|_{L^2; M} \lesssim (1 + s)^{-1} (1 + |q|)^{-3/2-k} |F(t)|_{L^2; M}. \] (394)
Furthermore, by Corollary 8.0.6, we have that
\[ |\nabla^k L^l \nabla (\alpha(t, x), \rho(t, x), \sigma(t, x))|_{L^2; M} \lesssim (1 + s)^{-1} (1 + |q|)^{-3/2-k} |F(t)|_{L^2; M}. \] (395)

The desired exterior region estimates (375a) and (375b) now follow from (393)–(394) and (395)–(396),
\[ |\nabla^k L^l \nabla (\alpha(t, x), \rho(t, x), \sigma(t, x))|_{L^2; M} \lesssim (1 + s)^{-1} (1 + |q|)^{-3/2-k} |F(t)|_{L^2; M}, \] (397)
\[ |\nabla^k L^l \nabla (\alpha(t, x), \rho(t, x), \sigma(t, x))|_{L^2; M} \lesssim (1 + s)^{-1} (1 + |q|)^{-3/2-k} |F(t)|_{L^2; M}. \] (398)

To prove (376b), we insert the estimates (375a) and (375b) into the right-hand side of (384), use the expressions (385)–(387) to estimate the nonlinear terms, and use Corollary 8.0.6 to translate back and forth between rotational Lie derivative estimates and \( r^{-1} \)-weighted \( \nabla \) derivative estimates.

To prove (376a), we first integrate inequality (382) over the exterior region to obtain the following inequality:
\[ \int_{|y|^{1+t/2} \geq \sqrt{\sum_{i=0}^{M-1} |y|^2 |\nabla^i L^l \alpha(t, y)|_{L^2; M-l}^2 + |y|^2 \sum_{i=0}^{M-1} |y|^2 |\nabla^i L^l \alpha(t, y)|_{L^2; M-l}^2} } d^3 y \lesssim |F(t)|_{L^2; M}. \] (399)

For each \( l' + m = 0, \ldots, M - 2 \) and each rotational multi-index \(|l| \leq m\), we define \( U_{l', t}(t, x) \) as
\[ r^{l'+1} L^{l'} \alpha. \] Arguing as in our proof of (391), we deduce that
\[ \int_{|y|^{1+t/2} \geq \sqrt{\sum_{i=0}^{M-2} |y|^2 |\nabla^i L^l \alpha(t, y)|_{L^2; M-l}^2 + |y|^2 |\nabla^i L^l \alpha(t, y)|_{L^2; M-l}^2} } d^3 y \lesssim |F(t)|_{L^2; M}. \] (400)

Therefore, using (379), we conclude that in the exterior region, we have that
\[ |U_{l', t}(t, x)| \lesssim r^{-3/2} |F(t)|_{L^2; M}. \] (401)
Summing over all \(|l| \leq m\) and using (165c) plus (401), we conclude that the following inequality holds in the exterior region:
\[ |\nabla^k L^l \alpha(t, x)|_{L^2; M} \lesssim r^{-5/2-l} |F(t)|_{L^2; M}. \] (402)
The desired inequality (366a) now follows from (402) and Corollary 8.0.6, as in our proof of (391) and (398).
XI. ENERGY ESTIMATES FOR THE MBI SYSTEM

The goal of this section is to prove the most important estimate in the article: an integral inequality for the norm $\| F(t) \|_{L^2;N}$. The inequality, which is the conclusion of the next proposition, is the crux of the proof of Theorem 1.

Proposition 11.0.1 (Integral inequality for $\| F(t) \|_{L^2;N}^2$): Let $N \geq 4$ be an integer. Assume that $F$ is a classical solution to the MBI system (87a) and (87b) existing on the slab $[0, T] \times \mathbb{R}^3$. Then there exist constants $\epsilon' > 0$ and $C > 0$ such that if $\sup_{t \in [0, T]} \| F(t) \|_{L^2;N} \leq \epsilon'$, then the following inequality holds for $t \in [0, T]$:

$$
\| F(t) \|_{L^2;N}^2 \leq C \left( \| F(0) \|_{L^2;N}^2 + \int_{t=0}^t \frac{1}{1 + \tau^2} \| F(\tau) \|_{L^2;N}^2 \, d\tau \right). \tag{403}
$$

The proof of Proposition 11.0.1 will follow easily from the following lemma and its corollary.

Lemma 11.0.2 (Pointwise bound for $\left| \nabla_\mu (j_\mu^{LZ} F(t, \chi)) \right|$): Let $N \geq 4$ be an integer. Assume that $F$ is a classical solution to the MBI system (87a) and (87b) existing on the slab $[0, T] \times \mathbb{R}^3$. For each $I \leq N$, let $j_\mu^{LZ} F \equiv \nabla_\mu \mathcal{K}_2 \frac{\mathcal{L}_2^2 F}{\mathcal{L}_2 F}$ be the energy current current (279) constructed out of the variation $\mathcal{F} \equiv \mathcal{L}_2^2 F$ and the background $F$. Let $E_{[N/2]+2}[F] = E_{[N/2]+2}[F(t)]$ be the energy defined in Definition 8.0.6. Then there exists a constant $C > 0$ such that $\| F \|_{L^2;[N/2]+2} \leq \epsilon$, then the following pointwise estimate holds on $[0, T] \times \mathbb{R}^3$:

$$
\left| \nabla_\mu (j_\mu^{LZ} F(t, \chi)) \right| \lesssim \frac{E_{[N/2]+2}^2[F(t)]}{1 + t^2} \| F(t) \|_{L^2;N}^2. \tag{404}
$$

Corollary 11.0.3: Assume the hypotheses of Lemma 11.0.2. Then there exists a constant $\epsilon > 0$ such that if $\| F \|_{L^2;[N/2]+2} \leq \epsilon$, then the following estimate holds for $t \in [0, T]$:

$$
\int_{\mathbb{R}^3} \left| \nabla_\mu (j_\mu^{LZ} F(t, \chi)) \right|^2 \, d^3 \chi \lesssim \frac{E_{[N/2]+2}^2[F(t)]}{1 + t^2} \| F(t) \|_{L^2;N}^2. \tag{405}
$$

Proof: Corollary 11.0.3 follows from integrating inequality (404) over $\Sigma_t$. \hfill $\Box$

We will now use the corollary of the lemma to prove the proposition; we will subsequently provide a proof of the lemma.

Proof of Proposition 11.0.1: Using the definition (281) of $E_N[F(t)]$, the fact that $[N/2] + 2 \leq N$, the divergence theorem, (292b), Corollary 10.0.3, and the smallness assumption on $\| F \|_{L^2;[N/2]+2}$, we have that

$$
\frac{d}{dt} \left( E_N^2[F(t)] \right) = \sum_{|I| \leq N} \int_{\mathbb{R}^3} \partial_\mu (j_\mu^{LZ} F(t, \chi)) \, d^3 \chi = \sum_{|I| \leq N} \int_{\mathbb{R}^3} \nabla_\mu (j_\mu^{LZ} F(t, \chi)) \, d^3 \chi \tag{406}
$$

$$
\lesssim \frac{E_{[N/2]+2}^2[F(t)]}{1 + t^2} \| F(t) \|_{L^2;N}^2 \lesssim E_N^2[F(t)] \frac{1}{1 + t^2}. \tag{407}
$$

Inequality (403) now follows from integrating inequality (406) from 0 to $t$ and using (292b) again. \hfill $\Box$

We now return to the proof of Lemma 11.0.2.

Proof of Lemma 11.0.2: First, we note that by the smallness assumption $\| F \|_{L^2;[N/2]+2} \leq \epsilon$, together with Corollary 10.0.2, we have that

$$
\| F \|_{L^2;[N/2]} \leq C(1 + s)^{-1}(1 + |q|)^{-3/2} \| F \|_{L^2;[N/2]+2} \leq C \epsilon (1 + s)^{-1}(1 + |q|)^{-3/2}. \tag{407}
$$
The above inequality is more than sufficient to guarantee that if $\epsilon$ is sufficiently small, then the hypotheses of Proposition 8.0.7 and of all of the lemmas and corollaries of Sec. IX are satisfied; we will make use of these results in our argument below.

By (187a)–(187b) and (340), we have that

$$\begin{align*}
\left| \nu_\mu \left( J^{(\mu)}_\nu \left[ L^I_2 F \right] \right) \right| &\lesssim \left| H^{(\mu\nu\xi)}_\Delta \nu_\mu \left( L^I_2 F \right) \right| - L^I_2 \left( H^{(\mu\nu\xi)}_\Delta \nu_\mu \left( L^I_2 F \right) \right) \left| L^I_2 F \right| \kappa \lambda\right| \\
&\quad + \left| \nu_\mu \left( H^{(\mu\nu\xi)}_\Delta \nu_\mu \left( L^I_2 F \right) \right) \left| L^I_2 F \right| \kappa \lambda\right| + \left| \kappa \lambda \right| \nu_\mu \left( H^{(\mu\nu\xi)}_\Delta \nu_\mu \left( L^I_2 F \right) \right) \left| L^I_2 F \right| \kappa \lambda\right|.
\end{align*}$$

(408)

Inequality (404) now follows from (408), Corollary 9.0.6, Lemmas 9.0.7, 9.0.8, and Corollary 10.0.2.

As an example, we describe the estimate of the first term on the right-hand side of (408) in more detail. To estimate this term, we first note that by Corollary 9.0.6, we have that

$$\sum_{|I| \leq N} \left| H^{(\mu\nu\xi)}_\Delta \nu_\mu \left( L^I_2 F \right) \right| - L^I_2 \left( H^{(\mu\nu\xi)}_\Delta \nu_\mu \left( L^I_2 F \right) \right) \left| L^I_2 F \right| \kappa \lambda\right| \lesssim \sum_{|I| \leq N} \left( 1 + |q| \right) \left( 1 + s \right) \left| L^I_2 F \right| \kappa \lambda\right| \lesssim \sum_{|I| \leq N} \left( 1 + q \right) \left( 1 + s \right) \left| L^I_2 F \right| \kappa \lambda\right|$$

(409)

By Corollary 10.0.2 and (292b), we have that

$$\sum_{|I| \leq N} \left( 1 + q \right) \left( 1 + s \right) \left| L^I_2 F \right| \kappa \lambda\right| \lesssim \sum_{|I| \leq N} \left( 1 + q \right) \left( 1 + s \right) \left| L^I_2 F \right| \kappa \lambda\right| \lesssim \sum_{|I| \leq N} \left( 1 + q \right) \left( 1 + s \right) \left| L^I_2 F \right| \kappa \lambda\right| \lesssim \sum_{|I| \leq N} \left( 1 + q \right) \left( 1 + s \right) \left| L^I_2 F \right| \kappa \lambda\right|$$

(410)

while by the definition of $\cdot \cdot$, we have that

$$\sum_{|I| \leq N} \left( 1 + q \right) \left( 1 + s \right) \left| L^I_2 F \right| \kappa \lambda\right| \lesssim \sum_{|I| \leq N} \left( 1 + q \right) \left( 1 + s \right) \left| L^I_2 F \right| \kappa \lambda\right| \lesssim \sum_{|I| \leq N} \left( 1 + q \right) \left( 1 + s \right) \left| L^I_2 F \right| \kappa \lambda\right|$$

(411)

It thus follows from (410) and (411) that the right-hand side of (409) is bounded by the right-hand side of (404). All other terms can be estimated in a similar fashion, and we omit the details.

**XII. LOCAL WELL-POSEDNESS AND THE CONTINUATION PRINCIPLE**

In this section, we briefly discuss local well-posedness for the MBI system. With the exception of the availability of Proposition 7.4.4, the material presented here is very standard. For the purposes of our global existence theorem, which is proved in Sec. XIII, the most important fact presented in
this section is the continuation principle: it shows that a priori control over the norm \( \| \mathcal{F}(t) \|_{L^2;N} \) is sufficient to deduce global existence.

**Proposition 12.0.1 (Local well-posedness and a continuation principle):** Let \( N \geq 4 \) be an integer, and let \((\mathcal{B}, \mathcal{D})\) be a pair of initial data one-forms that are tangent to the Cauchy hypersurface \( \Sigma_0 \), that satisfy the constraints (152a) and (152b), and that satisfy \( \|(\mathcal{B}, \mathcal{D})\|_{H^N} < \infty \). Here, \( H^N \) is the weighted Sobolev norm defined in (326). Then these data launch a unique classical solution \( \mathcal{F} \) to the MBI system existing on a non-trivial spacetime slab of the form \([0, T) \times \mathbb{R}^3\). The solution has the following regularity properties:

\[
\mathcal{F} \in C^{N-2}([0, T) \times \mathbb{R}^3),
\]

\[
(B, D) \in C^{N-2}([0, T) \times \mathbb{R}^3) \cap \bigcap_{k=0}^{k=N-2} C^k([0, T), H^N-k_1),
\]

where \((B, D)\) are the electromagnetic one-forms defined in (146a).

Let \( T_{\text{max}} \) be the supremum over all times \( T \) such that the solution exists classically on \([0, T) \times \mathbb{R}^3\) and such that the regularity properties (412a) and (412b) hold. Then either \( T_{\text{max}} = \infty \), or one of the following two breakdown scenarios must occur:

1. \( \lim_{t \uparrow T_{\text{max}}} \| \mathcal{F}(t) \|_{L^2;N} = \infty \)
2. There exists a sequence \((t_n, \mathcal{X}_n)\) with \( t_n < T_{\text{max}} \) such that \( \lim_{n \to \infty} \ell_{(MBI)}[\mathcal{F}(t_n, \mathcal{X}_n)] = 0 \),

where \( \ell_{(MBI)}[\mathcal{F}] = (1 + \tilde{q}_1[\mathcal{F}] - \tilde{q}_2^2[\mathcal{F}])^{1/2} \) is the function of \( \mathcal{F} \) defined in (80), and \( \| \mathcal{F}(t) \|_{L^2;N} \) is the norm defined in (280).

**Remark 12.0.1:** The classification of the two breakdown scenarios is known as a “continuation principle.”

**Remark 12.0.2:** (152b) and (152a) are the only two constraints on the data. Furthermore, it is easy to see that there exist data that verify the constraints. For example, if \( \mathcal{B}, \mathcal{D} \) are the curl of smooth vectorfields on \( \mathbb{R}^3 \), then \((\mathcal{B}, \mathcal{D})\) will verify the constraints.

Since Proposition 12.0.1 is rather standard, we do not provide a full proof, but instead refer to the reader to, e.g., Chap. VI of Ref. 27, Refs. 41, 59, 53 and 57, and Chap. 16 of Ref. 61 for the missing details concerning local well-posedness, and, e.g., Ref. 56 for the ideas behind the continuation principle. The crucial point is the availability of Proposition 7.4.4, which can be used to derive estimates for solutions to the linearized MBI system, that is, the equations of variation. More specifically, in constructing the local solution of Proposition 12.0.1, one typically uses an iteration argument or a contraction mapping argument. Both methods involve an analysis of solutions to the equations of variation. We remark that during an iteration scheme, in the equations of variation, one can think of the background \( \mathcal{F} \) as the “current” iterate, and \( \mathcal{F} \) as the “next” one. Both methods require uniform estimates of weighted Sobolev norms of \( \mathcal{F} \), and these uniform estimates can be derived using energy currents and the ideas contained in the proof of Proposition 11.0.1. In particular, suitable energy estimates for solutions to the equations of variation (186a) and (186b) can be derived by using energy currents \( j^\mu_{\text{local}}(\mathcal{F}) \) defined by

\[
j^\mu_{\text{local}}(\mathcal{F}) \stackrel{\text{def}}{=} -(1 + s^2) \tilde{Q}^\mu_{\text{local}}(\mathcal{F}),
\]

where \( \tilde{Q}^\mu_{\text{local}}(\mathcal{F}) \) is the modified canonical stress from (194), and \( X^\mu_{\text{local}} \) is the vectorfield defined in Proposition 7.4.4. Now by Proposition 7.4.4, if \( \mathcal{F} \in \mathcal{R} \), where \( \mathcal{R} \) is a compact subset of the regime of hyperbolicity \( \mathcal{X} \) (see Definition 7.4.1), then we have that

\[
| \mathcal{F} |^2 \leq (1 + s^2)|\mathcal{F}|^2 \leq C_{\mathcal{R}} j^0_{\text{local}}(\mathcal{F}).
\]

By Remark 5.8.3, in terms of the state-space variables \((B, D)\), this domain comprises the set finite values of \((B, D)\). On the other hand, using the simple inequalities \((1 + q^2)|\mathcal{F}|^2 \leq |\mathcal{F}|^2 \) and \( 1 + s^2 \)
\( \lesssim (1 + r^2)(1 + q^2) \), together with the fact that \( | \tilde{Q} | \lesssim | \mathcal{F} |^2 \), we deduce that
\[
J_{local}^0 [\mathcal{F}] \leq C^{-1}(1 + s^2) | \mathcal{F} |^2 \leq C^{-1}(1 + r^2)(1 + q^2) | \mathcal{F} |^2 \lesssim (1 + r^2) | \mathcal{F} |^2 .
\] (415)

Consequently, if we define the energy \( \mathcal{E}_{local;N}(\mathcal{F}(t)) \) by
\[
\mathcal{E}_{local;N}^2 [\mathcal{F}(t)] \equiv \sum_{|I| \leq N} \int_{\mathbb{R}^3} J_{local}^0 [L_{\mathcal{F}}^2 \mathcal{F}] \, d^3 x,
\] (416)
then (414) and (415) imply that
\[
\| \mathcal{F}(t) \|_{L_2;N} \lesssim \mathcal{E}_{local;N}(\mathcal{F}(t)) \lesssim (1 + t) \| \mathcal{F}(t) \|_{L_2;N} .
\] (417)

We remark that the implicit constants in (417) depend on \( \mathcal{K} \).

We now illustrate the fundamental energy estimate that can be used to deduce the desired local well-posedness result. We set \( N = 0 \) for simplicity, and consider a solution \( \mathcal{F} \) to the MBI equations of variation (186a) and (186b) with the initial data \(( \tilde{B}, \tilde{D} )\). Then from (195), (198), (393), (414), and (417), the divergence theorem, and the Cauchy-Schwarz inequality for integrals, it follows (as in our proof of Proposition 11.0.1) that
\[
\frac{d}{dt} \left( \mathcal{E}_{local}^2 [\mathcal{F}(t)] \right) \lesssim f(\mathcal{K}; \| \mathcal{F}(t) \|_{L_2}; \| \nabla \mathcal{F}(t) \|_{L_2}) \times \int_{\mathbb{R}^3} (1 + r^2 + |x|^2) | \tilde{F}(t, x) |^2 + (1 + r^2 + |x|^2) | \hat{F}(t, x) | ( | \mathcal{J}(t, x) | + | \mathcal{J}(t, x) | ) \, d^3 x
\lesssim f(\mathcal{K}; \| \mathcal{F}(t) \|_{L_2}; \| \nabla \mathcal{F}(t) \|_{L_2}) \left( \mathcal{E}_{local}^2 [\mathcal{F}(t)] + \mathcal{E}_{local;N} [\mathcal{F}(t)] \right) \| \mathcal{J}(t) \|_{H^1} + \| \mathcal{J}(t) \|_{H^1},
\] (418)
where \( H^1 \) is the weighted Sobolev norm defined in (326), \( \mathcal{J}_{\lambda, \nu}, \mathcal{J}^\nu \) are the inhomogeneous terms on the right-hand sides of (186a) and (186b), and \( f(\mathcal{K}; \cdot) \) can be chosen to be a positive, increasing, continuous function of its arguments. We remark that the implicit constants in (418) depend on \( t \), so that inequality (418) could be false for large times. We also remark that similar inequalities could be deduced for \( N \geq 1 \), and that the inhomogeneous terms would be controlled in the \( H^1 \) norm.

The availability of inequality (418) for solutions to the equations of variation is the fundamental reason that Proposition 12.0.1 holds. From (418), Gronwall’s inequality, and appropriate weighted Sobolev estimates for the inhomogeneous terms, it can be shown that \( \mathcal{E}_{local}^2 [\mathcal{F}(t)] \) can be uniformly bounded in terms of \( \| (\tilde{B}, \tilde{D}) \|_{H^1} \), if \( t \) is sufficiently small. Similar inequalities can be deduced for the higher order energies \( \mathcal{E}_{local;N} [\mathcal{F}(t)] \). As mentioned above, this is the main step in deducing local well-posedness for the nonlinear equations; the remaining details can be found in the aforementioned references.

There is one additional step in the proof of local well-posedness that we will comment on, namely, the issue of showing that \( \mathcal{E}_{local;N}^2 [\mathcal{F}(0)] \) is uniformly bounded by \( \| (\tilde{B}, \tilde{D}) \|_{H^1} \), whenever \( N \geq 4 \) and \( \mathcal{F} \) is a solution to the equations of variation. To accomplish this rather tedious step, one can first express the equations of variation in terms of \( (E, B) \) and \( (\tilde{E}, \tilde{B}) \), and inhomogeneous terms, where \( (E, B) \) and \( (\tilde{E}, \tilde{B}) \) are the electromagnetic decompositions of \( \mathcal{F} \) and \( \mathcal{F} \), respectively (see Sec. VH). One would then use weighted Sobolev multiplication estimates, as in our proof of Lemma 8.1.2, to deduce that
\[
\| (\tilde{B}, \tilde{D}) \|_{H^1} < \infty \Rightarrow \| \mathcal{F}(0) \|_{L_2;N} \lesssim \mathcal{F}(0) \|_{H^1},
\] (419)
where \( \mathcal{F} \) can be chosen to be a positive, increasing, continuous function of \( t \) by (417), the desired uniform bound for \( \mathcal{E}_{local;N}^2 [\mathcal{F}(0)] \) then follows from (419). To deduce (419), we have assumed that both \( \mathcal{F} \) and \( \mathcal{F} \) have the same initial data, and that \( \sum_{n=0}^{N}(1 + r^2)^{n+1} \left( | \nabla (\mathcal{F}(t)) |_{\Sigma_0} + | \nabla (\mathcal{F}(t)) |_{\Sigma_0} \right) \), the relevant Sobolev-Moser norm for \( \mathcal{F} \) during a proof of (419), can be bounded by a positive, increasing, continuous function of \( \| (\tilde{B}, \tilde{D}) \|_{H^1} \). In practice, during an iteration scheme, this argument would need to be slightly modified: for technical reasons, typical iteration schemes involve a slightly different smoothing of the initial data at each stage, so that the initial data change slightly from iterate to iterate. We remark that the data are smoothed for several reasons, one reason being that
during the iteration process, it is convenient to work with classically differentiable functions, rather than distributions.

XIII. SMALL-DATA GLOBAL EXISTENCE FOR THE MBI SYSTEM

In this section, we provide a proof of our main theorem. The global existence aspect of our theorem will be an easy consequence of the energy inequality of Proposition 11.0.1 and the continuation principle of Proposition 12.0.1, while the decay aspect will follow directly from Proposition 10.0.1 (i.e., the global Sobolev inequality).

Theorem 1 (Main theorem: small-data global existence): Let $N \geq 4$ be an integer, and let $(\hat{B}, \hat{D})$ be a pair of initial data one-forms that are tangent to the Cauchy hypersurface $\Sigma_0$ and that satisfy the constraints (152a) and (152b). Then there exists a constant $\epsilon_0 > 0$ such that if $\| (\hat{B}, \hat{D}) \|_{H^N} \leq \epsilon_0$, then these data launch a unique classical solution $F$ to the Maxwell-Born-Infeld system (87a) and (87b) existing on the spacetime slab $[0, \infty) \times \mathbb{R}^3$. Furthermore, there exists a constant $C_* > 0$ such that

$$\| F(t) \|_{L^2;N} \leq C_* \| (\hat{B}, \hat{D}) \|_{H^N}$$

holds for all $t \geq 0$. Here, $\| \cdot \|_{H^N}$ is the weighted Sobolev space defined in (326), while $\| \cdot \|_{L^2;N}$ is the weighted integral norm defined in (280). In addition, the null components $\alpha, \alpha, \rho, \sigma$ of $F$, which are defined in Sec. V C, decay according to the rates given by Proposition 10.0.1.

Proof: We will show that if $\| (\hat{B}, \hat{D}) \|_{H^N}$ is sufficiently small, then neither of the two breakdown scenarios from Proposition 12.0.1 occur. To this end, let $\epsilon' > 0$ be the small constant from the conclusion of Proposition 11.0.1. Choose a positive constant $\epsilon''$ such that $0 < \epsilon'' < \epsilon'$ and such that

$$\| F(t) \|_{L^2;N} \leq \epsilon'' \Rightarrow \inf_{\lambda \in \mathbb{R}^4} \lambda (\max(B), \max(D)) \geq 1/2,$$

where $\lambda (\max(B))$ is defined in (80); this is possible by Sobolev embedding. Define

$$T_* = \sup \{ t \geq 0 \mid \text{The solution exists on} [0, t] \text{ and } \| F(t) \|_{L^2;N} \leq \epsilon'' \}. \quad (421)$$

By the local existence aspect of Proposition 12.0.1 and inequality (329), we have that $T_* > 0$ if $\| (\hat{B}, \hat{D}) \|_{H^N}$ is sufficiently small. Applying Proposition 11.0.1, we conclude that the following inequality holds for $t \in [0, T_*)$:

$$\| F(t) \|_{L^2;N} \leq C \left\{ \| F(0) \|_{L^2;N}^2 + \int_{t_0}^t \frac{1}{1+\tau^2} \left\| F(\tau) \right\|_{L^2;N}^2 \ d\tau \right\}. \quad (422)$$

Applying Gronwall’s inequality to (422), and using Lemma 8.1.2, we conclude that the following inequality holds for $t \in [0, T_*)$:

$$\| F(t) \|_{L^2;N}^2 \leq \| F(0) \|_{L^2;N}^2 \exp \left( \int_{t_0}^\infty \frac{C}{1+\tau^2} \right) \leq C_*^2 \| (\hat{B}, \hat{D}) \|_{H^N}^2. \quad (423)$$

Now if $C_* \| (\hat{B}, \hat{D}) \|_{H^N} < \epsilon''$, then the continuation principle of Proposition 12.0.1 and inequality (423) together imply that $T_* = \infty$. Furthermore, inequality (420) is a direct consequence of (423).

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Appendix A: Data Norms

In Appendix A, we state the lemmas and corollaries that are used in Sec. VIII A to relate the smallness condition on the inherent data \((\hat{F}, \hat{D})\) to a smallness condition on \(\| F(0) \|_{C^2(N)}\). The lemmas we state are slight extensions of Lemmas 2.4 and 2.5 of Ref. 9, while the corollaries are easy (and non-optimal) consequences of the lemmas; we leave their proofs as exercises for the reader. Throughout the Appendix, we abbreviate \(C^N_\delta \equiv C^N_\delta(\mathbb{R}^3)\), \(H^N_\delta \equiv H^N_\delta(\mathbb{R}^3)\), etc. Furthermore, \(\nabla\) denotes the Levi-Civita connection corresponding to the standard Euclidean metric \(g\) on \(\mathbb{R}^3\), and we equip \(\mathbb{R}^3\) with standard rectangular coordinates \(x\).

Lemma A-2 (Lemma 2.4 of Ref. 9): Let \(N, N' \geq 0\) be integers, and let \(\delta, \delta'\) be real numbers subject to the constraints \(N' < N - 3/2\) and \(\delta' < \delta + 3/2\). Assume that \(v \in H^N_\delta\). Then \(v \in C^N_{\delta'}\), and

\[
\|v\|_{C^N_{\delta'}} \leq C \|v\|_{H^N_\delta}. \tag{A1}
\]

Lemma A-3 (Slight extension of Lemma 2.5 of Ref. 9): Let \(N_1, \cdots, N_p \geq 0\) be integers, and let \(\delta_1, \cdots, \delta_p\) be real numbers. Suppose that \(v_i \in H^N_{\delta_i}\) for \(i = 1, \cdots, p\). Assume that the integer \(N\) satisfies

\[
0 \leq N \leq \min \{N_1, \cdots, N_p\} \quad \text{and} \quad N \leq \sum_{i=1}^p N_i - (p - 1)3/2,
\]

and that \(\delta < \sum_{i=1}^p \delta_i + (p - 1)3/2\). Then

\[
\prod_{i=1}^p v_i \in H^N_\delta,
\]

(A2)

and the multiplication map

\[
H^N_{\delta_1} \times \cdots \times H^N_{\delta_p} \rightarrow H^N_\delta, \quad (v_1, \cdots, v_p) \mapsto \prod_{i=1}^p v_i
\]

(A3)

is continuous.

Corollary A-4: Let \(N \geq 2\) be an integer, and let \(\delta \geq 0\). Assume that \(v_i \in H^N_{\delta_i}\) for \(i = 1, \cdots, p\), and that \(m_i \geq 0\) are integers satisfying \(\sum_{i=1}^p m_i \leq N\). Then

\[
(1 + |x|^{2(\delta + \sum_{i=1}^p m_i)})^{1/2} \prod_{j=1}^p \sum_{m_j} v_j \in L^2 \tag{A4}
\]

and

\[
\left\| (1 + |x|^{2(\delta + \sum_{i=1}^p m_i)})^{1/2} \prod_{j=1}^p \sum_{m_j} v_j \right\|_{L^2} \lesssim \prod_{j=1}^p \| v_j \|_{H^{N_{\delta_j}}}. \tag{A5}
\]

Corollary A-5: Let \(N \geq 2\) be an integer, and let \(\delta \geq 0\). Assume that \(\mathcal{R}\) is a compact set, that \(\mathcal{R} \subset C^N(\mathbb{R})\) is a function, and that \(v_1\) is a function satisfying \(v_1(\mathbb{R}^3) \subset \mathcal{R}\). Assume further that \(v_1, v_2 \in H^N_{\delta}\). Then \((\mathcal{R} \circ v_1)v_2 \in H^N_{\delta}\), and

\[
\| (\mathcal{R} \circ v_1)v_2 \|_{H^N_{\delta}} \leq C(N) \left\{ \| v_2 \|_{H^N_{\delta}} \sum_{j=0}^N \| \mathcal{R}^{(j)} \|_{C^0} \| v_1 \|_{H^N_{\delta}} \right\}. \tag{A6}
\]
In the above inequality, \( \mathcal{F}^{(j)} \) denotes the array of all \( j \)th order partial derivatives of \( \mathcal{F} \) with respect to its arguments, and \( |\mathcal{F}^{(j)}|_K \overset{\text{def}}{=} \sup_{v \in K} |\mathcal{F}^{(j)}(v)| \).

**APPENDIX B: NOTATION AND CONVENTIONS**

In Appendix B, we collect together for convenience much of the notation that was introduced throughout the article.

1. **Constants**

We use the symbol \( C \) to denote a generic positive constant that is free to vary from line to line. In general, \( C \) can depend on many quantities, but in the small-solution regime that we consider in this article, \( C \) can be chosen uniformly. Sometimes it is illuminating to explicitly indicate one of the quantities \( Q \) that \( C \) depends on; we do by writing \( C_Q \) or \( C(Q) \).

If \( A \) and \( B \) are two quantities, then we often write \( A \lesssim B \) to mean that “there exists a \( C > 0 \) such that \( A \leq CB \).” Furthermore, if \( A \lesssim B \) and \( B \lesssim A \), then we often write \( A \approx B \).

2. **Indices**

- Lowercase Latin indices \( a, b, j, k, \) etc., take on the values 1, 2, 3.
- Greek indices \( \kappa, \lambda, \mu, \nu, \) etc., take on the values 0, 1, 2, 3.
- Accented indices \( \tilde{\kappa}, \kappa', \) etc., are used in the same way as unaccented indices.
- Uppercase Latin indices \( A, B, \) etc., take on the values 1, 2 and are used to enumerate the two orthogonal null frame vectors tangent to the spheres \( S_{r,t} \).
- Indices are lowered and raised with the Minkowski metric \( g_{\mu\nu} \) and its inverse \( (g^{-1})^{\mu\nu} \).
- Repeated indices are summed over.
- We sometimes use parentheses to distinguish indices that are labels from coordinate indices; e.g., the “0” in \( T_(0) \) is a labeling index. Note in particular that *parentheses are not used to denote symmetrization* in this article.

3. **Coordinates**

- \( \{x^\mu\}_\mu = 0, 1, 2, 3 \) are the *spacetime coordinates*; in our fixed inertial coordinate system only, we use the notation \( t = x^0, \quad x = (x^1, x^2, x^3) \).
- Relative to our inertial coordinate system, \( r = |x| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \) denotes the radial coordinate.
- \( q = r - t, \quad s = r + t \) are the *null coordinates*.

4. **Surfaces**

Relative to the inertial coordinate system:

- \( C_s^- = \{(\tau, y) \mid |y| + t = s\} \) are the *ingoing null cones*; the null coordinate \( s = r + t \) takes on the constant value \( s_0 \) along \( C_{s_0}^- \).
- \( C_q^+ = \{(\tau, y) \mid |y| - t = q\} \) are the *outgoing null cones*; the null coordinate \( q = r - t \) takes on the constant value \( q_0 \) along \( C_{q_0}^+ \).
- \( \Sigma_t = \{(\tau, y) \mid \tau = t\} \) are the *constant time slices*.
- \( S_{r,t} = \{(\tau, y) \mid \tau = t, \quad |y| = r\} \) are the Euclidean spheres.
5. Metrics and volume forms

- $g$ denotes the standard Minkowski metric on $\mathbb{R}^{1+3}$; in our fixed inertial coordinate system, $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.
- $e$ denotes the standard Euclidean metric on $\mathbb{R}^{1+3}$; in our fixed coordinate system, $e_{\mu\nu} = \text{diag}(1, 1, 1, 1)$.
- $e^{-1}$ denotes the inverse of the standard Euclidean metric on $\mathbb{R}^{1+3}$; in our fixed inertial coordinate system, $(e^{-1})^{\mu\nu} = \text{diag}(1, 1, 1, 1)$.
- $g$ denotes the first fundamental form of $\Sigma_t$; in our fixed inertial coordinate system, $g_{\mu\nu} = \text{diag}(0, 1, 1, 1)$.
- $\hat{g}$ denotes the first fundamental form of $S_{r,t}$; relative to an arbitrary coordinate system, $\hat{g}_{\mu\nu} = g_{\mu\nu} + \frac{1}{2}(L_{\mu}L_{\nu} + L_{\nu}L_{\mu})$, where $L, L$ are defined in Sec. B9.
- $e_{\mu\nu\lambda} = |\text{det}(g)|^{1/2} g_{\mu\nu\lambda}$ denotes the volume form of $g$; $[0123] = 1 = -[0123]$, etc.
- $e^{\mu\nu\lambda} = -|\text{det}(g)|^{-1/2} [\mu\nu\lambda]$, where $[\mu\nu\lambda]$ denotes the intrinsic curl of a vectorfield $\lambda$.
- $e_{\mu\nu\lambda} = e_{\mu\nu\lambda}(T_{(0)})^\mu$ denotes the volume form of $g$, where $T_{(0)}$ is defined in Sec. B8.
- $\hat{g}_{\mu\nu} = \frac{1}{2}e_{\mu\nu\lambda}L^sL^s$ denotes the volume form of $\hat{g}$.

6. Hodge duality

For an arbitrary two-form $F_{\mu\nu}$:

- $*F_{\mu\nu} = \frac{1}{2}g_{\mu\beta}g_{\nu\gamma}\hat{g}^{\alpha\lambda\beta\gamma}F_{\alpha\lambda} = -\frac{1}{2}|\text{det}(g)|^{-1/2}g_{\mu\beta}g_{\nu\gamma}[\mu\nu\lambda]F_{\lambda\beta}$ denotes the Hodge dual of $F_{\mu\nu}$ with respect to the spacetime metric $g$.

7. Derivatives

- In an arbitrary coordinate system $\{x^\alpha\}_{\mu = 0, 1, 2, 3}$, $\partial_\mu = \frac{\partial}{\partial x^\mu}$, $\nabla_\mu = \nabla_{\frac{\partial}{\partial x^\mu}}$.
- $\nabla^X$ denotes the Levi-Civita connection corresponding to $g$.
- $\nabla_X$ denotes the Levi-Civita connection corresponding to $g$.
- $\nabla^{X^a}$ denotes the Levi-Civita connection corresponding to $\hat{g}$.
- In our fixed inertial coordinate system, $\partial_r = \omega^a\partial_a$ denotes the radial vectorfield, where $\omega^a = x^a/r$.
- If $X$ is a vectorfield and $\phi$ is a function, then $X\phi = X^a\partial_a\phi$.
- $\nabla_X$ denotes the differential operator $X^a\nabla_a$.
- If $X$ is tangent to $\Sigma_t$, then $\nabla_X$ denotes the differential operator $X^a\nabla_a$.
- If $X$ is tangent to $S_{r,t}$, then $\nabla_X$ denotes the differential operator $X^a\nabla_a^X$.
- $\nabla_{(n)}U$ denotes the $n$th covariant derivative tensorfield of the tensorfield $U$.
- $\nabla_{(n)}U$ denotes the $n$th covariant derivative tensorfield of a tensorfield $U$ tangent to the hypersurfaces $\Sigma_t$.
- $\nabla_{(n)}U$ denotes the $n$th covariant derivative tensorfield of a tensorfield $U$ tangent to the spheres $S_{r,t}$.
- $\text{div} U = g^{\mu\nu}\nabla_\mu U_\nu$ denotes the intrinsic divergence of a vectorfield $U$ tangent to the hypersurfaces $\Sigma_t$.
- $(\text{curl} U)^\nu = e^{\nu\alpha\beta} \nabla_\beta U^\alpha$ are the components of the intrinsic curl of a vectorfield $U$ tangent to the hypersurfaces $\Sigma_t$.
- $\text{div} U = g^{\mu\nu}\nabla_\mu U_\nu$ denotes the intrinsic divergence of a vectorfield $U$ tangent to the spheres $S_{r,t}$.
- $(\text{curl} U)^\nu = \mathcal{E}_{\nu}^\alpha \nabla_\alpha U^\lambda$ denotes the intrinsic curl of a vectorfield $U$ tangent to the spheres $S_{r,t}$.
- $\mathcal{L}_X$ denotes the Lie derivative with respect to the vectorfield $X$.
- $[X,Y] = (\mathcal{L}_X Y)^\mu = X^a\partial_a Y^\mu - Y^a\partial_a X^\mu$ denotes the Lie bracket of the vectorfields $X$ and $Y$.
- For $Z \in \mathbb{Z}$, $\mathcal{L}_Z = \mathcal{L}_Z + 2c_Z$ denotes the modified Lie derivative, where the constant $c_Z$ is defined in Sec. B8.
- $\mathcal{L}_{U^a}U$, and $\mathcal{L}_{U^a}U$, $\nabla^a_{(n)}U$, respectively, denote an $n$th order iterated Lie, iterated modified Lie, and iterated covariant derivative of the tensorfield $U$ with respect to vectorfields belonging...
8. Minkowski conformal Killing fields

Relative to the inertial coordinate system \( \{x^\mu\}_{\mu=0,1,2,3} = (t, \mathbf{x}) \):

- \( T_{(\mu)} = \partial_\mu, (\mu = 0, 1, 2, 3) \), denotes a translation vectorfield.
- \( \Omega_{(j)} = x_j \partial_k - x_k \partial_j, (1 \leq j < k \leq 3) \), denotes a rotation vectorfield.
- \( \Omega_{(0)} = -t \partial_j - x_j \partial_t, (j = 1, 2, 3) \), denotes a Lorentz boost vectorfield.
- \( S = x^\kappa \partial_\kappa \) denotes the scaling vectorfield.
- \( K_{(\mu)} = -2x_\mu S + g_{\kappa\lambda} x^\kappa x^\lambda \partial_\mu, (\mu = 0, 1, 2, 3) \) denotes an acceleration vectorfield.
- \( \mathcal{K} = K_{(0)} + T_{(0)} \) denotes the Morawetz vectorfield.
- \( \mathcal{F} = \{T_{(\mu)}\}_{0 \leq \mu \leq 3} \).
- \( \mathcal{O} = \{\Omega_{(12)}, \Omega_{(13)}, \Omega_{(23)}\} \).
- \( \mathcal{Z} = \{T_{(\mu)}, \Omega_{(\mu\nu)}, S\}_{1 \leq \mu \leq \nu \leq 3} \).
- \( T, \mathcal{O}, \) and \( \mathcal{Z} \) are the Lie algebras generated by \( \mathcal{F}, \mathcal{O}, \) and \( \mathcal{Z} \), respectively.
- For \( Z \in \mathcal{Z} \), \( \pi_{\mu\nu} = \nabla_\mu Z_\nu + \nabla_\nu Z_\mu = c_Z g_{\mu\nu} \), denotes the deformation tensor of \( Z \), where \( c_Z \) is a constant.
- Commutation properties with the Maxwell-Maxwell term \( \left[ \left( g^{-1}\right)^{\mu\nu} (g^{-1})^{\nu\lambda} - (g^{-1})^{\nu\lambda} (g^{-1})^{\lambda\mu} \right] \nabla_\mu F_{\kappa\lambda} \left( \mathcal{K}^2 \right)^{\lambda\nu} - \left( g^{-1}\right)^{\nu\lambda} (g^{-1})^{\lambda\mu} \right] \nabla_\mu C_{\kappa\lambda}^A F_{\kappa\lambda} \).

9. Null frames

- \( L = \partial_t - \partial_\kappa \) denotes the ingoing null vectorfield generating the \( C^- \) and transversal to the \( C^+ \).
- \( L = \partial_t + \partial_\kappa \) denotes the outgoing null vectorfield generating the \( C^+ \) and transversal to the \( C^- \).
- \( e_A \), \( (A = 1, 2) \), denotes a pair of orthonormal vectorfields spanning the tangent space of the spheres \( S_{r, t} \).
- The set \( \mathcal{L} = \{L\} \) contains only \( L \).
- The set \( \mathcal{F} = \{L, e_1, e_2\} \) denotes the frame vectors tangent to the \( C^+_q \).
- The set \( \mathcal{U} = \{L, L, e_1, e_2\} \) denotes the entire null frame.

10. Null frame decomposition

- For an arbitrary vectorfield \( X \) and frame vector \( U \in \mathcal{U} \), we define \( X_U = X_\mu U^\mu \), where \( X_\mu = g_{\mu\kappa} X^\kappa \).
- For an arbitrary vectorfield \( X = X^\kappa \partial_\kappa = X^L L + X^A A + X^e e_A \), where \( X^L = -\frac{1}{2} X_L, X^A = X_A, X^e = X_e \).
- For an arbitrary pair of vectorfields \( X, Y \):

\[
g(X, Y) = X^e Y_e = -\frac{1}{2} X_L Y_L - \frac{1}{2} X_L Y_L + \delta_{AB} X_A Y_B.
\]

If \( \mathcal{F} \) is any two-form, its null components are

- \( \alpha_{\mu}[\mathcal{F}] = \alpha_{\mu} = g^{\nu\lambda} \mathcal{F}_{\nu\lambda} L^\mu \).
- \( \alpha_{\nu}[\mathcal{F}] = \alpha_{\nu} = g_{\mu\kappa} \mathcal{F}_{\nu\kappa} L^\mu \).
- \( \rho[\mathcal{F}] = \rho = \frac{1}{2} \mathcal{F}_{\kappa\lambda} L^\kappa L^\lambda \).
- \( \sigma[\mathcal{F}] = \sigma = \frac{1}{2} \mathcal{F}_{\kappa\lambda} L^\kappa L^\lambda \).
11. Null Forms

For arbitrary two-forms \( \mathcal{F}, \mathcal{G} \):

- \( Q_1(\mathcal{F}, \mathcal{G}) = \mathcal{F}^\wedge \mathcal{G}_{\kappa \lambda} \),
- \( Q_2(\mathcal{F}, \mathcal{G}) = *\mathcal{F}^\wedge \mathcal{G}_{\kappa \lambda} = \delta_{AB} \mathcal{A}_{\kappa \lambda}[\mathcal{F}] \mathcal{A}_{\kappa \lambda}[\mathcal{G}] - 2\rho[\mathcal{F}]\rho[\mathcal{G}] + 2\sigma[\mathcal{F}]\sigma[\mathcal{G}]. \)

12. Electromagnetic decompositions

Given a two-form \( \mathcal{F} \) and its associated MBI Maxwell tensor \( \mathcal{M}_{\mu \nu} = \mathcal{E}_{(MBI)}(\mathcal{F}_{\mu \nu} + \gamma(2)\mathcal{F}_{\mu \nu}) \), its electromagnetic components relative to an arbitrary coordinate system are

- \( E_\mu = \mathcal{F}_{\mu \kappa} T^\kappa_s. \)
- \( B_\mu = -*\mathcal{F}_{\mu \kappa} T^\kappa_s. \)
- \( D_\mu = -\mathcal{M}_{\mu \kappa} T^\kappa_s. \)
- \( H_\mu = -\mathcal{M}_{\mu \kappa} T^\kappa_s. \)
- \( \mathcal{E}_\mu = \mathcal{F}_{\mu \kappa} S^\kappa_s. \)
- \( \mathcal{B}_\mu = *\mathcal{F}_{\mu \kappa} S^\kappa_s. \)

13. Seminorms, norms, and energies

For an arbitrary tensor \( U \) of type \( _a^b \), and \( A \in \{ \mathcal{T}, \mathcal{O}, \mathcal{Z} \} \):

- \( |U|^2 = \sum_{V \in \mathcal{V}, W \in \mathcal{W}} [V^s W^t F_{\kappa \lambda}]. \)
- \( |\nabla U|^2 = \sum_{V \in \mathcal{V}, W \in \mathcal{W}} [V^s W^t \nabla Y F_{\kappa \lambda}]. \)

For an arbitrary two-form \( \mathcal{F} \) with null components \( \tilde{\mathcal{F}} \), \( \tilde{\alpha} \), \( \rho \), \( \sigma \); and \( A \in \{ \mathcal{T}, \mathcal{O}, \mathcal{Z} \} \):

- \( \mathcal{F}^2 = \frac{1}{4} \left[ (1 + q^2)|\tilde{\mathcal{F}}|^2 + (1 + s^2)|\tilde{\alpha}|^2 + (2 + q^2 + s^2)(\rho^2 + \sigma^2) \right]. \)
- \( \mathcal{F}_{\mathcal{A}}^2 = \sum_{|I| \leq N} |\mathcal{L}_A \mathcal{F}|^2. \)
- \( \|\mathcal{F}(t)\|^2_{L^2 S} = \int_{\mathbb{R}^3} \|\mathcal{F}(t, x)\|^2_{L^2 S} d^3 x. \)

For an arbitrary tensorfield \( U \) defined on the Euclidean space \( \Sigma \) with Euclidean coordinate system \( \Sigma \):

- \( \|U\|^2_{H^a} = \sum_{n=0}^N \int_{\Sigma} (1 + |x|^{2(4+a)}) |\nabla_n U(x)|^2 d^3 x \) is a weighted Sobolev norm of \( U \).
- \( \|U\|^2_{C^a} = \sum_{n=0}^N \sup_{x \in \Sigma} (1 + |x|^{2(4+a)}) |\nabla_n U(x)|^2 \) is a weighted pointwise norm of \( U \).

For arbitrary two-forms \( \mathcal{F} \) and \( \mathcal{F}' \):

- \( H^{\mu\nu\kappa\lambda} \Delta_{\mu\nu} \mathcal{F}_{\kappa \lambda}, \) where \( H^{\mu\nu\kappa\lambda} \) depends on \( \mathcal{F} \), is the principal term in the equations of variation (186b). 
- \( \tilde{\mathcal{Q}}_\mu[\mathcal{F}, \mathcal{F}'] = H^{\mu\nu\kappa\lambda} \mathcal{F}_{\kappa \lambda} \mathcal{F}'_{\nu \lambda} - \frac{1}{4} \delta_\mu^A H^{\mu\nu\kappa\lambda} \mathcal{F}_{\kappa \lambda} \mathcal{F}'_{\nu \lambda} \) is the modified canonical stress tensor.
- \( J^\mu_N[\mathcal{F}] = -\tilde{\mathcal{Q}}_\mu[\mathcal{K}, \mathcal{F}] \) is the energy current (used during the proof of global existence) constructed from the variation \( \mathcal{F} \), the background \( \mathcal{F} \), and the Morawetz-type vectorfield \( \mathcal{K} = \frac{1}{2} \left[ (1 + s^2) L + (1 + q^2) L \right]. \)
- \( \mathcal{E}_N[\mathcal{F}(t)] = \int_{|t| \leq N} J^\mu_N[\mathcal{L}_2 \mathcal{F}] d^3 x \) is the square of the \( N \)th order energy of \( \mathcal{F} \).
14. Function spaces and the regularity of maps

- $H^N$ is the set of all distributions $f$ such that $\|f\|_{H^N} < \infty$.
- $C^N$ is the set of all functions $f$ such that $\|f\|_{C^N} < \infty$.
- $C^k([0, T) \times \mathbb{R}^3)$ denotes the set of $k$-times continuously differentiable functions on $[0, T) \times \mathbb{R}^3$.
- If $X$ is a function space, then $C^k([0, T), X)$ denotes the set of $k$-times continuously differentiable maps from $[0, T)$ to $X$.


