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Microscopic model for the boson integer quantum Hall effect

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In two dimensions strongly interacting bosons in a magnetic field can form an integer quantum Hall state. This state has a bulk gap, no fractional charges or topological order in the bulk, but nevertheless quantized Hall transport and symmetry-protected edge excitations. Here we study a simple microscopic model for such a state in a system of two-component bosons in a strong orbital magnetic field. We show through exact-diagonalization calculations that the model supports the boson integer quantum Hall ground state in a range of parameters.

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Recently there has been considerable theoretical activity on the possibility of symmetry-protected topological (SPT) insulating phases of interacting bosons in diverse dimensions. These are generalizations of the celebrated free fermion topological insulators$^1$ to systems of bosons. As in the noninteracting limit a boson system is trivial, strong interactions are necessary to reach any insulating phase, including the topological ones. Thus studies of bosonic insulators force us to think of topological insulation without the crutch of free fermions and hand topology appropriate for noninteraction fermion systems. It is expected that the insights from these studies will be useful in thinking about the interplay of strong correlations with topological insulation in electronic systems.

In the interacting context we seek phases of bosonic matter that have a bulk gap, have no “intrinsic” topological order or fractionalized excitations, but nevertheless have protected boundary excitations. In space dimension $d = 1$ the familiar Haldane spin chain provides a well-established example of such a state. Recently progress was made in classifying all such gapped 1$d$ systems based on the concept of group cohomology.$^2$–$^5$ In $d > 1$ a cohomology classification has also been proposed$^6$ of all gapped bosonic systems with no “intrinsic” topological order which predicts the existence of many SPT states. Physical properties of these phases are not readily accessed through this framework and have been addressed through other methods in both two$^7$–$^10$ and three$^{11}$–$^{14}$ dimensions. States beyond the cohomology classification are also known$^{1}$,$^{13}$,$^{15}$ to exist (in $d = 3$).

One of the most important questions that remains open in this field is the identification of experimentally relevant microscopic models for these SPT phases in $d > 1$. Here we address this question for a prototypical SPT phase in $d = 2$, namely an integer quantum Hall state of bosons. We show that a simple and realistic Hamiltonian for a system of two-component bosons in a strong magnetic field provides a concrete realization of this state. The physical properties of this state have been studied in Refs. 8, 9, 16, and 17. The electrical Hall conductivity $\sigma_{xy}$ was shown$^9$ to be quantized to be an even integer (see Ref. 9 for a simple argument). At the edge to the vacuum there is a charged chiral edge mode and a counterpropagating neutral mode. Despite being nonchiral this edge structure is protected so long as charge conservation symmetry is preserved. This boson integer quantum Hall state provides a prototypical example of a symmetry-protected topological insulator in boson systems in two space dimensions. Reference 9 proposed that two-component bosons in strong orbital magnetic fields at filling factor $\nu = 1$ for each component provide a suitable platform to realize the boson integer quantum Hall state (bIQH). A number of potential competing states were also identified.

In this Rapid Communication we study the simplest Hamiltonian for such a system through exact-diagonalization studies. We find strong evidence for the stability of the boson integer quantum Hall state.

We consider a bilayer system of $N$ interacting bosons and $N_\theta$ flux quanta. All our calculations are performed in the lowest Landau level and the filling factor is defined as $\nu_i = N_i/N_\theta$. The bilayer index is equivalent to a spin 1/2. The number of particles per layer is respectively $N^1$ and $N^\perp$. We define the spin projection as $S_z = (N^1 - N^\perp)/2$. We denote by $z_i^\uparrow, z_i^\downarrow$ the complex coordinates of the two boson species. The Hamiltonian projected in the lowest Landau level reads

$$H = g_s \sum_{i<j} \delta^{(2)}(z_i^\uparrow - z_j^\downarrow) + g_d \delta^{(2)}(z_i^\downarrow - z_j^\uparrow) + g_d \sum_{i,j} \delta^{(2)}(z_i^\uparrow - z_j^\downarrow).$$

When $g_s = g_d$ the Hamiltonian has an extra pseudospin SU(2) symmetry which rotates the two species of bosons into one another. The ground state of this model was previously studied using exact diagonalization in Refs. 18 and 19 with an emphasis on total filling factor $\nu_i = \frac{k}{2}$. Here we focus instead on $\nu_i = 2$ which is a suitable platform to realize the bIQH state. Reference 9 showed that prototypical wave functions for the bIQH state is a pseudospin singlet. Thus we may expect the bIQH state to be stabilized, if at all, near the $g_s = g_d$ point. A competing pseudospin singlet state is a member of a family of non-Abelian spin singlet (NASS) states$^{20}$ for two-component bosons at various total fillings $\nu_i = \frac{k}{2}$ ($k$ integer). Our calculations below show that the bIQH state wins over the NASS state at total filling factor $\nu_i = 1 + 1 = 2$. When the ratio $\frac{g_s}{g_d} = 0$, the two components are decoupled and at $\nu = 1$ each. With delta function repulsion there is strong evidence$^{21, 23}$ that each component is in the non-Abelian Pfaffian (Moore-Read) state (PF).$^{24}$ This decoupled PF $\times$ PF...
The bIQH state is stable in a range of \( \nu_t \) estimated to be between \( \sim 0.8 \) and \( \sim 1.3 \) from exact-diagonalization calculations with 12 and 16 particles. PS refers to the phase-separated state which appears for large \( \nu_t \). For small \( \nu_t \) the decoupled Pfaffian state Pf \( \times Pf \) is stable.

In the other limit when \( \nu_t = \infty \) we expect phase separation into puddles with each puddle consisting of bosons living in one of the two layers with twice the average density (hence at filling factor one of the two layers). Thus the bIQH state seems quite robust in this model. The phase diagram obtained from our work is summarized in Fig. 1.

A wave function for the bIQH state, similar to one previously introduced in the context of the fractional quantum Hall (FQH) for fermions at \( \nu = 2/3 \), was proposed in Ref. 9:

\[
\Psi_{\text{bfu}} = P_{LLL} \left[ \prod_{i<j} |z_i^\uparrow - z_j^\uparrow|^2 \prod_{i<j} |z_i^\downarrow - z_j^\downarrow|^2 \right],
\]

where \( P_{LLL} \) denotes the projection onto the lowest Landau level. As there is no intrinsic topological order there is a unique ground state on the torus.

The competing NASS state can be built up starting with the \((m,m,n)\) Halperin wave function given by

\[
\Psi_{(m,m,n)} = \prod_{i<j} (z_i^\uparrow - z_j^\uparrow)^p \prod_{i<j} (z_i^\downarrow - z_j^\downarrow)^p \prod_{i,j} (z_i^\uparrow - z_j^\downarrow)^p.
\]

The filling factor is \( \nu_t = 2/(m + n) \). Note that the \((1,1,1)\) Halperin is the densest zero-energy state for our model Hamiltonian in Eq. (1) when \( g_s, g_d > 0 \). From the Halperin wave function, one can build a series of NASS states at filling \( \nu_t = 2/(m + n) \). This is done by dividing the particles into \( k \) groups, writing a Halperin \((m;m,m - 1)\) state for each group, and

\[
\Psi_{m} = S \left[ \prod_{i=0}^{k-1} \Psi_{(m,m,m-1)}(z_i^\uparrow, z_{i+1}^\uparrow), z_i^\downarrow, z_{i+1}^\downarrow, \ldots, z_{(i+1)\mod k}^\downarrow \right],
\]

where \( S \) is the symmetrization operator. For \( m = 2,n = 1 \), these are the NASS states introduced by Ref. 20. The NASS state is \( (2+2k+1) \) degenerate on the torus geometry.

For our numerical calculations, we consider the torus geometry with a square aspect ratio. The choice of the torus instead of the sphere allows us to avoid the bias of the so-called shift: In many cases, such a bias prevents the direct comparison of possible candidates in finite-size calculations. Moreover, each model state has a well-defined nontrivial degeneracy on the torus, providing a simple signature that we can extract from the energy spectrum. We performed calculations using the translation symmetry of the torus along the \( y \) direction. Thus our states are only labeled by the \( k_x \) momentum which can be expressed as an integer between 0 and \( N_b - 1 \). We first consider the SU(2) symmetric point (i.e., \( g_s = g_d \)). We have looked at several system sizes up to 18 particles. A typical energy spectrum is shown in Fig. 2(a) for \( N = 16 \) at filling factor \( \nu = 2 \). We clearly observe a unique ground state at \( k_x = 0 \) and with total spin \( S = 0 \). Such a signature is the one expected for an IQH state at this filling factor. There is no trace left of the Pf \( \times Pf \) that should occur in the limit \( g_d = 0 \). Its ninefold degenerate manifold (five states in momentum sector \( k_x = 0 \) and four states in momentum sector \( k_x = 4 \)) is completely lifted. All the energy spectra at \( \nu = 2 \) that we have computed share exactly the same features as this \( N = 16 \) example. In Fig. 3, we give the behavior of the neutral gap \( \Delta \) (i.e., without changing the number of particles or the number of flux quanta) as a function of the system size. While a thermodynamical extrapolation is not straightforward due to the size effects, it looks convincing that \( \Delta \) will stay finite. Note
that contrary to most of the other FQH phases, the gap seems to slightly increase with increasing \( N \).

The hallmark of the FQH effect is the unique nature of its excitations which are realized by inserting or removing flux quanta or particles. In the case of the integer quantum Hall at \( \nu = 1 \), removing an electron leads to a number of degenerate ground states that exactly matches the number of orbitals. For many of the model states of the FQH effect, this allows us to predict how many states should appear in the low-energy spectrum using the Haldane exclusion principle.\(^{26}\) Each counting is a signature of the considered phase. Figure 2(b) shows the low-energy part of the spectrum when we remove one particle from the case described in Fig. 2(a). There is a set of low-energy states with spin \( S = \frac{1}{2} \), one per momentum sector. In this example, we thus get 16 states. This is exactly what would be obtained for the bIQH state when removing 1 particle from a finite system 16 fermions. Similar results are obtained when adding one particle. This is a clear indication that the low-energy physics of our model is equivalent to the bIQH state. Other ways to nucleate excitations, such as adding or removing flux quanta, generate many more states. Due to the energy splitting in finite-size calculations, the signatures in the energy spectrum are not as clear as in the case of adding or removing one particle.

We have tried to look at the particle entanglement spectrum\(^{27, 28}\) of the ground state. Unfortunately, we were not able to extract any clear signature. Note that the bIQH phase is a composite fermion-like state with reverse flux attachment and such a behavior is expected: Such states (like the fermionic spin-polarized 2/3 state) have a particle entanglement spectrum that looks like a generic wave function for the same system size (i.e., the corresponding reduced density matrix does have either an exponentially large number of zero eigenvalues or an entanglement gap). We have also looked at the sphere geometry at the shift given by the model wave function of Eq. (3). For every system size, we have observed a spin-single state with a total angular momentum equal to zero and a large gap.

We now turn to the phase diagram when tuning the ratio \( g_d/g_s \), Figs. 4(a) and 4(b) for two system sizes, \( N = 12 \) and \( N = 16 \). Note that the Pf × Pf phase occurs when \( N \) is a multiple of 4, and the NASS state occurs when \( N \) is a multiple of 6. So \( N = 12 \) is the only case where we can have the three phases. In the \( g_d/g_s = 0 \) limit, we observe a ninefold low-energy manifold that corresponds to the Pf × Pf phase. However, one cannot distinguish it for \( N = 16 \). There is finite splitting of the threefold-degenerate MR ground state when this phase is realized through the two-body delta interaction. Thus when having two copies of this system in finite size, the splitting of the total ninefold low-energy manifold can be larger than the gap of a single copy.

The NASS state with \( k = 3 \) and \( m = 2 \) is a potential candidate at \( \nu_l = 2 \). A first signature of this state should be \( \nu_l = 1 \) where Refs. 18 and 19 found old evidence for the \( k = 2 \) NASS state.

In the phase diagram, the most prominent phase is related to the one discussed for the SU(2) symmetric point \( g_d/g_s = 1 \). For both system sizes, the gap is stable over a large region of \( g_d/g_s \), that include this specific point. To get a more quantitative estimate, we can compute the overlap \(|\langle \Psi_{gs/g_s=1}|\Psi_{gs/g_s}\rangle|^2\) between the ground state at any value of \( g_d/g_s \) and the ground state at \( g_d/g_s = 1 \). For \( N = 12 \), this overlap is higher than 0.9 for 0.80 \( \leq g_d/g_s \leq 1.25 \) [i.e., centered around the peak in Fig. 4(a)]. For \( N = 16 \), this range becomes 0.75 \( \leq g_d/g_s \leq 1.40 \) (for this system size, the Hilbert space dimension is \( 5.2 \times 10^6 \)). At the peak value of gap around \( g_d/g_s \simeq 1.7 \), the overlap is still 0.78.
In summary we have studied a simple microscopic boson model that shows strong evidence for an integer quantum Hall ground state with no intrinsic topological order. This state is a prototypical example of a symmetry-protected topological insulator of bosons in two space dimensions, and is an interacting boson analog of the free fermion topological insulators. The existence of this interacting dominated topological insulator phase in such a simple model provides hope that other such boson insulators in both two6,8 and three11 dimensions may also be realized in fairly simple models.

Clearly a natural experimental context to seek a realization of the boson integer quantum Hall state is in ultracold atoms. The delta function repulsion is realistic and natural in cold-atom systems. Realizing an orbital magnetic field big enough to be in the quantum Hall regime is a long-standing challenge in the field. We hope that the results of this Rapid Communication provide further impetus to meet this challenge.

Note added. Recently, we became aware of two related papers.29,30 In particular, we recover the same scaling of the gap on the torus geometry as Ref. 29.

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