Majorana flat bands and unidirectional Majorana edge states in gapless topological superconductors

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Majorana flat bands and unidirectional Majorana edge states in gapless topological superconductors

Chris L. M. Wong, Jie Liu, K. T. Law, and Patrick A. Lee

I. INTRODUCTION

A topological superconductor (TS) has a bulk superconducting gap and topologically protected gapless boundary states. Majorana fermions in TSs are under intense theoretical and experimental studies due to the possibility of realizing Majorana fermions in these systems, which act as their own antiparticles and obey non-Abelian statistics. Majorana fermions in TSs are topologically protected, in the sense that the Majorana fermions cannot be removed by perturbations unless the bulk energy gap is closed or certain symmetries are broken.

Remarkably, recent development shows that Majorana fermions exist in systems where the bulk is gapless. For example, Majorana edge states (MESs) which flat dispersion can be found in two-dimensional (2D) nodal $d_{xy} + ip$-wave superconductors which respect time-reversal symmetry. It is also shown that zero energy Majorana flat bands (MFBs) can appear on the surface of three-dimensional (3D) time-reversal invariant noncentrosymmetric superconductors which have topologically stable line nodes in the bulk.

In this Rapid Communication, we show that an in-plane magnetic field can drive a fully gapped $p \pm ip$ topological superconductor into a gapless phase which supports Majorana flat bands (MFBs). Unlike previous examples, the MFBs in the gapless regime are protected from disorder by a chiral symmetry. In addition, novel unidirectional Majorana edge states (MESs) which propagate in the same direction on opposite edges appear when the chiral symmetry is broken by Rashba terms. Unlike the usual chiral or helical edge states, unidirectional MESs appear only in systems with a gapless bulk. The MFBs and the unidirectional MESs induce nearly quantized zero bias conductance in tunneling experiments.

We start with a Bogoliubov–de Gennes (BdG) Hamiltonian which describes a two-dimensional noncentrosymmetric superconductor with both spin-triplet $p_x + ip_y$-wave, spin-singlet $s$-wave pairing, and Rashba spin-orbit coupling in the presence of a magnetic field:

$$H_p(k) = \begin{pmatrix} \xi(k) + V \cdot \sigma & \hat{\Delta}(k) \\ -\hat{\Delta}^*(k) & -\xi^T(-k) - V \cdot \sigma^* \end{pmatrix}. \quad (1)$$

Here $\xi(k) = [-t(\cos k_x + \cos k_y) - \mu] \sigma_0 - \alpha_R (-\sin k_x \sigma_y + \sin k_y \sigma_x)$ is the sum of the kinetic energy and the Rashba spin-orbit coupling. $V$ describes the Zeeman coupling of the electrons with an external magnetic field, and $\hat{\Delta}(k) = [\Delta_s + (d(k) \cdot \sigma)](i\sigma)$ is the superconducting gap function. We first assume that the spin-singlet pairing amplitude $\Delta_s$ and the Rashba spin-orbit coupling $\alpha_R$ are zero. The spin-triplet

![Diagram of a superconductor with a magnetic field](image)

FIG. 1. (Color online) (a) A schematic picture of a $p \pm ip$-wave superconductor subject to an in-plane magnetic field $V_y$. A tunnel junction and a normal lead $N$ is attached to the superconductor. (b) The energy spectrum of a $p \pm ip$ superconductor in the topologically nontrivial regime. Periodic boundary conditions in the $x$ direction and open boundary conditions in the $y$ direction are assumed. The parameters are $t = 12 \Delta_p$, $\mu = 3 \Delta_p - 2t$, $\Delta_s = 0$, $\alpha_R = 0$, and $V_y = 0$. (c) Same parameters as (b), except $V_y = 0.7 \Delta_p$. The bulk energy gap is closed in this regime.
pairing vector is chosen as \( \mathbf{d}(k) = \Delta_p(\sin k_y, \sin k_x, 0) \) such that the Hamiltonian describes a two-dimensional, helical, \( p \pm ip \)-wave superconductor where \( \Delta_p \) is a constant. When \( \mathbf{V} = 0 \), the Hamiltonian respects both time-reversal symmetry \( T = U_T K \) with \( U_T^{-1} H_T^* (k) U_T = H_T (k) \) and particle-hole symmetry \( P = U_P K \) with \( U_P^{-1} H_p^* (k) U_P = -H_p (-k) \). Here, \( K \) is the complex conjugate operator, \( U_T = i \sigma y \otimes \sigma y \) and \( U_P = \sigma y \otimes \sigma y \) such that \( T^2 = -1 \) and \( P^2 = 1 \).

According to symmetry classification, the above Hamiltonian in the absence of an external magnetic field belongs to the DIII class which can be topologically nontrivial. In the topologically nontrivial regime where \( |\sigma| < |\Delta_p| \) and \( |\Delta_p| > |\Delta_s| \), the superconductor possesses gapless counterpropagating helical MEs. The energy spectrum in the topologically nontrivial regime is shown in Fig. 1(b).

In the rest of this section, we show that the \( p \pm ip \) superconductor responds to an in-plane magnetic field in an anomalous way as described in the Introduction.

To be specific, we suppose a magnetic field is applied in the \( y \) direction such that \( \mathbf{V} = (0, V_y, 0) \). In the presence of a magnetic field, the time-reversal symmetry \( T = U_T K \) is broken. However, the Hamiltonian satisfies a time- and space-reversal-like symmetry \( T_{id} = U_{Tid} K \) such that \( T_{id}^{-1} H(k, k_y) T_{id} = H(k_x, -k_y) \), where \( U_{Tid} = \sigma y \otimes \sigma x \). Moreover, the Hamiltonian satisfies a particle-hole-like symmetry \( P_{id} = U_{Pid} K \) such that \( P_{id}^{-1} H(k_x, k_y) P_{id} = -H(k_x, -k_y) \) with \( U_{Pid} = \sigma y \otimes \sigma x \). Since the symmetry operators operate on \( k_y \) only and \( k_x \) is unchanged, one may regard \( k_x \) as a tuning parameter and the Hamiltonian can be written as \( H_{k_x} (k_y) \). As \( H_{k_x} (k_y) \) respects the symmetries \( T_{id} \) and \( P_{id} \) with \( T_{id}^2 = P_{id}^2 = 1 \), \( H_{k_x} (k_y) \) is a BDI class Hamiltonian which can be classified by an integer.

To classify the Hamiltonian \( H_{k_x} (k_y) \) with \( k_x \) as a tuning parameter, we note that as a result of the \( T_{id} \) and \( P_{id} \) symmetries, \( H_{k_x} (k_y) \) satisfies the chiral symmetry \( S_{id} = T_{id} P_{id} \) with

\[
S_{id}^{-1} H(k_x, k_y) S_{id} = -H(k_x, k_y).
\]

In this case, \( H_{k_x} (k_y) \) can be off-diagonalized in the basis which diagonalizes \( S_{id} \) such that

\[
\tilde{H}_{k_x} (k_y) = \begin{pmatrix} 0 & A_{k_x} (k_y) \\ A_{k_x}^*(k_y) & 0 \end{pmatrix}.
\]

Defining the quantity

\[
z(k) = e^{i\theta(k)} = \text{Det} [A_{k_x} (k_y)] / |\text{Det} [A_{k_x} (k_y)]|,
\]

the winding number of \( \theta (k) \), can be used as the topological invariant which characterizes the Hamiltonian \( H_{k_x} (k_y) \). The winding number \( N_{\text{BDI}} \) can be written as\(^{12,21}\)

\[
N_{\text{BDI}} = -\frac{i}{\pi} \int_{k_y = 0}^{k_y = \pi} \frac{dz(k_y)}{z(k_y)}.
\]

Using \( A_{k_x} (k_y) \) obtained from \( H_{k_x} (k_y) \), we have \( |N_{\text{BDI}}| = 1 \) when

\[
\mathcal{M}(k_x, k_y) = 0, \quad \mathcal{M}(k_x, k_y) = \pi < 0,
\]

where

\[
\mathcal{M}(k_x, k_y) = [\mu + r (\cos k_x + \cos k_y)]^2 + \Delta_p^2 \sin^2 k_x - V_y^2,
\]

assuming that \( V_y \) and \( \Delta_p \) are nonzero. In the range of \( k_x \) where \( N_{\text{BDI}} = 1 \), the Hamiltonian \( H_{k_x} (k_y) \) is topologically nontrivial. For a \( p \pm ip \) superconductor with periodic boundary conditions in the \( x \) direction and open boundary conditions in the \( y \) direction, there are zero energy Majorana modes localized on the edges of the system when Eq. (6) is satisfied. Therefore, MFBs appear in the corresponding parameter regime.

The evolution of the energy spectrum of a \( p \pm ip \) superconductor as a result of an increasing in-plane magnetic field is shown in Figs. 1 and 2. First, an in-plane magnetic field reduces the bulk gap as shown in Fig. 1(c). Second, after the bulk gap is closed, MFBs appear for a finite range of \( k_x \) where \( |N_{\text{BDI}}| = 1 \) [Fig. 2(a)]. Third, by further increasing the magnetic field, the bulk gap at \( k_y = 0 \) is closed [Fig. 2(b)]. Fourth, by increasing the magnetic field even further, the energy crossing at \( k_y = 0 \) disappears and only a MFB remains [Fig. 2(c)]. It is important to note that the MFBs appear when the bulk is gapless. The bulk energy spectrum of a \( p \pm ip \)-wave superconductor corresponding to Fig. 2(a) is shown in Fig. 3(a). It is evident that there are nodal points in the bulk spectrum when MFBs appear.

The nodal points in Fig. 2(a) are the projection of the bulk nodal points on the \( k_y \) axis in Fig. 3(a), similar to the cases in intrinsic gapless TSs.\(^{12,15,19}\) Both the nodal points in the bulk spectrum as well as the MFBs are protected by the topological invariant \( N_{\text{BDI}} \). In other words, the MFBs and the nodal points in the bulk appear whenever \( N_{\text{BDI}} \) is nontrivial for some range of \( k_x \). The results in this section apply to helical superconductors and superfluids with \( d \)-vector \( \mathbf{d}(k) = (\sin k_x, \sin k_y, 0) \) as well. One example of such a helical superfluid is helium 3B phase.

III. UNIDIRECTIONAL MAJORANA EDGE STATES

It is shown above that MFBs appear when \( N_{\text{BDI}} = 1 \) for a finite range of \( k_x \) and the MFBs are protected by the symmetries \( P_{\mu} \) and \( T_{id} \). However, \( s \)-wave pairing and Rashba terms break the chiral symmetry \( S_{id} \) in Eq. (2) and lift the zero energy modes to finite energy as shown in Fig. 2(d). In the
case of adding $s$-wave and Rashba terms to Fig. 2(c), the MFB acquires a finite slope and unidirectional MESs appear at the sample edge as shown in Fig. 4(a). A schematic picture of the unidirectional MESs is shown in the inset.

We point out that the right moving edge modes are compensated by extra left moving modes in the bulk, so the current on the edge is canceled by a backflow current in the bulk. Since a bulk backflow current is required to compensate for the edge current, the unidirectional edge states can only appear in systems with a gapless bulk. This is different from chiral and helical edge states which appear in systems with a bulk gap. In the presence of the $s$-wave pairing and Rashba spin-orbit coupling, the Hamiltonian in Eq. (1) describes noncentrosymmetric superconductors such as CePt$_3$Si, CeIrSi$_3$, and CeRhSi$_3$, and these materials are candidates for realizing the unidirectional edge states.

Another interesting finding is that the unidirectional MESs can appear in the absence of $p \pm ip$-wave pairing. The energy spectrum of an $s$-wave superconductor with Rashba terms and finite $V_y$ is shown in Fig. 4(b). It can be shown that the unidirectional edge states appear when

$$\mathcal{M}_s(0,0,\mathcal{M}_s(0,\pi) < 0,$$

where

$$\mathcal{M}_s(k_x, k_y) = [\mu + t(\cos k_x + \cos k_y)]^2 + \Delta_y^2 - V_y^2.$$ 

The systems with pure $s$-wave pairing and Rashba terms can be realized by inducing $s$-wave superconductivity in semiconductors as demonstrated in recent experiments.22-24

IV. ANDREEV REFLECTION AND EFFECTS OF DISORDER

It has been shown in previous works that Majorana fermions induce resonant Andreev reflection at the junction between a normal lead and a fully gapped TS.25,26 However, resonant Andreev reflection may not happen when the bulk is gapless due to the nonvanishing direct tunneling amplitudes from the normal lead to the gapless superconductor. In this section, we calculate the zero bias conductance (ZBC) of a junction between a normal lead and a TS as a function of the in-plane magnetic field strength. It is found that MFBs and unidirectional MESs induce nearly quantized ZBC even when the bulk is gapless and in the presence of disorder.

A schematic picture of the experimental setup is shown in Fig. 1(a). A normal lead is coupled to an edge of the TS to form a tunnel junction. Using the lattice Green’s function method,27-29 we calculate the direct tunneling amplitude and the Andreev reflection amplitude of the tunnel junction.

![Figure 5](image-url)
Figure 5(a) shows the ZBC as a function of $V_y$ for a $p \pm ip$-wave superconductor in the presence of on-site disorder. To understand the results, we note that time-reversal symmetry is preserved and the system is fully gapped at $V_y = 0$, the $4e^2/h$ quantization of ZBC is the property of a DIII class TS which has two Majorana zero modes on the edge. As $V_y$ increases, time-reversal symmetry is broken and the ZBC is suppressed by disorder. Further increasing $V_y$ closes the bulk gap and there is a large jump in the ZBC. This jump is due to the contribution from the Andreev reflection caused by the MFBs and the direct tunneling caused by the gapless bulk. This can be clearly seen from the $V_y$ dependence of $T$ in Fig. 5(a). It is interesting to note that the final ZBC is almost quantized at $2/e$.

To confirm this, the energy spectrum of the Majorana fermions from the flat band. Andreev reflection caused by the large number of independent conducting channels in the normal lead. This is due to the presence of only one zero energy edge mode on the edge when a MFB becomes an unidirectional edge state in this regime.

This can be clearly seen from the $V_y$ dependence of $T$ in Fig. 5(a). It is interesting to note that the final ZBC is almost quantized at $2/e$.

In our case, the zero energy edge states localized on the edge, is always nonzero when MFBs appear. Therefore, the net chirality number of an edge, which is the sum of the chirality numbers of all the zero energy states localized on the same edge of the sample have the same chirality. The chirality of an eigenstate of $H_k$ can be written as

$$\frac{i V_y - i \Delta \frac{\partial}{\partial y} + \Delta \frac{\partial}{\partial x}}{\Delta^2},$$

Using the ansatz $[0, 0, \alpha, \beta, -]T \sin(a + \beta) e^{-\gamma(y - \mu) + ikx}$ and $[\alpha, \beta, 0, 0]T \sin(a + \beta) e^{-\gamma(y - \mu) + ikx}$, we find two zero energy eigenstates for the Hamiltonian at $k_y = 0$ with $a = [2(2t + \mu) - \Delta^2_p + 2V_t]i/\gamma$ and $b = \Delta^{2 -}$.

Interestingly, when $V_y > |2t + \mu|$, $a_\pm$ becomes imaginary and the corresponding wave function is not normalizable. As a result, only one zero energy mode with chirality $+1$ is left at $k_x = 0$ as shown in Fig. 2(c). It can be shown similarly that for large $V_y$, all the zero energy modes localized on one edge have the same chirality. As a result, the net chirality number on an edge is not zero and the zero energy modes cannot be lifted by local perturbations which preserve the chirality number. Since $V_y$ does not break the chiral symmetry, the MFBs can appear in the presence of $V_y$. However, $V_y$ breaks the chiral symmetry that an out-of-plane magnetic field can destroy the MFBs.

### V. ROBUSTNESS OF MAJORANA FLAT BANDS

It is evident from Fig. 3(b) that the MFBs are robust against disorder. In this section, we argue that the MFBs are protected by the chiral symmetry $S_{id}$ in Eq. (2).

In our case, the zero energy edge states localized on the same edge of the sample have the same chirality. The chirality of an eigenstate of $S_{id}$ is defined as the eigenvalue of the state with respect to $S_{id}$ which is always $+1$ or $-1$. Therefore, the net chirality number of an edge, which is the sum of the chirality numbers of all the zero energy states localized on the edge, is always nonzero when MFBs appear.

Moreover, it can be shown that the number of stable zero energy modes on an edge equals to the net chirality number of the edge. Since on-site disorder does not break the chiral symmetry and cannot change the net chirality number, the number of stable zero energy modes cannot be changed by disorder. In contrast, in previous examples of MFBs, the net chirality number on each edge is zero.

To illustrate that the net chirality number on an edge is nonzero, we solve the eigenstates of the Hamiltonian in Eq. (2).

In the presence of an edge parallel to the $x$ direction at $y = 0$ and assuming that the superconductor is in the positive $y$ plane, $A_k, (k_y)$ can be written as

$$i V_y - i \Delta \frac{\partial}{\partial y} + \Delta \frac{\partial}{\partial x} - t(2 + \frac{i \gamma}{2 \gamma} + \frac{i \gamma}{2 \gamma}) - \mu$$

Using the ansatz $[0, 0, \alpha, \beta, -]T \sin(a + \beta) e^{-\gamma(y - \mu) + ikx}$ and $[\alpha, \beta, 0, 0]T \sin(a + \beta) e^{-\gamma(y - \mu) + ikx}$, we find two zero energy eigenstates for the Hamiltonian at $k_y = 0$ with $a_\pm = [2t + \mu - \Delta^2_p + 2V_t]i/\gamma$ and $b_\pm = \Delta^{2 -}$. As expected, the decay length $b_\pm$ equals the superconducting coherence length. These two eigenstates are manifested as zero energy modes at $k_x = 0$ in Figs. 2(a) and 2(b).

Interestingly, when $V_y > |2t + \mu|$, $a_\pm$ becomes imaginary and the corresponding wave function is not normalizable. As a result, only one zero energy mode with chirality $+1$ is left at $k_x = 0$ as shown in Fig. 2(c). It can be shown similarly that for large $V_y$, all the zero energy modes localized on one edge have the same chirality. As a result, the net chirality number on an edge is not zero and the zero energy modes cannot be lifted by local perturbations which preserve the chirality number. Since $V_y$ does not break the chiral symmetry, the MFBs can appear in the presence of $V_y$. However, $V_y$ breaks the chiral symmetry that an out-of-plane magnetic field can destroy the MFBs.

### VI. CONCLUSION

We show that an in-plane magnetic field can drive a $p \pm ip$-wave superconductor to a gapless phase which supports chiral symmetry protected MFBs. In the presence of $s$-wave pairing and Rashba terms, unidirectional MESs appear.

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