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Constructing symmetric topological phases of bosons in three dimensions via fermionic projective construction and dyon condensation

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Recently, there is a considerable study on gapped symmetric phases of bosons that do not break any symmetry. Even without symmetry breaking, the bosons can still be in many exotic new states of matter, such as symmetry-protected topological (SPT) phases, which are short-range entangled and symmetry-enriched topological (SET) phases, which are long-range entangled. It is well known that noninteracting fermionic topological insulators are SPT states protected by time-reversal symmetry and U(1) fermion number conservation symmetry. In this paper, we construct three-dimensional exotic phases of bosons with time-reversal symmetry and boson number conservation U(1) symmetry by means of fermionic projective construction. We first construct an algebraic bosonic insulator, which is a symmetric bosonic state with an emergent U(1) gapless gauge field. We then obtain many gapped bosonic states that do not break the time-reversal symmetry and boson number conservation via proper dyon condensations. We identify the constructed states by calculating the allowed electric and magnetic charges of their excitations, as well as the statistics and the symmetric transformation properties of those excitations. This allows us to show that our constructed states can be trivial SPT states (i.e., trivial Mott insulators of bosons with symmetry), nontrivial SPT states (i.e., bosonic topological insulators), and SET states (i.e., fractional bosonic topological insulators). In nontrivial SPT states, the elementary monopole (carrying zero electric charge but unit magnetic charge) and elementary dyon (carrying both unit electric charge and unit magnetic charge) are fermionic and bosonic, respectively. In SET states, intrinsic excitations may carry fractional charge.

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I. INTRODUCTION

A quantum ground state of a many-boson system can be in a spontaneous-symmetry-breaking state, or a topologically ordered (TO) state [1–3]. A TO state is defined by the following features: ground-state degeneracy in a spontaneously-symmetry-breaking state, or a topologically nontrivial closed manifold [1–3], emergent fermionic/anyonic excitations [4,5], or chiral gapless edge excitations [6,7]. If, in addition to a TO, the ground state also has a symmetry, such a state will be referred as a “symmetry-enriched topological (SET) phase.”

Recently, it was predicted that even if the bosonic ground state does not break any symmetry and has a trivial TO, it can still be in a nontrivial phase called bosonic symmetry-protected topological phase (SPT) [10–12]. Since the bosonic SPT phases do not have TOs, a systematic description/construction of those SPT phases was obtained via group cohomology theory [10–12]. Many new SPT phases were predicted/constructed with all possible symmetries and in any dimensions. In the following, we also refer all gapped phases of bosons that do not break the symmetry (including SPT and SET) as “topological phases.”

To realize bosonic TO phases or SPT phases, the interaction is crucial, since without interactions, bosons always tend to condense trivially. Weak interactions in most cases only lead to superfluid states. This fact hinders the conventional perturbation approach if we want to realize TO or SPT phases. One useful approach is via the exactly soluble models, as in the string-net approach [8,9] and the group cohomology approach [10–12]. Recently, many other approaches were proposed, which are based on field theory, topological invariants, critical theory of surface, topological response theory, etc [13–21,23–25]. A quite effective approach for strongly interacting systems is the “projective construction” [26–37]. Some appealing advantages of the projective construction are that (i) it can generate many useful trial wave functions for many-body systems and (ii) fractional charge/statistics and emergent gauge fields can be constructed effectively. In other words, it is quite easy to obtain TO states by using the projective construction. However, it has been recently realized that the projective construction is also helpful in constructing bosonic SPT states [22,38–41].

Roughly speaking, in the so-called “fermionic projective construction,” each bosonic operator of a given boson system is split into a product of fermionic parton operators. Different kinds of partons can individually form different mean-field ground states. The Hilbert space of partons is larger than the physical Hilbert space \( H_{\text{phys}} \) of the initial boson model. The physical ground state of the boson system is realized by doing Gutzwiller projection. In other words, the direct product of multiple mean-field ground states is projected into \( H_{\text{phys}} \) in which the multiple partons are glued back into a physical boson on each site. In terms of path integral formulation, such a gluing process is done by introducing fluctuating internal gauge fields that couple to partons. The gauge degrees of freedom can be in Coulomb phase, Higgs phase (e.g., residual Z_\( \lambda \) gauge symmetry), or confined phase. In Coulomb phase, gapless photon excitations close the bulk gap, while discrete gauge symmetry in Higgs phase generates TO.

To obtain SPT states (gapped, symmetric, without TO), we need to consider the confined phase of the internal gauge fields, where the gauge fluctuations are very strong and solitonic excitations (e.g., monopoles and dyons in 3D and instantons in 2D) are allowed. In 2D SPT construction, we have the well-known \( K \)-matrix Chern-Simons formalism, which is a “controllable program” to avoid the discussion of the...
confinement problem [41] during the fermionic projection construction. For example, one can compute the determinant of $K$ matrix to probe TO and even classify SPT states in 2D with Abelian symmetry group [14]. However, $K$-matrix Chern-Simons formalism is only applicable in 2D. The question is this: in the 3D fermionic projective construction, is there an “efficient program” that can lead to SPT states in a controllable way? In the present work, we will handle this problem by focusing on 3D SPT states with U(1) symmetry (boson particle number conservation) and time-reversal symmetry ($Z_2$). Here, $Z_2^3 = \{I, T\}$ where $I$ is identity while $T^2 = I$. We will refer those phases as bosonic topological insulators (BTI). If a SET state also contains these two symmetry groups, we call it fractional BTI ($f$BTI).

From our proposed algorithm, we can even learn more than the SPT construction itself. It is known that two fermions near a Fermi surface can have BCS superconducting instability under certain conditions. Various pairing symmetries and other pairing dynamical details may generically lead to various gapped superconducting states. In the fermionic projective construction, we may ask a similar question: by driving a gapless Coulomb phase into a gapped confinement phase and keeping some global symmetries, are there more than one type of confinement phases that the quantum many-body system eventually enters? In this paper, we will see that there are indeed many different kinds of confinement phases that are featured by different choices of monopole (or dyon) condensations without breaking symmetry. These different phases are finally classified into trivial SPT, nontrivial SPT, and SET states. We note that recently symmetry-breaking patterns induced by dyon condensations are discussed in the context of topological Mott insulators [42].

We specially choose the symmetry group $U(1) \times Z_2^3$ in this paper. One direct motivation is that three-dimensional noninteracting fermionic topological insulators (TI) [43] are well-understood and also have $U(1) \times Z_2^3$ symmetry group. TI is classified by $Z_2$, i.e., only one type of nontrivial state. Trivial and nontrivial TI states can be further elegantly labeled by the so-called “axionic $\Theta$ angle” in the electromagnetic response action $\mathcal{S}_{EM} = \int d^4x \epsilon^{\mu\nu\rho\sigma} A_\mu \partial_\nu A_\rho \partial_\sigma (A_\mu$ is the external electromagnetic gauge field). $\Theta = 0$ ($\pi$) corresponds to the trivial (nontrivial) phase [44]. It is interesting to ask whether there exists a bosonic version of TI, i.e., BTI and $f$BTI via the fermionic projective construction and how about the physical properties? Reference [45] applied the fermionic projective construction approach in which the boson creation operator is split into a singlet pair of spin-1/2 fermions. It is assumed that the fermions are described by a nontrivial TI mean-field ansatz that explicitly breaks the internal SU(2) gauge symmetry down to $Z_2$. The resultant physical ground state is a SET state admitting a fractional $\Theta$ angle and emergent $Z_2$ TO. By definition, this bosonic insulator is an $f$BTI, following Ref. [46] where a fermionic version in the presence of strong interactions is proposed.

In the present work, the underlying boson model contains four kinds of charge-1 bosons with $U(1) \times Z_2^3$ symmetry in three dimensions [i.e., Eq. (1)]. In the fermionic projective construction, each boson is split into two different fermions ($f_1$, $f_2$) carrying “spin-1/2.” $f_1$ and $f_2$ carry $\alpha$ and $(1 - \alpha)$ electric charge, respectively (see Table I).

| Table I. Assignment of EM electric charge and $a_\mu$-gauge charge. |
|------------------|------------------|
| Particle | EM electric charge | $a_\mu$-gauge charge |
| $f_1$ | $\alpha$ | +1 |
| $f_2$ | $1 - \alpha$ | −1 |
| $b$ | −1 | 0 |

To ensure that the mean-field Ansätze of fermions respect symmetry before projection, we assume that mean-field Ansätze of the fermions describe a fermionic gapped phase with $\theta$-angle $\theta_1$ for $f_1$ fermions and $\theta_2$ for $f_2$ fermions. We assume $\theta_1 = \theta_2 = 0$ or $\theta_1 = \theta_2 = \pi$ where the two fermions form the same trivial band insulator state or the same topological insulator states. We will use $(\theta_1, \theta_2, \alpha)$ to label those mean-field Ansätze. Due to the projective construction, an internal U(1) gauge field $a_\mu$ exists and is gapless. So our construction (at this first step) leads to gapless insulating states of the bosons after the projection. We call such a state algebraic bosonic insulator.

To obtain gapped insulating states of the bosons, we shall push the internal gauge field into its confined phases, where quantum fluctuations are very strong leading to a proliferation of a certain dyon. There are many different kinds of confined phases that correspond to proliferation of different dyons. These dyons may carry many quantum numbers including: fermion numbers of $f_1$, $f_2$, magnetic charge and gauge charge of the internal gauge field, magnetic charge and electric charge of an external electromagnetic gauge field. The latter is assumed to be compact such that a magnetic monopole is naturally allowed although the electromagnetism in our world is noncompact so far. We will use $(l, s)$ to label those different proliferated (or condensed) dyons that do not break the U(1) and time reversal symmetries.

After a symmetric dyon condensate $(l, s)$ is selected and the charge assignment $\alpha$ is fixed, we may construct a gapped topological phase with $U(1) \times Z_2^3$ symmetry. We find that the dyon condensation breaks the “gauge symmetry” of shifting $\alpha$ (dubbed “$\alpha$-gauge symmetry”), so that different $\alpha$ will generally lead to different bosonic states (cf. Sec. II C). Thus those topological phases are eventually labeled by $(\theta_1, \theta_2, \alpha)$ or $(l, s)$.

<table>
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<tr>
<th>Table II. Some concrete examples of nontrivial BTI phases in three dimensions. Each BTI state is labeled by five numbers $(\theta_1, \theta_2, \alpha, l, s)$. At $(\theta_2) \equiv \theta_2 \equiv 0$ or $\pi$, denoting the trivial or nontrivial TI phases of the fermion $f_1$ ($f_2$). $f_1$ and $f_2$ carry $\alpha$ and $(1 - \alpha)$ electric charge of external electromagnetic field, respectively.</th>
<th></th>
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</thead>
<tbody>
<tr>
<td>$(\theta_1, \theta_2, \alpha)$</td>
<td>Dyon condensate $(l, s)$</td>
</tr>
<tr>
<td>$(0, 0, 1)$</td>
<td>$(1, 1)$</td>
</tr>
<tr>
<td>$(0, 0, 1)$</td>
<td>$(3, 1)$</td>
</tr>
<tr>
<td>$(\pi, \pi, -\frac{1}{2})$</td>
<td>$(1, 1)$</td>
</tr>
<tr>
<td>$(\pi, \pi, \frac{1}{2})$</td>
<td>$(1, 1)$</td>
</tr>
</tbody>
</table>
TABLE III. Topological phases of bosons with $U(1)\times Z_2^s$ symmetry in three dimensions [labeled by $(\theta_1, \theta_2, \alpha = 1/2, l, s)$]. All symmetric dyons are labeled by two integers $(l, s)$. In each mean-field Ansatz, different kinds of dyon condensations lead to, generally, different topological phases (trivial SPT, nontrivial SPT, or SET). If $s = l = 0$, i.e., there is no dyon condensation, the resultant symmetric state is the algebraic bosonic insulator (ABI) state, which is gapless and can be viewed as a parent state of all topological phases before condensing some dyons. If $s = 0, l \neq 0$, the condensed dyon will break $U(1)$ symmetry, rendering a symmetry-breaking phase. To get symmetric gapped phases, $s \neq 0$ has been required in the table. The physical interpretation of $s$ and $l$ is the following. In the mean-field Ansatz $(\theta_1, \theta_2) = (0, 0)$, the condensed dyon with $U(1)\times Z_2^s$ symmetry is a composite of $s$ monopoles of internal gauge field and $l$ physical bosons. In the mean-field Ansatz $(\theta_1, \theta_2) = (\pi, \pi)$, the condensed dyon with $U(1)\times Z_2^s$ symmetry is a composite of $s$ monopoles of internal gauge field, $l$ physical bosons, and $s f_2$ fermions. One “physical boson” is equal to one $f_1$ fermion plus one $f_2$ fermion. “None” in a given entry means that the topological phase does not exist in the corresponding mean-field Ansatz. All trivial SPT have Witten effect with $\theta = 0 \mod(4\pi)$ and all nontrivial SPT have Witten effect with $\theta = 2\pi \mod(4\pi)$. The discussion on Witten effect of SET will be presented in Sec. V where trivial f/BTI and nontrivial f/BTI are defined and classified. The mean-field Ansatz $(0, \pi)$ always breaks time-reversal symmetry.

<table>
<thead>
<tr>
<th>Mean-field Ansatz $(\theta_1, \theta_2)$</th>
<th>Trivial SPT (trivial Mott insulator of bosons)</th>
<th>Nontrivial SPT (BTI: bosonic topological insulator)</th>
<th>SET (BTI: fractional bosonic topological insulator)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0,0)$</td>
<td>${l/s \in \mathbb{Z}, l \neq 0} \cup {l = 0, s = \pm 1}$</td>
<td>None</td>
<td>${l/s \notin \mathbb{Z}}$; Especially, ${l = 0,</td>
</tr>
<tr>
<td>$(\pi, \pi)$</td>
<td>${l/s \in \mathbb{Z}}$</td>
<td>None</td>
<td>${l/s \notin \mathbb{Z}}$ (if $l/s = -1/2$, an additional $Z_{</td>
</tr>
</tbody>
</table>

As well as $(l, s)$ (see Tables II–IV), we have to choose certain special values of $\alpha$ to preserve the $U(1)$ and time reversal symmetries (see Fig. 1). The excitation spectrum above a topological phase is formed by the so-called “deconfined dyons” that have trivial mutual statistics with the condensed dyon.

To understand whether the state supports TO, we have to examine a subset of excitations, namely intrinsic excitations (defined as those excitations that have zero magnetic charge of the external electromagnetic field). Noting that the underlying model is formed by charge-1 bosons, if there are intrinsic excitations that either carry fractional electric charge of the external electromagnetic field or carry fermionic statistics, the state must be a TO state. The reason we only consider intrinsic excitations is that the symmetry group $U(1)$ is specific to the conservation of electric charge of the external electromagnetic field. The magnetic charge of the external electromagnetic field is not an intrinsic quantity of the underlying boson system. For example, a nontrivial TI state in a background compact electromagnetic field allows a composite excitation that contains $1/2$ electric charge and one magnetic charge of

TABLE IV. Topological phases of bosons with $U(1)\times Z_2^s$ symmetry in three dimensions (for a generic $\alpha$ sequence), labeled by $(\theta_1, \theta_2, \alpha, l, s)$. A state is $U(1)\times Z_2^s$ symmetric if the parameters $l, s, \alpha$ satisfy the conditions in this table. We see that if $\alpha = 1/2$, all allowed SPT states are trivial in any mean-field Ansatz consistent with Table III. All SET states can be further classified into trivial f/BTI (without Witten effect) and nontrivial f/BTI (admitting Witten effect). The mean-field Ansatz $(0, \pi)$ always breaks time-reversal symmetry.

<table>
<thead>
<tr>
<th>Mean-field Ansatz $(\theta_1, \theta_2)$</th>
<th>Trivial SPT (trivial Mott insulator of bosons)</th>
<th>Nontrivial SPT (BTI: bosonic topological insulator)</th>
<th>SET (BTI: fractional bosonic topological insulator)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0,0)$</td>
<td>${l/s = \text{odd,} \alpha = \text{half-odd}} \cup {l/s = \text{even}, l \neq 0, 2\alpha = \text{integer}} \cup {l = 0, s = \pm 1, 2\alpha = \text{integer}}$</td>
<td>${l/s = \text{odd,} \alpha = \text{integer}}$</td>
<td>${l = 0,</td>
</tr>
<tr>
<td>$(\pi, \pi)$</td>
<td>${l/s \in \mathbb{Z}, \alpha = -\frac{1}{2} = \text{even}}$</td>
<td>${l/s \in \mathbb{Z}, \alpha = -\frac{1}{2} = \text{odd}}$</td>
<td>${l/s = -1/2, \alpha = \text{half-odd}}$; $Z_{</td>
</tr>
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\[\{l/s \notin \mathbb{Z}\}, \text{e.g., } l/s = -\frac{1}{2}; \alpha = \text{half-odd} \longrightarrow \text{trivial f/BTI with } \Theta = 0 \mod \frac{4\pi}{9}\]

\[\text{Witten effect; } \alpha = \text{integer} \longrightarrow \text{nontrivial f/BTI with } \Theta = \frac{2\pi}{9} \mod \frac{4\pi}{9}\]

\[\text{Witten effect}\]
the external electromagnetic field, but TI cannot be interpreted to a TO state supporting fractional excitations. It is well known that as a noninteracting fermionic gapped state, TI is not a TO state for sure, and, all "intrinsic excitations" do not carry fractional electric charge of the external electromagnetic field. The 1/2 fractional charge is actually induced by nonzero magnetic charge.

In addition to the above analysis of deconfined dyons, if a topological phase contains deconfined discrete gauge symmetry (e.g., Z2 gauge theory), the topological phase must be a TO state regardless of the properties of deconfined dyons. This state can be viewed as a realization of a Higgs phase of an internal gauge field, which will be discussed later.

Based on the above clarification, we have the following quantitative criterion that will be frequently utilized in this work. (1) If a topological phase respects global symmetry [i.e., U(1) × Z2] and contains deconfined discrete gauge symmetry (e.g., Z2 gauge theory), the topological phase must be a SET state where both TO and symmetry are present. (2) If a topological phase respects global symmetry and there are intrinsic excitations that either carry fractional electric charge of the external electromagnetic field or carry fermionic statistics, the topological phase is a SET state (i.e., a BTI). Otherwise, the state is a nontrivial BTI (i.e., a SET). Otherwise, the state is a nontrivial SPT (i.e., a BTI). Otherwise, the state is a nontrivial SPT (i.e., a BTI).

II. FERMIONIC PROJECTIVE CONSTRUCTION OF MANY-BOSON STATE WITH U(1) × Z2 SYMMETRY

A. Definition of boson operators

We will use a system with four kinds of electric charge-1 bosons in three dimensions. Those bosons are described by four boson operators. We split the boson operators into two different spin-1/2 fermions:

\[
(b_1, b_2, b_3, b_4) = (f_{1\uparrow}, f_{2\uparrow}, f_{1\downarrow}, f_{2\downarrow}, f_{1\downarrow}, f_{2\uparrow}).
\]

The fermionic projective construction of the four bosons implies that the underlying bosons are of hard-core nature since the dimension of the bosonic Hilbert space at each lattice site \(i\) is truncated to be finite and exchange of two bosons at different sites does not generate a fermionic sign. As a result, at the very beginning, the underlying boson model on the lattice must be a correlated bosonic system. All possible ground states with boson charge conservation symmetry U(1) and time-reversal symmetry \(Z_2^T\) are what we shall look for in this paper.

The physical ground-state wave function \(|GS\rangle\) in \(H_{\text{phys}}\) can be written in terms of direct product of fermions’ mean-field Ansätze, subject to Gutzwiller projection:

\[
|GS\rangle = \hat{P}_G(|\Psi(f_1)\rangle \otimes |\Psi(f_2)\rangle),
\]

where \(\hat{P}_G\) is the Gutzwiller projection operator, which enforces that the total number of \(f_1\) is equal to that of \(f_2\) at each site in the physical (projected) Hilbert space \(H_{\text{phys}}\). \(|\Psi(f_1)\rangle\) and \(|\Psi(f_2)\rangle\) are mean-field Ansätze for the ground states of \(f_1\) and \(f_2\), respectively. In the present work, we assume that both of \(f_1\) and \(f_2\) form mean-field Ansätze with band structures that respect \(U(1) \times Z_2^T\) symmetry. Such band structures are of TI classified by \(Z_2\), i.e., one trivial and one nontrivial states. It is thus instructive to separately study the physical ground states in two different classes: (1) both are trivial and (2) both are nontrivial. Other mean-field Ansätze explicitly break time-reversal symmetry already at mean-field level.

B. Definition of symmetry transformations

1. Time-reversal symmetry (\(T^2 = I\))

Under time reversal, the above fermions transform as the usual spin-1/2 fermions, but with an additional exchange \(f_{1\sigma} \leftrightarrow f_{2\sigma}\), and the bosons transform as

\[
b_1 \rightarrow -b_4, \quad b_2 \rightarrow b_3, \quad b_3 \rightarrow b_2, \quad b_4 \rightarrow -b_1.
\]

For instance, \(b_2 = f_{1\uparrow} f_{2\uparrow} \rightarrow -f_{1\downarrow} f_{2\downarrow} \rightarrow -f_{2\downarrow} f_{1\uparrow} = f_{1\uparrow} f_{2\uparrow}\) such that \(b_2\) is unchanged, where

\[
① \text{represents } f_{1\uparrow} \rightarrow -f_{1\downarrow}, f_{1\uparrow} \rightarrow f_{1\downarrow}, f_{2\uparrow} \rightarrow -f_{2\downarrow}, f_{2\downarrow} \rightarrow f_{2\uparrow}\]

② represents exchange of labels: 1 ↔ 2. In this projective construction, there is an internal U(1) gauge field, \(a_0\) minimally coupled to \(f_1, f_2\). Let us define a “pseudospinor” \(\mathbf{B} = (b_1, b_2, b_3, b_4)^T\). Equation (3) also indicates the time-reversal symmetry matrix that acts on the pseudospinor, which has the
following form:

\[
T = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix},
\]

which has determinant \(\text{Det}(T) = -1\). After two consecutive transformations \((T^2)\) in a unit matrix of four dimensions, the pseudospinor \(B\) remains invariant, implying that the time reversal we consider here is \(T^2 = 1\), where \(T\) and an identity \(I\) form the time-reversal symmetry group \(\mathbb{Z}_2 = \{I, T\}\). As a side note, we emphasize that the four-boson model shown in Eq. (1) is the minimal choice of \(T^2 = 1\) bosonic system in the present fermionic projective construction. If this choice is not adopted, one has to consider a more complicated choice, e.g., \(b_{\alpha}(s,\tau,\sigma,\sigma_\alpha) = f_{\alpha s} f_{\beta s} f_{\gamma s} f_{\delta s}\) where there are four spin-\(1/2\) partonic fermions (spin indices \(\sigma_1 \cdots \sigma_4 = \uparrow, \downarrow\)) leading to a pseudospinor with 16 components.

The assignment of gauge charges carried by \(f_s\) \((s = 1, 2)\) is shown in Table I. \(f_1\) and \(f_2\) carry \(+1\) and \(-1\) gauge charges of \(a_\mu\), respectively, such that all physical boson operators are invariant under \(a_\mu\) gauge transformation.

2. Boson number conservation \(U(1)\) and the charge assignment

Each boson carries \(+1\) fundamental electric charge of the external electromagnetic (EM) field \(A_\mu\) such that one can make the following assignment for fermions shown in Table I: \(f_1\) and \(f_2\) carry \(\alpha\) and \(1 - \alpha\) EM electric charge of \(A_\mu\), respectively. Here, \(\alpha\) is a real number whose value should not alter the vacuum expectation value of EM gauge-invariant operators. More precisely, \(\alpha\) is not a defining parameter of the underlying boson model. Rather, it is introduced in the projective construction at ultraviolet (UV) scale. When we only change \(\alpha\), the projected wave function should not change once the projection is done exactly at lattice scale. As a side note, one can check that U(1) symmetry here corresponds to conservation of total number of bosons, i.e., \(\sum_i (B_i B_i) = \sum_i (b_i b_i + b_{\overline{i}} b_{\overline{i}})\) is constant.

C. Residual \(\alpha\)-gauge symmetry after dyon condensation

The above discussion about \(\alpha\) suggests that \(\alpha\) is a pure gauge degree of freedom (or more precisely: a gauge redundancy). We conclude that before the dyon condensation, there is an \(\alpha\)-gauge symmetry, which is defined as \(\alpha \to \alpha + \lambda\), where \(\lambda\) is any real number. Later, we will see that the dyon condensation can break such an \(\alpha\)-gauge symmetry; just like the Higgs condensation can break the usual "gauge symmetry." However, we believe that dyon condensation does not break all the \(\alpha\)-gauge symmetry: shifting \(\alpha\) by any integer remains to be a "gauge symmetry" even after the dyon condensation. The physical consideration behind this statement is that the EM charge quantization is unaffected by any integer shift at all, and, such an integer shift is nothing but redefinition of field variables.

In this paper, we will show that in the mean-field Ansatz \((\theta_1,\theta_2) = (0,0)\) case, all topological phases, including symmetry-protected topological phases (SPT) and symmetry-enriched topological phases (SET), satisfy this statement (cf. Sec. V). In other words, after \(\alpha \to \alpha + 1\), the calculated properties of the topological phases are unaffected.

However, the statement is invalid in the mean-field Ansatz \((\theta_1,\theta_2) = (\pi,\pi)\) via our continuum effective field theory approach (cf. Sec. V). After \(\alpha \to \alpha + 1\), the physical properties are changed. In this case, it appears that \(\alpha\)-gauge symmetry with any odd integer shift is broken by the dyon condensation. Shifting \(\alpha\) by any even integer remains to be a "gauge symmetry," after the dyon condensation.

At the moment, we do not understand why dyon condensations for \((\theta_1,\theta_2) = (0,0)\) and \((\theta_1,\theta_2) = (\pi,\pi)\) lead to different \(\alpha\)-gauge symmetry breaking. However, we would like to point out that our field theoretic treatment on dyons is established in the continuum limit of space-time. More rigorous approach, however, should involve the regularization procedures of dyon fields on lattice. For example, dyons are not point particles at all on lattice. Rather, dyons are regularized on a dual hypercubic lattice of space-time where magnetic charge and electric charge are put on dual sublattices. This more careful lattice consideration may allow us to understand how dyons condensation may break the \(\alpha\)-gauge symmetry and in which way the \(\alpha\)-gauge symmetry is broken. We leave this issue to future work.

III. GENERAL PROPERTIES OF DYONS

A. Quantum numbers of dyons

The projective construction is a very natural way to obtain topological phases with TO since at the very beginning the fermionic degrees of freedom and internal gauge fields are introduced at UV scale. To obtain SPT phases, we must prohibit the emergence of TO, by at least considering the confined phase of the internal gauge field, where the dyons of the internal gauge field play a very important role. For the purpose of probing the EM response, the nondynamical EM field is applied and is assumed to be compact. Thus a dyon may carry gauge (electric) charges and magnetic charges of both internal gauge field and EM field. The terms “gauge (electric) charge” and “magnetic charge” are belonging to both gauge fields, while, for the EM field, we specify the charges by adding “EM” to avoid confusion. A dyon can also include \(f_1\) and \(f_2\) fermions, resulting in nonzero “fermion number.” Thus a generic dyon is labeled by a set of quantum numbers that describe those gauge charges, magnetic charges, and fermion numbers. Specially, a monopole is defined as a special dyon which does not carry any kind of gauge (electric) charges.

To describe those dyon excitations systematically, let us assume that each fermion \((f_i)\) couples to its own gauge field \(A_{\mu}^{f_i}\) with "+1" gauge charge. In fact, \(A_{\mu}^{f_i}\) are combinations of \(a_\mu\) and \(A_\mu\) (cf. Table I):

\[
A_{\mu}^{f_1} \equiv a_\mu + \alpha A_\mu, \quad A_{\mu}^{f_2} \equiv -a_\mu + (1 - \alpha) A_\mu.
\]

A dyon can carry the magnetic charges in \(A_{\mu}^{f_2}\) gauge groups, which are labeled by \(\{N_m^{(s)}\} \in \mathbb{Z}(s = 1, 2)\). These two magnetic charges \(\{N_m^{(s)}\}\) are related to the magnetic charge \(N_m^{(s)}\) in \(a_\mu\) gauge group and magnetic charge \(N_M\) in \(A_\mu\) gauge group in the following way:

\[
N_m^{(1)} = N_m^{(s)} + \alpha N_M, \quad N_m^{(2)} = -N_m^{(s)} + (1 - \alpha) N_M.
\]
where the EM magnetic charge $N_M$ is integer-valued as usual: $N_M \in \mathbb{Z}$. For this reason, the quantization of magnetic charge $N_m^a$ of $a_{\mu}$, gauge group is determined by two integers: “$N_m^{(1)}$” and “$N_M^a$”, via Eq. (6) with a given $\alpha$. The following relations are useful:

$$N_M = N_m^{(1)} + N_m^{(2)}, \quad N_m^a = (1-\alpha)N_m^{(1)} - \alpha N_m^{(2)}.$$  

(7)

A dyon can also carry the fermion numbers of $f_1, f_2$ denoted by $(N_f^{(1)}), s = 1, 2$. They are related to magnetic charges in the following way:

$$N_f^{(1)} = n_f^{(1)} + \frac{\theta_1}{2\pi} N_m^{(1)}, \quad N_f^{(2)} = n_f^{(2)} + \frac{\theta_2}{2\pi} N_m^{(2)},$$  

(8)

where the $\theta$-related terms are polarization electric charge clouds due to Witten effect [47,48] and $n_f^{(i)}$ are integer-valued, indicating that integer numbers of fermions are able to be trivially attached to the dyon. The nature of “polarization” is related to the fact that this charge cloud does not contribute quantum statistics to dyons [50]. $\theta_1$ and $\theta_2$ determine the topology of fermionic band structures of $f_1$ and $f_2$, respectively, if symmetry group $U(1) \times \mathbb{Z}_2^f$ is implemented. For example, $\theta_1 = 0$ if $f_1$ forms a trivial TI Ansatz and $\theta_1 = \pi$ if $f_1$ forms a nontrivial TI Ansatz.

B. Time-reversal transformation of dyons, gauge fields, and Lagrangians

To see whether the ground state breaks symmetry or not, it is necessary to understand how the symmetry acts on dyon labels (all transformed symbols are marked by “~”):

$$\overline{N_f^{(1)}} = N_f^{(2)}, \quad \overline{N_f^{(2)}} = N_f^{(1)}, \quad \overline{N_M} = -N_M, \quad \overline{N_m^a} = N_m^a,$$  

(9)

(10)

where Eq. (9) holds by definition. In Eq. (10), the EM magnetic charge’s sign is reversed as usual, which is consistent to reverse the sign of the EM gauge potential $A$, which is a polar vector:

$$\overline{A} = -A.$$  

(11)

The second formula in Eq. (10) can be understood in the following way. A single $f_1$ fermion couples to $A$ and $a$ with $+1$ and $-1$ coupling constants, respectively, as shown in Table I. A single $f_2$ fermion couples to $\overline{A}$ and $\overline{a}$ with $-\alpha$ and $-1$ coupling constants, respectively, and, $f_2$ fermion couples to $A$ and $a$ with $-1 + \overline{\alpha}$ and $+1$ coupling constants, respectively. The second step is to exchange the two fermions as defined in Sec. II B. By definition, $f_2$ (i.e., the new $f_2$ fermion after time-reversal transformation) should couple to $\overline{A}$ and $\overline{a}$ with $-\alpha$ and $-1$ coupling constants respectively, which results in $(1-\alpha)\overline{A} - \overline{a} = -\alpha \overline{a} + \alpha a$. Likewise, $f_1$ couples to $A$ and $a$ with $\alpha$ and $+1$ coupling constants, such that $\alpha A + a = (-1 + \overline{\alpha}) \overline{A} + \overline{a}$. Overall, we obtain the following rule by using Eq. (11):

$$\overline{a} = a,$$  

(13)

which requires the relation of magnetic charges $\overline{N_m^a} = N_m^a$ in a self-consistent manner as shown in Eq. (10).

Based on the above results, one may directly derive the transformation rules obeyed by other quantum numbers:

$$\overline{N_m^{(1)}} = -N_m^{(2)}, \quad \overline{N_m^{(2)}} = -N_m^{(1)}, \quad \overline{n_f^{(1)}} = n_f^{(2)} + \frac{\theta_1 + \theta_2}{2\pi} N_m^{(2)}, \quad \overline{n_f^{(2)}} = n_f^{(1)} + \frac{\theta_1 + \theta_2}{2\pi} N_m^{(1)}.$$  

(14)

Suppose that $N_A$ and $N_a$ are the bare EM electric charge and $a_{\mu}$-gauge charge carried by dyons (cf. Table I):

$$N_A = \alpha N_f^{(1)} + (1-\alpha) N_f^{(2)}, \quad N_a = N_f^{(1)} - N_f^{(2)}.$$  

(15)

We have

$$\overline{N_A} = N_A, \quad \overline{N_a} = -N_a,$$  

(16)

by noting that $\overline{\alpha} = 1 - \alpha$.

By definition, the curls of dual gauge potentials ($\overline{A}, \overline{a}$) contribute electric fields ($\overline{E}, \overline{E}^s$). Therefore the dual gauge potentials should obey the same rules as electric fields under time-reversal transformation, and electric fields should also obey the same rules as electric charges ($N_A, N_a$) in a consistent manner such that the dual gauge potentials are transformed in the following way:

$$\overline{A} = \overline{\overline{A}}, \quad \overline{a} = -\overline{\overline{a}}.$$  

(17)

The four formulas in Eqs. (11), (13), and (17) are transformation rules obeyed by the spatial components of the gauge potentials. The time components of the gauge potentials ($A_0, a_0, \overline{A}_0, \overline{a}_0$) obey the following rules:

$$\overline{A}_0 = A_0, \quad \overline{a}_0 = -a_0,$$  

(18)

$$\overline{A}_0 = -\overline{A}_0, \quad \overline{a}_0 = \overline{a}_0,$$  

(19)

by adding an overall minus sign in each of Eqs. (11), (13), and (17).

On the other hand, let us consider the effective Lagrangian that describes the dyon dynamics. Let us start with a general dyon $\phi$ and try to understand its time-reversal partner $\overline{\phi}$. The effective Lagrangian term $L_{\text{kin}}$ that describes the kinetic energy of $\phi$ can be written as (up to a quadratic level)

$$L_{\text{kin}}[\phi] = \frac{1}{2m} \left| (-i \nabla + N^a \overline{a} + N_A \overline{A} + N_m^a \overline{\overline{a}} + N_M \overline{\overline{A}}) \phi \right|^2.$$  

(20)

Here, we are performing time-reversal transformation in field theory action such that we keep all real-valued numerical
coefficients \( (N^a, N_A, \ldots) \) but change all field variables. By using Eqs. (11), (13), (17), and (19) and noting that \( -i \nabla = i \nabla \), we obtain the result
\[
\mathcal{L}_{km}[\bar{\phi}] = \frac{1}{2m} \left| (i \nabla + N^a a + N_A A + N^a a + N_M \tilde{A}) \phi \right|^2
\]
\[
= \frac{1}{2m} \left| (-i \nabla - N^a a + N_A A + N^a a - N_M \tilde{A}) \phi \right|^2.
\]

The time component is similar:
\[
\mathcal{L}_t[\phi] = \frac{1}{2m} \left| (i \partial_t + N^a a_0 + N_A A_0 + N^a a_0 + N_M \tilde{A}_0) \phi \right|^2.
\]

After \( Z_2^T \) operation,
\[
\mathcal{L}_t[\bar{\phi}] = \frac{1}{2m} \left| (i \partial_t - N^a a_0 + N_A A_0 + N^a a_0 - N_M \tilde{A}_0) \phi \right|^2.
\]

C. Mutual statistics and quantum statistics

One of the important properties of dyons is their 3D “mutual statistics.” Two dyons with different quantum numbers may perceive a nonzero quantum Berry phase mutually. More specifically, let us fix one dyon (“\( \phi_1 \)”) at the origin and move another dyon “\( \phi_2 \)” (labeled by symbol with primes) along a closed trajectory that forms a solid angle \( \Omega \) with respect to the origin. Under this circumstance, one can calculate the Berry phase that is added into the single-particle wave function of \( \phi_2 \):
\[
\text{Berry phase} = \frac{1}{2} \left[ \sum_s N^{(s)}_m N^{(s)'}_f - \sum_s N^{(s)'}_m N^{(s)}_f \right] \Omega.
\]

If the Berry phase is nonvanishing for any given \( \Omega \), i.e., \( \sum_s N^{(s)}_m N^{(s)'}_f \neq \sum_s N^{(s)'}_m N^{(s)}_f \), these two dyons then have a nontrivial “mutual statistics.” The physical consequence of mutual statistics is the following. If the confined phase of the internal gauge field is formed by a condensate of dyon \( \phi_1 \), all other allowed deconfined particles (i.e., the particles that may form the excitation spectrum with a finite gap) must have trivial mutual statistics with respect to \( \phi_1 \), i.e.,
\[
\sum_s N^{(s)}_m N^{(s)'}_f = \sum_s N^{(s)'}_m N^{(s)}_f.
\]

Otherwise, they are confined by infinite energy gap. There are two useful corollaries: (i) it is obvious that a particle has a trivial mutual statistics with respect to itself, and (ii) we also note that \( N^{(s)}_f \) and \( N^{(s)'}_f \) may be replaced by integers \( n^{(s)}_f \) and \( n^{(s)'}_f \), respectively, by taking Eq. (8) into consideration. As a result, the criterion of trivial mutual statistics (24) may be equivalently expressed as
\[
\sum_s N^{(s)}_m n^{(s)'}_f = \sum_s N^{(s)'}_m n^{(s)}_f.
\]

On the other hand, it is also crucial to determine the quantum statistics of a generic dyon. A generic dyon can be viewed as \( N^{(s)}_m \) magnetic charges of \( A^{(s)}_\mu \) gauge field attached by \( n^{(s)'}_f \) fermions. The quantum statistics of such a dyon is given by
\[
\text{Sgn} = \prod_s (-1)^{N^{(s)}_m n^{(s)'}_f} (-1)^{n^{(s)'}_f},
\]
where \(+/-\) represents bosonic/fermionic [50]. The first part, \( (-1)^{N^{(s)}_m n^{(s)'}_f} \), is due to the interaction between the magnetic charge \( N^{(s)}_m \) and the gauge charge \( n^{(s)'}_f \) of the dyon. The polarization electric charges due to Witten effect do not attend the formation of internal angular momentum of electric-magnetic composite according to the exact proof by Goldhaber et al. [50] so that \( n^{(s)}_f \) instead of \( N^{(s)}_f \) is put in Eq. (26). One may also express \( n^{(s)}_f \) in terms of “\( N^{(s)}_m - \frac{\theta}{2\pi} N^{(s)}_f \).” After this replacement, it should be kept in mind that both \( \langle N^{(s)}_f \rangle = \frac{N^{(s)}_f}{N^{(s)}_m} \) and \( N^{(s)}_m \) are integer-valued and \( N^{(s)}_f \) can be any real number in order to ensure that \( n^{(s)}_f \) are integer valued.

The second part, \( (-1)^{n^{(s)'}_f} \), is due to the Fermi statistics from the attachment of \( n^{(s)'}_f \) fermions. Alternatively, the quantum statistics formula (26) can be reorganized into the following Sgn \( \equiv (-1)^\Gamma \):
\[
\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3
\]

with
\[
\Gamma_1 = N_M \left[ an^{(1)}_f + (1 - \alpha) n^{(2)}_f \right],
\]
\[
\Gamma_2 = N_M \left[ n^{(1)}_f - n^{(2)}_f \right],
\]
\[
\Gamma_3 = n^{(1)}_f + n^{(2)}_f,
\]
in which \( \Gamma_1 \) and \( \Gamma_2 \) are contributed from the two gauge groups \( A_\mu \) and \( a_\mu \), respectively. \( \Gamma_3 \) is from the fermionic sign carried by the attached fermions. The notation “\( \equiv \)” here represents that the two sides of the equality can be different up to any even integer.

IV. TOPOLOGICAL PHASES WITH SYMMETRY: \( \alpha = 1/2 \)

A. Algebraic bosonic insulators: parent states of topological phases

Let us use the projective construction to study an exotic gapless bosonic insulator (without the dyon condensation), which is called algebraic bosonic insulator (ABI) and can be viewed as a parent state of gapped symmetric phases (i.e., SPT and SET phases).

For keeping time-reversal symmetry at least at mean-field level, we will only focus on \( (\theta_1, \theta_2) = (0,0) \) and \( (1,1) \). The ABI state does not break the \( U(1) \times Z_2^T \) symmetry since all possible dyons (each dyon and its time-reversal partner) are included without condensation. However, the bulk is gapless since it contains an emergent gapless \( U(1) \) gauge boson described by \( a_\mu \). The emergent \( U(1) \) gauge bosons are neutral. In addition to the emergent \( U(1) \) gauge bosons, ABI also contains many dyon excitations, which may carry fractional electric charges \( [N^{(s)}_f] \) in Eq. (8) and emergent Fermi statistics [determined by Eq. (26)]. Since all electrically charged excitations are gapped, such a phase is an electric insulator. At mean-field level, if \( \theta_1 = \theta_2 = \pi \), a gapless surface state emerges, which is described by Dirac fermions. Beyond mean-field theory, those gapless surface Dirac fermions (possibly with Fermi energies away from the Fermi surface) can form gapless surface states (possibly with Fermi energies away from the Fermi surface)
from the nodes) will interact with the emergent U(1) gauge fields that live in 3 + 1 dimensions. We note that the internal U(1) gauge field $a_\mu$ has strong quantum fluctuations, and its “fine structure constants” are of order 1. It is possible (relying on the physical boson Hamiltonian) that the internal U(1) gauge field is driven into a confined phase of gauge theory due to too strong quantum fluctuations. Due to the strong quantum fluctuations, the internal U(1) gauge-field configuration will contain many monopoles and even more general dyons. The ABI discussed above is realized as an unstable gapless fixed-point residing at the boundary between Coulomb and confined phases. ABI finally flows into a strongly coupled fixed point of a confined phase by energetically condensing a bosonic dyon and thus opening a bulk gap. In this case, it is possible that some nontrivial topological (gapped) phases with a global symmetry (including SPT and SET states) may be constructed in this confined phase, which is featured by dyon condensations. As we have seen, our ABI has many kinds of dyons. Relying on the details of the physical boson Hamiltonian, different dyon condensations may appear. Different dyon condensations will lead to many different confined phases.

In the following, we will set $a = 1/2$ and focus on looking for dyon condensations that generate a bulk spectral gap, and, most importantly, respect $U(1)\times Z_2$ symmetry. All topological phases are summarized in Table III. We should note that in each mean-field Ansatz, only one dyon whose quantum numbers are self-time-reversal invariant is condensed to form a topological phase. As a matter of fact, two time-reversal conjugated dyons can condense simultaneously, still without breaking time-reversal symmetry. However, this situation is trivially back to the single dyon condensate for the reason that the two dyons are exactly the same once the trivial mutual statistics between them is considered. (The details can be found in Appendix A.)

B. Standard labeling and defining properties of topological phases

Before moving on to topological phases of boson systems, we need to quantitatively define trivial SPT, nontrivial SPT, and SET states based on physically detectable properties in EM thought experiments (compactness of EM field is assumed).

Each dyon is sufficiently determined by four independent quantum numbers in ABI state. The total number of independent quantum numbers will be decreased to three in a specific topological phase where the condensed dyon provides a constraint on the four quantum numbers as we will see later. There are many equivalent choices of labeling. In the following, we choose these four integer-valued quantum numbers $[N_M, N_M^{(1)}, N_f^{(1)}, N_f^{(2)}]$ to express the final key results of a given mean-field Ansatz, such as quantum statistics and the total EM electric charge of excitations. We call it “standard labeling.” Based on these four integers, we can obtain $N_m^{(2)}$, $N_f^{(1)}$, and $N_f^{(2)}$ via Eqs. (7) and (8). As a result, $N_A$ and $N_a$ can be determined by Eq. (15). In each mean-field Ansatz, we will unify all key results by using the standard labeling.

A trivial SPT state has the following properties: (1) quantum statistics: $\Gamma_1 = N_M N_E$; (2) quantization condition: (i) $N_M \in \mathbb{Z}$, $N_E \in \mathbb{Z}$ and (ii) at least one excitation exists for any given integer combination $(N_M, N_E)$; and (3) TO does not exist. The first two conditions (quantum statistics plus quantization condition) define a “charge lattice” formed by two discrete data points, $N_M (y \text{ axis})$ and $N_E (x \text{ axis})$. Differing from $N_A$, which is “bare EM electric charge,” $N_E$ is the “total EM electric charge” in which possible dynamical screening arising from the ground state is taken into consideration. In an experiment, $N_E$ is detectable, while $N_A$ is not.

In the trivial SPT state here, the charge lattice corresponds to the “trivial Witten effect” phenomenon. It rules out TO with fractional electric charges for intrinsic excitations and TO with fermionic intrinsic excitations. The trivial Witten effect implies the elementary EM monopole $(N_M = 1, N_E = 0)$ is bosonic while the elementary EM dyon $(N_M = 1, N_E = 1)$ is fermionic. This charge lattice is shown in Fig. 2.

A nontrivial SPT state has the following properties: (1) quantum statistics: $\Gamma_2 = N_M (N_M - N_M) = N_M N_E - N_M$; (2) quantization condition: (i) $N_M \in \mathbb{Z}$, $N_E \in \mathbb{Z}$ and (ii) at least one excitation exists for any given integer combination $(N_M, N_E)$; (3) TO does not exist. The first two conditions here correspond to the charge lattice with the “nontrivial Witten effect with $\Theta = 2\pi \text{ mod}(4\pi)$” phenomenon. It also rules out TO with fractional electric charges for intrinsic excitations and TO with fermionic intrinsic excitations. The nontrivial Witten effect implies the elementary EM monopole $(N_M = 1, N_E = 0)$ is fermionic, while the elementary EM dyon $(N_M = 1, N_E = 1)$ is bosonic. This charge lattice is shown in Fig. 3. This statistical transmutation has been recently discussed in Ref. [21].

If the state with symmetry supports a charge lattice that cannot be categorized into both of trivial and nontrivial SPT state, it must be an SET state. A constructed topological phase is a time-reversal symmetric state if the following three conditions are satisfied. (1) Condition I: the dyon condensate is $Z_2$-symmetric. The selected condensed dyon is
FIG. 3. Illustration of the charge lattice of nontrivial SPT. Excitations with \(-7 \leq N_E \leq 7\) and \(-7 \leq N_M \leq 7\) are plotted. The solid circles (open circles) denote bosonic (fermionic) statistics. The elementary EM monopole \((N_M = 1, N_E = 0)\) is fermionic, while the elementary EM dyon \((N_M = 1, N_E = 1)\) is bosonic.

self-time-reversal symmetric (time-reversal pair condensates are not possible, see Appendix A). (2) **Condition II**: the charge lattice is mirror-symmetric about \(x\) axis. On the charge lattice, the distribution of sites, quantum statistics and excitation energy are mirror-symmetric about \(x\) axis. More specifically, \((N_E,N_M)\) and \((N_E,-N_M)\) are simultaneously two sites of the charge lattice. At each site, there are many excitations that are further labeled by the third quantum number [e.g., \(N_m^{(1)}\)] in addition to the given \(N_E\) and \(N_M\). Each excitation \([N_{E,N_M,N_m^{(1)}}]\) has a counterpart \([N_{E,-N_M,N_m^{(1)}}]\) with the same quantum statistics and the same excitation energy, and vice versa. (3) **Condition III**: \(\alpha\)-gauge equivalence condition. \(2\alpha = \text{integer in the mean-field Ansatz } (0,0); \alpha = \text{half-odd in the mean-field Ansatz } (\pi,\pi)\). This condition and condition II determine \(\alpha\) altogether. Details of proof and a related discussion on this condition are presented in Secs. II C and V.

We simply say that the charge lattice is mirror-symmetric if condition II is satisfied. These three conditions lead to time-reversal invariance of the whole excitation spectrum.

### C. Mean-field Ansatz \((\theta_1,\theta_2) = (0,0)\)

#### 1. Dyon condensation with symmetry

Let us first consider the simplest starting point: the mean-field Ansatz with \((\theta_1,\theta_2) = (0,0)\). In other words, both of fermions \((f_1, f_2)\) are trivial TI. In this case,

\[
[N_f^{(1)}]_c = [n_f^{(1)}]_c \in \mathbb{Z}, \quad [N_f^{(2)}]_c = [n_f^{(2)}]_c \in \mathbb{Z}
\]

according to Eq. (8). Hereafter, we use the subscript “\(c\)” to specify all symbols related to the condensed dyon \(\phi_c\).

Excitations \(\phi\) are labeled by symbols without subscript \(c\). Thus, in the present mean-field Ansatz \((\theta_1,\theta_2) = (0,0)\), the quantum numbers of excitations take values in the following domains:

\[
N_f^{(1)} = n_f^{(1)} \in \mathbb{Z}, \quad N_f^{(2)} = n_f^{(2)} \in \mathbb{Z},
\]

\[
N^{a} = N_f^{(1)} - N_f^{(2)} = n^{(1)}_f - n^{(2)}_f \in \mathbb{Z}.
\]

Condition I further restricts \([N^{a}]_c = [N^{(2)}_f]_c\). Therefore a general dyon with time-reversal symmetry is labeled by two integers \(l\) and \(s\), i.e.,

\[
(N^{a})_c = s, \quad [N^{(1)}_f]_c = l, \quad [N^{(2)}_f]_c = -l, \quad (N_M)_c = 0.
\]

According to Eq. (6), where \([N^{a}]_c \in \mathbb{Z}\) and \((N_M)_c = 0\), \((N^{a})_c\) is also an integer. Such a time-reversal symmetric dyon is always bosonic since \((\Gamma)_c = 0\), according to Eq. (27).

Most importantly, by definition, the EM field here is a probe field such that once the EM field is switched off, the physical ground state (formed by the dyon considered here) should not carry EM magnetic charge. Therefore, \((N_M)_c\) vanishes, which is also required by time-reversal symmetry. Other quantum numbers of the condensed dyon are straightforward:

\[
(N^{a})_c = l, \quad [N^{(1)}_m]_c = s, \quad [N^{(2)}_m]_c = -s,
\]

\[
(N^{a})_c = 0, \quad [n^{(1)}_f]_c = l, \quad [n^{(2)}_f]_c = -l.
\]

But will such a dyon condensed state respect the EM electric U(1) symmetry and behave like an insulator? To answer this question, according to Sec. III B, let us write down the effective Lagrangian of the condensed dyon \(\phi_c\) in real time (only spatial components are written here for simplicity—the time components are similar):

\[
\mathcal{L}_{\text{kin}}[\phi_c] = \frac{1}{2m}((-i \vec{\nabla} + s\vec{a} + l\vec{A})\phi_c)^2 - V(\phi_c).
\]

where \(V(\phi_c)\) is a symmetric potential energy term, which energetically stabilizes the bosonic condensate.

In the dyon condensed state \(\phi_c \neq 0\), the internal gauge field \(a_\mu\) is gapped and satisfies

\[
s\vec{a} = -l\vec{A},
\]

which indicates that the internal gauge field cannot fluctuate freely and is locked to the nondynamical EM background. We see that the dyon condensation does not generate the \(\vec{A}^2\) term if

\[
s \neq 0.
\]

This requirement may be understood in the following way. If \(s = l = 0\), there is no dyon condensation, which is nothing but the ABI state discussed in Sec. IV A. If \(s = 0\) and \(l \neq 0\), Eq. (34) reduces to \(\vec{A} = 0\), which is the consequence of mass term \(\vec{A}^2\), a fingerprint of superconductor/superfluid with broken U(1). This case is nothing but condensation of \(l\) bosons (carrying EM electric charge \(l\)), which breaks U(1) symmetry down to \(Z_l\) discrete symmetry spontaneously [\(Z_l\) represents complete breaking of U(1)]. In the following, in order to preserve U(1) symmetry and consider dyon condensation, we will restrict our attention to \(s \neq 0\). Thus the dyon condensed state indeed respects the EM electric U(1) symmetry and represents a fully gapped insulator.
To construct excitations “$\phi$” (including intrinsic excitations and test particles), one must trivialize the mutual statistics between the excitation considered and $\phi_c$ such that the excitation is a deconfined particle, which is observable in the excitation spectrum. According to Eq. (25), the mutual statistics between $\phi_c$ and an excitation $\phi$ (labeled by quantum numbers without subscript “c”) is trivialized by the following formula:

$$N_m^{(1)} l + \left[ N_m - N_m^{(1)} \right] l = sn^{(1)}_f - sn^{(2)}_f,$$  \hspace{1cm} (36)

which constrains the quantum numbers of excitations leading to three independent labels instead of four. It may be equivalently expressed as

$$\frac{l}{s} N_m = n^{(1)}_f - n^{(2)}_f = N^{(1)}_f - N^{(2)}_f = N^a, \hspace{1cm} (37)$$

where the definition (15) is applied, and, the condition (35) is implicit. Therefore all excitations in the present mean-field Ansatz can be uniquely labeled by $[N_M, N^{(1)}_m, N^{(2)}_m]$ in the standard labeling, while $n^{(1)}_f$ is determined by Eq. (37).

Meanwhile, due to the screening effect shown in Eq. (34), the total EM electric charge $N_E$ is the sum of $N_A$ [which is equal to $\alpha N^{(1)}_f + (1 - \alpha) N^{(2)}_f$ according to Eq. (15)] and an additional screening part:

$$N_E = N_A - \frac{l}{s} N^a_m. \hspace{1cm} (38)$$

In the standard labeling, $N_E$ is expressed as (details of derivation are present in Appendix B)

$$N_E = n^{(2)}_f + \frac{l}{s} N^{(1)}_m + 2\alpha \frac{l}{s} N_M. \hspace{1cm} (39)$$

In fact, the condensed dyon has a trivial mutual statistics with itself. Thus the total EM electric charge of the condensed dyon can also be calculated via Eq. (38): \[(N_E)_c = (N_A)_c - \frac{l}{s} N^a_m \equiv L - l - l = 0, \hspace{1cm} (40)\]

which indicates that the condensation indeed does not carry total EM electric charge and U(1) symmetry is exactly unbroken.

Under time-reversal symmetry transformation, $N_E$ has the following property:

$$\bar{N}_E \equiv n^{(3)}_f - \frac{l}{s} N^{(1)}_m + 2\alpha \frac{l}{s} N_M = N_E, \hspace{1cm} (41)$$

where Eqs. (10) and (14) are applied and $\bar{\alpha} = 1 - \alpha$ due to the exchange of $f_1$ and $f_2$. Since $[N_M, n^{(2)}_f, n^{(1)}_m]$ is an excitation [and thus $(N_E, N_M)$ is a site on the charge lattice], we can prove that $[N_M, n^{(2)}_f, n^{(1)}_m]$ is also an excitation [and thus $(N_E, -N_M)$ is a site on the charge lattice] by justifying that $[N^{(2)}_m, N^{(1)}_m]$ satisfies the trivial mutual statistics condition (37). Required by the time-reversal invariant mean-field Ansatz we considered, the two dyons have the same excitation energy, such that Eq. (41) indicates that both the excitation energy and site distribution are mirror-symmetric at arbitrary $\alpha$.

We have selected a time-reversal invariant dyon condensate $\phi_c$ but will the excitation spectrum (i.e., charge lattice) respect time-reversal symmetry? According to Sec. IV B, in order to preserve time-reversal symmetry, one must also require that the charge lattice is mirror-symmetric about $x$ axis (including site distribution, quantum statistics, and excitation energy). As we have proved that the site distribution and excitation energy are already mirror-symmetric, shown in Eq. (41), the subsequent task is to examine whether the quantum statistics is mirror-symmetric.

Generally, we expect that only a sequence of $\alpha$ is allowed. $\alpha = 1/2$ satisfies condition III. In the following discussion of Sec. IV, we will only focus on $\alpha = 1/2$, which is the simplest choice in every mean-field Ansatz. We will leave the discussion on the general $\alpha$ sequence to Sec. V.

Now we turn to the discussion of quantum statistics of excitations. According to Eq. (26), the statistics sign in the present mean-field Ansatz (both $\theta_1 = \theta_2 = \alpha = 1/2$) can be obtained (details of derivation are present in Appendix C):

$$\Gamma = N_M N_E + \left[ 2 N^{(1)}_m - N^{(2)}_m + 1 \right] \frac{l}{s} N_M. \hspace{1cm} (42)$$

where $N_E$ can be expressed as

$$N_E = n^{(2)}_f + \frac{l}{s} [N^{(2)}_m - N^{(1)}_m], \hspace{1cm} (43)$$

by plugging $\alpha = 1/2$ into Eq. (39). In Eq. (42), $N_E$ is explicitly written in order to compare $\Gamma$ with the trivial Witten and nontrivial Witten effects defined in Sec IV B. One may also replace $N_E$ in Eq. (42) by Eq. (43), rendering an equivalent expression of Eq. (42):

$$\Gamma = N_M \left[ n^{(2)}_f + \frac{l}{s} [N^{(1)}_m + 1] \right]. \hspace{1cm} (44)$$

2. Different topological phases via different condensed dyons

a. Bosonic intrinsic excitations. With the above preparation, let us study the nature of the $U(1) \times Z_2^T$-symmetric topological (gapped) phases constructed via the condensed dyon $\phi_c$, mainly based on Eqs. (37), (38), (42), and (43). From Eq. (42), all intrinsic excitations (carrying zero $N_M$) are bosonic, which rules out all fermionic intrinsic excitations in the underlying boson system. We also note that the $f_1$ fermions all have nontrivial “mutual statistics” with the $\phi_c$ dyon, and thus those fermionic excitations are confined. Up to now, the only requirement on topological phases with symmetry $s \neq 0$ is to understand different topological phases via different condensed dyons, one needs to study the physical properties (quantum statistics, total EM electric charge) of all possible excitations constrained by Eq. (37).

\[ \{ l/s \in Z \} \]. Let us first focus on the parameter regime defined by $\frac{l}{s} \in Z$. In this case, Eq. (37) allows excitations carrying arbitrary integer $N_M$ and arbitrary integer $N^{(1)}_m$. In other words, an arbitrarily given integer $N_M$ can ensure that $N^a$ in the right-hand side of Eq. (37) is integer-valued required by Eq. (30). The other two quantum numbers $N^{(1)}_m, N^{(2)}_m$ are still unconstrained and thus can take arbitrary integer values and $N_E$ can also take arbitrary integer values due to Eq. (43). To conclude, if $l/s \in Z$, for any given integer combination $[N_M, N^{(1)}_m, N^{(2)}_m]$, there exists at least one excitation. Thus, for any given integer combination $(N_E, N_M)$, there exists at least one excitation on the charge lattice. A useful corollary is that for all intrinsic excitations ($N_M = 0$), $N_E$ is always
integer-valued, which rules out intrinsic excitations carrying fractional EM electric charge (namely, fractional intrinsic excitation).

Then we will look for the general solutions of \((l,s)\), which admit a trivial or a nontrivial Witten effect. To look for the general solutions, which admit \(\Gamma_2\), we solve the equation \(\Gamma - \Gamma_2 = 0\), i.e.,

\[
\left[2N_m^{(1)} - N_M + 1\right] \frac{l}{s} N_M - N_M = 0,
\]

where \(N_m^{(1)}\) and \(N_M\) are arbitrary integers if \(l/s \in \mathbb{Z}\). More precisely, if \(N_M\) is an arbitrary even integer, we require that \(N_M^{(1)} = 2k + 1\), where \(k\) is an arbitrary integer, \(\left[\frac{k}{2}\right]\left[2N_m^{(1)} - N_M + 1\right] = 2k + 1\) must be odd. In other words, \(l/s = \frac{k}{2}\) must be integer-valued. This requirement is not new and is nothing but our starting point. On the other hand, if \(N_M\) is an arbitrary odd integer, i.e., \(N_M = 2k + 1\), we will have \(l/s = \frac{k}{2}\) must be odd for an arbitrary integer \(N_m^{(1)}\). However, it is obviously even. Therefore there is no solution admitting a nontrivial SPT state.

Likewise, to look for a solution that gives \(\Gamma_0\), we solve the equation \(\Gamma - \Gamma_0 = 0\). It is easily obtained that the requirement \(l/s = \frac{k}{2}\) is an integer, which is our starting point here. Therefore, if the two integers \(l\) and \(s\) satisfy that \(l/s = \frac{k}{2}\) is an integer, such a choice \((l/s)\) is a solution that admits a state with a trivial Witten effect. The state is a trivial SPT state by definition if \(TO\) does not exist. One can check that the quantum statistics and sites are mirror-symmetric about \(x\) axis. In addition, the distribution of excitation energy is also mirror-symmetric due to Eq. (41).

Overall, the charge lattice is indeed mirror-symmetric and thus satisfies condition II.

In fact, we can also directly derive the trivial Witten effect result by reformulating \(\Gamma\) in Eq. (42) to \(\Gamma \equiv N_k NE + \left[2N_m^{(1)} - N_M + 1\right] \frac{l}{s} N_M - N_M N_E\) where the last term in \(\Gamma\) is always even once \(l/s \in \mathbb{Z}\).

The trivial Witten effect only rules out TO with fractional intrinsic excitations and TO with fermionic intrinsic excitations. Other TO patterns are still possible. Meanwhile, we note that actually the excitations can also come from a pure gauge sector, in addition to the matter field sector (i.e., dyons) considered above. If \(l = 0, s = 2, \frac{l}{s} = 0\), the internal gauge symmetry \(U(1)\) of \(U(1)\times Z_s\) dynamical gauge field \(\phi\) is not fully broken but broken down to \(Z_{2s} \times Z_s\) gauge symmetry according to Eq. (33), which renders \(Z_{2s}\) TO in three dimensions in the presence of global symmetry \(U(1)\). The low-energy field theory of this TO pattern is the topological BF theory of level \(s\). The ground-state degeneracy (GSD) on a three-torus is \(\left|s\right|^3\). Therefore, to get a trivial SPT state, which does not admit any TO by definition, one should further restrict the two integers \((l,s)\) satisfying \(\frac{l}{s} = \frac{k}{2}\) and \(l = 0, s = 1\) and further determine the quantum statistics of the excitations should be consistently fractional, i.e., \(e^* = \frac{k}{q}\). Therefore the dyon excitations have self-consistently included fractional intrinsic excitations in response to the new quantization sequence of the EM magnetic charge \(N_M\). In this sense, the topological phase labeled by \((l,s) = 1\) contains TO (emergence of fractional intrinsic excitations) with global symmetry, i.e., a SET state.

Generally, we may parametrize \(l/s = k' + \frac{l'}{q} \in \mathbb{Z}\), where \(k', q \in \mathbb{Z}\), \(q > p > 0\), and \(gcd(p,q) = 1\). The allowed excitations constrained by Eq. (37) enforce \(N_M\) quantization as

\[
N_M = qk,
\]

where \(k \in \mathbb{Z}\), i.e., \(N_M = \pm q, \pm 2q, \pm 3q, \ldots\) Thus the effective fundamental EM electric charge unit \(e^*\) of intrinsic excitations should be consistently fractional, i.e., \(e^* = \frac{k}{q}\). On the other hand, Eq. (43) shows that \(NE\) of all excitations may be fractional as a multiple of \(1/q\):

\[
N_E = k_1 - pk_2/q,
\]

where the two integer variables \(k_1, k_2\) are introduced,

\[
k_1 \equiv N_f^{(2)} + k\left[qk - N_m^{(1)}\right] + pk, \quad k_2 \equiv N_m^{(1)},
\]

to replace \(N_f^{(1)}\) and \(N_m^{(1)}\). The first term in Eq. (47) is always integer-valued, but the second term may be fractional as a multiple of \(1/q\) by noting that \(N_m^{(1)}\) is an arbitrary integer and \(q > p > 0\). By setting \(k = 0\), we find that \(N_E\) may be fractional with unit \(e^* = 1/q\). In short, the state constructed here must be a SET state with fractional intrinsic excitations.

Next, we will focus on the quantum statistics of all excitations defined in Eq. (42). In Eq. (42), the term including \(N_m^{(1)}\) can be removed since \(2N_m^{(1)} N_M = 2N_m^{(1)}(qk + p)k = 0\). Therefore \(\Gamma\) is uniquely determined by \(N_M, N_E, N_f^{(1)}\) in the following formula up to the quantization conditions (46) and (47):

\[
\Gamma = N_M N_E - N_M(N_M - 1) \frac{l}{s}.
\]

In short, by giving three arbitrary integers \((k, k_1, k_2)\), one can determine \(N_E\) and \(N_M\) via Eqs. (47) and (46), respectively, and further determine the quantum statistics of the excitations labeled by \((k, k_1, k_2)\). One can plot the quantum statistics in the charge lattice formed by discrete data \(N_M\) and \(N_E\). We call this charge lattice with TO the “charge lattice I.” One can check that

For instance, if \(l = 1, s = 3\), the allowed value of \(N_M\) should take \(N_M = 3k\) with \(k \in \mathbb{Z}\), i.e., \(N_M = 0, \pm 3, \pm 6, \pm 9, \ldots\) in order to ensure the right-hand side of Eq. (37) is integer-valued. This quantization sequence is different from the sequence \((0, \pm 1, \pm 2, \pm 3, \ldots)\) we are familiar with in the vacuum. By recovering full units (each boson carries a fundamental charge unit \(e\), the EM magnetic charge \(\frac{2}{3}e\) can be reexpressed as \(\frac{2}{3}k\), where \(h\) is the Planck constant and the effective fundamental EM electric charge unit \(e\) of intrinsic excitations is fractional: \(e^* = \frac{k}{q}\). This fractional fundamental EM electric charge unit implies that the \(U(1)\times Z_s\)-symmetric ground state constructed via condensing the dyon \(\phi\) labeled by \((l,s) = (1,3)\) in the mean-field Ansatz \(\theta_1 = \theta_2 = 0\) admits fractional intrinsic excitations (which carry fractional EM electric charge), a typical signature of TO. Interestingly, from Eq. (43), we find that \(N_E\) of excitations with \(N_M = 0\) indeed can take a fractional value. Therefore the dyon excitations have self-consistently included fractional intrinsic excitations in response to the new quantization sequence of the EM magnetic charge \(N_M\). In this sense, the topological phase labeled by \((l,s) = 1\) contains TO (emergence of fractional intrinsic excitations) with global symmetry, i.e., a SET state.
the quantum statistics and sites are mirror-symmetric about \( x \) axis. In addition, the distribution of excitation energy is also mirror-symmetric due to Eq. (41). Overall, the charge lattice is indeed mirror-symmetric and thus satisfies condition II. For example, if \( 1/s = k' + 1/3 \), the charge lattice \( I \) is shown in Fig. 4.

In short, this new SET labeled by \( l/s = k' + p/q \) with \( k',p,q \in \mathbb{Z}, q > p > 0, \gcd(q,p) = 1 \) (coined “dyonic TO” in order to distinguish it from TO which arises from the gauge sector, e.g., \( Z_{\text{l}} \) TO) has the following key properties. (1) All excitations are uniquely labeled by three arbitrary integers \( (k,k'_1,k'_2) \), which are related to the standard labeling via Eq. (48). (2) The total EM electric charge \( N_E \) of intrinsic excitations is fractional with unit \( e^* = 1/q \) given by \( N_E = \frac{k_1 - \frac{l}{2}k_2}{q} \). The allowed EM magnetic charge \( N_M \) is quantized at \( q \). (3) The quantum statistics of excitations is uniquely determined by \( N_M \) and \( N_E [\text{cf. Eq. (49)}] \). (4) All intrinsic excitations are bosonic.

Before closing the analysis of the present mean-field Ansatz, a possible confusion should be clarified. The TI state of free fermions admits \( \Theta = \pi \). Witten effect and thus \( N_E = n + \frac{1}{2} N_M \) (\( n \) is integer number of attached fermions) may be fractional. However, this fractional EM electric charge is due to the presence of external EM magnetic monopole. To diagnose the TO of the ground state, we should restrict our attention to “intrinsic excitations,” which requires \( N_M = 0 \) by definition, as we are doing in the present work. Therefore, indeed, the total EM electric charge of any intrinsic excitation in the TI is nonfractional and no TO exist.

In summary, in the mean-field Ansatz with \( (\theta_1,\theta_2) = (0,0) \), all symmetric gapped phases (condensed dyons with symmetry) are labeled by two integers \( (s,l) \). Physically, a condensed dyon with symmetry labeled here is a composite of \( l \) physical bosons (formed by \( l \) \( f_1 \) fermions and \( l \) \( f_2 \) fermions) and \( s \) unit magnetic monopoles of internal gauge field \( a_{\mu} \). The state is a trivial SPT state, i.e., a trivial Mott insulator of bosons with \( U(1) \times Z_l^2 \) symmetry if the two integers \( (l,s) \) satisfy \( \{ l \in \mathbb{Z}, l \neq 0, s \neq 0 \} \cup \{ l = 0, s = \pm 1 \} \). The state is a SET state if the two integers \( (l,s) \) either (i) satisfy \( l = 0, s \geq 2 \), which corresponds to the SET state with \( Z_{\text{l}} \) TO, or (ii) satisfy \( l/s \notin \mathbb{Z} \), which corresponds to a new SET—dyonic TO.

We note that if the fermions \( f_1 \) and \( f_2 \) in our construction are replaced by two bosons both of which are in trivial Mott insulator states of bosons (i.e., \( \theta_1 = \theta_2 = 0 \) and \( \alpha = 1/2 \)), there exists a solution for a nontrivial SPT state. For example, \( l = s = 1 \) is a candidate of nontrivial SPT state, i.e., a nontrivial BTI state, which corresponds to condensation of one physical boson attached to a magnetic monopole of internal gauge field \( a_{\mu} \) [22]. The charge lattice is shown in Fig. 3. This can be verified easily with the same technique shown above. The only difference is that in this bosonic projective construction, the terms \( \prod_{\alpha} (-1)^{\nu_{\alpha}} \) in the quantum statistics formula (26) should be removed for the reason that attached particles are bosonic.

D. Mean-field Ansatz \((\theta_1,\theta_2) = (\pi,\pi)\)

1. Dyon condensation with symmetry

Following the same strategy, we can also consider the mean-field Ansatz with \( (\theta_1,\theta_2) = (\pi,\pi) \) where both fermions are in nontrivial TI states. In this case, condition I restricts \( |N_f^{(1)}|_c = |N_f^{(2)}|_c \). Therefore a general dyon with time-reversal symmetry is labeled by two integers \( l \) and \( s \):

\[
\begin{align*}
(N_m^{(a)})_c &= s, \\
[N_f^{(1)}]_c &= l + \frac{s}{2}, \\
[N_f^{(2)}]_c &= l + \frac{s}{2}, \\
(N_M)_c &= 0.
\end{align*}
\]

One can check that the dyon condensate is time-reversal symmetric and \( (N_m^{(a)})_c \) is quantized at integer with any given \( \alpha \) in the present mean-field Ansatz, and such a general dyon is always bosonic since \( (\Gamma)_c \) = even integer. Most importantly, by definition, the EM field here is a probe field such that by switching off EM field, the physical ground state should not carry EM magnetic charge. So, \( (N_M)_c \) must be vanishing, which is also required by time-reversal symmetry. Other quantum numbers of the condensed dyon are straightforward:

\[
\begin{align*}
(N_A)_c &= l + \frac{s}{2}, \\
(N_m^{(1)})_c &= s, \\
[N_m^{(2)}]_c &= -s, \\
(N^n)_c &= 0, \\
[n_f^{(1)}]_c &= l, \\
[n_f^{(2)}]_c &= l + s.
\end{align*}
\]

But will such a dyon condensed state respect the EM electric \( U(1) \) symmetry and behave like an insulator? To answer this question, according to Sec. III B, let us write down the effective Lagrangian of the condensed dyon \( \phi_c \) in real time (only spatial components are written here for simplicity and time component is similar):

\[
\mathcal{L}_{\text{kin}} = \frac{1}{2m} \left| \left( -i \nabla + s \mathbf{a} + \left( l + \frac{s}{2} \right) \mathbf{A} \right) \phi_c \right|^2 - V(\phi_c),
\]

where \( V(\phi_c) \) is a symmetric potential energy term that energetically stabilizes the bosonic condensate.
In the dyon condensed state $\phi_c \neq 0$, the internal gauge field $a_\alpha$ is gapped and satisfies
\[ s\tilde{a} = -\left( l + \frac{s}{2} \right) A, \] (53)
which indicates that the internal gauge field cannot fluctuate freely and is locked to the nondynamical EM background. We also require that
\[ s \neq 0 \] (54)
in the following in order that the dyon condensation does not generate the $A^1$ term. This requirement may be understood in the following way. If $s = l = 0$, there is no dyon condensation, which is nothing but the ABI state discussed in Sec. IV A. If $s = 0$ and $l \neq 0$, Eq. (53) reduces to $A = 0$, which is the consequence of mass term $A^2$, a fingerprint of superconductor/superfluid with broken $U(1)$. This case is nothing but a condensation of $l$ bosons (carrying EM electric charge $l$), which breaks $U(1)$ symmetry down to $Z_{2l}$ discrete symmetry spontaneously [$Z_l$ represents complete breaking of $U(1)$]. In the following, in order to preserve $U(1)$ symmetry and consider dyon condensation, we will restrict our attention to $s \neq 0$. Thus the dyon condensed state indeed respects the EM electric $U(1)$ symmetry and represents a fully gapped insulator.

It should be noted that the surface fermionic gapless excitations described by two $f_1$ Dirac fermions are also confined by the $\phi_c$ condensation. The confinement behaves like a strong attraction between $f_1$ and $f_2$ fermions, which may turn the surface into a superconducting state.

To construct excitations "$\phi^\alpha"$ (including intrinsic excitations and test particles), one must trivialize the mutual statistics between the excitation $\phi$ and condensed dyon $\phi_c$ such that the excitation is a deconfined particle that is observable in the excitation spectrum. According to Eq. (25), the mutual statistics between an excitation $\phi$ labeled by quantum numbers without subscript "c" and $\phi_c$ is trivialized by the following formula:
\[ (N_{m_l}^{(1)}) + \left( N_{-m} - N_{m_l}^{(1)} \right)(l+s) = sn_{f_1} - sn_{f_2}, \] (55)
which leads to
\[ n_{f_1}^{(1)} - n_{f_2}^{(2)} = \left( l + \frac{s}{2} \right) N_M - N_{m_l}^{(1)}. \] (56)

Note that by considering Eqs. (8), (15), and (56), we obtain an alternative expression of Eq. (56):
\[ N^\alpha = \left( l + \frac{s}{2} \right) N_M. \] (57)

Equation (56) is an important and unique constraint on the quantum numbers of excitations $\phi_c$ constructed above the condensed dyon $\phi_c$. In other words, a dyon is served as a deconfined particle (i.e., an excitation with a finite gap) above the condensed dyon $\phi_c$ if its four quantum numbers are constrained by Eq. (56). Therefore all excitations in the present mean-field Ansatz can be uniquely labeled by $[N_{m_l}^{(1)}, n_{f_1}^{(2)}, n_{f_2}^{(2)}]$ in the standard labeling, while $n_{f_1}^{(2)}$ is determined by Eq. (56).

Meanwhile, due to the screening effect shown in Eq. (53), the total EM electric charge $N_E$ is sum of $N_A$ and a screening part:
\[ N_E = N_A - \frac{l + \frac{s}{2} N_m^a}. \] (58)

In the standard labeling, $N_E$ is expressed as (details of derivation are present in Appendix D)
\[ N_E = -\left( \frac{l}{s} + 1 \right) N_{m_l}^{(1)} + n_{f_l}^{(2)} + \left( 2\pi l + \alpha + 1 \right) N_M \] (59)
which indicates that the condensation indeed does not carry total EM electric charge such that $U(1)$ symmetry is exactly unbroken.

Under time-reversal symmetry transformation, $N_E$ has the following property:
\[ N_E = -\left( \frac{l}{s} + 1 \right) N_{m_l}^{(1)} + n_{f_l}^{(2)} + \left( 2\pi l + \alpha + 1 \right) N_M = N_E. \] (60)

Since $[N_{m_l}^{(1)}, n_{f_l}^{(2)}, n_{f_2}^{(2)}]$ is an excitation [and thus $(N_{E}, N_M)$ is a site on the charge lattice], we can prove that $[N_{m_l}^{(1)}, n_{f_l}^{(2)}, n_{f_2}^{(2)}]$ is also an excitation [and thus $(N_E, -N_M)$ is a site on the charge lattice] by justifying that $[N_{m_l}^{(1)}, n_{f_l}^{(2)}, n_{f_2}^{(2)}]$ satisfies the trivial mutual statistics condition (37). Required by the time-reversal invariant mean-field Ansatz we considered, the two dyons have the same excitation energy, such that Eq. (41) indicates that both the excitation energy and site distribution are mirror-symmetric at arbitrary $\alpha$. Likewise, we choose the simplest case: $\alpha = 1/2$ in this section. Plugging $\alpha = 1/2$ into Eq. (59), we obtain
\[ N_E = -\left( \frac{l}{s} + 1 \right) N_{m_l}^{(1)} + n_{f_l}^{(2)} + \left( \frac{l}{s} + 1 \right) N_M. \] (62)

Now we turn to the discussion of quantum statistics of excitations. According to Eq. (26), the statistics sign in the present mean-field Ansatz ($\theta_1 = \theta_2 = \pi, \alpha = 1/2$) is (details of the derivation are present in Appendix E)
\[ \Gamma \equiv N_M \left[ N_{m_l}^{(1)} + 1 \right] \left( \frac{l}{s} + 1 \right) + N_M n_{f_2}^{(2)}. \] (63)

2. Different topological phases via different condensed dyons

With the above preparation, let us study the nature of the $U(1) \times Z_2^l$-symmetric topological (gapped) phases constructed above the condensed dyon $\phi_c$, mainly based on Eqs. (56), (62), and (63).

a. Bosonic intrinsic excitations. From Eq. (63), all intrinsic excitations with zero $N_M$ are bosonic [excitations with $N_M = 0$ always exist in Eq. (56)], which rules out all fermionic intrinsic excitations in the underlying boson system. We also note that the $f_1$ fermions all have nontrivial mutual statistics with the
ϕe, dyon, and thus those fermionic excitations are confined. Up to now, the only requirement on topological phases with symmetry is s ≠ 0. To understand different topological phases via different condensed dyons, one needs to study the physical properties (quantum statistics, total EM electric charge) of all possible excitations constrained by Eq. (56).

b. {l/s ∈ ℤ}. Let us first focus on the parameter regime defined by \( \frac{1}{2} ∈ ℤ \). In this case, Eq. (56) allows excitations carrying arbitrary integers \( N_E, N_m^{(1)}, \) and \( n_f^{(2)}, n_s^{(1)} \) is uniquely fixed by Eq. (56), and \( N_E \) in Eq. (62) is also fixed and can also take arbitrary integer values.

We may reformulate \( \Gamma \) in Eq. (63) by means of \( l/s ∈ ℤ \) (details of derivation are present in Appendix G):
\[
\Gamma \doteq N_M N_E . \tag{64}
\]
Due to Eq. (56), \( n_f^{(2)} \) and \( N_m^{(1)} \) can be still arbitrarily integer-valued, and \( n_f^{(1)} \) is fixed once \( N_m^{(1)}, N_M, \) and \( n_f^{(2)} \) are given. Thus \( N_E \) in Eq. (62) can take any integer. There exists at least one excitation for any given integer combination \( (N_E, N_M) \). In short, if the two integers \( l \) and \( s \) satisfy that \( \frac{1}{2} \) is an integer, such a choice \((l, s)\) is a solution that admits a state with a trivial Witten effect. The state is a trivial SPT state. One can check that the quantum statistics and sites are mirror-symmetric about \( x \) axis. In addition, the distribution of excitation energy is also mirror-symmetric due to Eq. (61). Overall, the charge lattice is indeed mirror-symmetric and thus satisfies condition II.

c. \( \{ l/s \notin ℤ \} \). We note that \( l/s = -1/2 \) is a special point where the internal gauge symmetry \( U(1) \) is broken down to \( Z_\alpha \) gauge symmetry according to Eq. (52). It leads to \( Z_{\alpha} \) TO in three dimensions in the presence of global symmetry \( U(1)\times Z^\alpha_2 \). In the following, we will not consider this point.

For a general parameter choice in \( l/s \notin ℤ \), we will see that there is a dyonic TO, which is defined as TO arising from the dyons. Generally, we may parametrize \( l/s = k' + \frac{2}{q} \), where \( k', p, q ∈ ℤ, q > p > 0, \gcd(p, q) = 1 \) (gcd: greatest common divisor). Plugging \( l/s = k' + \frac{2}{q} \) into Eq. (56), we find that \( N_M \) must be quantized at \( q \) in all allowed excitations constrained by Eq. (56). That is,
\[
N_M = qk , \tag{65}
\]
where \( k ∈ ℤ \), i.e., \( N_M = 0, ± q, ±2q, ±3q, \ldots \). On the other hand, Eq. (62) shows that \( N_E \) of all excitations may be fractional as a multiple of \( l/q \):
\[
N_E = k_1 - pk_2/q , \tag{66}
\]
where the two integer variables \( k_1, k_2 \) are introduced and related to the standard labeling in the following way:
\[
k_1 \equiv (k' + 1)q k + pk + n_f^{(2)} - (k' + 1)N_m^{(1)} , \tag{67}
\]
\[
k_2 \equiv N_m^{(1)} . \tag{68}
\]
Due to Eq. (56), \( n_f^{(2)} \) and \( N_m^{(1)} \) can be still arbitrarily integer-valued, and \( n_f^{(1)} \) is fixed once \( N_m^{(1)}, N_M, \) and \( n_f^{(2)} \) are given. Thus the new variables \( k_1 \) and \( k_2 \) can be any integers. Hereafter, all excitations are labeled by the three independent integers \((k, k_1, k_2)\). Using these new labels, we see that \( N_E \) of intrinsic excitations \((k = 0)\) can be still fractional according to Eq. (66), which does not depend on \( k \). It indicates that the state constructed here is a SET state with fractional intrinsic excitations. A useful observation from Eq. (66) is that \( N_E \) can also take any integer once \( k_2 = q \).

Next, we will focus on the quantum statistics of all excitations defined in Eq. (63). In the present parameter regime, \( \Gamma \) can be expressed as (details of derivation are present in Appendix G)
\[
\Gamma \doteq N_M N_E - \frac{l}{s} N_M (N_M - 1) . \tag{69}
\]

Therefore \( \Gamma \) is uniquely determined by \( N_E, N_M \) in the following formula up to the quantization conditions (65) and (66). In short, by giving three arbitrary integers \((k, k_1, k_2)\), one can determine \( N_E \) and \( N_M \) via Eqs. (66) and (65), respectively, and further determine the quantum statistics of the excitations labeled by \((k, k_1, k_2)\). One can plot the quantum statistics in the charge lattice expanded by discrete variables \( N_M \) and \( N_E \), which is same as charge lattice I discussed in Sec. IV C 2 and shown in Fig. 4 \((l/s = 1/3)\). One can check that the quantum statistics and sites are mirror-symmetric about \( x \) axis. In addition, the distribution of excitation energy is also mirror-symmetric due to Eq. (61). Overall, the charge lattice is indeed mirror-symmetric and thus satisfies condition II.

In summary, in the mean-field Ansatz with \((\theta_1, \theta_2) = (\pi, \pi)\), all symmetric gapped phases (condensed dyons with symmetry) are labeled by two integers \((l, f)\). Physically, a condensed dyon with symmetry labeled here is a composite of \( l \) physical bosons (formed by \( f_1 \) fermions and \( f_2 \) fermions), \( f_2 \) fermions, and \( s \) unit magnetic monopoles of internal gauge field \( a_\mu \). If \( l/s ∈ ℤ \), the ground state is a trivial SPT state, i.e., trivial Mott insulator of bosons. In the parameter regime \( l/s \notin ℤ, l/s \neq -1/2 \), the ground state is a SET state with Dyonic TO (fractional intrinsic excitations). If \( l/s = -1/2 \), the ground state is a SET state with \( Z_{\alpha} \) TO.

V. TOPOLOGICAL PHASES WITH SYMMETRY: GENERAL \( \alpha \)-SEQUENCE

A. Main results

In Sec. IV, we have obtained many topological phases based on dyon condensations (see Table III). The value of \( \alpha \) is chosen to be \( \alpha = 1/2 \) such that both \( f_1 \) and \( f_2 \) carry 1/2 EM electric charge (see Table I). Such a choice preserves the time-reversal symmetry. If we choose \( \alpha \) to be some other values, the time-reversal symmetry may be broken. However, \( \alpha = 1/2 \) is not the only value that potentially preserves time-reversal symmetry. In the following, we shall study the general \( \alpha \) sequence, which respects time-reversal symmetry. Since the mean-field Ansatz \((0, \pi)\) always breaks time-reversal symmetry, we only consider the other two Ansätze. The main results are summarized in Table IV. We note that the results in Table III can be obtained by taking \( \alpha = 1/2 \) in Table IV.

B. Mean-field Ansatz \((\theta_1, \theta_2) = (0, 0)\)

In this mean-field Ansatz, according to the general statement in Sec. II C, two allowed values of \( \alpha \) must be differed from each other by any integer, which is required by the charge quantization argument. Let us consider \( \bar{\alpha} = 1 - \alpha \) and \( \alpha \), where \( \bar{\alpha} \) is the time-reversal transformed \( \alpha \) shown in Eq. (12).
The requirement \( \bar{\alpha} - \alpha = \text{any integer} \) is equivalent to the constraint \( 2\alpha = \text{integer} \), which is nothing but condition III. This is the first constraint we obtained on the domain value of \( \alpha \).

We have obtained the total EM electric charge \( N_E \) in the standard labeling and in the presence of \( \alpha \) [cf. Eq. (39)]. To approach a mirror-symmetric charge lattice, we require that the site distribution, excitation energy, and quantum statistics are mirror-symmetric (cf. condition II).

1. \( 1/s \in \mathbb{Z} \)

Let us first consider \( 1/s \in \mathbb{Z} \) such that \( N_M \) is arbitrarily integer-valued due to the constraint (37). We assume that the mirror site of \([N_M,n_f^{(2)},N_m^{(1)}]\) is labeled by \([-N_M,n_f^{(2)},N_m^{(1)}]\). In order that the mirror site does exist on the charge lattice, the integer solutions \([n_f^{(2)},N_m^{(1)}]\) of the following equation must exist for any given integer \( N_M \) [see Eq. (39)]:

\[
N_E = n_f^{(2)} - \frac{1}{s} N_m^{(1)} - 2\alpha N_M,
\]

which is equivalent to

\[
[n_f^{(2)} - n_f^{(2)}] - \frac{1}{s} [N_m^{(1)} - N_m^{(1)}] = 4\alpha N_M,
\]

by means of Eq. (39). Therefore the mirror-symmetric site distribution requires that \( 4\alpha \) is integer. To construct symmetric topological phases, we need to further check the quantum statistics in the presence of \( \alpha \) (details of derivation are present in Appendix H):

\[
\Gamma = N_M \left( N_E - 2\alpha N_M + \frac{1}{s} \right).
\]

The quantum statistics is mirror-symmetric if

\[
N_M \left( N_E - 2\alpha N_M + \frac{1}{s} \right) \equiv (-N_M) \left[ N_E - 2\alpha \left( -N_M + \frac{1}{s} \right) \right],
\]

i.e., \( 2N_M N_E \equiv 0 \).

In other words, \( 2N_M N_E \) must be always even.

If \( 4\alpha \) is odd, Eq. (39) indicates that \( N_E \) is half-odd integer if we take \( N_M = 1 \). As a result, Eq. (73) is not satisfied. Therefore, in order to guarantee mirror-symmetric distribution of quantum statistics, we need to consider a stronger condition: \( 4\alpha \) is even, i.e., \( 2\alpha \) is integer. In Eq. (41), we have already proved that energy is mirror-symmetric for any \( \alpha \), so that we conclude that to obtain a mirror-symmetric charge lattice (i.e., condition II), we need \( 2\alpha \) is integer. Under this condition as well as \( 2\alpha \) is integer, we may obtain trivial SPT states and nontrivial SPT states summarized in Table IV by comparing \( \Gamma \) with the standard trivial Witten effect and nontrivial Witten effect defined in Sec. IV B. As usual, one should pay attention to the emergence of \( Z_{\bar{\alpha}} \) TO if \( l = 0 \) and \( |s| \geq 2 \) although charge lattice formed by deconfined dyons is same as a trivial SPT state. Strikingly, we obtain BTI states, which are completely absent in Table III where \( \alpha = 1/2 \) is fixed.

Since \( N_M \in \mathbb{Z} \), \( \frac{1}{s} \in \mathbb{Z} \), and \( \frac{1}{s} N_M \equiv \frac{1}{s} (N_M)^2 \), one may rewrite Eq. (72) as \( \Gamma \equiv N_M (N_E - 2\alpha N_M + \frac{1}{s} N_M) \equiv N_M (N_E - \frac{2\alpha}{s} N_M) \). The minimal periodicity of \( \Theta \) is \( 4\pi \) because \( \Gamma \) is invariant after \( 4\pi \) shift. As a result, a \( \Theta \) angle can be formally defined as

\[
\Theta \equiv -2\pi \frac{l}{s} + 4\pi \frac{l}{s} \alpha \mod(4\pi),
\]

from which we see that the \( \Theta \) angle is linearly related to \( \alpha \).

2. \( 1/s \notin \mathbb{Z} \)

Generally, we may parametrize \( 1/s = k' + p \), where \( k',p,q \in \mathbb{Z}, |q| > p > 0 \). In this case, \( N_M \) is quantized at \( qk \) as shown in Eq. (46). To guarantee mirror-symmetric site distribution, the integer solutions \([n_f^{(2)},N_m^{(1)}]\) of Eq. (71) must exist for any given \( N_M = qk \):

\[
[n_f^{(2)} - n_f^{(2)}] - k' + p \frac{1}{s} (N_m^{(1)} - N_m^{(1)}) = 4\alpha(qk' + p)k,
\]

by means of Eq. (39). Equation (72) is also valid when \( 1/s \notin \mathbb{Z} \) by noting that \(-2N_Ml\) is still even integer in deriving the fourth line of Appendix H. Therefore Eq. (73) is also valid when \( 1/s \notin \mathbb{Z} \).

A general discussion on Eq. (75) and (73) is intricate. Let us take a simple example: \( 1/s = 1/3 \), i.e., \( k' = 0, q = 3, p = 1 \). The right-hand side of Eq. (75) becomes \( 4ak \). To obtain the integer solutions \([n_f^{(2)},N_m^{(1)}]\) for any given integers \([k,N_m^{(1)},n_f^{(2)}]\), a constraint on \( \alpha \) is necessary: \( \alpha = \text{integer}/12 \).

Under this condition, Eq. (73) leads to a stronger condition: \( 6\alpha = k_0 \), where \( k_0 \) is an integer. It guarantees mirror-symmetric distribution of both sites and quantum statistics. As we have proved, energy is already mirror-symmetric due to Eq. (41). Overall, to obtain a mirror-symmetric charge lattice (i.e., condition II), we need \( \alpha = \text{integer}/6 \). Keeping in mind that \( 2\alpha \) is integer is required by condition III, the two conditions altogether still give \( 2\alpha \) is integer.

Since \( N_M/3 \in \mathbb{Z} \), \( \frac{1}{s} \equiv 1/3 \), and \( \left( n_f^{(2)} + \frac{s}{3} N_M \right) / (N_M)^2 \), one may rewrite Eq. (72) as \( \Gamma \equiv N_M (N_E - \frac{2\alpha}{s} N_M + \frac{1}{s} N_M) \equiv N_M (N_E - \frac{2\alpha}{s} N_M) \). The minimal periodicity of \( \Theta \) is \( 2\pi / 9 \) because \( \Gamma \) is invariant after \( 2\pi / 9 \) shift. As a result, a \( \Theta \) angle can be formally defined:

\[
\Theta \equiv -\frac{2\pi}{9} + \frac{4\pi}{3} \alpha \mod\left(\frac{4\pi}{9}\right),
\]

from which we see that the \( \Theta \) angle is linearly related to \( \alpha \). All SET states have fractional intrinsic excitations. We can further classify these states into two categories: one is \( \Theta = 0 \mod\left(\frac{2\pi}{9}\right) \) with \( \alpha = \text{half-odd}, \) and another is \( \Theta = \frac{2\pi}{9} \mod\left(\frac{2\pi}{9}\right) \) with \( \alpha = \text{integer} \). In comparison to the trivial and nontrivial SPT states, we call the former “trivial fBTT” and the latter “nontrivial fBTT” via investigating the Witten effect.

C. Mean-field Ansatz \((\theta_1,\theta_2) = (\pi,\pi)\)

In this mean-field Ansatz, we have obtained the total EM electric charge \( N_E \) in the standard labeling and in the presence of \( \alpha \) [cf. Eq. (59)]. To approach a mirror-symmetric charge
lattice, we require that the site distribution and quantum statistics are mirror-symmetric.

1. $l/s \in \mathbb{Z}$

Let us first consider $l/s \in \mathbb{Z}$, such that $N_M$ is arbitrarily integer-valued due to the constraint (56). We assume that the mirror site of $[N_M,n_f^{(2)},N_m^{(1)}]$ is labeled by $[-N_M,n_f^{(2)},N_m^{(1)}]$. In order that the mirror site does exist in the charge lattice, the integer solutions $[n_f^{(2)},N_m^{(1)}]$ of the following equation must exist for any given integer $N_M$:

\[ N_E = -\left(\frac{l}{s} + 1\right)N_m^{(1)} + n_f^{(2)} - \left(2\alpha \frac{l}{s} + \alpha + \frac{1}{2}\right)N_M, \]

(77)

which is equivalent to

\[ [n_f^{(2)} - n_f^{(2)}] = \left(\frac{l}{s} + 1\right)[N_m^{(1)} - N_m^{(1)}] = 2\left(2\alpha \frac{l}{s} + \alpha + \frac{1}{2}\right)N_M, \]

(78)

by means of Eq. (59). Therefore the mirror-symmetric site distribution requires that $2 \times (2\alpha \frac{l}{s} + \alpha + \frac{1}{2}) = \text{integer}$. To construct symmetric topological phases, we need to further check the quantum statistics in the presence of $\alpha$ (details of derivation are present in Appendix I):

\[ \Gamma = N_M \left[ N_E - \left(2\alpha \frac{l}{s} + \alpha + \frac{1}{2}\right)N_M - \left(\frac{l}{s} + 1\right)\right]. \]

(79)

Therefore the quantum statistics is mirror-symmetric if

\[ N_M \left[ N_E - \left(2\alpha \frac{l}{s} + \alpha + \frac{1}{2}\right)N_M - \left(\frac{l}{s} + 1\right)\right] \equiv -N_M \left[ N_E + \left(2\alpha \frac{l}{s} + \alpha + \frac{1}{2}\right)N_M - \left(\frac{l}{s} + 1\right)\right], \]

i.e., $2N_MN_E \equiv 0$.

(80)

In other words, $2N_MN_E$ must be even always. If $2 \times (2\alpha \frac{l}{s} + \alpha + \frac{1}{2})$ is odd, Eq. (59) indicates that $N_M$ is half-odd integer if we take $N_M = 1$. As a result, Eq. (80) is not satisfied. Therefore, in order to guarantee mirror-symmetric distribution of quantum statistics, we need to consider a stronger condition: $2 \times (2\alpha \frac{l}{s} + \alpha + \frac{1}{2}) = \text{even}$, i.e., $(2\alpha \frac{l}{s} + \alpha + \frac{1}{2}) = \text{integer}$. In Eq. (61), we have already proved that energy is mirror-symmetric for any $\alpha$, so that we can conclude that the charge lattice is mirror-symmetric if the stronger condition $2\alpha \frac{l}{s} + \alpha + \frac{1}{2} = \text{integer}$ is satisfied.

Since $N_M \in \mathbb{Z}$, $\frac{l}{s} \in \mathbb{Z}$ and $(\frac{l}{s} + 1)N_M \equiv (\frac{l}{s} + 1)(N_M^2)$, one may rewrite Eq. (79) as $\Gamma \equiv N_M[N_E - (2\alpha \frac{l}{s} + \alpha + \frac{1}{2})N_M - (\frac{l}{s} + 1)N_M] \equiv N_M(N_E - \frac{10}{3\pi}N_M)$. The minimal periodicity of $\Theta$ is $4\pi$ because $\Gamma$ is invariant after $\frac{4\pi}{3\pi}$ shift. As a result, a $\Theta$ angle can be formally defined:

\[ \Theta \equiv 2\pi \left(\frac{l}{s} + \frac{3}{2}\right) + 4\pi \left(\frac{l}{s} + \frac{1}{2}\right) \alpha \mod (4\pi), \]

(81)

from which we see that the $\Theta$ angle is linearly related to $\alpha$. From this $\Theta$ formula, we realize that shifting $\alpha$ by an odd integer will change trivial (nontrivial) SPT to nontrivial (trivial) SPT. Therefore we arrive at the statement in Sec. II C. Thus two allowed values of $\alpha$ must be differed from each other by an even integer. Let us consider $\alpha = 1 - \alpha$, where $\alpha$ is the time-reversal transformed $\alpha$ shown in Eq. (12). The requirement $\alpha - \alpha = \text{any even integer}$ is equivalent to the constraint $\alpha = \frac{1}{2}$, which is nothing but condition III. Under this condition as well as the conditions obtained from mirror symmetric charge lattice, we may obtain trivial SPT states and nontrivial SPT states summarized in Table IV by comparing $\Gamma$ with the standard trivial Witten effect and nontrivial Witten effect defined in Sec. IV B. Strikingly, we obtain BTI states, which are completely absent in Table III where $\alpha = 1/2$ is fixed.

2. $l/s \notin \mathbb{Z}$

Generally, we may parametrize $l/s = k' + \frac{p}{q}$, where $k', p, q \in \mathbb{Z}$, $q > p > 0$, gcd($p, q) = 1$ (gcd: greatest common divisor). In this case, $N_M$ is quantized at $\pi k$ as shown in Eq. (65). To guarantee mirror-symmetric site distribution, the integer solutions $[n_f^{(2)},N_m^{(1)}]$ of (78) must exist for any given $N_M = \pi k$:

\[ [n_f^{(2)} - n_f^{(2)}] = \left(k' + 1 + \frac{p}{q}\right)[N_m^{(1)} - N_m^{(1)}] = [4\pi(\pi k + p) + 2\pi k + p]k \]

(82)

by means of Eq. (59). Equation (79) is also valid when $l/s \notin \mathbb{Z}$ by noting that $-2N_M(\frac{p}{q} + 1)[N_m^{(1)} + 1]$ is still even integer in deriving the fourth line of Appendix I. Therefore Eq. (80) is also valid when $l/s \notin \mathbb{Z}$.

A general discussion on Eqs. (82) and (80) is intricate. Let us take a simple example: $l/s = 1/3$, i.e., $k' = 0, q = 3, p = 1$. The right-hand side of Eq. (82) becomes $(10\alpha + 3)k$. To obtain the integer solutions $[n_f^{(2)},N_m^{(1)}]$ for any given integers $[k,N_m^{(1)},n_f^{(2)}]$, a constraint on $\alpha$ is necessary: $10\alpha + 3 = \text{integer}/3$, i.e., $\alpha = k_0 - \frac{9}{10}$ where $k_0$ is an integer. Under this condition, Eq. (80) leads to a stronger condition: $\alpha = \frac{2k_0 - 9}{10}$, which guarantees mirror-symmetric distributions of both sites and quantum statistics. As we have proved, excitation energy is already mirror-symmetric due to Eq. (61). By further considering condition III, $\alpha$ is finally restricted to $\alpha = \text{half-odd}$.

Since $N_M/3 \in \mathbb{Z}$, $\frac{l}{s} = 1/3$, and $(\frac{l}{s} + 1)N_M = 4N_M/3 \equiv 0$, one may rewrite Eq. (79) as $\Gamma \equiv N_M[N_E - (\frac{2}{3} + \frac{1}{2})N_M] \equiv N_M(N_E - \frac{10}{3\pi}N_M)$. The minimal periodicity of $\Theta$ is $\frac{4\pi}{3\pi}$ because $\Gamma$ is invariant after $\frac{4\pi}{3\pi}$ shift. As a result, a $\Theta$ angle can be formally defined:

\[ \Theta \equiv \pi + \frac{10\pi}{3} \alpha \mod \left(\frac{4\pi}{9}\right). \]

(83)

from which we see that the $\Theta$ angle is linearly related to $\alpha$. All SET states have fractional intrinsic excitations. We can further classify these states into two categories: one is $\Theta = 0 \mod (\frac{4\pi}{3\pi})$ with $\alpha - \frac{1}{2} = \text{even}$ and another is $\Theta = \frac{2\pi}{3\pi} \mod (\frac{4\pi}{3\pi})$ with $\alpha - \frac{1}{2} = \text{odd}$. In comparison to the trivial and nontrivial SPT states, we call the former “trivial fBTI” and the latter “nontrivial fBTI” via investigating the Witten effect.
As usual, one should pay attention to the emergence of $Z_{s[1]}$ TO if $l/s = -1/2$. One may also examine whether there is a Witten effect if $l/s = -1/2$ in addition to $Z_{s[1]}$ TO. Following the same procedure, we obtain that

$$\Theta \equiv \frac{\pi}{2} \text{ mod}(\pi)$$

and $\alpha$ is restricted to $\alpha = \text{half-odd}$. At this special point, $l/s = -1/2$, we find that the Witten effect is independent on $\alpha$ and the state is a nontrivial $\text{FTI}$. Thus, it is necessary to construct new features instead of the Witten effect, e.g., 2D surface properties. In other words, it is necessary to construct more new features instead of the Witten effect, e.g., 2D surface properties. In other words, it is necessary to construct more new features instead of the Witten effect, e.g., 2D surface properties.

VI. CONCLUSION

In conclusion, we used fermionic projective construction and dyon condensation to construct many three-dimensional SPT and SET states with time-reversal symmetry and $U(1)$ boson number conservation symmetry. Without dyon condensation, we obtained an algebraic bosonic insulator, which contains an emergent $U(1)$ gapless photon excitation. Then we assumed the internal $U(1)$ gauge field to fluctuate strongly and form one of many confined phases characterized by different dyon condensations. After a dyon condensate that preserves the $U(1) \times Z^2$ symmetry is selected properly, the excitation spectrum (formed by deconfined dyons) above this dyon condensate is entirely determined. The symmetric dyon condensate determines the quantization conditions of EM magnetic charge and EM electric charge of excitations. It also determines the quantum statistics (boson/fermion) and excitation energy. By calculating these properties, we then obtained SPT and SET states summarized in Tables II–IV. The basic process of this construction approach is shown in Fig. 1.

In short, we presented an “efficient program” for constructing SPT states in 3D symmetric topological phases; we also found that confined phases of the internal gauge field with global symmetry can be further classified into many different phases. There are some interesting and direct directions for future work.

1. Classification via projective construction and dyon condensation. The definition of nontrivial SPT states, i.e., bosonic topological insulators (BTI), is only related to the nontrivial Witten effect (i.e., $\Theta = 2\pi$) as shown in Sec. IV B. As shown in Ref. [19], classification of an SPT state with a certain symmetry corresponds to looking for a complete set of “topological invariants.” In this paper, we only consider one $Z_2$ topological invariant, which distinguishes the physical properties of trivial/nontrivial Witten effect, meaning that it is potentially possible some trivial SPT states we found in this paper actually are nontrivial and are characterized by new features instead of the Witten effect, e.g., 2D surface properties. In other words, it is necessary to construct more topological invariants to completely distinguish all SPT states.

There are some clues. Firstly, in this paper, we have systematically shown how to construct a charge lattice that respects symmetry by means of fermionic projective construction and dyon condensation. We expect that, in addition to Witten effect, more information (i.e., more topological invariants) can be extracted from more complete analysis of charge lattice. Secondly, we may consider SPT states with merely time-reversal symmetry. In other words, these states are protected sufficiently by time-reversal symmetry while the boson number conservation symmetry $U(1)$ does not play any role. Literally, these states are also SPT states with $U(1) \times Z^2$, although $U(1)$ here is not necessary. Therefore one may consider new mean-field Ansätze for fermions and try to find new SPT states.

2. Surface theory and bulk topological field theory via projective construction and dyon condensation. SPT states have a quite trivial bulk but the surface may admit many nontrivial physical properties that are absent in trivial Mott insulator states. It has been recently shown that classifying surface topological order may provide the answer to classifying the BTI bulk [15,51]. Indeed, the surface detectable features may be tightly connected to the complete set of topological invariants that we shall look for. For instance, a nontrivial Witten effect is indeed related to the surface quantum Hall effect (by breaking time-reversal symmetry on the surface) with anomalous quantization of Hall conductance that cannot be realized in 2D U(1) SPT [49,51]. In short, it is interesting for future work on a surface theoretical description via the present fermionic projective construction and dyon construction. Beside the SPT states constructed in this paper, we also constructed many SET states in which fractional intrinsic excitations exist (defined as excitations with zero EM magnetic charge). A full understanding of these topologically ordered states with symmetry is interesting from the perspective of (3 + 1)D topological quantum field theory (TQFT) descriptions and fixed-point lattice Hamiltonian realizations. In addition, realistic model Hamiltonians that can realize all the topological phases constructed in this paper are quite interesting; we leave the task of constructing such Hamiltonians to future work.

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APPENDIX A: CONDENSATION OF TWO TIME-REVERSAL CONJUGATED DYONS LEADS TO A SINGLE DYON CONDENSATE

The most general Ansatz for the quantum numbers of two time-reversal conjugated dyons is shown in Table. V. We see that there are four numbers ($l_1,l_2,s,t$)—enough to label two dyons, one of which is the time-reversal partner of the other.

In both $\phi_1$ and $\phi_2$ are condensed, the mutual statistics between them must be trivialized, i.e.,

$$l_1[s-(1-\alpha)t]+l_2(-s-\alpha t)$$

$$=l_2(s+\alpha t)+l_1[-s+(1-\alpha)t],$$

(A1)

which leads to

$$l_1[s-(1-\alpha)t]+l_2(-s-\alpha t) = 0.$$  

(A2)

We note that the $U(1)$ symmetry of the original boson system is generated by conserved EM electric charge rather than EM
magnetic charge. The EM field here is nondynamical so that the dyon condensates that form the physical ground state should have zero EM magnetic charge. Therefore we have \( t = 0 \), and the above trivial mutual statistics condition becomes

\[
(l_1 - l_2)s = 0.
\]

To satisfy this condition, \( l_1 = l_2 \) or \( s = 0 \). However, \( s \) must be nonzero in order to preserve U(1) symmetry. The reason is that once \( s = 0 \), the condensation will break U(1) since it is EM electric charged. The reasonable choice is \( l_1 = l_2 \). This choice leads to the fact that \( \phi_1 = \phi_2 \), i.e., a condensate of one kind of dyon.

### APPENDIX B: DERIVATION OF EQ. (39)

\[
N_E = \alpha N_f^{(1)} + \frac{l}{s} N_m^{a} - l N_m^{a}
\]

\[
= \alpha n_f^{(1)} + (1 - \alpha) n_f^{(2)} - \frac{l}{s} [N_m^{(1)} - \alpha N_M]
\]

\[
= \alpha \left[ \frac{l}{s} N_M + n_f^{(2)} \right] + (1 - \alpha) n_f^{(2)} - \frac{l}{s} N_m^{(1)}
\]

\[
= n_f^{(2)} + 2 \alpha \frac{l}{s} N_M - \frac{l}{s} N_m^{(1)}.
\]

In deriving the first line, Eqs. (15) and (38) are applied. In deriving the second line, Eq. (6) is applied. In deriving the third line, Eq. (37) is applied.

### APPENDIX C: DERIVATION OF EQ. (42)

\[
\Gamma \doteq N_M \left[ \frac{1}{2} n_f^{(1)} + \frac{1}{2} N_f^{(2)} \right] + N_m^{a} [N_f^{(1)} - N_f^{(2)}] + N_f^{(1)} + N_f^{(2)}
\]

\[
= N_M \left[ \frac{1}{2} N_f^{(1)} + \frac{1}{2} N_f^{(2)} \right] + N_m^{a} [N_f^{(1)} - N_f^{(2)}] + N_f^{(1)} - N_f^{(2)}
\]

\[
= N_M N_f^{(1)} + (N_m^{a} + 1) N_f^{(2)}
\]

\[
= N_M N_f^{(1)} + (N_m^{a} + 1) N_f^{(2)} + N_m^{a} [N_f^{(1)} - N_f^{(2)}]
\]

\[
= N_M N_f^{(1)} + (N_m^{a} + 1) N_f^{(2)}
\]

\[
= N_M N_f^{(1)} + (2 N_m^{a} + 1) \frac{l}{s} N_M
\]

\[
= N_M N_f^{(1)} + (2 N_m^{a} + 1) \frac{l}{s} N_M.
\]

In deriving the second line, an even integer \( -2 N_f^{(2)} \) is added. In deriving the fifth line, Eqs. (37) and (38) are applied. In deriving the last line, Eq. (6) is applied.

### APPENDIX D: DERIVATION OF EQ. (59)

\[
N_E = N_A - \frac{l}{s} N_m^{a}
\]

\[
= \alpha \left[ n_f^{(1)} + \frac{1}{2} N_m^{a} \right] + (1 - \alpha) \left[ n_f^{(2)} + \frac{1}{2} [N_M - N_m^{(1)}] \right]
\]

\[
- \left( \frac{l}{s} + \frac{1}{2} \right) [N_m^{(1)} - \alpha N_M]
\]

\[
= \alpha \left[ n_f^{(2)} + \left( \frac{l}{s} + 1 \right) N_M - N_m^{(1)} + \frac{1}{2} N_m^{(1)} \right]
\]

\[
+ (1 - \alpha) \left[ n_f^{(2)} + \frac{1}{2} [N_M - N_m^{(1)}] \right]
\]

\[
- \left( \frac{l}{s} + \frac{1}{2} \right) [N_m^{(1)} - \alpha N_M]
\]

\[
= \left( \frac{l}{s} + 1 \right) N_m^{(1)} + n_f^{(2)} + \left( 2 \alpha \frac{l}{s} + \alpha + \frac{1}{2} \right) N_M.
\]

In deriving the third line, Eq. (56) is applied.

### APPENDIX E: DERIVATION OF EQ. (63)

\[
\Gamma \doteq \frac{1}{2} N_M \left[ n_f^{(1)} + n_f^{(2)} \right] - N_m^{a} [n_f^{(1)} - n_f^{(2)}] + n_f^{(1)} + n_f^{(2)}
\]

\[
\doteq \frac{1}{2} N_M \left[ n_f^{(1)} + n_f^{(2)} \right] - N_m^{a} [n_f^{(1)} - n_f^{(2)}] + n_f^{(1)} - n_f^{(2)}
\]

\[
= \left( \frac{l}{s} + 1 \right) N_m^{(1)} + N_m^{a} n_f^{(2)}
\]

\[
= \left( \frac{l}{s} + 1 \right) N_m^{(1)} + N_m^{(1)} + N_m^{a} n_f^{(2)}
\]

\[
= \left( \frac{l}{s} + 1 \right) N_m^{(1)} + N_m^{a} n_f^{(2)}.
\]

In deriving the first line, Eq. (27) is applied. In deriving the second line, an even integer \( -2 n_f^{(2)} \) is added. In deriving the fourth line, the first formula in Eq. (6) is applied with \( \alpha = 1/2 \). In deriving the fifth line, Eq. (56) is applied. In deriving the last line, the even integer \( -N_m^{(1)} [N_m^{(1)} + 1] \) is removed.

### APPENDIX F: DERIVATION OF EQ. (64)

\[
\Gamma \doteq N_M \left[ N_m^{(1)} + 1 \right] \left( \frac{l}{s} + 1 \right) + N_m^{a} n_f^{(2)}
\]

\[
\doteq -N_M \left[ N_m^{(1)} + 1 \right] \left( \frac{l}{s} + 1 \right) + N_m^{a} n_f^{(2)}
\]
\[ N_M \left[ -\left( \frac{l}{s} + 1 \right) N_m^{(1)} + n_f^{(2)} \right] - \left( \frac{l}{s} + 1 \right) \]
\[ = N_M \left[ N_E - \left( \frac{l}{s} + 1 \right) N_M - \left( \frac{l}{s} + 1 \right) \right] \]
\[ = N_M \left[ N_E - \left( \frac{l}{s} + 1 \right) \right] \]
\[ = N_M N_E, \quad (F1) \]

where an even integer \(-2N_M(N_m^{(1)} + 1)(\frac{l}{s} + 1)\) is added in the second line. In the fourth line, Eq. (62) is applied. In the last line, \(N_M(N_M + 1)(\frac{l}{s} + 1)\) is removed since it is always even.

**APPENDIX G: DERIVATION OF EQ. (69)**

\[ \Gamma = N_M \left[ N_m^{(1)} + 1 \right] \left( \frac{l}{s} + 1 \right) + N_M n_f^{(2)} \]
\[ = N_M \left[ -\left( \frac{l}{s} + 1 \right) N_m^{(1)} + n_f^{(2)} \right] - \left( \frac{l}{s} + 1 \right) \]
\[ = N_M \left[ N_E - \left( \frac{l}{s} + 1 \right) N_M - \left( \frac{l}{s} + 1 \right) \right] \]
\[ = N_M N_E - \left( \frac{l}{s} + 1 \right) N_M(N_M + 1) \]
\[ = N_M N_E - \frac{l}{s} N_M(N_M + 1) \]
\[ = N_M N_E - \frac{l}{s} N_M(N_M - 1). \quad (G1) \]

where an even integer \(-2N_M(N_m^{(1)} + 1)(\frac{l}{s} + 1) = -2qk(N_m^{(1)} + 1)(\frac{l}{s} + 1) = -2k(N_m^{(1)} + 1)(p + q)\) is added in the second line. In deriving the fourth line, Eq. (62) is applied. In deriving the sixth line, the even integer \(-N_M(N_M + 1) = -qk(qk + 1)\) is removed. In deriving the last line, an even integer \(2\frac{l}{s}N_M = 2(qk + p)k\) is added.

**APPENDIX H: DERIVATION OF EQ. (72)**

\[ \Gamma = \left[ N_m^{(1)} + 1 \right] n_f^{(1)} + \left[ N_m^{(2)} + 1 \right] n_f^{(2)} \]
\[ = \left[ N_m^{(1)} + 1 \right] \left[ \frac{l}{s} N_M + n_f^{(2)} \right] + \left[ N_M - N_m^{(1)} + 1 \right] n_f^{(2)} \]
\[ = N_M \left[ \frac{l}{s} N_m^{(1)} + \frac{l}{s} + n_f^{(2)} \right] \]
\[ = N_M \left[ -\frac{l}{s} N_m^{(1)} + \frac{l}{s} + n_f^{(2)} \right] \]
\[ = N_M \left( N_E - 2\frac{l}{s} N_M + \frac{l}{s} \right). \quad (H1) \]

In deriving the second line, Eqs. (37) and (7) are applied. In deriving the third line, the even integer \(2n_f^{(2)}\) is removed. In deriving the fourth line, an even integer \(-2N_M\frac{l}{s}N_m^{(1)}\) is added. In deriving the last line, Eq. (39) is applied.

**APPENDIX I: DERIVATION OF EQ. (79)**

\[ \Gamma = \left[ N_m^{(1)} + 1 \right] n_f^{(1)} + \left[ N_m^{(2)} + 1 \right] n_f^{(2)} \]
\[ = \left[ N_m^{(1)} + 1 \right] \left[ \frac{l}{s} N_M - N_m^{(1)} + n_f^{(2)} \right] \]
\[ + \left[ N_M - N_m^{(1)} + 1 \right] n_f^{(2)} \]
\[ = N_M \left[ \frac{l}{s} + 1 \right] \left[ N_m^{(1)} + 1 \right] + n_f^{(2)} \]
\[ = N_M \left[ \left( \frac{l}{s} + 1 \right) \left[ N_m^{(1)} + 1 \right] + n_f^{(2)} \right] \]
\[ = N_M \left[ \left( \frac{l}{s} + 1 \right) N_m^{(1)} + 1 + n_f^{(2)} \right] \]
\[ = N_M \left[ N_E - 2\alpha \frac{l}{s} + \alpha + \frac{1}{2} \right] N_M - \left( \frac{l}{s} + 1 \right). \quad (I1) \]

In deriving the second line, Eqs. (56) and (7) are applied. In deriving the third line, the even integers \(2n_f^{(2)}\) and \(-[N_m^{(1)} + 1]N_m^{(1)}\) are removed. In deriving the fourth line, an even integer \(-2N_M\frac{l}{s}N_m^{(1)}\) is added. In deriving the last line, Eq. (59) is applied.