Casimir forces beyond the proximity approximation

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The proximity force approximation (PFA) relates the interaction between closely spaced, smoothly curved objects to the force between parallel plates. Precision experiments on Casimir forces necessitate, and spur research on, corrections to the PFA. We use a derivative expansion for gently curved surfaces to derive the leading curvature modifications to the PFA. Our methods apply to any homogeneous and isotropic materials; here we present results for Dirichlet and Neumann boundary conditions and for perfect conductors. A Padé extrapolation constrained by a multipole expansion at large distance and our improved expansion at short distances, provides an accurate expression for the sphere-plate Casimir force at all separations.

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First introduced by Derjaguin [1], the proximity force approximation (PFA) relates forces between gently curved objects at close separations to the corresponding interactions between flat surfaces over an area set by the local radii of curvature. Originally developed in the context of surface adhesion and colloids, the PFA has found application as far afield as the study of inter-nuclear forces and fission [2, 3]. In particular, the PFA has long been applied to Casimir and other fluctuation forces that are short range and therefore dominated by the surface proximity. Despite its nearly universal use to parameterize Casimir forces in experimentally accessible geometries like a sphere opposite a plate [4, 5], not even the first correction to the PFA in the ratio of the inter-surface separation to the scale of curvature, \( d/R \), is tractable at the values of \( R/d \). While conceptual threads connected to this approach can be traced back to earlier multipole-scattering methods [6], concrete analytical [7] and numerical [8] results are relatively recent. Indeed, the experimentally most relevant set-up of a sphere and plate was only treated in 2008 [9]. The scattering method is most powerful at large separation, providing analytic expressions for a multipole expansion in the ratio \( R/d \) and numerical results for \( R/d \lesssim 10 \), but becomes intractable at the values of \( d/R \sim 10^{-3} \), needed to study the PFA. In a feat of mathematical dexterity, Bordag and Nikolaev (BN) summed the scattering series to obtain the first correction to the PFA for the cylinder/plane and sphere/plane geometries, initially for a scalar field obeying Dirichlet (D) boundary conditions (bc) [11], and later for the electromagnetic (EM) field with ideal metallic bc [12]. Intriguingly, whereas the correction to the force in the former case for both geometries is analytic in \( d/R \), the sphere/plane EM case was predicted to include logarithmic corrections \( (d/R) \ln(d/R) \). While the BN PFA corrections for cylinders (where there are only analytical corrections), were independently verified by Teo [13], the results of Ref. [11, 12] for the experimentally interesting sphere/plane EM case remain unconfirmed.

A recent work by Fosco et al. [14] provides a quite promising perspective on Casimir PFA corrections. Reinterpreting perturbative corrections to parallel plate forces [15, 16], they propose a gradient expansion in the local separation between surfaces for the force between gently curved bodies. They implement their program for scalar fields and confirm that their general approach reproduces the BN cylinder and sphere/plane correction for D bc. Inspired by this work, the current paper is organized as follows: (i) We show that the extension of
Ref. [14] to two curved surfaces is constrained by tilt invariance of the reference plane (from which the two separations are measured). This provides a stringent test of the self-consistency of perturbative results. (ii) Going beyond the scalar fields and D bc of Ref. [13], we compute the form of the gradient expansion for Neumann (N), mixed D/N, and EM (perfect metal) boundaries. Interestingly, we find that the EM correction must co-occur with the sum of D and N corrections. (iii) We reproduce previous results for cylinders [13] with D, N, and mixed D/N conditions, and the sphere with D bc. However, we do not confirm the BN predictions for the sphere/plane geometry [12] either with N or EM bc. (iv) As a further test, we introduce a Padé approximant for sphere/plane geometry [12] either with N or EM bc. (iv) By exploiting Eqs. (2) we see that, to second order in the gradient expansion, the two-surface problem reduces to the simpler problem of a single curved surface facing a plane, since $U$, $\beta_1$ and $\beta_2$ can be determined in that case and $\beta_\chi$ follows from Eqs. (2). Therefore in Eq. (1) we set $H_1 = 0$ and define $\beta_2(H) \equiv \beta(H)$. By a simple generalization of Ref. [14], where the problem is studied for a scalar field subjected to D bc on both plates, we can determine the exact functional dependence of $\beta(H)$ on $H$ by comparing the gradient expansion Eq. (1) to a perturbative expansion of the Casimir energy around flat plates, to second order in the deformation. For this purpose, we take $\Sigma$ to be a planar surface and decompose the height of the curved surface as $H(x) = d + h(x)$, where $d$ is chosen to be the distance of closest separation. For small deformations $|h(x)/d| \ll 1$ we can expand $E[0,d+h]$ as:

$$E[0,d+h] = A U(d) + \mu(d) \tilde{h}(0) + \int \frac{d^2k}{(2\pi)^2} G(k;d) |\tilde{h}(k)|^2 ,$$

where $A$ is the area, $k$ is the in-plane wave-vector and $h(k)$ is the Fourier transform of $h(x)$. The kernel $G(k;d)$ has been evaluated by several authors, for example in Ref. [16] for a scalar field fulfilling D or N bc on both plates, as well for the EM field satisfying ideal metal bc on both plates. More recently, $G(k;d)$ was evaluated in Ref. [17] for the EM field with dielectric bc. For a deformation with small slope, the Fourier transform $\tilde{h}(k)$ is peaked around zero. Since the kernel can be expanded at least through order $k^2$ about $k = 0$ [14], we define

$$G(k;d) = \gamma(d) + \delta(d) k^2 + \cdots .$$

(4)

For small $h$, the coefficients in the derivative expansion can be matched with the perturbative result. By expanding Eq. (1) in powers of $h(x)$ and comparing with the perturbative expansion to second order in both $h(k)$ and $k^2$, we obtain

$$U'(d) = \mu(d), \quad U''(d) = 2\gamma(d), \quad \beta(d) = \frac{\delta(d)}{U(d)},$$

(5)

Arbitrariness in the choice of $\Sigma$ constrains the $\beta$-coefficients. Invariance of $E$ under a parallel displacement of $\Sigma$ requires that all the $\beta$’s depend only on the height difference $H$, and not on the individual heights $H_1$ and $H_2$. The $\beta$-coefficients are further constrained by the invariance of $E$ with respect to tilting the reference plate $\Sigma$. Under a tilt of $\Sigma$ by an infinitesimal angle $\epsilon$ in the $(x,z)$ plane, the height profiles $H_i$ change by $\delta H_i = -\epsilon (x + H_i \frac{\partial H_i}{\partial z})$, and the invariance of $E$ implies

$$2 (\beta_1(H) + \beta_2(H)) + 2 \beta_\chi(H) + H \frac{d \log U}{dH} - 1 = 0 ,$$

$$\beta_\perp(H) = 0 .$$

(2)

Hence the non-vanishing cross term $\beta_\chi$ is determined by $\beta_1$, $\beta_2$ and $U$. Therefore in Eq. (1) we set $H_1 = 0$ and define $\beta_2(H) \equiv \beta(H)$. By a simple generalization of Ref. [14], where the problem is studied for a scalar field subjected to D bc on both plates, we can determine the exact functional dependence of $\beta(H)$ on $H$ by comparing the gradient expansion Eq. (1) to a perturbative expansion of the Casimir energy around flat plates, to second order in the deformation. For this purpose, we take $\Sigma$ to be a planar surface and decompose the height of the curved surface as $H(x) = d + h(x)$, where $d$ is chosen to be the distance of closest separation. For small deformations $|h(x)/d| \ll 1$ we can expand $E[0,d+h]$ as:

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$$U'(d) = \mu(d), \quad U''(d) = 2\gamma(d), \quad \beta(d) = \frac{\delta(d)}{U(d)},$$

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with prime denoting a derivative. Using the above relation, we computed the coefficient $\beta$ for the following five cases: a scalar field obeying D or N bc on both surfaces, or D bc on the curved surface and N bc on the flat, or vice versa, and for the EM field with ideal metal bc. Since in all these cases the problem involves no other length apart from the separation $d$, $\beta$ is a pure number. $\beta_D$ was computed in [14], and found to be $\beta_D = 2/3$. We find $\beta_N = 2/3(1 - 30/\pi^2)$, $\beta_{DN} = 2/3$, $\beta_{ND} = 2/3 - 80/7\pi^2$ (where the double subscripts denote the curved surface and flat surface bc respectively), and $\beta_{EM} = 2/3(1 - 15/\pi^2)$. Upon solving Eqs. (2) one then finds $\beta_\infty = 2 - \beta_1 - \beta_2$, where $\beta_1$ and $\beta_2$ are chosen to be both equal to either $\beta_D$, $\beta_N$ or $\beta_{EM}$, for the case of identical bc on the two surfaces, or rather $\beta_1 = \beta_{DN}$ and $\beta_2 = \beta_{ND}$ for the case of a scalar field obeying mixed ND bc. While we obtain agreement for D conditions, our results for $\beta$ in the case of N and EM conditions disagree with Refs. [11, 12], which finds $\beta_N = 2/3 - 5/\pi^2$ and $\beta_{EM} = -2.1$, multiplied by logarithmic terms in the latter case that we do not obtain. We note that our value of $\beta_{EM}$ is equal to the average of $\beta_D$ and $\beta_N$. Using the well known results $U(d) = -\alpha \pi^2 \hbar c/(1440 d^3)$, where $\alpha_D = \alpha_N = \alpha_{EM}/2 = 1$ and $\alpha_{DN} = \alpha_{ND} = -7/8$ in Eq. (1), it is easy to verify that to second order in the gradient expansion the ideal-metal EM Casimir energy for two arbitrary surfaces is equal to the sum of the D and N Casimir energies in the same geometry.

Using the values for $\beta$ and $\beta_\infty$ obtained above, it is possible to evaluate the leading correction to PFA, by explicit evaluation of Eq. (6) for the desired profiles. For example, for two spheres of radii $R_1$ and $R_2$, both with the same bc for simplicity, we obtain:

$$E = E_{\text{PFA}} \left[ 1 - \frac{d}{R_1 + R_2} + (2\beta - 1) \left( \frac{d}{R_1} + \frac{d}{R_2} \right) \right],$$

where $E_{\text{PFA}} = -(\alpha \pi^2 \hbar c R_1 R_2)/(1440 d^3(R_1 + R_2))$. The corresponding formula for the sphere-plate case can be obtained by taking one of the two radii to infinity. Similarly, for two parallel circular cylinders with different radii and our results, not given here for brevity, fully agree with those derived very recently by Teo [13]. Finally we consider two circular cylinders whose axes are inclined at a relative angle $\theta$.

$$E = -\frac{\alpha \pi^3 \hbar c \sqrt{R_1 R_2}}{720 d \sin \theta} \left[ 1 + \left( \beta - \frac{3}{8} \right) \frac{d}{R_1 + R_2} \right].$$

again assuming the same bc on both surfaces.

Surprisingly, a hyperboloid/plate configuration, $h(r) = \sqrt{R^2 + \lambda^2 r^2} - R$ leads to a PFA correction proportional to $(2\beta + 1/\lambda^2)d/R$, which vanishes for the EM case if $\lambda = 1.20$. To better understand what geometric features of the surface determine the PFA corrections, we considered a general profile, expanded around the point of shortest separation, including cubic and quartic terms in addition to quadratic terms representing its curvature. The cubic terms – representing an asymmetry absent in the shapes considered above – actually result in a correction term that scales as $\sqrt{d/R}$. The more standard correction proportional to $d/R$ depends on both cubic and quartic but no higher order terms in the expansion of the profile. The quartic coefficients in fact have opposite signs for the sphere and hyperboloid, and it is this difference that accounts for the possibility of a vanishing PFA correction (for the hyperboloid with EM bc).

![FIG. 1: Casimir energy normalized to PFA energy as a function of $d/R$ for the sphere-plate configuration: Numerical data (dots, from Ref. [11]), Padé approximants (solid curves) and the first correction to the PFA (dashed lines). Also shown are the linear slopes (dashed-dotted lines) computed by BN [11, 12].](image)

The disagreement between our results for N and EM bc and those of BN [11, 12] in the prototype configuration of sphere and plate calls for a comparison with independent results. Precise numerical results based on an extrapolation of a spherical multipole expansion exist for $d/R \gtrsim 0.1$ [10]. Also, at large separations analytic asymptotic expansions (AE) of the rescaled force $f = R^2 F/(\hbar c) \sim \sum_{j=1}^{\infty} f_j r^{j_0 + j}$ in powers of $r = R/d$ are available for sufficiently large $n$, where $j_0 = 2$ for D and $j_0 = 4$ for N and EM conditions, respectively. We resum the AE for $f$, match it with our improved PFA result using the method of Padé approximants, and compare the resulting interpolation with the numerical results of Ref. [10]. We introduce a Padé approximant $f_{[M/ N]}(r)$ that agrees with $n$ terms of the AE for small $r$ and with the leading two terms $\sim r^3$, $r^2$ for large $r$ given by Eq. (6). This requires $N = M - 3$ whereas $M$ is determined by
are uniquely determined. The energy, obtained from $\tilde{f}_{M/M-3}(r)$ by integration over distance is shown with $n = 7, 13$ and $15$ for D, N and EM bc, respectively, in Fig. 1 together with our result of Eq. (1) for the sphere-plate case ($R_1 = R$, $R_2 \to \infty$) and the numerical results of Ref. [10]. The difference between Padé approximants and the numerical data varies between 0.009% and 0.062% (D), 0.048% and 0.42% (N), 0.009% and 0.248% (EM). This agreement is quite remarkable given that the Padé approximants are obtained independent of the numerical data. The Padé approach may thus be valuable for Casimir interactions between slowly varying surfaces at all separations starting from expansions valid at small and large distances respectively. Figure 1 also shows the BN results for the first correction to the PFA.

The Padé approximant for the force at small $d/R$ and integration yields for the energy the expansion

$$E_{PFA} = 1 + \theta_1 \frac{d}{R} + \theta_2 \left( \frac{d}{R} \right)^2 \log \frac{d}{R} + \mathcal{O} \left( \left( \frac{d}{R} \right)^3 \right)$$

with $\theta_1 = 2\beta - 1$ and $\theta_2 = 0.08$ for D, $\theta_2 = -24.01$ for N and $\theta_2 = -4.52$ for EM conditions. The appearance of a logarithm at second order in Eq. (9) follows from our assumption that the force can be expanded in powers of $d$ at small $d$ including a term $\sim 1/d^2$. A Padé approximation to the energy would yield no $1/d$ term in the force. To check that the $d^2 \log d$ term in the energy is necessary, we have tried a Padé analysis of the energy, which of course contains no logarithms, and found much poorer agreement with the numerical data (with fractional differences of the order of 100 times larger than those given above for the Padé approximant for the force and moreover a pole in the approximant). One could perform Padé approximations to derivatives of the force which might turn out to be even more accurate but this would only change the functional form (logarithmic factors) of Eq. (9) at third order in $d/R$. As an independent test of our result $\theta_1 = 2\beta - 1$ we fitted the functional form of Eq. (9) to the numerical data reported in Ref. [10] [12]. The corresponding values for $\theta_1$ are summarized in Table I. When comparing numerical and analytic values for $\theta_1$ one must consider that they are limited to $d/R \gtrsim 0.1$ with somewhat better accuracy for the D and EM case. Finally, we note that the presence of the logarithm at order $(d/R)^2$ indicates that the gradient expansion for the energy might break down at next order.

Based on the derivative expansion proposed by Fosco et al. [14], we have described a systematic treatment of the first non-trivial corrections to the PFA. While this paper focused on ideal boundary conditions, the scattering amplitudes for material with arbitrary EM response

<table>
<thead>
<tr>
<th>BOUNDARY CONDITION</th>
<th>$\theta_1$ FROM FIT TO NUMERICAL DATA</th>
<th>$\theta_1$ FROM GRADIENT EXPANSION</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>+0.36</td>
<td>1/3</td>
</tr>
<tr>
<td>N</td>
<td>-2.99</td>
<td>1/3 $- 40/\pi^2 = -3.72$</td>
</tr>
<tr>
<td>EM</td>
<td>-1.62</td>
<td>1/3 $- 20/\pi^2 = -1.69$</td>
</tr>
</tbody>
</table>

(frequency dependent $\epsilon(\omega)$, $\mu(\omega)$, isotropic or not, local or not) have been computed in the gradient expansion and can be used to compute improved PFA corrections in experimentally relevant situations of sphere/plate as well as corrugated plates and other setups.

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[18] We note that in [10] [12] numerical data have been fitted to a function that differs from Eq. (9) by the absence of the log, leading to different values for $\theta_1$.