Symmetry-protected topological invariants of symmetry-protected topological phases of interacting bosons and fermions

The MIT Faculty has made this article openly available. Please share how this access benefits you. Your story matters.

<table>
<thead>
<tr>
<th>Citation</th>
<th>Wen, Xiao-Gang. “Symmetry-Protected Topological Invariants of Symmetry-Protected Topological Phases of Interacting Bosons and Fermions.” Phys. Rev. B 89, no. 3 (January 2014). © 2014 American Physical Society</th>
</tr>
</thead>
<tbody>
<tr>
<td>As Published</td>
<td><a href="http://dx.doi.org/10.1103/PhysRevB.89.035147">http://dx.doi.org/10.1103/PhysRevB.89.035147</a></td>
</tr>
<tr>
<td>Publisher</td>
<td>American Physical Society</td>
</tr>
<tr>
<td>Version</td>
<td>Final published version</td>
</tr>
<tr>
<td>Accessed</td>
<td>Mon Dec 17 22:02:36 EST 2018</td>
</tr>
<tr>
<td>Citable Link</td>
<td><a href="http://hdl.handle.net/1721.1/88947">http://hdl.handle.net/1721.1/88947</a></td>
</tr>
<tr>
<td>Terms of Use</td>
<td>Article is made available in accordance with the publisher’s policy and may be subject to US copyright law. Please refer to the publisher’s site for terms of use.</td>
</tr>
<tr>
<td>Detailed Terms</td>
<td></td>
</tr>
</tbody>
</table>
Symmetry-protected topological invariants of symmetry-protected topological phases of interacting bosons and fermions

Xiao-Gang Wen
Perimeter Institute for Theoretical Physics, Waterloo, Ontario, Canada N2L 2Y5
and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

(Received 30 April 2013; revised manuscript received 18 November 2013; published 31 January 2014)

Recently, it was realized that quantum states of matter can be classified as long-range entangled states (i.e., with nontrivial topological order) and short-range entangled states (i.e., with trivial topological order). We can use group cohomology class $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ to systematically describe the SRE states with a symmetry $SG$ [referred as symmetry-protected trivial (SPT) or symmetry-protected topological (SPT) states] in $d$-dimensional space-time. In this paper, we study the physical properties of those SPT states, such as the fractionalization of the quantum numbers of the global symmetry on some designed point defects and the appearance of fractionalized SPT states on some designed defect lines/membranes. Those physical properties are SPT invariants of the SPT states which allow us to experimentally or numerically detect those SPT states, i.e., to measure the elements in $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ that label different SPT states. For example, 2 + 1-dimensional bosonic SPT states with $Z_n$ symmetry are classified by a $Z_n$ integer $m \in \mathcal{H}^2(Z_n, \mathbb{R}/\mathbb{Z}) = Z_n$. We find that $n$ identical monodromy defects, in a $Z_n$ SPT state labeled by $m$, carry a total $Z_n$ charge $2m$ (which is not a multiple of $n$ in general).

DOI: 10.1103/PhysRevB.89.035147

PACS number(s): 71.27.+a, 02.40.Re

I. INTRODUCTION

A. Beyond symmetry breaking and quantum entanglement

Landau symmetry-breaking theory [1–3] was regarded as the standard theory to describe all phases and phase transitions. However, in 1989, through a theoretical study of chiral spin liquid [4,5] in connection with high $T_c$ superconductivity, we realized that there exists a new kind of order: topological order [6–8]. Topological order cannot be characterized by the local order parameters associated with the symmetry breaking. Instead, it is characterized/defined by (a) the robust ground-state degeneracy that depends on the spatial topologies [6,7] and (b) the modular representation of the degenerate ground states [8,9], just like superfluid order is characterized/defined by zero viscosity and quantized vorticity. In some sense, the robust ground-state degeneracy and the modular representation of the degenerate ground states can be viewed as a type of “topological order parameters” for topologically ordered states. Those “topological order parameters” are also referred as topological invariants of topological order.

We know that, microscopically, superfluid order originates from boson or fermion-pair condensation. Then, what is the microscopic origin of topological order? Recently, it was found that, microscopically, topological order is related to long-range entanglement [10,11]. In fact, we can regard topological order as pattern of long-range entanglement [12] defined through local unitary (LU) transformations [13–15]. The notion of topological orders and quantum entanglement leads to a point of view of quantum phases and quantum phase transitions (see Fig. 1) [12]: For gapped quantum systems without any symmetry, their quantum phases can be divided into two classes: short-range entangled (SRE) states and long-range entangled (LRE) states.

SRE states are states that can be transformed into direct product states via LU transformations. All SRE states can be transformed into each other via LU transformations, and thus all SRE states belong to the same phase (see Fig. 1(a)). LRE states are states that cannot be transformed into direct product states via LU transformations. There are LRE states that cannot be connected to each other through LU transformations. Those LRE states represent different quantum phases, which are nothing but the topologically ordered phases. Chiral spin liquids [4,5], fractional quantum Hall states [16,17], $Z_2$ spin liquids [18–20], non-Abelian fractional quantum Hall states [21–24], etc., are examples of topologically ordered phases.

Topological order and long-range entanglement, as truly new phenomena, even require new mathematical language to describe them. It appears that tensor category theory [12,13,25,26] and simple current algebra [21,27] may be part of the new mathematical language. Using the new language, we have developed a systematic and quantitative theory for nonchiral topological orders in two-dimensional (2D) interacting boson and fermion systems [12,13,26]. Also for chiral 2D topological orders with only Abelian statistics, we find that we can use integer $K$ matrices to classify them [28–33].

B. Short-range entangled states with symmetry

For gapped quantum systems with symmetry, the structure of phase diagram is much richer [see Fig. 1(b)]. Even SRE states now can belong to different phases, which include the well-known Landau symmetry-breaking states. However, even SRE states that do not break any symmetry can belong to different phases, despite that they all have trivial topological order and vanishing symmetry-breaking order parameters. The 1D Haldane phase for spin-1 chain [34–37] and topological insulators [38–43] are nontrivial examples of SRE phases that do not break any symmetry. We refer this kind of phases as symmetry-protected trivial (SPT) phases or symmetry-protected topological (SPT) phases [36,37]. Note that the SPT phases have no long-range entanglement and have trivial topological orders.

It turns out that there is no gapped bosonic LRE state in 1 + 1D (i.e., topological order does not exist in 1 + 1D) [14]. So all 1D gapped bosonic states are either symmetry-breaking states...
The gauge approach can also be applied to fermion systems: we would like to point out that the gauge approach can also be applied to fermion systems: gauging the symmetry group \( H = (SG, GG) \) allows us to fully characterize the SPT phases. This realization led to a complete classification of all 1 + 1D gapped bosonic quantum phases [44–46].

In Refs. [47–49], the classification of 1 + 1D SPT phases is generalized to any dimensions.

For gapped bosonic systems in \( d \) space-time dimensions with an on-site symmetry group \( SG \), the SPT phases that do not break the symmetry are described by the elements in \( \mathcal{H}^d(SG, \mathbb{R}/\mathbb{Z}) \), the group cohomology class of \( SG \).

Such a systematic understanding of SPT states was obtained by thinking of those states as “trivial” SRE states rather than topologically ordered states. The group cohomology theory predicted several new bosonic topological insulators and bosonic topological superconductors, as well as many other new quantum phases with different symmetries and in different dimensions. This led to an intense research activity on SPT states [50–72].

What are the “topological order parameters” or more precisely SPT invariants that can be used to characterize SPT states? One way to characterize SPT states is to gauge the on-site symmetry and use the introduced gauge field as an effective probe for the SPT order [73]. This is the main theme of this paper. After we integrate out the matter fields, a nontrivial SPT phase will lead to a nontrivial quantized gauge topological term [54]. So one can use the induced gauge topological terms, such as the “topological order parameters” or SPT invariants, to characterize the SPT phases. It turns out that the quantized gauge topological terms for gauge group \( SG \) is also classified by the same group cohomology class \( \mathcal{H}^d(SG, \mathbb{R}/\mathbb{Z}) \). Thus, the gauge probe allows us to fully characterize the SPT phases. We use the structure of \( \mathcal{H}^d(SG, \mathbb{R}/\mathbb{Z}) \) as a guide to help us to construct the SPT invariants for the SPT states. Another general way to obtain SPT invariants is to study boundary states, which is effective for both topological order [74–76] and SPT order [44,56].

We would like to point out that the gauge approach can also be applied to fermion systems:

We can use the elements in \( \mathcal{H}^d(G, \mathbb{R}/\mathbb{Z}) \) to characterize fermionic SPT states [77] in \( d \) space-time dimensions with a full symmetry group \( G \) (see Sec. III D 1).

However, it is not clear if every element in \( \mathcal{H}^d(G, \mathbb{R}/\mathbb{Z}) \) can be realized by fermionic systems or not. It is also possible that two different elements in \( \mathcal{H}^d(G, \mathbb{R}/\mathbb{Z}) \) may correspond to the same fermionic SPT state. Despite the incomplete result, we can still use \( \mathcal{H}^d(G, \mathbb{R}/\mathbb{Z}) \) to guide us to construct the SPT invariants for fermionic SPT states.

### C. Long-range entangled states with symmetry

For gapped LRE states with symmetry, the possible quantum phases should be even richer than SRE states. We may call those phases symmetry enriched topological (SET) phases. At moment, we do not have a classification or a systematic description of SET phases. However, we have some partial results.

Projective symmetry group (PSG) was introduced in 2002 to study the SET phases [78–80]. The PSG describes how the quantum numbers of the symmetry group \( SG \) get fractionalized on the gauge excitations [79]. When the gauge group \( GG \) is Abelian, the PSG description of the SET phases can be expressed in terms of group cohomology: The different SET states with symmetry \( SG \) and gauge group \( GG \) can be (partially) described by a subset of \( \mathcal{H}^d(SG, GG) \) [81].

One class of SET states in \( d \) space-time dimensions with global symmetry \( SG \) is described by weak-coupling gauge theories with gauge group \( GG \) and quantized topological terms (assuming the weak-coupling gauge theories are gapped, which can happen when the space-time dimension \( d = 3 \) or when \( d > 3 \) and the gauge group \( GG \) is finite). Those SET states (i.e., the quantized topological terms) are described by the elements in \( \mathcal{H}^d(PSG, \mathbb{R}/\mathbb{Z}) \) [59,82], where the group PSG is an extension of \( SG \) by \( GG \): \( SG \rightarrow PSG/\rightarrow GG \). Or, in other words, we have a short exact sequence,

\[
1 \rightarrow GG \rightarrow PSG \rightarrow SG \rightarrow 1.
\]

We denote PSG as \( PSG = GG \times SG \). Many examples of the SET states can be found in Refs. [50,78,83–85].

Although we have a systematic understanding of SPT phases and some of the SET phases in terms of \( \mathcal{H}^d(SG, \mathbb{R}/\mathbb{Z}) \) and \( \mathcal{H}^d(PSG, \mathbb{R}/\mathbb{Z}) \), those constructions do not tell us how to experimentally or numerically measure the elements in \( \mathcal{H}^d(SG, \mathbb{R}/\mathbb{Z}) \) or \( \mathcal{H}^d(PSG, \mathbb{R}/\mathbb{Z}) \) that label the different SPT or SET phases. We do not know, even given an exact ground-state wave function, how to determine which SPT or SET phase the ground state belongs to.

In this paper, we address this important question. We find physical ways to the detect different SPT/SET phases and to measure the elements in \( \mathcal{H}^d(SG, \mathbb{R}/\mathbb{Z}) \) or \( \mathcal{H}^d(PSG, \mathbb{R}/\mathbb{Z}) \). This is achieved by gauging the symmetry group \( SG \) (i.e., coupling the \( SG \) quantum numbers to a \( SG \) gauge potential \( A^{SG} \)). Note that \( A^{SG} \) is treated as a nonfluctuating probe field. By studying the topological response of the system to various \( SG \) gauge configurations, we can measure the elements in \( \mathcal{H}^d(SG, \mathbb{R}/\mathbb{Z}) \) or \( \mathcal{H}^d(PSG, \mathbb{R}/\mathbb{Z}) \). Those topological responses are the measurable SPT invariants (or “topological order parameters”) that characterize the SPT/SET phases. Tables I and II and the SPT invariant statements in the paper describe the many constructed SPT invariants and represent the main results of the paper.
<table>
<thead>
<tr>
<th>Symmetry Dimension</th>
<th>Labels</th>
<th>Symmetry-protected topological invariants</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_2$</td>
<td>$m \in \mathcal{H}^1(Z_2) = Z_2$</td>
<td>$n$ identical monodromy defects have a total $Z_2$ charge $2m$.</td>
</tr>
<tr>
<td>$U(1)$</td>
<td>$m \in \mathcal{H}^1(U(1)) = \mathbb{Z}$</td>
<td>A monodromy defect has a statistics $\theta = 2\pi(\frac{m}{2} + \frac{\text{mod}, m}{2})$ [59,73,82].</td>
</tr>
<tr>
<td></td>
<td>$m_1 \in \mathcal{H}^1(Z_4) \otimes \mathcal{H}^1(Z_2) = Z_{(1,2)}$</td>
<td>The degenerate states at an end of chain form the $m_1$th projective representation of $Z_{n_1} \times Z_{n_2}$. The $Z_{n_1}$ flux through the 1D circular space induces a $Z_{n_2}$ charge $m + {n_1,n_2} \times \text{integer}$.</td>
</tr>
<tr>
<td>$Z_{n_1} \times Z_{n_2}$</td>
<td>$m_0 \in \mathcal{H}^1(Z_{n_2}) = Z_{n_2}$</td>
<td>The statistics of $Z_{n_1}$ monodromy defects: $\theta_1 \equiv \frac{2\pi m_0}{n_1}$ [59,73,82]. The statistics of $Z_{n_2}$ monodromy defects: $\theta_2 \equiv \frac{2\pi m_0}{n_2}$. The mutual statistics between $Z_{n_1}$ and $Z_{n_2}$ monodromy defects: $\theta_2 \equiv \frac{2\pi m_0}{n_2}$.</td>
</tr>
<tr>
<td></td>
<td>$m_2 \in \mathcal{H}^1(Z_{n_1}) \otimes \mathcal{H}^1(Z_{n_2}) = Z_{(1,2)}$</td>
<td>$n_1$ identical $Z_{n_2}$ monodromy defects carry $2m_2 Z_{n_2}$ charges and $m_2 + {n_1,n_2} \times \text{integer} Z_{n_2}$ charges. $n_2$ identical $Z_{n_2}$ monodromy defects carry $2m_2 Z_{n_2}$ charges and $m_2 + {n_1,n_2} \times \text{integer} Z_{n_1}$ charges.</td>
</tr>
<tr>
<td></td>
<td>$m_3 \in \mathcal{H}^1(Z_{n_1}) \otimes \mathcal{H}^1(Z_{n_2}) = Z_{(1,2)}$</td>
<td>A $Z_{n_1}$ monodromy line defect will carry gapless/degnerate edge states of the $m_3$th $2 + 1$D bosonic $Z_{n_2}$ SPT states. A $Z_{n_2}$ monodromy line defect will carry gapless/degnerate edge states of the $m_3$th $2 + 1$D bosonic $Z_{n_2}$ SPT states.</td>
</tr>
<tr>
<td>$U(1) \times Z_2$</td>
<td>$m_0 \in \mathcal{H}^1(U(1)) = \mathbb{Z}$</td>
<td>A $\pi$-flux line of the $U(1)$ will carry gapless/degnerate edge states of the $2 + 1$D bosonic $Z_2$ SPT state, if $m_1 = 1$.</td>
</tr>
<tr>
<td></td>
<td>$m_3 \in \mathcal{H}^1(Z_2) = Z_2$</td>
<td>Same as the $U(1)$ or the $Z_2$ SPT states in 2 + 1D.</td>
</tr>
<tr>
<td></td>
<td>$m_2 \in \mathcal{H}^1(Z_2) \otimes \mathcal{H}^1(U(1)) = Z_2$</td>
<td>A $\pi$-flux line of the $U(1)$ will carry gapless/degnerate edge states of the $2 + 1$D bosonic $Z_2$ SPT state, if $m_1 = 1$.</td>
</tr>
<tr>
<td>$U(1) \times Z_2^T$</td>
<td>$m_2 \in \mathcal{H}^1(Z_2) \otimes \mathcal{H}^1(U(1)) = Z_2$</td>
<td>A $\pi$-flux line of the $U(1)$ will carry gapless/degnerate edge states of the $2 + 1$D bosonic $Z_2$ SPT state, if $m_1 = 1$.</td>
</tr>
<tr>
<td></td>
<td>$m_4 \in \mathcal{H}^1(Z_2) \otimes \mathcal{H}^1(U(1)) = Z_2$</td>
<td>A $\pi$-flux line of the $U(1)$ will carry gapless/degnerate edge states of the $2 + 1$D bosonic $Z_2$ SPT state, if $m_1 = 1$.</td>
</tr>
<tr>
<td>$U(1) \times Z_2^T$</td>
<td>$m_2 \in \mathcal{H}^1(Z_2) \otimes \mathcal{H}^1(U(1)) = Z_2$</td>
<td>A $\pi$-flux line of the $U(1)$ will carry gapless/degnerate edge states of the $2 + 1$D bosonic $Z_2$ SPT state, if $m_1 = 1$.</td>
</tr>
<tr>
<td></td>
<td>$m_4 \in \mathcal{H}^1(Z_2) \otimes \mathcal{H}^1(U(1)) = Z_2$</td>
<td>A $\pi$-flux line of the $U(1)$ will carry gapless/degnerate edge states of the $2 + 1$D bosonic $Z_2$ SPT state, if $m_1 = 1$.</td>
</tr>
</tbody>
</table>

**TABLE I.** Symmetry-protected topological invariants for various bosonic SPT states. Here the flux, the monodromy defects and the monopoles are always the minimal ones. \(\{n_1,n_2\}\) is the smallest common multiple of \(n_1,n_2\), \(\{n_1,n_2\}\) is the largest common divisor of \(n_1,n_2\). Also, \(\mathcal{H}^1(G) \equiv \mathcal{H}^1(G,R/\mathbb{Z})\). The previously known results are indicated by the references.
II. SPT INVARIANTS OF SPT STATES: A GENERAL DISCUSSION

Because of the duality relation between the SPT states and the SET states described by weak-coupling gauge theories [59,73,82] (see Appendix E), in this paper, we mainly discuss the physical properties and the SPT invariants of the SPT state. The physical properties and the topological invariants of the SET states can be obtained from the physical properties and the SPT invariants of corresponding SPT states via the duality relation.

A. Gauge the symmetry and twist the space

Let us consider a system with symmetry group $G$ in $d$ space-time dimensions. The ground state of the system is a SPT state described by an element $\nu_d$ in $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$. However, how do we physically measure $\nu_d$? Following the idea of Ref. [73], we propose to measure $\nu_d$ by “gauging” the symmetry group $G$, i.e., by introducing a gauge potential $A_\mu(x')$ to couple to the quantum numbers of $G$. The gauge potential $A_\mu$ is a fixed probe field, not a dynamical field. We would like to consider how the system responds to various gauge configurations described by $A_\mu$. We show that the topological responses allow us to fully measure the cocycle $\nu_d$ that characterizes the SPT phase, at least for the many cases considered. Those topological responses are the SPT invariants that we are looking for.

There are several topological responses that we can use to construct SPT invariants.

1. We set up a time-independent $G$ gauge configuration $A_\mu(x')$. If the gauge configuration is invariant under a subgroup $GG$ of $G$, $A_\mu(x') = h^{-1}A_\mu(x')h$, $h \in GG$, then we can study the conserved $GG$ quantum number of the ground state under such a gauge configuration. Sometimes, the ground states may be degenerate and form a higher dimensional representation of $GG$.

In particular, the time-independent $G$ gauge configuration may be chosen to be a monopole-like or other soliton-like gauge configuration. The quantum number of the unbroken symmetry carried by those defects can be SPT invariants of the SPT states.

We can also remove $n$ identical regions $D(i), i = 1, \ldots, n$, from the space $M_{d-1}$ to get a $(d-1)$-dimensional manifold $M'_{d-1}$ with $n$ “holes.” Then we consider a flat $G$ gauge configuration $A_\mu(x')$ on $M'_{d-1}$ such that the gauge fields near the boundary of those holes, $\partial D(i)$, are identical. We then measure the conserved $GG$ quantum number on the ground state for such $G$ gauge configuration. We that the $GG$ quantum number may not be multiples of $n$, indicating a nontrivial SPT phase.

2. We may choose the space to have a form $M_k \times M_{d-k-1}$ where $M_k$ is a closed $k$-dimensional manifold or a closed $k$-dimensional manifold with $n$ identical holes. $M_{d-k-1}$ is a closed $(d-k-1)$-dimensional manifold. We then put a $G$ gauge configuration $A_\mu(x')$ on $M_k$, or a flat $G$ gauge configuration on $M_k$ if $M_k$ has $n$ holes. In the large $M_{d-k-1}$ limit, our system can be viewed as a system in $(d-k-1)$-dimensional space with a symmetry $GG$, where $GG \subset G$ is formed by the symmetry transformations that leave the $G$ gauge configuration invariant. The ground state of the system is a SPT state characterized by cocycles in $\mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})$. The mapping from the gauge configurations on $M_k$ to $\mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})$ is our SPT invariant.

3. We can have a family of $G$ gauge configurations $A_\mu(x')$ that have the same energy. As we go around a loop in such a family of $G$ gauge configurations, the corresponding ground states will generate a geometric phase (or non-Abelian geometric phases if the ground states are degenerate). Sometimes, the (non-Abelian) geometric phases are also SPT invariants which allow us to probe and measure the cocycles. One such type of the SPT invariants is the statistics of the $G$ gauge vortices in $2+1D$ or monopoles in $3+1D$.

4. The above topological responses can be measured in a Hamiltonian formulation of the system. In the imaginary-time path-integral formulation of the system where the space-time manifold $M_d$ can have an arbitrary topology, we can have a most general construction of SPT invariants. We simply put a nearly flat $G$ gauge configuration on a closed space-time manifold $M_d$ and evaluate the path integral. We obtain a partition function $Z(M_d, A_\mu)$ which is a function of the space-time topology $M_d$ and the nearly flat gauge configuration $A_\mu$. In the limit of the large volume $V = \lambda^d V_0$ of the space-time (i.e., $\lambda \to \infty$), $Z(M_d, A_\mu)$ has a form (assuming we only scale the space-time volume without any change in shape)

$$Z(M_d, A_\mu) \propto e^{-\sum_{i=1}^n f_i \pi_{xy} Z_{\text{top}}(M_d, A_{\mu})}, \tag{2}$$

where $Z_{\text{top}}(M_d, A_{\mu})$ is independent of the scaling factor $\lambda$. $Z_{\text{top}}(M_d, A_{\mu})$ is a SPT invariant that allows us to fully measure the elements in $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ that describe the SPT phases [54,86,87]. In fact, $Z_{\text{top}}(M_d, A_{\mu})$ is the partition function for the pure topological term $W_{\text{top}}^\mu(g, A)$ in Eq. (E3).

We would like to point out that if $Z_{\text{top}}(M_d, A_{\mu})$ contains a Chern-Simons term (i.e., $Z_{\text{top}}(M_d, A_{\mu}) = e^{i2\pi \kappa_{xy}}$), then it describes a SPT phase that is labeled by an element in the free part of $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$. If $Z_{\text{top}}(M_d, A_{\mu})$ is a topological term whose value is independent of any small perturbations of $A_{\mu}$, then it describes a SPT phase that is labeled by an element in the torsion part of $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ [54].
B. Cup product, K"unneth formula, and SPT invariants

The cohomology class $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z}) \cong \mathcal{H}^{d+1}(G, \mathbb{Z})$ is not only an Abelian group. The direct sum of $\mathcal{H}^d(G, \mathbb{Z})$, $\mathcal{H}^d(G, \mathbb{Z}) = \oplus \mathcal{H}^d(G, \mathbb{Z})$, also has a cup product that makes $\mathcal{H}^d(G, \mathbb{Z})$ into a ring:

$$\nu_{d_1} \cup \nu_{d_2} = \nu_{d_1+d_2}, \quad \nu_{d_1+d_2} \in \mathcal{H}^{d_1+d_2}(G, \mathbb{Z}),$$
$$\nu_{d_1} \in \mathcal{H}^{d_1}(G, \mathbb{Z}), \quad \nu_{d_1} \in \mathcal{H}^{d_1}(G, \mathbb{Z}).$$

We also have the K"unneth formula,

$$\mathcal{H}^d(GG \times SG, \mathbb{R}/\mathbb{Z}) = \bigoplus_{k=0}^d \mathcal{H}^k[S\mathbb{G}, \mathcal{H}^{d-k}(SG, \mathbb{R}/\mathbb{Z})] = \bigoplus_{k=0}^d \mathcal{H}^k(GG, \mathcal{H}^{d-k}(SG, \mathbb{R}/\mathbb{Z})) = \bigoplus_{k=0}^d \mathcal{H}^{d-k}(SG, \mathbb{R}/\mathbb{Z}).$$

[see Eqs. (C15) and (C17)]. Both of the above two results relate cocycles at higher dimensions to cocycles at lower dimensions. The structures of the cup product and K"unneth formula give us some quite direct hints on how to construct SPT invariants that probe the cocycles. For example, consider a SPT state with symmetry $GG \times SG$, which is labeled by an element in $\mathcal{H}^d(GG \times SG, \mathbb{R}/\mathbb{Z})$. If such an element belong to $\mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{H}^{d-k}(SG, \mathbb{R}/\mathbb{Z})$ or $\mathcal{H}^d(GG, \mathbb{R}/\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{H}^{d-k}(SG, \mathbb{R}/\mathbb{Z})$, then we can choose the space-time to have a topology $M_1 \times M_{d-k}$. Next we try to design a defect that couple to $\mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})$, or $\mathcal{H}^d(GG, \mathbb{R}/\mathbb{Z})$ on $M_1$ and try to measure the response described by $\mathcal{H}^{d-k}(SG, \mathbb{R}/\mathbb{Z})$ or $M_{d-k}$. [See also SPT invariant (Sec. IV D 1) and (Sec. IV E 2).] For simple examples, see Secs. IV D 1 and IV E 2.

In this paper, we review some known SPT invariants, such as Hall conductance and defect statistics, for some simple SPT states, such as $2+1D$ $U(1) \times U(1)$, $U(1) \times Z_n$, and $Z_m \times Z_n$ SPT states [50,51,59,71–73] and the $3+1D$ $U(1) \times Z^{2}_f$ SPT state [70]. We also introduce some additional SPT invariants, such as total quantum number of the identical monodromy defects that can be created on a closed space and the dimension reduction of SPT states, for those simple SPT states. We compare those SPT invariants to the structures of the cup product formula. This comparison helps us to understand the relation between the cup product formula and the SPT invariants.

We discuss SPT invariants in many examples of SPT states, starting from simple ones. Each example offers a few new features than the previous example. We hope that, through those examples, we will build some intuitions for constructing SPT invariants for general SPT states. Such intuition and understanding, in turn, help us to construct new SPT invariants for more complicated SPT states [see SPT invariants, Secs. IV D 1 and IV E 2]. The new understanding allows us to construct SPT invariants for more general SPT states in $1+1D$, $2+1D$, and $3+1D$. The main results are summarized in Table I.

III. SPT INVARIANTS OF SPT STATES WITH SIMPEL SYMMETRY GROUPS

A. Bosonic $Z_n$ SPT phases

1. $0+1D$

In 1D space-time, the bosonic SPT states with symmetry $Z_n = \{g^n\} = e^{2\pi i/n}[k = 0, \ldots, n-1]$ are described by the cocycles in $\mathcal{H}^1(Z_n, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$. How do we measure the cocycles in $\mathcal{H}^1(Z_n, \mathbb{R}/\mathbb{Z})$? What are the measurable SPT invariants that allow us to characterize the $Z_n$ SPT states?

One way to construct a SPT invariant is to gauge the $Z_n$ global symmetry in the action that describes that SPT state and obtain a $Z_n$-gauge theory $L(g_i, h_{ij})$, where $h_{ij} \in Z_n$ is the $Z_n$-gauge “connection” on the link connecting vertices $i$ and $j$, and $g_i \in Z_n$ is the “matter” field that describes the SPT state (if we set $h_{ij} = 1$). Due to the gauge invariance, $L(g_i, h_{ij})$ has a form $L(g_i, h_{ij}) = L(g_i^{-1} h_{ij} g_j)$ [see Eq. (E9)].

After integrating out the “matter” fields $g_i$, we obtain a SPT invariant which appears as a topological term in the $Z_n$-gauge theory $Z_{top}(M_d, A_\mu) = Z_{top}(M_d, h_{ij})$. (Note that, in a $Z_n$ gauge theory, $h_{ij}$ is the gauge “connection” $A_\mu$.) The $Z_n$-gauge topological term can be expressed in term of cocycles $\omega_{ij}$,

$$Z_{top}(S_1, A_\mu) = e^{i 2\pi \sum_{\{i\}} \omega_{ij} h_{ij+1}^{i+1}},$$

where we have assumed that the space-time is a circle $S_1$ formed by a ring of vertices labeled by $i$.

In fact, before we integrate out that matter field $g_i$, the partition function for an ideal fixed-point SPT Lagrangian is given by [see Eq. (E9)]

$$Z(S_1, A_\mu) = \sum_{\{g\}} e^{i 2\pi \sum_{\{i\}} \omega_{ij} (g_{i+1}^{i+1} h_{ij+1}^{i+1})},$$

where $\sum_{\{g\}}$ sums over all the $g_i$ configurations on $S_1$. Since $e^{i 2\pi \sum_{\{i\}} \omega_{ij} (g_{i+1}^{i+1} h_{ij+1}^{i+1})}$ is independent of $\{g\}$, we can integrate out $g_i$ easily and obtain Eq. (5).

A $Z_n$-gauge configuration on $S_1$ is given by $Z_n$ group elements $h_{i+1}^{i+1}$ on each link $(i, i + 1)$. We may view the cocycle $\omega_{ij}$ as a “discrete differential form” and use the differential form notion to express the above topological action amplitude (which is also a $Z_n$-gauge topological term):

$$Z_{top}(S_1, A_\mu) = e^{i 2\pi \int_{S_1} \omega_{ij} h^{i+1}_{ij+1}} = 1$$

if $h_{i+1}^{i+1} = g^{i+1} g^{i-1}_i$ is a pure $Z_n$ gauge.

The cocycles in $\mathcal{H}^1(Z_n, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$ are labeled by $m = 0, \ldots, n-1$, with $m = 0$ corresponding to the trivial cocycle. The $m$th cocycle is given by

$$\omega_{ij} (g^{k}) = \text{mod}(mk/n, 1).$$

We note that the above cocycle $\omega_{ij} (h_{i+1}^{i+1})$ is a torsion element in $\mathcal{H}^1(Z_n, \mathbb{R}/\mathbb{Z})$. It gives rise to a quantized topological term $Z_{top}(S_1, A_\mu)$:

$$e^{i 2\pi \int_{S_1} \omega_{ij} h_{ij+1}^{i+1}} = e^{i 2\pi m/k}$$

if $\prod_{i} h_{i+1}^{i+1} = g^{k}$. (10)
Such a partition function is a SPT invariant. Its nontrivial dependence on the total $Z_n$ flux through the circle, $g^{(k)}_i = \prod_{j} H_{i,j+1}$, implies that the SPT state is nontrivial.

The above partition function also implies that the ground state of the system carries a $Z_n$ quantum number $m$. Thus, the nontrivial $Z_n$ quantum number of the ground state $m \neq 0$ also measures the nontrivial cocycle in $\mathcal{H}^1(Z_n, \mathbb{R}/\mathbb{Z})$.

2. Monodromy defect

In 3D space-time, the bosonic $Z_n$ SPT states are described by the cocycles in $\mathcal{H}^1(Z_n, \mathbb{R}/\mathbb{Z}) = Z_n$. To find the SPT invariants for such a case, let us introduce the notion of monodromy defect [73].

Let us assume that the 2D lattice Hamiltonian for a SPT state with symmetry $G$ has a form (see Fig. 2)

$$ H = \sum_{(ijk)} H_{ijk}, $$

where $\sum_{(ijk)}$ sums over all the triangles in Fig. 2 and $H_{ijk}$ acts on the states on site $i$, site $j$, and site $k$: $|g_{i,j,k}\rangle$. (Note that the states on site $i$ are labeled by $g_i \in G$.) $H$ and $H_{ijk}$ are invariant under the global $G$ transformations.

Let us perform a $G$ transformation only in the shaded region in Fig. 2. Such a transformation will change $H$ to $H'$. However, only the Hamiltonian terms on the triangles $(ijk)$ across the boundary are changed from $H_{ijk}$ to $H'_{ijk}$. Since the $G$ transformation is a unitary transformation, $H$ and $H'$ have the same energy spectrum. In other words, the boundary in Fig. 2 (described by $H'_{ijk}$'s) does not cost any energy.

Now let us consider a Hamiltonian on a lattice with a “cut” (see Fig. 3),

$$ \tilde{H} = \sum_{(ijk)} H_{ijk} + \sum_{(ijk)}^{\text{cut}} H_{ijk}', $$

where $\sum'_{(ijk)}$ sums over the triangles not on the cut and $\sum^{\text{cut}}_{(ijk)}$ sums over the triangles that are divided into disconnected pieces by the cut. The triangles at the ends of the cut have no Hamiltonian terms. We note that the cut carries no energy. Only the ends of the cut cost energies. Thus, we say that the cut corresponds to two monodromy defects. The Hamiltonian $\tilde{H}$ defines the two monodromy defects.

We would also like to point out that the above procedure to obtain $\tilde{H}$ is actually the “gauging” of the $G$ symmetry. $\tilde{H}$ is a gauged Hamiltonian that contain a $G$ vortex-antivortex pair at the ends of the cut.

To summarize, a system with on-site symmetry $G$ can have many monodromy defects, labeled by the group elements that generate the twist along the cut. When $G$ is singly generated, we call the monodromy defect generated by the natural generator of $G$ as elementary monodromy defect. In this case, other monodromy defects can be viewed as bound states of several elementary monodromy defects. In the rest of this paper, we only consider the elementary monodromy defects.

3. $2 + 1D$: Total $Z_n$ charge of $n$ identical monodromy defects

The SPT invariant to detect the cocycle in $\mathcal{H}^1(Z_n, \mathbb{R}/\mathbb{Z})$ is the $Z_n$ quantum number of $n$ identical monodromy defects created by the twist $g^{(1)} \in Z_n$ (see Fig. 3). Note that the monodromy defects created by $g^{(1)}$ are the elementary monodromy defects. Other elementary monodromy defects can be viewed as bound states of the elementary monodromy defects. Also note that the monodromy defects or the $Z_n$ vortices are identical and correspond to the same kind of triangles.

Since $\mathcal{H}^1(Z_n, \mathbb{R}/\mathbb{Z}) = Z_n$, the $2 + 1D$ $Z_n$ SPT states are labeled by $m = 0, \ldots, n - 1$, with the corresponding 3-cocycle given by

$$ \omega_3(g^{(k_1)}_i, g^{(k_2)}_j, g^{(k_3)}_k) = e^{i \frac{2\pi}{n} k_1(k_2+k_3)-(k_1+k_3)l_n}, $$

$$ g^{(k)} = e^{i \frac{2\pi}{n} k}, $$

where $[k]_n$ is a shorthand notation for $[k]_n \equiv \text{mod}(k, n)$. (14)

In Appendix F2, we show the following.

**SPT invariant 1.** $n$ identical monodromy defects generated by $g^{(1)}$ twist in $2 + 1D$ $Z_n$ SPT states on a torus always carry a total $Z_n$ charge $2m$ if the $Z_n$ SPT states are described by the $m$th cocycle in $\mathcal{H}^1(Z_n, \mathbb{R}/\mathbb{Z})$.

When $n = \text{odd}$, we find that the total $Z_n$ charge of $n$ identical monodromy defects allows us to completely characterize the $2 + 1D$ $Z_n$ SPT states. However, when $n = \text{even}$, the total $Z_n$ charge of $n$ identical monodromy defects only allows us to...
distinguish \(n/2\) different \(Z_n\) SPT states. The \(m\) and \(m + \frac{n}{2}\) \(Z_n\) SPT states give rise to the same total \(Z_n\) charge and cannot be distinguished this way.

We would like to point out that when constructing the above SPT invariant, we have assumed that the system has an additional translation symmetry, although the existence of the \(Z_n\) SPT states does not require the translation symmetry. We use the translation symmetry to make identical monodromy defects, which allows us to construct the above SPT invariant.

4. 2+1D: The statistics of the monodromy defects

To construct new SPT invariant that can distinguish \(m\) and \(m + \frac{n}{2}\) \(Z_n\) SPT states, we consider the statistics of the (elementary) monodromy defects [73]. To compute the statistics of the monodromy defects we use the duality relation between the \(Z_n\) SPT states and the twisted \(Z_n\)-gauge theory discovered by Levin and Gu [73]. The \((2+1)D\) \(Z_n\) gauge theory can be studied using \(U(1) \times U(1)\) Chern-Simons theory [50,51,59,83].

The \(Z_n\) SPT states are described by \(\mathcal{H}^3(Z_n, \mathbb{R}/\mathbb{Z}) = \{m|m = 0, \ldots, n - 1\}\). Thus, the \(Z_n\) integer \(m\) labels different \(2+1D\) \(Z_n\) SPT states. The dual gauge theory description of the \(Z_n\) SPT state (labeled by \(m\)) is given by

\[
\mathcal{L} + W_{\text{top}} = \frac{1}{4\pi} K_{1IJ} a_{IJ} \partial_{\nu} a_{IJ} + \cdots,
\]

with

\[
K = \begin{pmatrix} -2m & n & 0 \\ n & 0 & 0 \end{pmatrix}.
\]

The \(K\) matrix with \(K_{1I} = -2m\) correspond to the 3-cocycle in Eq. (13) [59]. Note that, here, \(a_{IJ}\) are dynamical gauge fields whose charges are quantized as integers. They are not the fixed probe gauge fields which are denoted by capital letter \(A_{IJ}\). Two \(K\) matrices \(K_1\) and \(K_2\) are equivalent \(K_1 \sim K_2\) (i.e., give rise to the same theory) if \(K_1 = U^t K_2 U\) for an integer matrix with \(\text{det}(U) = \pm 1\). We find that \(K(m) \sim K(m + n)\). Thus, only \(m = 0, \ldots, n - 1\) give rise to nonequivalent \(K\) matrices.

A particle carrying \(l_1 a_1^I\) charge will have a statistics

\[
\theta_1 = \pi l_1 (K^{-1})^{IJ} l_1 J.
\]

A particle carrying \(l_1 a_1^I\) charge will have a mutual statistics with a particle carrying \(l_1 a_1^I\) charge:

\[
\theta_{IJ} = 2\pi l_1 (K^{-1})^{IJ} l_1 J.
\]

A particle with a unit of \(Z_n\) charge is described by a particle with a unit \(a_\mu^I\) charge. Using

\[
K^{-1} = \frac{1}{n^2} \begin{pmatrix} 0 & n \\ n & 2m \end{pmatrix},
\]

we find that the \(Z_n\) charge (the unit \(a_\mu^I\) charge) is always bosonic.

The \(Z_n\) monodromy defect in the original theory corresponds to \(2\pi/n\) flux in \(a_\mu^I\), since the unit \(a_\mu^I\) charge corresponds to the \(Z_n\) charge in the original theory. We note that a particle carrying \(l_1 a_1^I\) charge created a \(2\pi\) flux in \(a_\mu^I\). So a unit \(a_\mu^I\) charge always represents a \(Z_n\) monodromy defect. However, such a \(Z_n\) monodromy defect may not be a pure \(Z_n\) monodromy defect. It may carry some additional \(Z_n\) charges.

Since the \(Z_n\) monodromy defect corresponds to \(2\pi/n\) flux in \(a_\mu^I\), by itself, a single monodromy defect is not an allowed excitation. However, \(n\) identical \(Z_n\) monodromy defects (i.e., \(n\) particles that each carries a unit \(a_\mu^I\) charge) correspond to \(2\pi\) flux in \(a_\mu^I\), which is an allowed excitation. Then, what is the total \(Z_n\) charge of \(n\) identical \(Z_n\) monodromy defects (i.e., \(n\) units of \(a_\mu^I\) charges)? We note that \(n\) units of \(a_\mu^I\) charges can be viewed as a bound state of a particle with \((l_1, l_2) = (-2m, n)\) \(a_1^I\) charges and a particle with \((l_1, l_2) = (2m, 0)\) \(a_1^I\) charges. The particle with \((l_1, l_2) = (2m, 0)\) \(a_1^I\) charges is a trivial excitation that carries zero \(Z_n\) charge, since \((l_1, l_2) = (-2m, n)\) is a row of the \(K\) matrix. The particle with \((l_1, l_2) = (2m, 0)\) \(a_1^I\) charges carries \(2m Z_n\) charges. Thus, \(n\) identical \(Z_n\) monodromy defects (described by \(n\) particles, each carrying a unit \(a_\mu^I\) charge) have \(2m\) total \(Z_n\) charges, which agrees with the result obtained the in last section.

A particle that carries a unit \(a_\mu^I\) charge is only one way to realize the \(Z_n\) monodromy defect. A generic \(Z_n\) monodromy defect that may carry a different \(Z_n\) charge corresponds to \(I^M = (l^M_1, 1) a_\mu^I\) charge. The statistics of such generic \(Z_n\) monodromy defect is

\[
\theta_M = \pi (I^M)^T K^{-1} I^M = 2\pi \left(\frac{1}{n} + \frac{m}{n^2}\right).
\]

We find the following.

SPT invariant 2. The statistical angle \(\theta_M\) of an elementary monodromy defect is a SPT invariant that allows us to fully characterize the \(2+1D\) bosonic \(Z_n\) SPT states [73]. In particular, \(n m, \pi/n, \pi\) where \(m \in \mathcal{H}^3(Z_n, \mathbb{R}/\mathbb{Z}) = Z_n\) labels the different \(Z_n\) SPT states.

We note that such a SPT invariant can fully detect the 3-cocycles in \(\mathcal{H}^3(Z_n, \mathbb{R}/\mathbb{Z})\).

5. \(Z_n\)-gauge topological term in \(2+1D\)

Just like the \(0+1D\) case, we can also construct a SPT invariant and probe the 3-cocycles in \(\mathcal{H}^3(Z_n, \mathbb{R}/\mathbb{Z})\) by gauging the global \(Z_n\) symmetry. After integrating out the matter fields, we obtain a \(Z_n\)-gauge topological term. Such a \(Z_n\)-gauge topological term correspond to a 3-cocycle \(\omega_3\) in \(\mathcal{H}^3(Z_n, \mathbb{R}/\mathbb{Z})\), which describes the \(Z_n\) SPT states. In fact, the \(Z_n\)-gauge topological term can be directly expressed in terms of the 3-cocycle \(\omega_3(h_{ij})\) (using the differential form notation in Appendix A4),

\[
e^{i2\pi \int_{M^3} \omega_3(h_{ij})},
\]

where \(M^3\) is the 3D space-time and \(h_{ij}\) the \(Z_n\)-gauge “connection” in the link \(ij\). Such a \(Z_n\)-gauge topological term is a generalization of the Chern-Simons term to a discrete group \(Z_n\).

6. \(4+1D\)

We can also generalize the above construction to \(5D\) spacetime, where \(Z_n\) SPT states are described by \(\mathcal{H}^4(Z_n, \mathbb{R}/\mathbb{Z}) = Z_n\). We choose the \(4+1D\) space-time to have a topology \(M_4 \times M_4\), where \(M_4\) and \(M_4\) are two closed \(2+1D\) and \(2D\) manifolds. We then create \(n\) identical \(Z_n\) monodromy defects on \(M_2\). In the large \(M_3\) limit, we may view our \(4+1D\) \(Z_n\) SPT
state on space-time $M_1 \times M_2$ as a $2+1$D $Z_n$ SPT state on $M_1$, which is described by $\mathcal{H}^4(Z_n,\mathbb{R}/\mathbb{Z})$. We have the following.

**SPT invariant 3.** In a $2+1$D $Z_n$ SPT state labeled by $m \in \mathcal{H}^4(Z_n,\mathbb{R}/\mathbb{Z}) = Z_n$ on space-time $M_1 \times M_2$, $n$ identical $Z_n$ vortices (i.e., $Z_n$ monodromy defects) on $M_2$ induce a $2+1$D $Z_n$ SPT state labeled by $3m \in \mathcal{H}^4(Z_n,\mathbb{R}/\mathbb{Z})$ on $M_2$ in the small $M_2$ limit.

We show the above result when we discuss the $U(1)$ SPT states in 4+1D (see Sec. III B 3).

In Sec. III A 3, we discussed how to detect the cocycles in $\mathcal{H}^4(Z_n,\mathbb{R}/\mathbb{Z})$ by creating $n$ identical $Z_n$ monodromy defects on $M_2$ and then measuring the $Z_n$ charge of the ground state. So the cocycles in $\mathcal{H}^4(Z_n,\mathbb{R}/\mathbb{Z})$ can be measured by creating $n$ identical $Z_n$ monodromy defects on $M_2$ and $n$ identical $Z_n$ monodromy defects on $M_2$. Then we measure the $Z_n$ charge of the corresponding ground state.

The above construction of $Z_n$ SPT invariant is motivated by the following mathematical result. First, $\mathcal{H}^{2k+1}(Z_n,\mathbb{R}/\mathbb{Z}) \cong \mathcal{H}^{2k+2}(Z_n,\mathbb{Z})$. The generating cocycle $c_{2k+2}$ in $\mathcal{H}^{2k+2}(Z_n,\mathbb{Z})$ can be expressed as a wedge product $c_{2k+2} = c_2 \wedge c_2 \wedge \cdots \wedge c_2$, where $c_2$ is the generating cocycle in $\mathcal{H}^2(Z_n,\mathbb{Z})$. Since $\mathcal{H}^2(Z_n,\mathbb{Z}) \cong \mathcal{H}^4(Z_n,\mathbb{R}/\mathbb{Z})$, we can replace one $c_2$ in $c_{2k+2} = c_2 \wedge c_2 \wedge \cdots \wedge c_2$ with $\theta_1$ in $\mathcal{H}^4(Z_n,\mathbb{R}/\mathbb{Z})$ and write $c_{2k+2} = \theta_1 \wedge c_2 \wedge \cdots \wedge c_2$. Note that $c_2 \wedge \cdots \wedge c_2$ describes the topological gauge configuration on 2k-dimensional space, while $\theta_1$ describes the 1D representation of $Z_n$. This motivates us to use a $Z_n$ gauge configuration on 2k-dimensional space to generate a nontrivial $Z_n$ charge in the ground state. In the next section, we use a similar idea to construct the SPT invariant for bosonic $U(1)$ SPT states.

**B. Bosonic $U(1)$ SPT phases**

1. **$0+1$D**

In 1D space-time, the bosonic SPT states with symmetry $U(1) = \{e^{i\theta}\}$ are described by the cocycles in $\mathcal{H}^4(U(1),\mathbb{R}/\mathbb{Z}) = Z$. Let us first study the SPT invariant from the topological partition function.

A nontrivial cocycle in $\mathcal{H}^4(U(1),\mathbb{R}/\mathbb{Z}) = Z$ labeled integer $m$ is given by

$$\omega_1(e^{i\theta}) = e^{im\theta}. \quad (22)$$

Let us assume the space-time is a circle $S_1$ formed by a ring of vertices labeled by $i$. A flat $U(1)$-gauge configuration on $S_1$ is given the $U(1)$ group elements $e^{i\theta_{i,i+1}}$ on each link $(i,i+1)$. The topological part of the partition function for such a flat $U(1)$-gauge configuration is determined by the above cocycle $\omega_1$:

$$Z_{top}(S_1,A_\mu) = e^{2\pi i \sum_i \omega_1(g_{i,i+1})}. \quad (23)$$

We note that the above $\omega_1(g_{i,i+1})$ is a free element in $\mathcal{H}^4(U(1),\mathbb{R}/\mathbb{Z})$. So it gives rise to a Chern-Simons-type topological term $Z_{top}(S_1,A_\mu)$:

$$Z_{top}(S_1,A_\mu) = e^{im \sum_i \theta_{i,i+1}} = e^{im \oint A_\mu}. \quad (24)$$

where $A$ is the $U(1)$-gauge potential one form. [Note that $\oint A$ is the $U(1)$ Chern-Simons term in 1D, and Eq. (7) can be viewed as a discrete 1D Chern-Simons term for $Z_n$-gauge theory.] Such a partition function is a SPT invariant. When $m \neq 0$, its nontrivial dependence on the total $U(1)$ flux through the circle, $\sum_i \theta_{i,i+1} = \oint dt A_\theta = \oint A_\mu$, implies that the SPT state is nontrivial.

The above partition function also implies that the ground state of the system carries a $U(1)$ quantum number $m$. Thus, the nontrivial $U(1)$ quantum number $m$ of the ground state also measures the nontrivial cocycle in $\mathcal{H}^4(U(1),\mathbb{R}/\mathbb{Z})$.

2. **$2+1$D**

In 3D space-time, the bosonic $U(1)$ SPT states are described by the cocycles in $\mathcal{H}^2(U(1),\mathbb{R}/\mathbb{Z}) = Z$. How do we measure the cocycles in $\mathcal{H}^2(U(1),\mathbb{R}/\mathbb{Z})$? One way is to “gauged” the $U(1)$ symmetry and put the “gauged” system on a 2D closed space $M_2$. We choose a $U(1)$-gauge configuration on $M_2$ such that there is a unit of $U(1)$ flux. Then we measure the $U(1)$ charge $q$ of the ground state on $M_2$. We show that $q$ is an even integer and $q/2 = m \in Z$ is the SPT invariant that characterizes the $U(1)$ SPT states. In fact, such a SPT invariant is actually the quantized Hall conductance.

**SPT invariant 4.** The SPT invariant for 2+1D bosonic $U(1)$ SPT phases is given by quantized Hall conductance which is quantized as even integers $\sigma = \frac{2m}{\pi}$, $m \in Z$. \[51-53,88\]

To show the above result, let us use the result that all 2+1D Abelian bosonic topological orders can be described by $U(1)$ Chern-Simons theory characterized by an even K matrix \[31\]:

$$L = \frac{1}{4\pi} K_{ij} a_i \partial_r a_j e^{i\nu r} + \frac{1}{2\pi} q_i A_i \partial_r a_j e^{i\nu r} + \cdots.$$ \(25\)

The SPT states have a trivial topological order and are special cases of 2+1D Abelian topological order. Thus, the SPT states can be described by even $K$ matrices with det$(K) = 1$ and a zero signature. In particular, we can use a $U(1) \times U(1)$ Chern-Simons theory to describe the $U(1)$ SPT state \[51,88\], with the $K$ matrix and the charge vector $q$ given by \[28,29,31\]:

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} (m) \end{pmatrix}, \quad m \in Z. \quad (26)$$

Note that, here, $a_{1\mu}$ are dynamical gauge fields. They are not fixed probe gauge fields, which are denoted by capital letter $A_{\mu}$. The Hall conductance is given by

$$\sigma_{xy} = (2\pi)^{-1} q^T K^{-1} q = \frac{2m}{2\pi}. \quad (27)$$

If we write the topological partition function as $Z_{top}(M_\mu,A_{\mu}) = e^{\int d^4x L_{top}}$, the above Hall conductance implies that the topological partition function is given by a 3D Chern-Simons term [obtained from (25) by integrating out 214’s]

$$L_{top} = \frac{2m}{4\pi} A_i \partial_r a_j e^{i\nu r} = \frac{2m}{4\pi} A F. \quad (28)$$

where $F$ is the $U(1)$ field strength two form. Note that, in comparison, Eq. (21) can be viewed as a discrete 3D Chern-Simons term for $Z_n$-gauge theory.

The above result can be generalized to other continuous symmetry groups. Following are examples.

**SPT invariant 5.** The SPT invariant for 2+1D bosonic SU(2) SPT phases is given by quantized spin Hall conductance which is quantized as half-integers $\sigma_{xy} = \frac{n\pi}{2}$, $m \in Z$. \[52\]
SYMMETRY-PROTECTED TOPOLOGICAL INVARIANTS OF . . .

PHYSICAL REVIEW B 89, 035147 (2014)

SPT invariant 6. The SPT invariant for 2 + 1D bosonic SO(3) SPT phases is given by quantized spin Hall conductance which is quantized as even integers \( \sigma_{xy} = \frac{2m}{e^2} \), \( m \in \mathbb{Z} \) [52].

3. 4 + 1D

In 5D space-time, the bosonic \( U(1) \) SPT states are labeled by an integer \( m \in \mathcal{H}^4(U(1), \mathbb{R}/\mathbb{Z}) = \mathbb{Z} \). Again, one can construct a SPT invariant to measure \( m \) by "gauging" the \( U(1) \) symmetry and putting the gauged system on a 4D closed space \( M_4 \). We choose a \( U(1) \)-gauge configuration on \( M_4 \) such that

\[
\int_{M_4} \frac{F^2}{8\pi^2} = 1, \tag{29}
\]

where \( F \) is the two-form \( U(1) \)-gauge field strength and \( F^2 = F \wedge F \) is the wedge product of differential forms. We then measure the \( U(1) \) charge \( q \) of the ground state induced by the \( U(1) \)-gauge configuration. Here the potential SPT invariant \( q \) must be an integer.

However, not all of the integer SPT invariants are realizable. We find that the bosonic \( U(1) \) SPT states can only be realized the SPT invariants \( q = 6m \). This is because, after integrating out the matter fields, the bosonic \( U(1) \) SPT states are described by the following \( U(1) \)-gauge topological term (see discussions in Sec. IV D 2):

\[
\mathcal{L}_{\text{top}} = \frac{m}{(2\pi)^2} AF^2. \tag{30}
\]

Such a topological term implies the following.

SPT invariant 7. \( \int_{M_4} \frac{F^2}{8\pi^2} = 1 \) gauge configuration on space \( M_4 \) will induce \( 6m \) \( U(1) \) charges for a bosonic 4 + 1D \( U(1) \) SPT state labeled by \( m \in \mathcal{H}^4(U(1), \mathbb{R}/\mathbb{Z}) = \mathbb{Z} \).

Thus, \( m/6 \) measures the cocycles in \( \mathcal{H}^4(U(1), \mathbb{R}/\mathbb{Z}) \).

Again, one can also construct another SPT invariant by putting the gauged system on a 4 + 1D space-time with topology \( M_2 \times M_3 \). We choose a \( U(1) \)-gauge configuration on \( M_2 \) such that

\[
\int_{M_2} \frac{F}{2\pi} = 1. \tag{31}
\]

In the large \( M_3 \) limit, we may view the 4 + 1D system on \( M_2 \times M_3 \) as a 2 + 1D system on \( M_3 \). The 4 + 1D Chern-Simons topological term [Eq. (30)] on \( M_2 \times M_3 \) reduces to a 2 + 1D Chern-Simons topological term on \( M_3 \):

\[
\mathcal{L}_{\text{top}} = \frac{3m}{2\pi} AF. \tag{32}
\]

Such a 2 + 1D Chern-Simons topological term implies that the 4 + 1D \( U(1) \) SPT on \( M_2 \times M_3 \) reduces to a 2 + 1D \( U(1) \) SPT labeled by \( 3m \) on \( M_3 \) in the large \( M_3 \) limit. To summarize, we present the following.

SPT invariant 8. In a 4 + 1D \( U(1) \) SPT state labeled by \( m \in \mathcal{H}^4(U(1), \mathbb{R}/\mathbb{Z}) = \mathbb{Z} \) on space-time \( M_3 \times M_2 \), \( 2\pi \) \( U(1) \) flux on \( M_2 \) induces a 2 + 1D \( Z_n \) SPT state on \( M_3 \) labeled by \( 3m \in \mathcal{H}^4(U(1), \mathbb{R}/\mathbb{Z}) \) in the large \( M_3 \) limit.

We may embed the \( Z_n \) group into the \( U(1) \) group and view the \( U(1) \) SPT states as a \( Z_n \) SPT state. By comparing the \( Z_n \) SPT invariants and the \( U(1) \) SPT invariants, we find that a \( U(1) \) SPT state labeled by \( m \in \mathcal{H}^4(U(1), \mathbb{R}/\mathbb{Z}) \) corresponds to a \( Z_n \) SPT state labeled by \( \mod(m,n) \in \mathcal{H}^4(Z_n, \mathbb{R}/\mathbb{Z}) \).

C. Bosonic \( Z^2 \) SPT phases

We have been constructing SPT invariants by gauging the on-site symmetry. However, since we do not know how to gauge the time-reversal symmetry \( Z^2 \), to construct the SPT invariants for \( Z^2 \) SPT phases, we have to use a different approach.

1. 1 + 1D

We first consider bosonic \( Z^2 \) SPT states in 1 + 1 dimensions, where \( Z^2 \) is the antiunitary time-reversal symmetry. The \( Z^2 \) SPT states are described by \( \mathcal{H}^2(Z^2, (\mathbb{R}/\mathbb{Z})_T) \), which is given by

\[
\mathcal{H}^2(Z^2, (\mathbb{R}/\mathbb{Z})_T) = \mathbb{Z}_2 = \{ m \}. \tag{33}
\]

Here \( (\mathbb{R}/\mathbb{Z})_T \) is the module \( \mathbb{R}/\mathbb{Z} \). The subscript \( T \) just stresses that the time-reversal symmetry \( T \) has a nontrivial action on the module \( \mathbb{R}/\mathbb{Z} \).

We see that \( m = 0,1 \) labels different 1 + 1D \( Z^2 \) SPT states. To measure \( m \), we put the system on a finite line \( L_1 \). At the end of the line, we get degenerate states that form a projective representation of \( Z^2 \), which is classified by \( \mathcal{H}^2(Z^2, (\mathbb{R}/\mathbb{Z})_T) \) [44–46]. We find the following.

SPT invariant 9. A 1 + 1D bosonic \( Z^2 \) SPT state labeled by \( m \) has a degenerate Kramer doublet at an open boundary if \( m = 1 \).

2. 3 + 1D

The 3 + 1D \( Z^2 \) SPT states are described by \( \mathcal{H}^4(Z^2, (\mathbb{R}/\mathbb{Z})_T) \), which is given by

\[
\mathcal{H}^4(Z^2, (\mathbb{R}/\mathbb{Z})_T) = \mathbb{Z}_2 = \{ m \}. \tag{34}
\]

References [56,64] have constructed several potential SPT invariants for the \( Z^2 \) SPT states. Here we give a brief review of those potential SPT invariants.

The first way to construct the potential SPT invariants is to consider a 3 + 1D \( Z^2 \) SPT state with a boundary. We choose the boundary interaction in such a way that the boundary state is gapped and does not break the symmetry. In this case, the 2 + 1D boundary state must be a topologically ordered state. It was shown in Refs. [56,64] that if the boundary state is a 2 + 1D \( Z^2 \) topologically ordered state [18,19] and if the \( Z_2 \) charge and the \( Z_2 \)-vortex excitations in the \( Z_2 \) topologically ordered state are both Kramer doublets under the time-reversal symmetry, then the 3 + 1D bulk \( Z^2 \) SPT state must be nontrivial. Also, if the boundary state is a 2 + 1D “all fermion \( Z_2 \) liquid” [56,64,65], then the 3 + 1D bulk \( Z^2 \) SPT state must be nontrivial as well. Both of the above two SPT invariants can be realized by 3 + 1D states that contain no topologically nontrivial particles [64].

The second way to construct the potential SPT invariants is to break the time-reversal symmetry explicitly at the boundary only. We break the symmetry in such a way that the ground state at the boundary is gapped without any degeneracy. Since there is no ground-state degeneracy, there are no excitations with fractional statistics at the boundary. We may also break the time-reversal symmetry in the opposite way to obtain the time-reversal partner of the above gapped nondegenerate ground state. Now, let us consider a domain wall between the above two ground states with opposite time-reversal symmetry breaking. Since there are no excitations with fractional
statistics at the boundary, the low-energy edge state on the domain wall must be a chiral boson theory described by an integer \( K \) matrix which is even and \( \det(K) = 1 \),

\[
\mathcal{L}_{1+1D} = \frac{1}{4\pi} [K_{I J} \partial_\phi \partial_{\phi_I} \phi_J - V_{I J} \partial_\phi \phi_I \partial_\phi \phi_J] + \sum_I \sum_{J = 1, 2} \left[ c_{I J} e^{i K_{I J} \phi_I} + \text{H.c.} \right], \tag{35}
\]

where the field \( \phi_I(x, t) \) is a map from the 1 + 1D space-time to a circle \( 2\pi \mathbb{R}/\mathbb{Z} \), and \( V \) is a positive-definite real matrix.

If we modify the domain wall, while keeping the surface state unchanged, we may obtain a different low-energy effective chiral boson theory on the domain wall described by a different even \( K \) matrix, \( K' \), with \( \det(K') = 1 \). We say the \( K' \) matrix is equivalent to \( K \). According to Ref. [89], the equivalent classes of even \( K \) matrices with \( \det(K) = 1 \) are given by

\[
K = K_{E_1} \oplus \cdots \oplus K_{E_n}, \tag{36}
\]

where \( K_{E_i} \) is the \( K \) matrix that describes the \( E_8 \) root lattice.

When \( K \) is a direct sum of even number \( n \) of \( K_{E_i} \)'s, such a domain wall can be produced by a pure 2D bosonic system, where the boundary ground state is the bosonic quantum Hall state described by a \( K \) matrix [28–33] that is a direct sum of \( n/2 \) \( K_{E_i} \)'s. The time-reversal partner is the bosonic quantum Hall state described by a \( K \) matrix that is a direct sum of \( n/2 \) \(-K_{E_i} \)'s. In this case, the edge state on the domain wall does not reflect any nontrivialness of \( 3 + 1 \)D bulk. So if \( K \) is a direct sum of even number \( n \) of \( K_{E_i} \)'s, it will represent a trivial potential SPT invariant.

When \( K \) is a direct sum of an odd number of \( K_{E_i} \)'s, then, there is no way to use a pure 2D bosonic system to produce such an edge state on the domain wall. Thus, if the domain wall between the time-reversal partners of boundary ground states is described by a \( 1 + 1 \)D boson theory with a \( K \) matrix \( K_{E_i} \) (or a direct sum of an odd number of \( K_{E_i} \)), then the \( 3 + 1 \)D bosonic \( Z_2 \) SPT state is nontrivial. It was suggested that such a \( K_{E_i} \) SPT invariant is the same as the all-fermion-\( Z_2 \)-liquid SPT invariant [56,64].

### D. Fermionic \( U(1) \) SPT phases

Although the SPT invariant described above is motivated by the group cohomology theory that describes the bosonic SPT states; however, the obtained SPT invariant can be used to characterize/define fermionic SPT phases.

The general theory of interacting fermionic SPT phases is not as well developed as the bosonic SPT states. (A general theory of free fermion SPT phases was developed in Refs. [90–92], which includes the noninteracting topological insulators [38–43,93] and the noninteracting topological superconductors [94–98].) The first attempt was made in Ref. [77], where a group super-cohomology theory was developed. However, the group super-cohomology theory can only describe a subset of fermionic SPT phases. A more general theory is needed to describe all fermionic SPT phases.

Even though the general theory of interacting fermionic SPT phases is not as well developed, this does not prevent us from using the same SPT invariants constructed by bosonic SPT states to study fermionic SPT states. We hope the study of the SPT invariants may help us to develop the more general theory for interacting fermionic SPT phases.

#### 1. Symmetry in fermionic systems

A fermionic system always has a \( Z_2 \) symmetry generated by \( P_f \equiv (-)^{N_f} \), where \( N_f \) is the total fermion number. Let us use \( G_f \) to denote the full symmetry group of the fermion system. \( G_f \) always contains \( Z_2^f \) as a normal subgroup. Let \( G_b \equiv G_f/Z_2^f \), which represents the "bosonic" symmetry. We see that \( G_f \) is an extension of \( G_b \) by \( Z_2^f \), described by the short exact sequence

\[
1 \rightarrow Z_2^f \rightarrow G_f \rightarrow G_b \rightarrow 1. \tag{37}
\]

People sometimes use \( G_b \) to describe the symmetry in fermionic systems and sometimes use \( G_f \) to describe the symmetry. Neither \( G_b \) nor \( G_f \) contains the full information about the symmetry properties of a fermion system. To completely describe the symmetry of a fermion system, we need to use the short exact sequence (37). However, for simplicity, we still use \( G_f \) to refer the symmetry in fermion systems. When we say that a fermion system has a \( G_f \) symmetry, we imply that we also know which \( Z_2^f \) is embedded in \( G_f \) as a normal subgroup. (Note that \( P_f \) always commutes with any elements in \( G_f \): \( [P_f, g] = 0, g \in G_f \).)

#### 2. SPT invariant for fermionic \( U(1) \) SPT phases

In this section, we discuss the SPT invariant for the simplest fermionic SPT states, which is a system with a full symmetry group \( G_f = U(1) \). The full symmetry group contains \( Z_2^f \) as a subgroup such that odd \( U(1) \) charges are always fermions. We use the SPT invariant developed in the last section to study fermionic SPT states with a \( U(1) \) symmetry in 3D space-time. To construct the SPT invariance, we first "gauge" the \( U(1) \) symmetry and then put the fermion system on a 2D closed space \( M_2 \) with a \( U(1) \) gauge configuration that carries a unit of the gauge flux \( \int_{M_2} \frac{F}{2\pi} = 1 \). We then measure the \( U(1) \) charge \( q \) of the ground state on \( M_2 \) induced by the \( U(1) \)-gauge configuration. Such a \( U(1) \) charge is a SPT invariant that can be used to characterize the fermionic \( U(1) \) SPT phases.

Do we have other SPT invariants? We may choose \( M_2 = S_1 \times S_1 \) (where \( S_d \) is a \( d \)-dimensional sphere). However, the \( S_1 \times S_1 \) (where \( S_1 \) is a \( 1 \)-dimensional line) do not have additional discrete topological \( U(1) \) gauge configurations except those described by the \( U(1) \) flux \( \int_{M_2} \frac{F}{2\pi} \) discussed above. We need discrete topological gauge configurations to induce discrete \( U(1) \) charges.] This suggests that we do not have other SPT invariant and the fermionic \( U(1) \) SPT states are described by integers \( Z \). In fact, the integer \( q \) is nothing but the integral quantized Hall conductance \( \sigma_{xy} = \frac{q}{2e^2} \).

The above just shows that every fermionic \( U(1) \) SPT state can be characterized by an integer \( q \). However, we do not know if every integer \( q \) can be realized by a fermionic \( U(1) \) SPT state or not. To answer this question, we note that a fermionic \( U(1) \) SPT state is an Abelian state. So it can be described by a \( U(1) \times \cdots \times U(1) \) Chern-Simons theory with an odd \( K \) matrix and a charge vector \( q \) [31]. Let us first assume that the \( K \) matrix is 2D. In this case, the fermionic \( U(1) \) SPT state
must be described by a $U(1) \times U(1)$ Chern-Simons theory in Eq. (25) with the $K$ matrix and the charge vector $q$ of the form [31]

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad q = \begin{pmatrix} 2m_1 + 1 \\ 2m_2 + 1 \end{pmatrix}, \quad m_{1,2} = \text{integers}.$$ (38)

We require the elements of $q$ to be odd integers since odd $U^f(1)$ charges are always fermions. The Hall conductance is given by

$$\alpha_{st} = (2\pi)^{-1} q^T K^{-1} q = \frac{4[m_1(m_1 + 1) - m_2(m_2 + 1)]}{2\pi}.$$ (39)

We find the following.

SPT invariant 10. The SPT invariant for $2 + 1$D fermionic $U^f(1)$ SPT phases is given by quantized Hall conductance which is quantized as 8 times integers $\alpha_{st} = \frac{8q_m}{2\pi}$, $m \in \mathbb{Z}$.

This result is valid even if we consider higher dimensional $K$ matrices.

It is interesting to see that the potential SPT invariants for bosonic $U(1)$ SPT states are integers (the integrally quantized Hall conductances). However, the actual SPT invariants are even integers. Similarly, the potential SPT invariants for fermionic $U^f(1)$ SPT states are also integers (the integrally quantized Hall conductances). However, the actual SPT invariants are 8 times integers.

E. Fermionic $Z^f_2$ SPT phases

Next we consider fermionic $Z^f_2$ SPT phases in $3D$ space-time. We find that the $2 + 1$D fermionic $Z^f_2$ SPT phases have two types of potential SPT invariants. However, so far we cannot find any fermionic SPT phases that give rise to nontrivial SPT invariants. This suggests that there is no nontrivial fermionic $Z^f_2$ SPT phases in 3D space-time. Let us use $fSPT^Z_2$ to denote the Abelian group that classifies the fermionic SPT phases with full symmetry group $G_f$ in $d$-dimensional space-time. The above result can be written as $fSPT^Z_2 = 0$.

Let us discuss the first potential SPT invariant. We again create two identical $Z^f_2$ monodromy defects on a closed 2D space. We then measure the $P_f$ quantum number $(-)^q$ for a ground state with the two identical $Z^f_2$ monodromy defects. So the potential SPT invariants $q$ are elements in $Z_2$. However, what are the actual SPT invariants? Can we realize the nontrivial SPT invariant $q = 1$?

We may view a fermion $U^f(1)$ SPT phase discussed above as a $Z^f_2$ SPT phase by viewing the $\pi U^f(1)$ rotation as $P_f$. In this case the SPT invariants $q_U$ for the $U^f(1)$ SPT phases become the SPT invariants $q$ for $Z^f_2$ SPT phases: $q = q_U \mod 2$. To see this result, we note that $q_U$ is the induced $U^f(1)$ charge by $2\pi U^f(1)$ flux. $2\pi U^f(1)$ flux can be viewed as two identical $Z^f_2$ vortices (each has $\pi U^f(1)$ flux). So the induced $Z^f_2$ charge is $q = q_U \mod 2$.

Since $q_U = 0 \mod 8$, fermionic $U^f(1)$ SPT phases always correspond to a trivial $Z^f_2$ SPT phase. We fail to get any nontrivial fermionic $Z^f_2$ SPT phases from the fermionic $U^f(1)$ SPT phases.

We would like to point out that the induced $P_f$ quantum numbers by two identical $Z^f_2$ monodromy defects are not the only type of SPT invariants. There exists a new type of SPT invariant for fermion systems.

SPT invariant 11. Two identical $Z^f_2$ monodromy defects may induce topological degeneracy [7], with different degenerate states carrying different $P_f$ quantum numbers.

This new type of SPT invariant is realized by a $p + i\omega$ state, where $2N$ identical $Z^f_2$ monodromy defects induce $2^N$ topologically degenerate ground states. Those topologically degenerate ground states are described by $2N$ Majorana zero modes which correspond to $N$ zero-energy orbitals for complex fermions [95,99]. However, the $p + i\omega$ state has an intrinsic topological order which is not a fermionic SPT state. So far we cannot find any fermionic SPT phases that give rise to nontrivial SPT invariants of the second type. Thus, we believe that $fSPT^Z_2 = 0$.

In $0 + 1D$, we have nontrivial fermionic SPT phases $fSPT^Z_1 = \mathbb{Z}_2$. The two fermionic SPT phases correspond to 0-dimensional ground state with no fermion and one fermion. One can also show that $fSPT^Z_2 = 0$, i.e., no nontrivial fermionic SPT phases in $1 + 1D$ [77].

IV. SPT INVARIANTS OF SPT STATES WITH SYMMETRY $G = GG \times SG$

A. Bosonic $U(1) \times U(1)$ SPT phases in $2 + 1D$.

In this section, we are going to discuss the SPT invariant for bosonic $U(1) \times U(1)$ SPT states in 3D space-time [48,49,51]. To construct the SPT invariance, we first gauge the $U(1) \times U(1)$ symmetry and then put the boson system on a 2D closed space $M_2$ with a $U(1) \times U(1)$ gauge configuration $(A_n, \tilde{A}_n)$ that carries a unit of the $U(1)$ gauge flux $\int_{M_2} F = 1$. We then measure the $U(1)$ charge $c_1$ and the $\tilde{U}(1)$ charge $c_2$ of the ground state. Next, we put another $U(1) \times \tilde{U}(1)$-gauge configuration on $M_2$ with a unit of the $\tilde{U}(1)$ gauge flux $\int_{M_2} F = 1$, then measure the $U(1)$ charge $c_1$ and the $\tilde{U}(1)$ charge $c_2$. We can use $c_i$ to form a two-by-two integer matrix $C$. So an integer matrix $C$ is a potential SPT invariant for bosonic $U(1) \times \tilde{U}(1)$ SPT phases in 3D space-time.

However, what are the actual realizable SPT invariants? To answer this question, let us consider a $U(1) \times U(1)$ Chern-Simons theory that describes the bosonic $U(1) \times \tilde{U}(1)$ SPT state.

$$\mathcal{L} = \frac{1}{4\pi} K_{IJK} a_I\partial_t a_J \epsilon^{IJK} + \frac{1}{2\pi} q_{1,1} A'_\mu \partial_\mu a_{1J} \epsilon^{\mu\lambda\nu} \epsilon^{IJK} + \frac{1}{2\pi} q_{2,1} A'_\mu \partial_\mu a_{1J} \epsilon^{\mu\lambda\nu} + \cdots,$$ (40)

with the $K$ matrix and two charge vectors $q_1, q_2$:

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad q_1 = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, \quad q_2 = \begin{pmatrix} k_3 \\ k_4 \end{pmatrix}, \quad k_i = \text{integers}.$$ (41)
The SPT invariant $C$ is given by

$$C = (q^T K^{-1} q_j).$$

(42)

Since stacking two SPT states with SPT invariants $C_1$ and $C_2$ gives us a SPT state with a SPT invariant $C_1 + C_2$, the actual SPT invariants form a vector space. We find that the actual SPT invariants form a 3D vector space with basis vectors

$$C_1 = \left( \begin{array}{c} 2 \\ 0 \\ 0 \end{array} \right), \quad C_2 = \left( \begin{array}{c} 0 \\ 0 \\ 2 \end{array} \right), \quad C_3 = \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right).$$

(43)

So the bosonic $U(1) \times \tilde{U}(1)$ SPT phases in 3D space-time are described by three integers $m_1 C_1 + m_2 C_2 + m_3 C_3$, which agrees with the group cohomology result $\mathcal{H}^3(U(1) \times \tilde{U}(1), \mathbb{R}/\mathbb{Z}) = \mathbb{Z}^3$.

**B. Fermionic $U(1) \times U^f(1)$ SPT phases in 2+1D**

Now let us discuss the SPT invariant for fermionic SPT states in 3D space-time, which has a full symmetry group $G_f = U(1) \times U^f(1)$ with $\mathbb{Z}^2$ as a subgroup where odd $U(1)$ charges are always fermions. To construct the SPT invariance, we again gauge the $U(1) \times U^f(1)$ symmetry and then put the fermion system on a 2D closed space $M_2$ with a $U(1) \times U^f(1)$ gauge configuration that carries a unit of the $U(1)$ gauge flux $\int_{M_2} \mathcal{F} = 1$. We then measure the $U(1)$ charge $c_{11}$ and the $U^f(1)$ charge $c_{12}$ of the ground state on $M_2$ induced by the $U(1)$ gauge flux. Next, we put another $U(1) \times U^f(1)$ gauge configuration on $M_2$ with a unit of the $U^f(1)$ gauge flux $\int_{M_2} \mathcal{F} = 1$, then measure the $U(1)$ charge $c_{21}$ and the $U^f(1)$ charge $c_{22}$. So an integer matrix $C$ formed by $c_{ij}$ is a potential SPT invariant for fermionic $U(1) \times U^f(1)$ SPT phases in 3D space-time.

However, what are the actual SPT invariants? Let us consider the following $U(1) \times U(1)$ Chern-Simons theory that describes the fermionic $U(1) \times U^f(1)$ SPT state,

$$\mathcal{L} = \frac{1}{4\pi} K_{ijkl} \partial_0 a_{ij} \partial_0 a_{kl} e^{\mu\nu} + \frac{1}{2\pi} q_{11} \Lambda_{ij} \partial_0 a_{ij} e^{\mu\nu} + \frac{1}{2\pi} q_{21} \Lambda_{ij} \partial_0 a_{ij} e^{\mu\nu} + \cdots,$$

(44)

with the $K$ matrix and two charge vectors $q_1, q_2$:

$$K = \left( \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right), \quad q_1 = \left( \begin{array}{c} m_1 \\ m_2 \end{array} \right), \quad q_2 = \left( \begin{array}{c} m_3 \\ m_4 \end{array} \right),$$

(45)

$m_{1,4}$ are odd integers.

The requirement "$m_{1,4}$ are odd integers" comes from the fact that odd $U^f(1)$ charges are always fermions. The SPT invariant $C$ is given by

$$C = (q^T K^{-1} q_j).$$

(46)

We find that the actual SPT invariants form a 3D vector space with basis vectors

$$C_1 = \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right), \quad C_2 = \left( \begin{array}{c} 0 \\ 0 \\ 8 \end{array} \right), \quad C_3 = \left( \begin{array}{c} 0 \\ 2 \\ 0 \end{array} \right).$$

(47)

So the fermionic $U(1) \times U^f(1)$ SPT phases in 3D space-time are also described by three integers $\mathbb{Z}^3$.

**C. A general discussion for the case $G = GG \times SG$**

With the above two simple examples to give us some intuitive pictures, here we would like to give a general discussion for $G = GG \times SG$ cases. In Appendix C, we show that [see Eq. (C15)]:

$$\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z}) = \oplus_{k=0}^d \mathcal{H}^k(SG, \mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})).$$

(48)

This means that we can use $(m_0, \ldots, m_d)$ to label each element of $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ where $m_i \in \mathcal{H}^i(SG, \mathcal{H}^{d-i}(GG, \mathbb{R}/\mathbb{Z}))$. Note that $m_0$ only involves the group cohomology of smaller groups, which may be simpler. Using the similar setup in the above two examples, here we would like to discuss how to physically measure each $m_i$.

First, we notice that $\mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})$ describes the bosonic SPT phases in $(d-k)$-dimensional space-time. To stress this point, we rewrite $\mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})$ as $b\mathcal{SPT}^{d-k}_{GG}$, and rewrite above decomposition as

$$\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z}) = \oplus_{k=0}^d \mathcal{H}^k(SG, b\mathcal{SPT}^{d-k}_{GG}).$$

(49)

Since $b\mathcal{SPT}^{d-k}_{GG}$ is a direct sum of $\mathbb{Z}$'s and $\mathbb{Z}_n$'s, $\mathcal{H}^k(SG, b\mathcal{SPT}^{d-k}_{GG})$ is direct sum of $\mathcal{H}^k(SG, \mathbb{Z})$'s and $\mathcal{H}^k(SG, \mathbb{Z}_n)$'s. Such a structure motivates the following construction of SPT invariants that allow us to measure $m_i$: We first gauge the $SG$ symmetry and create nontrivial gauge configurations described by "$\mathcal{H}^k(SG)$". Such gauge configurations will induce SPT invariants whose "value" is in $b\mathcal{SPT}^{d-k}_{GG} = \mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})$. Again, we would like to stress that the gauge potentials for $SG$ are treated as fixed classical background without any fluctuations.

To create suitable gauge configurations, we may choose the space-time manifold to have a form $M_k \times M_{d-k}$, where $M_k$ has $k$ dimensions and $M_{d-k}$ has $d-k$ dimensions. We assume the $SG$ gauge configuration to be constant on $M_{d-k}$. Such a $SG$ gauge configuration can be viewed as a gauge configuration on $M_k$. Now we assume that $M_k$ is very small, and our system can be viewed as a system on $M_{d-k}$ which has a $GG$ symmetry. The ground state of such a $GG$ symmetric system is $GG$ SPT state on $M_{d-k}$ which is labeled by an element in $b\mathcal{SPT}^{d-k}_{GG} = \mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})$. This way, we obtain a function $\tilde{m}_k$ that maps a $SG$ gauge configuration on $M_k$ to an element in $\mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})$. In the above, we have discussed how to measure such an element physically when $GG = U(1), \mathbb{Z}_n$.

We note that $m_k$ in $\mathcal{H}^k(SG, \mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z}))$ is a cocycle, which is denoted as $\omega_k$ in Appendix A 2. $\omega_k$ maps a $SG$ gauge configuration on a $k$ cell in $M_k$ to an element in $\mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})$. In fact, $\omega_k$ (or $m_k$) is given by

$$\omega_k(s_0, s_1, \ldots, s_{k-1}, k) \in \mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z}),$$

(50)

where $s_i \in SG$ live on the edges of the $k$ cell which describe a $SG$ gauge configuration on the $k$ cell. If we sum over the contributions from all the $k$ cells in $M_k$, we obtain the above $\tilde{m}_k$ function that maps an $SG$ gauge configuration on $M_k$ to an element in $\mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})$.

The key issue is that whether the function $\tilde{m}_k$ allows us to fully detect $m_k \in \mathcal{H}^k(SG, \mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z}))$, i.e., whether different $m_k$ always lead to different $\tilde{m}_k$. We can show that this is indeed the case using the classifying space. Let $BSG$ be the classifying space of $SG$. We know
that the group cocycles in $H^k(SG, H^{d-k}(GG, \mathbb{R}/\mathbb{Z}))$ can be one-to-one represented by the topological cocycles in $H^k(BSG, H^{d-k}(GG, \mathbb{R}/\mathbb{Z}))$. We know that a topological cocycle $m_k^B$ in $H^k(BSG, H^{d-k}(GG, \mathbb{R}/\mathbb{Z}))$ gives rise to a function that maps all the $k$ cycles in BSG to $H^{d-k}(GG, \mathbb{R}/\mathbb{Z})$. Such a function can fully detect the cocycle $m_k^B$ (i.e., different cocycles always lead to different mappings). We also know that each $k$ cycle in BSG can be viewed as an embedding map from a $k$-dimensional space-time $M_k$ to BSG, and each embedding map defines a $SG$ gauge configuration on $M_k$. Thus, the topological cocycle $m_k^B$ is actually a function that maps a $SG$ gauge configuration in space-time to $H^{d-k}(GG, \mathbb{R}/\mathbb{Z})$, and such a mapping can fully detect $m_k^B$. All the $k$ cycles in BSG can be continuously deformed into a particular type of $k$ cycle where all the vertices on the $k$ cycle occupy one point in BSG. The $m_k^B$ that maps the $k$ cycles to $H^{d-k}(GG, \mathbb{R}/\mathbb{Z})$ is a constant under such a deformation. $m_k^B$, when restricted on the $k$ cycles whose vertices all occupy one point, become the map $\tilde{m}_k$. This way, we show that the function $\tilde{m}_k$ can fully detect the group cocycles $m_k$ in $H^k(SG, H^{d-k}(GG, \mathbb{R}/\mathbb{Z}))$. This is how we fully measure $m_k$.

In the above we see that each embedding map from $k$-dimensional space-time $M_k$ to BSG defines a $SG$ gauge configuration on $M_k$. This relation tells us how to choose the $SG$ gauge configurations on $M_k$ so that we can fully measure $m_k$. We choose the $SG$ gauge configurations on $M_k$ that come from the embedding maps from $M_k$ to BSG such that the images are the nontrivial $k$ cycles in BSG.

D. An example with $SG = U(1)$ and $GG = U(1)$

I. 2 + 1D

Let us reconsider the bosonic SPT states with symmetry $G = U^{SG}(1) \times U^{GG}(1)$ (i.e., $SG = U(1) \equiv U^{SG}(1)$ and $GG = U(1) \equiv U^{GG}(1)$) in three space-time dimensions. Such SPT states are described by $H^3(G, \mathbb{R}/\mathbb{Z})$ with $G = U^{SG}(1) \times U^{GG}(1)$. We have

$$H^3(G, \mathbb{R}/\mathbb{Z}) = \bigoplus_{k=0}^3 H^k(U^{SG}(1), H^{d-k}(U^{GG}(1), \mathbb{R}/\mathbb{Z})) \oplus H^3(U^{SG}(1), \mathbb{R}/\mathbb{Z}).$$

with

$$H^3(U^{GG}(1), \mathbb{R}/\mathbb{Z}) = Z = \{ m_0 \},$$

$$H^2(U^{SG}(1), H^1(U^{GG}(1), \mathbb{R}/\mathbb{Z})) = Z = \{ m_1 \},$$

$$H^1(U^{SG}(1), \mathbb{R}/\mathbb{Z}) = Z = \{ m_2 \}.$$  

$m_0$ labels different 2 + 1D $U^{GG}(1)$ SPT states and $m_3$ labels different 2 + 1D $U^{SG}(1)$ SPT states. We discussed how to measure $m_0$ and $m_3$ in Sec. III B 2. Here we discuss how to measure $m_2$. The structure of the Künneth expansion directly suggests the way to construct the SPT invariant.

We first choose the space-time manifold to be $S_2 \times S_1$, where $S_n$ is a $n$-dimensional sphere. We gauge the $GG$ symmetry and consider a $GG$ gauge configuration with 1 unit of flux on $S_2$. The flux on $S_2$ corresponds to an element in $H^2(U^{GG}(1), \mathbb{R}/\mathbb{Z}) = H^2(U^{GG}(1), \mathbb{R}/\mathbb{Z})$. In the small $S_2$ limit, our system becomes a $0 + 1D U^{GG}(1)$ symmetric theory on $S_1$. The ground state of such a $0 + 1D$ theory is a SPT state described by $H^1(U^{SG}(1), \mathbb{R}/\mathbb{Z}) = Z$ which corresponds to the $U^{SG}(1)$ charge of the ground state. Such a charge happens to be $m_2$, which we intend to measure.

This example also suggests the following general construction of SPT invariant (see Sec. IV C).

SPT invariant 1. In order to measure $m \in H^{d-k}(GG, \mathbb{R}/\mathbb{Z}) \subset H^k(GG \times SG, \mathbb{R}/\mathbb{Z})$, we choose the space-time to have a topology $M_k \times M_{d-k}$. Next, we construct a $GG$ gauge configuration on $M_k$ that corresponds to an element of $H^k(GG, \mathbb{R}/\mathbb{Z})$. In the large $M_{d-k}$ limit, the system can be viewed as having a $d-k$ space-time dimension with $SG$ symmetry. Such a $d-k$ dimensional system is described by an element in $H^k(GG, \mathbb{R}/\mathbb{Z})$, which is the response to the $GG$ gauge configuration. Measuring the responses for all possible $GG$ gauge configurations allows us to measure $m \in H^{d-k}(GG, \mathbb{R}/\mathbb{Z}) \subset H^k(GG \times SG, \mathbb{R}/\mathbb{Z})$.

In the above example, we try to measure $m_2 \in H^1(U^{GG}(1), \mathbb{R}/\mathbb{Z}) \subset H^1(U^{GG}(1), \mathbb{R}/\mathbb{Z})$ by choosing the space-time to be $S_2 \times M_1$. The $U^{GG}(1)$ gauge configuration with 1 unit of flux on $S_2$ corresponds to an element in $H^2(U^{GG}(1), \mathbb{R}/\mathbb{Z}) = H^1(U^{GG}(1), \mathbb{R}/\mathbb{Z})$. The $U^{SG}(1)$ charge of the ground state corresponds to an element in $H^1(U^{SG}(1), \mathbb{R}/\mathbb{Z})$. This example illustrates the idea of using the Künneth formula (4) to construct SPT invariants.

In fact, if we also gauge the $U^{GG}(1)$ symmetry and integrate out the matter fields (described by $a_{\mu}$’s) in Eq. (40), $m_2$ corresponds to an induced topological Chern-Simons term in $U^{SG}(1) \times U^{GG}(1)$ gauge theory,

$$\mathcal{L} = \frac{m_2}{2\pi} A_{SG} F_{GG},$$  

with $A_{SG}$ the gauge potential one form for the $U^{SG}(1)$ gauge field and $F_{GG}$ is the field strength two form for the $U^{GG}(1)$ gauge field. Similarly, $m_0$ and $m_3$ also correspond to topological Chern-Simons terms in the $U^{SG}(1) \times U^{GG}(1)$ gauge theory

$$\mathcal{L} = \frac{m_0}{2\pi} A_{GG} F_{GG} + \frac{m_3}{2\pi} A_{SG} F_{GG}.$$  

So the topological partition function $Z_{top}(M_d, A_{\mu}) = e^{i \int \mathcal{L}_{top}}$ is given by

$$\mathcal{L}_{top} = \frac{m_0}{2\pi} A_{GG} F_{GG} + \frac{m_2}{2\pi} A_{SG} F_{GG} + \frac{m_3}{2\pi} A_{SG} F_{SG}.$$  

We see a direct correspondence between the Künneth expansion of the group cohomology and the gauge topological term.

If we turn on one unit of $U^{GG}(1)$ flux on $S_2$ (described by a background field $A_{GG}$), the above topological terms become (with $A_{SG} = \delta A_{GG} + A_{SG}$)

$$\mathcal{L}_{top} = \frac{m_0}{2\pi} \delta A_{GG} F_{GG} + O(\delta A_{GG}^2) + \cdots ,$$  

which implies that one unit of $U^{GG}(1)$ flux on $S_2$ will induce $2m_0$ units of $U^{GG}(1)$ charge. The factor $2$ agrees with the
result of even-integer-quantized Hall conductance obtained before.

2. $4 + 1D$

Next, we consider bosonic $U^{SG}(1) \times U^{GG}(1)$ SPT states in $4 + 1D$. The SPT states are described by

$$H^S(\mathbb{R}/\mathbb{Z}) = \mathbb{R}/\mathbb{Z} \otimes H^1(\mathcal{U}(1), \mathbb{R}/\mathbb{Z}) \oplus H^2(\mathbb{R}/\mathbb{Z}) \oplus H^3(\mathbb{R}/\mathbb{Z}),$$

with

$$H^2(\mathbb{R}/\mathbb{Z}) = \mathbb{Z} = \{m_0\},$$

$$H^3(\mathbb{R}/\mathbb{Z}) = \mathbb{Z} = \{m_2\},$$

$$H^4(\mathbb{R}/\mathbb{Z}) = \mathbb{Z} = \{m_4\}.$$

The topological terms labeled by $m_k$ are the Chern-Simons terms,

$$L_{top} = \frac{m_0}{(2\pi)^2} A_{GG}^2 F_{GG}^2 + \frac{m_2}{(2\pi)^2} A_{SG}^2 F_{SG}^2 + \frac{m_4}{(2\pi)^2} A_{SG}^2 F_{GG}^2.$$

which gives rise to the topological partition function $Z_{top}(M_d, A_\mu) = e^{i \int d^4x L_{top}}$.

Why must the topological terms take the above form? Here we give an argument by considering the following general topological terms with $U(1)$-gauge fields

$$L_{top} = \sum_{1 \leq i,j \leq M \in \mathbb{Z}} \frac{K_{IJM}}{(2\pi)^2} A^I F_J F_M.$$

First we assume $K_{IJM}$ are real numbers. Then we would like to show that, when $I \neq J \neq M$, $K_{IJM}$ must be quantized as integers. Otherwise, a gauge configuration of $\int_{M_d} \frac{1}{2\pi} F_{SG}^2 = 1$ in the 4D space $M_4$ will induce a fractional $A^I$ charge. Also, the quantization conditions on $K_{IJM}$ should be invariant under the $SL(k, \mathbb{Z})$ transformation $A^I \rightarrow U_{ij} A^I$, $U \in SL(k, \mathbb{Z})$. In this case, an integral $K_{IJM}$ for $I \neq J \neq M$ will generate an integral $K_{IJM}$ for general $I, J, M$. This leads us to believe that $K_{IJM}$ are quantized as integers for general $I, J, M$. So the topological terms must take the form as in Eq. (59).

Now let us go back to the $U^{SG}(1) \times U^{GG}(1)$ topological terms (59). We have discussed the measurement of $m_0$ and $m_2$ before in our discussion of $U(1)$ SPT states. To measure $m_2$, we choose a space-time manifold of a form $M_2 \times M_2 \times S_1$ (where $S_1$ is the time direction). We put a $SG$ gauge field on space $M_2$ such that $\int_{M_2} \frac{1}{2\pi} F_{SG}^2 = 1$. In the small $m_2$ limit, our theory reduces to a $GG$ gauge theory on $M_2 \times S_1$ described by $m_2$ in $H^S[U^{GG}(1), \mathbb{R}/\mathbb{Z}]$. We can then put a $GG$ gauge field on space $M_2$ such that $\int_{M_2} \frac{1}{2\pi} F_{SG}^2 = 1$. Such a configuration will induce $2m_2$ units of $U^{GG}(1)$ charges. In other words, a $SG$ gauge field on space $M_2$ such that $\int_{M_2} \frac{1}{2\pi} F_{SG}^2 = 1$ and a $GG$ gauge field on space $M_2$ such that $\int_{M_2} \frac{1}{2\pi} F_{GG}^2 = 1$ will induce $2m_2$ units of $U^{GG}(1)$ charges.

The $m_4$ term can be measured by putting a $SG$ gauge field on space $M_4$ such that $\int_{M_4} \frac{1}{2\pi} F_{SG}^2 = 1$. Such a $SG$ gauge configuration will induce $2m_4$ units of the $U^{GG}(1)$ charges. The $SG$ gauge configuration will also induce $6m_5$ units of the $U^{SG}(1)$ charges.

E. Bosonic $Z_n \times Z_n$ SPT states

1. $2 + 1D$

Next, let us consider SPT states with symmetry $G = Z_n \times Z_n$ in $2 + 1D$. Such a theory was studied in Refs. [51,59,72] using $U(1)$ Chern-Simons theory. The $Z_n \times Z_n$ SPT states are described by $H^3(Z_n \times Z_{n_2}, \mathbb{R}/\mathbb{Z})$, which has the decomposition [see (C15)]

$$H^3(Z_n \times Z_{n_2}, \mathbb{R}/\mathbb{Z}) = \mathbb{Z} = \{m_0\},$$

$$H^3(Z_n \times Z_{n_2}, \mathbb{R}/\mathbb{Z}) = \mathbb{Z} = \{m_2\},$$

$$H^3(Z_n \times Z_{n_2}, \mathbb{R}/\mathbb{Z}) = \mathbb{Z} = \{m_4\}.$$

where $(n_1, n_2)$ is the greatest common divider of $n_1$ and $n_2$. $m_0$ labels different $2 + 1D Z_n$ SPT states and $m_2$ labels different $2 + 1D Z_{n_2}$ SPT states. To measure $m_2$, we may create two identical $Z_n$ monodromy defects on a closed 2D space. We then measure the induced $Z_{n_2}$ charge, which measures $2m_2$. We can also measure the induced $Z_n$ charge, which measures $m_2$.

To understand why measuring the induced $Z_{n_2}$ charges and $Z_n$ charges allows us to measure $2m_2$ and $m_2$, let us start with the dual gauge theory description of the $Z_n \times Z_n$ SPT state. The total Lagrangian has a form

$$L + W_{top} = \frac{1}{4\pi} K_{IJM} a_{I\mu} \partial_{\lambda} a_{J\lambda} + \cdots,$$

with

$$K = \begin{pmatrix} -2m_3 & n_1 & -m_2 & 0 \\ n_1 & 0 & 0 & 0 \\ -m_2 & 0 & -2m_0 & n_2 \\ 0 & 0 & n_2 & 0 \end{pmatrix}.$$
A particle carrying $l_1 a_{\mu}^l$ charge will have a statistics
\[ \theta_1 = \pi l_1 (K^{-1})^{l_1 l_1}. \] (65)

A particle carrying $l_1 a_{\mu}^l$ charge will have a mutual statistics with a particle carrying $l_1 a_{\mu}^l$ charge:
\[ \theta_{l_1} = 2 \pi l_1 (K^{-1})^{l_1 l_1}. \] (66)

A particle with a unit of $Z_{n_1}$ charge is described by a particle with a unit $a_{\mu}^l$ charge. A particle with a unit of $Z_{n_2}$ charge is described by a particle with a unit $a_{\mu}^l$ charge. Using
\[ K^{-1} = \begin{pmatrix} 0 & \frac{1}{n_1} & 0 & 0 \\ \frac{1}{n_1} & \frac{1}{n_2} & \frac{m_0}{n_1 n_2} & 0 \\ 0 & 0 & 0 & \frac{1}{n_1} \\ \frac{m_0}{n_1 n_2} & \frac{1}{n_1} & \frac{m_0}{n_1 n_2} & -\frac{1}{n_2} \end{pmatrix}, \] (67)
we find that the $Z_{n_1}$ charge (the unit $a_{\mu}^l$ charge) and the $Z_{n_2}$ charge (the unit $a_{\mu}^l$ charge) are always bosonic.

The $Z_{n_1}$ monodromy defect in the original theory corresponds to $2\pi/n_1$ flux in $a_{\mu}^l$, since the unit $a_{\mu}^l$ charge corresponds to the $Z_{n_1}$ charge in the original theory. We note that a particle carrying $l_1 a_{\mu}^l$ charge created a $2l_1\pi/n_1$ flux in $a_{\mu}^l$. So a unit $a_{\mu}^l$ charge always represents a $Z_{n_1}$ monodromy defect. Similarly, a unit $a_{\mu}^l$ charge always represents a $Z_{n_2}$ monodromy defect.

Since a $Z_{n_1}$ monodromy defect corresponds to $2\pi/n_1$ flux in $a_{\mu}^l$, by itself, a single monodromy defect is not an allowed excitation. However, $n_1$ identical $Z_{n_1}$ monodromy defects (i.e., $n_1$ particles that each carries a unit $a_{\mu}^l$ charge) correspond to $2\pi$ flux in $a_{\mu}^l$, which is an allowed excitation. We note that $n$ units of $a_{\mu}^l$ charges can be viewed as a bound state of a particle with $(l_1, l_2, l_3, l_4) = (-2m_3, n_1, -m_2, 0) a_{\mu}^l$ charges and a particle with $(l_1, l_2, l_3, l_4) = (2m_3, 0, m_2, 0) a_{\mu}^l$ charges. The particle with $(l_1, l_2, l_3, l_4) = (-2m_3, n_1, -m_2, 0) a_{\mu}^l$ charges is a trivial excitation that carries zero ($Z_{n_2}$) charge. The particle with $(l_1, l_2, l_3, l_4) = (2m_3, 0, m_2, 0) a_{\mu}^l$ charges carries $2m_3 Z_{n_2}$ charges and $m_2 Z_{n_2}$ charges. Thus, we get the following.

**SPT invariant 13.** In a $2 + 1$D $Z_{n_1} \times Z_{n_2}$ bosonic SPT state labeled by $(m_0, m_2, m_3)$, $n_1$ identical elementary $Z_{n_1}$ monodromy defects have $2m_3$ total $Z_{n_1}$ charges and $m_2 + (n_1, n_2) \times \text{integer total} Z_{n_2}$ charges. Similarly, $n_2$ identical elementary $Z_{n_2}$ monodromy defects have $2m_0$ total $Z_{n_1}$ charges and $m_3 + (n_1, n_2) \times \text{integer total} Z_{n_2}$ charges.

We see that to probe $m_2 \in H^2(Z_{n_1}, H^1(Z_{n_2}, \mathbb{Z}/Z)) = H^2(Z_{n_1}, \mathbb{Z}/Z) \otimes H^1(Z_{n_2}, \mathbb{Z}/Z)$, we first turn on a $Z_{n_2}$ gauge configuration of $n_1$ identical $Z_{n_2}$ monodromy defects which corresponds to an element in $H^2(Z_{n_1}, \mathbb{Z}/Z) = H^2(Z_{n_1}, \mathbb{Z})$. We then measure the induced $Z_{n_2}$ charge which corresponds to an element in $H^2(Z_{n_2}, \mathbb{Z}/Z)$.

We note that, sometimes, the above SPT invariants cannot fully detect $m_0$ and $m_3$. More complete SPT invariants can be obtained from the statistics of the monodromy defects. Let $\theta_{l_1}$ be the statistic angle of the elementary $Z_{n_1}$ monodromy defect and $\theta_{l_2}$ be the statistic angle of the elementary $Z_{n_2}$ monodromy defect. Note that a generic elementary $Z_{n_1}$ monodromy defect is described by a particle with $(l_1, l_2, l_3, l_4) = (l_{Z_{n_1}}, 1, l_{Z_{n_2}}, 0) a_{\mu}^l$ charges and a generic elementary $Z_{n_2}$ monodromy defect is described by a particle with $(l_1, l_2, l_3, l_4) = (l_{Z_{n_1}}, 0, l_{Z_{n_2}}, 1) a_{\mu}^l$ charges, where $l_{Z_{n_1}}$ and $l_{Z_{n_2}}$ describe different $Z_{n_1, n_2}$ charges that a generic monodromy defect may carry. We find that an elementary $Z_{n_1}$, monodromy defect has a statistics
\[ \theta_{l_1} = 2\pi \left( \frac{m_3}{n_1} + \frac{l_{Z_{n_1}}}{n_1} \right). \] (68)

So $\theta_{l_1} \mod \frac{2\pi}{n_1} = 2\pi \frac{m_3}{n_1}$ is a SPT invariance. Similarly, $\theta_{l_2} \mod \frac{2\pi}{n_2}$ is also a SPT invariance. Let $\theta_{l_2}$ be the mutual statistical angle between an elementary $Z_{n_1}$ monodromy defect and an elementary $Z_{n_2}$ monodromy defect. We find that $\theta_{l_1} \mod \frac{2\pi}{n_1} = 2\pi \frac{m_3}{n_1}$ is a SPT invariance. Here $\{n, m\}$ is the smallest common multiple of $n$ and $m$. Therefore, the statistic of the monodromy defects gives us the following SPT invariants:

\[ \Theta = \left( \begin{array}{c} \theta_{l_1} \mod \frac{2\pi}{n_1} \\ \theta_{l_2} \mod \frac{2\pi}{n_2} \end{array} \right). \] (69)

We note that if we stack two SPT states with SPT invariants $(C, \Theta)$ and $(C', \Theta')$, we obtain a new SPT state with SPT invariants

\[ (C'', \Theta'') = (C, \Theta) + (C', \Theta'). \] (70)

**SPT invariant 14.** In a $2 + 1$D $Z_{n_1} \times Z_{n_2}$ bosonic SPT state labeled by $(m_0, m_2, m_3)$, the statistics/numerical-statistics matrix $\Theta$ can fully detect $m_0$, $m_2$, and $m_3$.

Just like the bosonic $U^G(1) \times U^G(1)$ SPT states can be characterized by the $U^G(1) \times U^G(1)$ Chern-Simons topological term [see Eq. (55)] after we gauge the global symmetry $U^G(1) \times U^G(1)$, the bosonic $Z_{n_1} \times Z_{n_2}$ SPT states can also be characterized by a $Z_{n_1} \times Z_{n_2}$ gauge topological term after we gauge the global $Z_{n_1} \times Z_{n_2}$ symmetry. The $Z_{n_1} \times Z_{n_2}$ gauge topological term is obtained by integrating out the matter fields in a background of $Z_{n_1} \times Z_{n_2}$ gauge configuration. In terms of the discrete differential forms (see Appendix A), the $Z_{n_1} \times Z_{n_2}$ gauge topological term can be written as

\[ \mathcal{L}_{\text{top}} = 2\pi m_0 \omega_3^{Z_{n_2}} + 2\pi m_2 \omega_{2,1}^{Z_{n_2}} + 2\pi m_3 \omega_{3}^{Z_{n_2}}, \] (71)

where $\omega_3^{Z_{n_2}} \in H^3(Z_{n_2}, \mathbb{R}/\mathbb{Z})$, $\omega_3^{Z_{n_1}} \in H^3(Z_{n_1}, \mathbb{R}/\mathbb{Z})$, and $\omega_{2,1}^{Z_{n_2}} \in H^2(Z_{n_2}, H^1(Z_{n_1}, \mathbb{R}/\mathbb{Z}))$. Compared to Eq. (55), the above can be viewed as discrete Chern-Simons terms for $Z_{n_1} \times Z_{n_2}$ gauge fields.

### 2. 1 + 1D

In the above examples, we see that measuring topological responses give rise to a complete set of SPT invariants which fully characterize the SPT states. We believe this is true in general. Next we use this idea to study the $Z_{n_1} \times Z_{n_2}$ SPT states in 1 + 1D and 3 + 1D.

The 1 + 1D bosonic $G = Z_{n_1} \times Z_{n_2}$ SPT states are described by $H^2(G, \mathbb{R}/\mathbb{Z})$, which has the following
decomposition [see Eqs. (C15) and (C17)]:

\[
\mathcal{H}^i(Z, R/Z) = \bigoplus_{k=0}^3 \mathcal{H}^k(Z, R/Z) = \mathcal{H}^i(Z, R/Z) \bigotimes \mathcal{H}^i(\mathbb{Z}_2, R/Z),
\]

(72)

To measure \( m_1 \), we choose the space to be \( S_1 \) and create a twist boundary condition on \( S_1 \) generated by \( g^{(1)} = e^{i2\pi/n_1} \in \mathbb{Z}_n \) [which corresponds to the generating element in \( \mathcal{H}^i(Z, R/Z) \)]. Then we measure the induced \( Z_{n_1} \) charge on \( S_1 \) (which is \( \mathcal{H}^i(Z, R/Z) \)). The physical meaning of the above decomposition is that the induced \( Z_{n_1} \) charge mod \( (n_1, n_2) \) is \( m_1 \) (for details see Sec. IV C). Thus, we get the following.

**SPT invariant 15.** In a 1 + 1D \( Z_{n_1} \times Z_{n_2} \) SPT state labeled by \( m_1 \in \mathcal{H}^i(Z_{n_1} \times Z_{n_2}, R/Z) \), a twist boundary condition on the space \( S_1 \) generated by \( g^{(1)} = e^{i2\pi/n_1} \in \mathbb{Z}_n \) will induce a \( Z_{n_2} \) charge \( m_1 + (n_1, n_2) \) in the ground state.

This example also suggests the following construction of SPT invariant (see Sec. IV C).

**SPT invariant 16.** In order to measure \( m \in \mathcal{H}^{i}(G, R/Z) \bigotimes_{\mathbb{Z}_2} \mathcal{H}^{d-k}(SG, R/Z) \subset \mathcal{H}^{d-k}(G \times SG, R/Z) \), we choose the space-time to have a topology \( M_k \times M_{d-k} \).

Next, we construct a \( G \) gauge configuration on \( M_k \) that corresponds to an element of \( \mathcal{H}^i(G, R/Z) \). In the large \( M_{d-k} \) limit, the system can be viewed as having a \( d - k \)-dimensional space-time with \( SG \) symmetry. Such a \( d - k \)-dimensional system is described by an element in \( \mathcal{H}^{d-k}(SG, R/Z) \), which is the response of the \( G \) gauge configuration. Measuring the responses for all possible \( G \) gauge configurations allows us to measure \( m \in \mathcal{H}^{d-k}(G, R/Z) \bigotimes_{\mathbb{Z}_2} \mathcal{H}^{d-k}(SG, R/Z) \).

In the above example, we try to measure \( m_1 \in \mathcal{H}^{i}(Z_{n_1} \times Z_{n_2}, R/Z) \) by choosing the space-time to be \( S_1 \times M_1 \). “A twist boundary condition on the space \( S_1 \) generated by \( g^{(1)} = e^{i2\pi/n_1} \in \mathbb{Z}_n \) is a \( Z_{n_2} \) gauge configuration that corresponds to the generator in \( \mathcal{H}^{i}(Z_{n_1} \times Z_{n_2}, R/Z) \). “The induced \( Z_{n_2} \) charge in the ground state” corresponds to an element in \( \mathcal{H}^{i}(Z_{n_1} \times Z_{n_2}, R/Z) \).

3. 3 + 1D

The 3 + 1D bosonic \( G = Z_{n_1} \times Z_{n_2} \) SPT states are described by \( \mathcal{H}^i(G, R/Z) \) with the decomposition [see Eq. (C15)]

\[
\mathcal{H}^i(G, R/Z) = \bigoplus_{k=0}^3 \mathcal{H}^k(Z, R/Z) \bigotimes \mathcal{H}^i(\mathbb{Z}_2, R/Z),
\]

(73)

with

\[
\mathcal{H}^i(Z, R/Z) = \mathcal{H}^i(Z, R/Z) \bigotimes \mathcal{H}^i(\mathbb{Z}_2, R/Z) = Z_{n_1} \times Z_{n_2} = \{m_1, m_2\},
\]

(74)

Motivated by the structure of the Kühneth expansion, we can construct SPT invariants in a similar way as we did for the 1 + 1D SPT state. For example, to measure \( m_1 \), we choose the space to be \( S_1 \times M_2 \). We then create a twist boundary condition on \( S_1 \) generated by \( g^{(1)} = e^{i2\pi/n_1} \in \mathbb{Z}_n \) [which corresponds to an element in \( \mathcal{H}^i(Z_{n_1} \times Z_{n_2}, R/Z) \)]. In the small \( S_1 \) limit, the SPT state on \( S_1 \times M_2 \) reduces to a SPT state on \( M_2 \), which is described by \( m_1 + (n_1, n_2) \times \text{integer} \in \mathcal{H}^i(Z_{n_1} \times Z_{n_2}, R/Z) \). The element \( m_1 + (n_1, n_2) \times \text{integer} \in \mathcal{H}^i(Z_{n_1} \times Z_{n_2}, R/Z) \) can be measured by the SPT invariants discussed in Sec. IV E 1. To summarize, we present the following.

**SPT invariant 17.** Consider a 3 + 1D \( Z_{n_1} \times Z_{n_2} \) SPT state labeled by \( m_1, m_2 \) on a space with topology \( M_2 \times S_1 \). Adding the minimal \( Z_{n_2} \) flux through \( S_1 \) will reduce the 3 + 1D \( Z_{n_1} \times Z_{n_2} \) SPT state to a 2 + 1D \( Z_{n_2} \), SPT state on \( M_2 \) labeled by \( m_1 + (n_1, n_2) \times \text{integer} \in \mathcal{H}^i(Z_{n_1} \times Z_{n_2}, R/Z) \). By symmetry, adding the minimal \( Z_{n_2} \) flux through \( S_1 \) will reduce the 3 + 1D \( Z_{n_1} \times Z_{n_2} \) SPT state to a 2 + 1D \( Z_{n_2} \) SPT state on \( M_2 \) labeled by \( m_1 + (n_1, n_2) \times \text{integer} \in \mathcal{H}^i(Z_{n_1} \times Z_{n_2}, R/Z) \).

Just like the bosonic \( U^{SGG}(1) \times U^{UGG}(1) \) SPT states can be characterized by the \( U^{SGG}(1) \times U^{UGG}(1) \) Chern-Simons topological term [see Eq. (55)] after we gauge the global symmetry \( U^{SGG}(1) \times U^{UGG}(1) \), the bosonic \( Z_{n_1} \times Z_{n_2} \) SPT states can also be characterized by a \( Z_{n_1} \times Z_{n_2} \) gauge topological term. If we gauge the global \( Z_{n_1} \times Z_{n_2} \) symmetry and integrate out the matter fields, we get a \( Z_{n_1} \times Z_{n_2} \) gauge topological term in 3 + 1D,

\[
\mathcal{L}_{top} = 2\pi m_1 \omega_{1,3} Z_{n_1} \times Z_{n_2} + 2\pi m_2 \omega_{3,1} Z_{n_1} \times Z_{n_2},
\]

(75)

where \( \omega_{1,3} Z_{n_1} \times Z_{n_2} \in \mathcal{H}^i(Z_{n_1} \times Z_{n_2}, R/Z) \) and \( \omega_{3,1} Z_{n_1} \times Z_{n_2} \in \mathcal{H}^i(Z_{n_2} \times Z_{n_1} \times Z_{n_2}, R/Z) \).

F. 2 + 1D Bosonic \( U(1) \times Z_2 \) SPT phases

In this section, we would like to consider SPT states with symmetry \( G = U(1) \times Z_2 \) in 2 + 1 dimensions. The \( U(1) \times Z_2 \) SPT states are described by \( \mathcal{H}^i(G, R/Z) \), which has the decomposition [see Eq. (C15)]

\[
\mathcal{H}^i(G, R/Z) = \bigoplus_{k=0}^3 \mathcal{H}^k(Z, R/Z) \bigotimes \mathcal{H}^i(\mathbb{Z}_2, R/Z),
\]

(76)

with

\[
\mathcal{H}^3(Z_2) = \mathcal{H}^i(U(1), R/Z) = \mathcal{H}^i(Z_2, R/Z) \bigotimes \mathcal{H}^i(U(1), R/Z) = \mathbb{Z}_2 \\
\mathcal{H}^3(Z_2) = \mathcal{H}^i(U(1), R/Z) = \mathbb{Z}_2 \bigotimes \mathcal{H}^i(Z_2, R/Z) = \mathbb{Z}_2 = \{m_3\},
\]

(77)

- \( m_0 \) labels different 2 + 1D \( U(1) \) SPT states and \( m_3 \) labels different 2 + 1DZ \( Z_2 \) SPT states, whose measurement were discussed before.

**SPT invariant 18.** To measure \( m_2 \in \mathcal{H}^i(Z_2, R/Z) \), we may create two identical \( Z_2 \) monodromy defects on a closed 2D space. We then measure the induced \( U(1) \) charge mod 2, which measures \( m_2 \).
This result can be obtained by viewing the $U(1) \times Z_2$ SPT states as $Z_2 \times Z_2$ SPT states and using the result in Sec. IV E 1. If we gauge the global $U(1) \times Z_2$ symmetry and integrate out the matter fields, we get a $U(1) \times Z_2$ gauge topological term in 2 + 1D,

$$L_{\text{top}} = \frac{m_0}{2\pi} AF + 2\pi m_2 \omega_{Z_2}^U(1) + 2\pi m_3 \omega_{Z_2}^A,$$  \hspace{1cm} (78)

where $\omega_{Z_2}^U(1) \in \mathcal{H}^2(Z_2, \mathcal{H}^1(U(1), \mathbb{R}/\mathbb{Z}))$ and $\omega_{Z_2}^A \in \mathcal{H}^3(Z_2, \mathbb{R}/\mathbb{Z})$. Also, $A$ and $F$ are the gauge potential one form and the field strength two form for the $U(1)$-gauge field. We can further rewrite the above $U(1) \times Z_2$ gauge topological term as

$$L_{\text{top}} = \frac{m_0}{2\pi} AF + m_2 \Omega_{Z_2}^U A + 2\pi m_3 \Omega_{Z_2}^A,$$  \hspace{1cm} (79)

where $\Omega_{Z_2}^U \in \mathcal{H}^2(Z_2, \mathbb{Z})$, which is viewed as a discrete differential two form (see Appendix A 4). $\Omega_{Z_2}^U A = \Omega_{Z_2}^A \wedge A$ is the wedge product of the differential forms.

G. Bosonic $U(1) \times Z_2^T$ SPT phases

In this section, we consider bosonic $U(1) \times Z_2^T$ SPT phases. The $U(1) \times Z_2$ SPT phases can be realized by time-reversal symmetric spin systems where the spin rotation symmetry is partially broken.

1. 1 + 1D

We first consider SPT states with symmetry $G = U(1) \times Z_2^T$ in 1 + 1 dimensions, where $Z_2^T$ is the antisymmetric time-reversal symmetry. The $U(1) \times Z_2^T$ SPT states are described by $\mathcal{H}^2(G, \mathbb{R}/\mathbb{Z})$, which has the decomposition [see Eq. (C15)]

$$\mathcal{H}^2(G, \mathbb{R}/\mathbb{Z}) = \bigoplus_{k=0}^1 \mathcal{H}^2(U(1), \mathcal{H}^{1-k}(Z_2^T, \mathbb{R}/\mathbb{Z})) \oplus \mathcal{H}^2(U(1), Z_2),$$  \hspace{1cm} (80)

with

$$\mathcal{H}^2(U(1), Z_2) = \mathcal{H}^1(U(1), \mathbb{R}/\mathbb{Z}) \otimes \mathcal{H}^0(Z_2^T, \mathbb{R}/\mathbb{Z}) \otimes \mathcal{H}^0(Z_2^T, \mathbb{R}/\mathbb{Z}) \otimes \mathcal{H}^0(Z_2^T, \mathbb{R}/\mathbb{Z}),$$  \hspace{1cm} (81)

$m_0$ labels different 1 + 1D $Z_2^T$ SPT states and $m_2$ labels different 1 + 1D $U(1)$ SPT states whose action amplitudes are real numbers (i.e., $\pm 1$). To measure $m_2$, we put the system on a finite line $L_1$. At an end of the line, we get discrete terms that form a projective representation of $U(1) \times Z_2^T$, which is classified by $\mathcal{H}^2(U(1) \times Z_2^T, \mathbb{R}/\mathbb{Z})$ [44–46]. We find the following.

SPT invariant 19. A 1 + 1D bosonic $U(1) \times Z_2^T$ SPT state labeled by $(m_0, m_2)$ has a degenerate Kramer doublet at an open boundary if $(m_0, m_2) = (1, 0)$ or a degenerate doublet of $U(1)$ charge $\pm 1/2$ if $(m_0, m_2) = (0, 1)$. The time-reversal transformation flips the sign of the $U(1)$ charge.

Another way to probe $m_2$ is to gauge the $U(1)$ symmetry. The $U(1) \times Z_2^T$ SPT states are described by the following gauge topological term (induced by integrating out the matter fields)

$$L_{\text{top}} = \frac{m_2}{2} F,$$  \hspace{1cm} (82)

where $F$ is the field strength two form for the $U(1)$-gauge field. Under $Z_2^T$ transformation,

$$A_0 \rightarrow -A_0, \quad A_i \rightarrow A_i, \quad F \rightarrow -F.$$  \hspace{1cm} (83)

[Note that under $Z_2^T$, the $U(1)$ charge changes sign.] Since $\int_M \frac{m_2}{2} F = m_2 \pi$ x integers, on any closed 1 + 1D space-time manifold $M_2$, the $Z_2^T$ symmetry requires $m_2$ to be quantized as an integer.

If the space-time $M_2$ has a boundary, the above topological term naively reduces to an effective Lagrangian on the boundary

$$L_{0+1D} = \frac{m_2^2}{2} A,$$  \hspace{1cm} (84)

where $A$ is the gauge potential one form. This is nothing but a 1D $U(1)$ Chern-Simons term with a fractional coefficient. However, such a 1D $U(1)$ Chern-Simons term breaks the $Z_2^T$ symmetry, since $A_0 \rightarrow -A_0$ under the time-reversal transformation. So only if the $Z_2^T$ symmetry is broken at the boundary can the topological term reduce to the above 1D Chern-Simons term on the boundary. We find the following.

SPT invariant 20. For a 1 + 1D $U(1) \times Z_2^T$ SPT state on a open chain, if the $Z_2^T$ symmetry is broken at the boundary, the boundary will carry a $U(1)$ charge $m_2/2$ or $-m_2/2$ mod 1.

If the $Z_2$ symmetry is not broken, we have an effective boundary theory,

$$L_{0+1D} = \frac{m_2 \sigma}{2} A + \mathcal{L}(\sigma),$$  \hspace{1cm} (85)

where the $\sigma(x)$ field only takes two values $\sigma = \pm 1$. We see that if $m_2 = 0$, the ground state of the 0 + 1D system is not degenerate (ground) $|\sigma = 1\rangle + |\sigma = -1\rangle$. If $m_2 = 1$, the ground states of the 0 + 1D system is degenerate with $|\sigma = \pm 1\rangle$ states carrying fractional $\pm 1/2$ $U(1)$ charges. Such states form a projective representation of $U(1) \times Z_2^T$.

2. 2 + 1D

Next, we consider SPT states with symmetry $G = U(1) \times Z_2^T$ in 2 + 1 dimensions. The $U(1) \times Z_2^T$ SPT states are described by $\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$, which has the decomposition [see Eqs. (C15) and (C19)]

$$\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z}) = \bigoplus_{k=0}^1 \mathcal{H}^3(U(1), \mathcal{H}^{1-k}(Z_2^T, \mathbb{R}/\mathbb{Z})) \oplus \mathcal{H}^3(U(1), Z_2^T),$$  \hspace{1cm} (86)

Thus, there is no nontrivial $U(1) \times Z_2^T$ SPT states in 2 + 1 dimensions.

3. 3 + 1D

Now we consider $U(1) \times Z_2^T$ SPT states in 3 + 1 dimensions, which are described by $\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$,

$$\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z}) = \bigoplus_{k=0}^1 \mathcal{H}^4(U(1), \mathcal{H}^{1-k}(Z_2^T, \mathbb{R}/\mathbb{Z})) \oplus \mathcal{H}^4(U(1), Z_2^T),$$  \hspace{1cm} (87)
with
\[ H^2(Z^T_1, \mathbb{R}/\mathbb{Z}) = Z_2 = \{m_0\}. \]
\[ H^2(U(1), H^2(Z^T_2, \mathbb{R}/\mathbb{Z})) = H^1(U(1), \mathbb{R}/\mathbb{Z}) \]
\[ \otimes_{\mathbb{Z}} H^2(Z^T_3, \mathbb{R}/\mathbb{Z}) \]
\[ = Z_2 = \{m_4\}. \]
\[ H^4(U(1), Z_2) = H^3(U(1), \mathbb{R}/\mathbb{Z}) \]
\[ \otimes_{\mathbb{Z}} H^2(Z^T_1, \mathbb{R}/\mathbb{Z}) \]
\[ = Z_2 = \{m_4\}. \] (88)

The elements in \( H^4(U(1) \times Z^T_2, \mathbb{R}/\mathbb{Z}) \) can also be labeled by a set of \( \{(m'_0, m'_1, m'_2, m'_3)\} \) (see Appendix D), where
\[ m'_0 \in H^0(Z^T_1, H^4(U(1), \mathbb{R}/\mathbb{Z})) = Z_1, \]
\[ m'_1 \in H^1(Z^T_1, H^4(U(1), \mathbb{R}/\mathbb{Z})) = H^1(Z^T_2, \mathbb{Z}) = Z_2, \]
\[ m'_2 \in H^2(Z^T_1, H^4(U(1), \mathbb{R}/\mathbb{Z})) = Z_1, \] (89)
\[ m'_3 \in H^3(Z^T_1, H^4(U(1), \mathbb{R}/\mathbb{Z})) = H^3(Z^T_2, \mathbb{Z}) = Z_2, \]
\[ m'_4 \in H^4(Z^T_1, H^4(U(1), \mathbb{R}/\mathbb{Z})) = H^4(Z^T_2, \mathbb{R}/\mathbb{Z}) = Z_2, \]

where \( H^4(U(1), \mathbb{R}/\mathbb{Z}) \) means that \( Z^T_2 \) has a nontrivial action on \( H^4(U(1), \mathbb{R}/\mathbb{Z}) \). Again, we see that \( H^4(U(1) \times Z^T_2, \mathbb{R}/\mathbb{Z}) \) is a Z_1-graded 1D manifold. We then put a U(1)-gauge configuration that carries a unit of the U(1) gauge flux \( \int_{M_2} F = 1 \) on \( M_2 \). In the large I_1 limit, we may view the system as a 1 + 1D system on I_1 with the same U(1) \( \times Z^T_2 \) symmetry (note that the U(1) flux does not break the \( Z^T_2 \) time-reversal symmetry). The resulting 1 + 1D U(1) \( \times Z^T_2 \) SPT state is classified by \( H^4(U(1) \times Z^T_2, \mathbb{R}/\mathbb{Z}) = Z_{2} \) discussed in Sec. IV G 1. For such a setup, a nonzero \( m_2 \) (and \( m_0 = m_4 = 0 \)) will give rise to a degenerate Kramer doublet at each end of the line I_1, which carry no U(1) charge. We find the following.

SPT invariant 21. In a 3 + 1D bosonic U(1) \( \times Z^T_2 \) SPT state labeled by \( (m_0, m_2, m_4) = (0, 1, 0) \), a U(1) monopole of unit magnetic charge will carry a U(1)-neutral degenerate Kramer doublet.

From Sec. IV G 1, we also know that the other kind of 1 + 1D U(1) \( \times Z^T_2 \) SPT states is characterized by the degenerate doublet states of U(1) charge \( \pm 1/2 \) at each end of the line I_1. One may wonder if a nonzero \( m_4 \) (and \( m_2 = m_0 = 0 \)) will give rise to such a 1 + 1D U(1) \( \times Z^T_2 \) SPT state on the line I_1? In the following, we argue that a nonzero \( m_4 = 1 \) does not give rise to a nontrivial 1 + 1D U(1) \( \times Z^T_2 \) SPT state.

As before, a way to probe \( m_4 \) is to gauge the U(1) symmetry. We believe that the U(1) \( \times Z^T_2 \) SPT states labeled by \( (m_0, m_2, m_4) = (0, 0, m_4) \) are described by the following U(1)-gauge topological term:
\[ \mathcal{L}_{\text{top}} = \frac{m_4 \pi}{(2\pi)^2} F^2. \] (90)

Under the \( Z^T_2 \) transformation, \( F^2 \rightarrow -F^2 \) and \( e^{i\phi} \frac{\alpha_0}{\sin^2 k} F^2 \rightarrow e^{-i\phi} \frac{\alpha_0}{\sin^2 k} F^2 \). Because \( \int_{M_1} \frac{m_4 \pi}{(2\pi)^2} F^2 = \pi m_4 \) \( \times \) integers, on any closed 3 + 1D orientable space-time manifold \( M_4 \), the \( Z^T_2 \) symmetry is not broken due to the fact that \( m_4 \) is an integer. \( m_4 \) \( \equiv \) odd describes the nontrivial 3 + 1D U(1) \( \times Z^T_2 \) SPT state, while \( m_4 = \) even describes the trivial SPT state.

If we put a U(1)-gauge configuration that carries a unit of the U(1) gauge flux \( \int_{M_1} F = 1 \) on \( M_2 \), the above 3 + 1D U(1)-gauge topological term (90) will reduce to a 1 + 1D U(1)-gauge topological term:
\[ \mathcal{L}_{\text{top}} = \frac{m_4 \pi}{(2\pi)^2} F^2. \] (91)

Compared to Eq. (82), we see that even \( m_4 = 1 \) will give rise to a trivial 1 + 1D U(1) \( \times Z^T_2 \) SPT state.

To measure \( m_4 \), we need to use the statistical effect discussed in Refs. [70,100,101].

SPT invariant 22. In a 3 + 1D bosonic U(1) \( \times Z^T_2 \) SPT state labeled by \( (m_0, m_2, m_4) = (0, 0, m_4) \), a dyon of the U(1) gauge field with \( [U(1) \text{ charge, magnetic charge}] = (q, m) \) has a statistics \( (−1)^{m_4−m_0} \) (where \( + \rightarrow \text{boson and } − \rightarrow \text{fermion})

If the space-time \( M_2 \) has a boundary, the topological term (90) reduces to an effective Lagrangian on the boundary,
\[ \mathcal{L}_{\text{top}} = \frac{m_4 \pi}{4\pi} A F. \] (92)

if the \( Z^T_2 \) time-reversal symmetry is broken on the boundary. The above is nothing but a 2 + 1D U(1) Chern-Simons term with a quantized Hall conductance \( \sigma_x = m_4 / 2 \pi \). We note that if a 2 + 1D state with \( U(1) \) symmetry has no topological order, a Hall conductance must be quantized as even integer \( \sigma_x = \text{even} / 2 \pi \). Thus, we get the following.

SPT invariant 23. In a 3 + 1D bosonic U(1) \( \times Z^T_2 \) SPT state labeled by \( (m_0, m_2, m_4) \), the gauged time-reversal symmetry-breaking boundary has a Hall conductance \( \sigma_x = \frac{m_4}{2\pi} + \text{even} \).

If the \( Z^T_2 \) symmetry is not broken, we actually have an effective boundary theory,
\[ \mathcal{L}_{\text{top}} = \frac{m_4 \pi}{4\pi} A F + \mathcal{L}(\sigma). \] (93)

where the \( \sigma(\chi) \) field only takes two values \( \sigma = \pm 1 \). The gapless edge states on the domain wall between \( \sigma = 1 \) and \( \sigma = -1 \) regions may give rise to the gapless boundary excitations on the 2 + 1D surface.

H. Bosonic \( Z_2 \times Z^T_2 \) SPT phases

In this section, we consider bosonic \( Z_2 \times Z^T_2 \) SPT phases. The \( Z_2 \times Z^T_2 \) SPT phases can be realized by time-reversal symmetric spin systems where the spin rotation symmetry is partially broken.

1. 1 + 1D

We first consider SPT states with symmetry \( G = Z_2 \times Z^T_2 \) in 1 + 1 dimensions. The \( Z_2 \times Z^T_2 \) SPT states are described by \( \mathcal{H}(G, \mathbb{R}/\mathbb{Z}) \), which has a decomposition [see Eq. (C15)],
\[ \mathcal{H}(G, \mathbb{R}/\mathbb{Z}) = \bigoplus_{k \in \mathbb{Z}} \mathcal{H}(Z_2, \mathcal{H}^{2-k}(Z^T_2, \mathbb{R}/\mathbb{Z})). \]
\[ = \mathcal{H}(Z^T_2, \mathbb{R}/\mathbb{Z}) \oplus \mathcal{H}(Z_2, Z_2). \] (94)
with
\[ H^2(Z_2, \mathbb{R}/\mathbb{Z}) = H^0(Z_2, \mathbb{R}/\mathbb{Z}) \otimes H^2(Z_2^T, \mathbb{R}/\mathbb{Z}) \]
\[ = Z_2 = \{m_0\}, \]
\[ H^2(Z_2, Z_2) = H^1(Z_2, \mathbb{R}/\mathbb{Z}) \otimes H^0(Z_2^T, \mathbb{R}/\mathbb{Z}) \]
\[ = Z_2 = \{m_2\}. \]  
(95)

\( m_0 \) labels different \( 1 + 1 \)D \( Z_2^T \) SPT states and \( m_2 \) labels different \( 1 + 1 \)D \( Z_2 \) SPT states whose action amplitudes are real numbers (i.e., \( \pm 1 \)). To measure \( m_k \), we put the system on a finite line \( I_1 \). At an end of the line, we get degenerate states that form a projective representation of \( Z_2 \times Z_2^T \), which is classified by \( H^2(Z_2, Z_2^T, \mathbb{R}/\mathbb{Z}) \) \cite{44-46}. We find the following.

**SPT invariant 24.** A \( 1 + 1 \)D bosonic \( Z_2 \times Z_2^T \) SPT state labeled by \( (m_0, m_2) \) has a degenerate Kramer doublet at an open boundary if \( (m_0, m_2) = (1, 0) \) or a degenerate doublet of \( Z_2 \) charge \( \pm 1/2 \) mod \( 2 \) if \( (m_0, m_2) = (0, 1) \). The time-reversal transformation flips the sign of the \( Z_2 \) charge.

The above result can also be understood by viewing the \( U(1) \times Z_2^T \) SPT phases discussed in the last section as \( Z_2 \times Z_2^T \) SPT phases.

### 2. \( 2 + 1 \)D

Next, we consider SPT states with symmetry \( G = Z_2 \times Z_2^T \) in \( 2 + 1 \) dimensions. The \( Z_2 \times Z_2^T \) SPT states are described by \( H^3(G, \mathbb{R}/\mathbb{Z}) \), which has the decomposition [see Eq. (C15)]

\[ H^3(G, \mathbb{R}/\mathbb{Z}) = \oplus_{\lambda=0}^{\lambda=1} H^3(U(1), \mathbb{R}/\mathbb{Z}) \]
\[ = H^3(Z_2, \mathbb{R}/\mathbb{Z}) \oplus H^3(Z_2^T, \mathbb{R}/\mathbb{Z}) \]
\[ = H^3(Z_2) \oplus H^3(Z_2^T) \]  
(96)

with [see Eq. (C18)]

\[ H^1(Z_2, \mathbb{R}/\mathbb{Z}) \otimes H^2(Z_2^T) \mathbb{R}/\mathbb{Z} \]
\[ = H^1(Z_2, \mathbb{R}/\mathbb{Z}) \otimes Z_2 \]
\[ = Z_2 = \{m_1\}, \]

\[ H^1(Z_2, Z_2) = H^1(Z_2, \mathbb{R}/\mathbb{Z}) \otimes H^0(Z_2^T, \mathbb{R}/\mathbb{Z}) \]
\[ = Z_2 = \{m_3\}. \]  
(97)

We note that \( m_3 \) actually describe the \( Z_2 \) SPT order in \( 2 + 1 \)D. Such a nontrivial \( Z_2 \) SPT order can survive even if we break the time-reversal symmetry. We can use the fractional statistics of the \( Z_2 \) monodromy defects to detect \( m_3 \) (see Sec. III A 4).

To detect/measure \( m_1 \) we can use the results in SPT invariant 16. We first choose the space-time topology to be \( S_1 \times M_2 \). We next add a \( Z_2 \) flux through \( S_1 \), which is an element in \( H^1(Z_2, \mathbb{R}/\mathbb{Z}) \). We then measure its response by measuring the induced \( Z_2^T \) SPT state on \( M_2 \), which is an element in \( H^2(Z_2^T, \mathbb{R}/\mathbb{Z}) \). Thus, we present the following.

**SPT invariant 25.** Consider a \( 2 + 1 \)D \( Z_2 \times Z_2^T \) SPT state. The \( Z_2 \) monodromy defect will carry a degenerate Kramer doublet if \( m_1 = 1 \) and no Kramer doublet if \( m_1 = 0 \).

### 3. \( 3 + 1 \)D

Now we consider \( Z_2 \times Z_2^T \) SPT states in \( 3 + 1 \) dimensions, which are described by \( H^4(G, \mathbb{R}/\mathbb{Z}) \),

\[ H^4(G, \mathbb{R}/\mathbb{Z}) = \oplus_{k=0}^{k=1} H^4(Z_2, \mathbb{R}/\mathbb{Z}) \]
\[ = H^4(Z_2, \mathbb{R}/\mathbb{Z}) \]
\[ \oplus H^4(Z_2, H^2(Z_2^T, \mathbb{R}/\mathbb{Z})) \oplus H^4(Z_2, Z_2), \]  
(98)

with

\[ H^4(Z_2, \mathbb{R}/\mathbb{Z}) = Z_2 = \{m_0\}, \]

\[ H^4(Z_2, H^2(Z_2^T, \mathbb{R}/\mathbb{Z})) = Z_2 = \{m_2\}, \]

\[ H^4(Z_2, Z_2) = Z_2 = \{m_4\}. \]  
(99)

\( m_0 \) labels different \( 3 + 1 \)D \( Z_2^T \) SPT states, and \( m_4 \) labels different \( 3 + 1 \)D \( Z_2 \) SPT states whose action amplitudes are real numbers (i.e., \( \pm 1 \)).

To detect \( m_2 \), we consider a \( 3 \)D space with topology \( M_2 \times I_1 \) where \( M_2 \) is a closed \( 2 \)D manifold and \( I_1 \) is a \( 1 \)D segment. In the large \( I_1 \) limit, we may view the system as a \( 1 + 1 \)D gapped state with \( Z_2^T \) symmetry. Let us assume that the end of \( I_1 \) does not carry degenerate Kramer doublet. We then put two identical \( Z_2 \) monodromy defects on \( M_2 \). In the large \( I_1 \) limit, we may view the system as a \( 1 + 1 \)D system on \( I_1 \) with the same \( Z_2 \times Z_2^T \) symmetry (note that the \( Z_2 \) monodromy defects do not break the \( Z_2^T \) time-reversal symmetry). The resulting \( 1 + 1 \)D \( U(1) \times Z_2^T \) SPT state is classified by \( H^4(Z_2 \times Z_2^T, \mathbb{R}/\mathbb{Z}) = Z_2^3 \) discussed in Sec. IV H 1. For such a setup, a nonzero \( m_2 \) (and \( m_0 = m_4 = 0 \)) will give rise to a degenerate Kramer doublet at each end of the line \( I_1 \) which carry no \( Z_2 \) charge. We find the following.

**SPT invariant 26.** In a \( 3 + 1 \)D bosonic \( Z_2 \times Z_2^T \) SPT state labeled by \( (m_0, m_2, m_4) = (0, 1, 0) \), two identical \( Z_2 \) monodromy defects on the surface of the sample will induce a \( Z_2 \)-neutral degenerate Kramer doublet.

To detect \( m_4 \), let us view the \( U(1) \times Z_2^T \) SPT state as a \( Z_2 \times Z_2^T \) SPT state and assume that a \( U(1) \times Z_2^T \) SPT state described by \( m_4 \) is a \( Z_2 \times Z_2^T \) SPT state described by the same \( m_4 \). [Note that the \( U(1) \times Z_2^T \) SPT states and the \( Z_2 \times Z_2^T \) SPT states are labeled by the same set of \( m_4 \) quantum numbers.] We have seen that if the space-time \( M_2 \) has a boundary, the gapped boundary of \( 3 + 1 \)D \( U(1) \times Z_2^T \) SPT state can have a quantized Hall conductance \( \sigma_{xy} = m_4/2\pi \) if the time-reversal symmetry is broken only at the boundary. Let assume that we have a fat \( U(1) \) flux on the surface. Such a fat \( U(1) \) flux will induce a \( U(1) \) charge \( m_4/2 \). Similarly, a fat \( U(1) \) flux will induce a \( U(1) \) charge \( -m_4/2 \). If we shrink the fat \( U(1) \) flux to a point, the \( U(1) \) flux will become the same \( Z_2 \) monodromy defect, which will have degenerate states with \( U(1) \) charge \( \pm m_4/2 \).

**SPT invariant 27.** Consider a gapped surface of the \( Z_2 \times Z_2^T \) SPT state that breaks the \( Z_2^T \) symmetry. If \( m_1 = 1 \), then a \( Z_2 \) monodromy line defect that ends on the surface will have two degenerate states whose \( Z_2 \) charges differ by \( 1 \), or the \( Z_2 \) monodromy line defect in the bulk is gapless.

We also know that for a \( U(1) \times Z_2^T \) SPT state, a gapped time-reversal symmetry-breaking boundary has a Hall
condactance \( \sigma_{xy} = 1/2\pi \). As a result, the symmetry-breaking domain wall will carry the edge excitations of the 2 + 1D \( U(1) \) SPT state with a Hall conductance \( \sigma_{xy} = 1/\pi \). We can view the \( U(1) \times Z_2^f \) SPT state as a \( Z_2 \times Z_2^f \) SPT state and the edge excitations of the 2 + 1D \( U(1) \) SPT state as the edge excitations of the 2 + 1D \( Z_2 \) SPT state. This way, we find the following.

**SPT invariant** 28. Consider a gapped surface of a \( Z_2 \times Z_2^f \) SPT state that breaks the \( Z_2^f \) symmetry. If \( m_4 = 1 \), then the symmetry-breaking domain wall will carry the edge excitations of the 2 + 1D \( Z_2 \) SPT state.

### I. 2 + 1D fermionic \( U(1) \times Z_2^f \) SPT phases

The fermionic \( U(1) \times Z_2^f \) SPT phases can be realized by systems with two types of fermions; one carries the \( U(1) \) charge and the other is neutral. To construct the SPT invariants for the fermionic \( U(1) \times Z_2^f \) SPT states, we again gauge the \( U(1) \times Z_2^f \) symmetry and then put the fermion system on a 2D closed space \( M_2 \) with a \( U(1) \times Z_2^f \) gauge configuration that carries a unit of the \( U(1) \) gauge flux \( \int_{M_2} \mathcal{F} = 1 \). Then we measure the \( U(1) \) charge \( c_{11} \) and the \( Z_2^f \) charge \( c_{12} \) of the ground state on \( M_2 \) induced by the \( U(1) \) gauge flux. Next, we put another \( U(1) \times Z_2^f \) gauge configuration on \( M_2 \) with no \( U(1) \) gauge flux but two identical \( Z_2^f \) vortices, then measure the \( U(1) \) charge \( c_{21} \mod 2 \) and the \( Z_2^f \) charge \( c_{22} \). So an integer matrix \( C \) formed by \( c_{ij} \),

\[
C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \mod 2. \tag{100}
\]

is a potential SPT invariant for fermionic \( U(1) \times Z_2^f \) SPT phases in 3D space-time.

However, which SPT invariants can be realized? What are the actual SPT invariants? One way to realize the fermionic \( U(1) \times Z_2^f \) SPT phases is to view them as the fermionic \( U(1) \times U(1) \) SPT phases discussed in Sec. IV B. Using the \( U(1) \times U(1) \) Chern-Simons theory for the fermionic \( U(1) \times U(1) \) SPT phases, we see that the SPT invariant

\[
C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{101}
\]

can be realized.

By binding the \( U(1) \)-charged fermion and neutral fermion to formug \( U(1) \)-charged boson, we can form other fermionic \( U(1) \times Z_2^f \) SPT phases through the bosonic \( U(1) \) SPT phases of the above bosonic bound states. This allows us to realize a SPT invariant,

\[
C' = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{102}
\]

which is twice that of \( C_1 \). This suggests that the realizable SPT invariants are \( C_1 \times \text{integers} \).

To summarize, some of the fermionic \( U(1) \times Z_2^f \) SPT phases are described by \( Z \) in 3D space-time, whose SPT invariant is \( C_1 \) times an integer. It is not clear if those are all the fermionic \( U(1) \times Z_2^f \) SPT phases. The integer \( Z \) that labels the fermionic \( U(1) \times Z_2^f \) SPT phases corresponds to the integer Hall conductance. This result should be contrasted with the result for the fermionic \( U^f(1) \) SPT phases discussed in Sec. III D, where the Hall conductance is quantized as 8 times integer.

### J. 2 + 1D fermionic \( Z_2 \times Z_2^f \) SPT states

Now let us consider fermionic SPT states with full symmetry \( Z_2 \times Z_2^f \) in 2 + 1 dimensions. This kind of fermionic SPT states were studied in Ref. [77] using group super-cohomology theory where four fermionic \( Z_2 \times Z_2^f \) SPT states (including the trivial one) were constructed. They were also studied in Refs. [71,102], where 8 SPT states were obtained (see also Refs. [103,104]). To construct SPT invariants for the fermionic \( Z_2 \times Z_2^f \) SPT states, we may create two identical \( Z_2 \) monodromy defects on a closed 2D space. We then measure the induced \( Z_2 \) charge \( c_{11} \) and the \( Z_2^f \) charge \( c_{12} \). We then create two identical \( Z_2^f \) monodromy defects and measure the induced \( Z_2 \) charge \( c_{21} \) and the \( Z_2^f \) charge \( c_{22} \). Note that \( c_{ij} = c_{ji} \mod 2 = 0.1 \). Thus, there are eight potential different SPT invariants described by a two-by-two symmetric integer matrix,

\[
C = \begin{pmatrix} c_{11} & c_{21} \\ c_{21} & c_{22} \end{pmatrix} \mod 2. \tag{103}
\]

More general SPT invariants can be obtained from the statistics of the monodromy defects. Let \( \theta_{11} \mod \pi \) be the statistic angle of the \( Z_2 \) monodromy defect and \( \theta_{22} \mod 2\pi \) be the statistic angle of the \( Z_2^f \) monodromy defect. Note that adding a \( Z_2 \) neutral fermion to a \( Z_2 \) monodromy defect will change its statistic angle by \( \pi \). So \( \theta_{11} \) is only well defined \( \mod \pi \). Adding a fermion to a \( Z_2^f \) monodromy defect will not change its statistic since a fermion always carries a nontivial \( Z_2 \) charge. So \( \theta_{22} \) is well defined \( \mod 2\pi \) but \( \theta_{22} \mod 2\pi \) is a mutual statistic angle \( \theta_{12} \mod \pi \). Note that adding a fermion to a \( Z_2 \) monodromy defect will change the mutual statistics angle \( \theta_{12} \mod \pi \), and thus \( \theta_{12} \mod \pi \) is well defined \( \mod \pi \). So the statistic of the monodromy defects gives us the following SPT invariants

\[
\Theta = \begin{pmatrix} \theta_{11} \mod \pi & \theta_{12} \mod \pi \\ \theta_{12} \mod \pi & \theta_{22} \mod 2\pi \end{pmatrix}. \tag{104}
\]

However, which values of the above SPT invariants can be realized by actual fermion systems? We may view the 2 + 1D fermionic \( U(1) \times U^f(1) \) SPT states discussed in Sec. IV B as fermionic \( Z_2 \times Z_2^f \) SPT states. The different \( U(1) \times U^f(1) \) SPT states can be obtained by stacking a fermion system where the \( Z_2 \)-charged fermions form a \( v = 1 \) integer quantum Hall state and the \( Z_2 \)-neutral fermions form a \( v = -1 \) double integer quantum Hall state. Such a \( (v = 1)/(v = -1) \) double integer quantum Hall state can realize the SPT invariants

\[
C_j = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mod 2, \tag{105}
\]

\[
\Theta_j = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{12} & \theta_{22} \end{pmatrix} = \begin{pmatrix} \pi/4 \mod \pi & \pi/2 \mod \pi \\ \pi/2 \mod \pi & 0 \mod 2\pi \end{pmatrix}. \tag{105}
\]
This because a monodromy defect of \( \mathbb{Z}_2 \) in the \((v = 1)/(v = -1)\) double integer quantum Hall state carries a \((\mathbb{Z}_2, \mathbb{Z}_2^l)\) charge \((1/2, 1/2)\) + integer and a statistics \(\theta_{11} = \pi/4 \mod \pi\), while a monodromy defect of \(\mathbb{Z}_2^l\) in the \((v = 1)/(v = -1)\) double integer quantum Hall state carries a \((\mathbb{Z}_2, \mathbb{Z}_2^l)\) charge \((1/2, 0)\) + integer and a statistics \(\theta_{22} = 0\). Also, moving a \(\mathbb{Z}_2\) monodromy defect around a \(\mathbb{Z}_2^l\) monodromy defect gives us a mutual statistics \(\theta_{12} = \pi/2 \mod 2\pi\).

If we assume that the fermions form bound states, we get a bosonic system with \(\mathbb{Z}_2\) symmetry. Such a bosonic system can realize a SPT invariant

\[
\mathbf{C}_2 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \mod 2,
\]

\[
\mathbf{\Theta}_2 = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{12} & \theta_{22} \end{pmatrix} = \begin{pmatrix} \pi/2 \mod \pi & 0 \mod \pi \\ 0 \mod \pi & 2\pi \end{pmatrix}.
\]

The calculation of \(\mathbf{C}_2\) was discussed in Sec. IIIA and the calculation \(\theta_{11}\) was given by Eq. (20). The other entries of \(\mathbf{\Theta}_2\) are obtained by noting the the \(\mathbb{Z}_2^l\) monodromy defect is trivial since the \(\mathbb{Z}_2^l\) symmetry acts trivially. We note that \((2\mathbf{C}_1, 2\mathbf{\Theta}_1) = (\mathbf{C}_2, \mathbf{\Theta}_2)\). So it is possible that the bosonic \(\mathbb{Z}_2\) SPT state is the same SPT state obtained by stacking two \((v = 1)/(v = -1)\) double integer quantum Hall states.

As we have mentioned that the SPT invariant \((C_1, \Theta_1)\) is realized by a fermion system where the \(\mathbb{Z}_2\)-charged fermions form a \(v = 1\) integer quantum Hall state and the \(\mathbb{Z}_2\)-neutral fermions form a \(v = -1\) integer quantum Hall state. We can have a new SPT invariant which is realized by a fermion system where the \(\mathbb{Z}_2\)-charged fermions form a \(p + ip\) superconducting state and the \(\mathbb{Z}_2\)-neutral fermions form a \(p - ip\) superconducting state [95,99]. We note that the \(\mathbb{Z}_2\) monodromy defects in the \((p + ip)/(p - ip)\) superconducting state will have non-Abelian statistics [95]. We cannot simply use \(\mathbf{\Theta}_1/2\) to describe their statistics. We also note that two \(\mathbb{Z}_2\) monodromy defects in the \((p + ip)/(p - ip)\) superconducting state have topological degeneracy [95,99], where the two degenerate states carry different \(\mathbb{Z}_2\) and \(\mathbb{Z}_2^l\) quantum numbers. We cannot simply use \(C_1/2\) to describe the induced \(\mathbb{Z}_2\) and \(\mathbb{Z}_2^l\) charges either.

Stacking four \((v = 1)/(v = -1)\) double integer quantum Hall states [one eight \((p + ip)/(p - ip)\) superconducting states] will give us a trivial fermionic \(\mathbb{Z}_2 \times \mathbb{Z}_2^l\) SPT state since \((4\mathbf{C}_1, 4\mathbf{\Theta}_1)\) is trivial. This agrees with the result obtained in Ref. [95].

Let us examine the assumption that the fermionic \(\mathbb{Z}_2 \times \mathbb{Z}_2^l\) SPT phases are described by \(m_k \in \mathcal{H}^\ell(\mathbb{Z}_2, \mathbb{Z}_2^l)\) \(k = 0, 1, 2\), and \(m_3 \in b\mathbb{SPT}^3_{\mathbb{Z}_2}\) (note that \(\mathbb{Z}_2\) does not contain \(\mathbb{Z}_2^l\) and is a symmetry for the bosonic two-fermion bound states discussed above). Using \(b\mathbb{SPT}^3_{\mathbb{Z}_2} = \mathbb{Z}_2\) and \(f\mathbb{SPT}^k_{\mathbb{Z}_2} = 0\) for \(k > 1\), we have

\[
m_0 = 0, \quad m_1 = 0,
\]

\[
m_2 \in \mathcal{H}^\ell(\mathbb{Z}_2, f\mathbb{SPT}^1_{\mathbb{Z}_2}) = \mathcal{H}^\ell(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2 \quad (107)
\]

\[
m_3 \in b\mathbb{SPT}^3_{\mathbb{Z}_2} = \mathcal{H}^\ell(\mathbb{Z}_2, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2.
\]

The above only gives us four different SPT states. So not all fermionic \(\mathbb{Z}_2 \times \mathbb{Z}_2^l\) SPT phases can be described by \(m_k \in \mathcal{H}^\ell(\mathbb{Z}_2, f\mathbb{SPT}^{k+1}_{\mathbb{Z}_2})\) \(k = 0, 1, 2\), and \(m_3 \in b\mathbb{SPT}^3_{\mathbb{Z}_2}\).

V. GAPLESS BOUNDARY EXCITATIONS OR DEGENERATE BOUNDARY STATES AS EXPERIMENTALLY MEASURABLE SPT INVARIANTS

In the above, we have discussed many SPT invariants for SPT states. However, those SPT invariants are designed for numerical calculations and can be probe by numerical calculations. They are hard to measure in real experiments. In this section, we argue the following.

SPT invariant 29. A nontrivial SPT state with symmetry \(G\), must have gapless boundary excitations or degenerate boundary states that transform nontrivially under the symmetry transformations, even when the symmetry is not spontaneously broken at the boundary.

These low-energy states can be probed by perturbations that break the symmetry.

The above result is proven for 2 + 1D SPT states in Ref. [47], which has a stronger form.

SPT invariant 30. A nontrivial 2 + 1D SPT state with symmetry \(G\), must have gapless boundary excitations that transform nontrivially under the symmetry transformations, even if the symmetry is not spontaneously broken at the boundary.

This is due to the fact there are no (intrinsic) topological orders in 1 + 1D. In the following, we present some arguments for the above result through a few simple examples. The new arguments are valid for higher dimensions.

A. Bosonic \(\mathbb{Z}_n\) SPT state in 2 + 1D

We have shown that, in a nontrivial 2 + 1D \(\mathbb{Z}_n\) SPT state labeled by \(m \in \mathcal{H}^\ell(\mathbb{Z}_n, \mathbb{R}/\mathbb{Z})\), \(m \neq 0\), \(n\) identical \(\mathbb{Z}_n\) monodromy defects will carry a total \(Z_n\) charge \(2m \mod n\) (see Sec. III A3). We may realize the \(n\) identical \(\mathbb{Z}_n\) monodromy defects through \(n\) large holes in the 2D space (see Fig. 4). Let us assume that the \(\mathbb{Z}_n\) symmetry is not spontaneously broken.

FIG. 4. (Color online) A \(\mathbb{Z}_2\)-gauge configuration with two identical holes on a torus that contains a \(\mathbb{Z}_2\) monodromy defect in each hole. Such a \(\mathbb{Z}_2\)-gauge configuration has \(U(-1) = -1\) (each yellow triangle contributes a factor of \(-1\)) (see Fig. 3).
at the edge of the holes. Then depending on if a hole contains a $Z_m$ monodromy defect or not, the $Z_n$ charge of the hole will be $2m/n$ or $0$.

In the large hole limit, adding a monodromy defect to a hole corresponds to twisting the boundary condition as we go around the edge of the hole. Such a twist of boundary condition costs zero energy in the large hole limit (since the branch cut of a monodromy defect costs no energy). If twisting the boundary condition around the edge changes the $Z_n$ charge on the edge by $2m/n$, then we change the $Z_n$ charge on the edge by $2m$ if we make n identical twists of the boundary condition around the edge. Since twists cost zero energy and $n$ twists are equivalent to no twist, this way, we show the following.

$SPT$ invariant 31. The edge of $2+1D$ $Z_n$ SPT state labeled by $m \in \mathcal{H}^3(Z_n,\mathbb{R}/\mathbb{Z})$ contains nearly degenerate ground states that carry different $Z_n$ charges (by $2m$) in the large edge limit.

According to the above result, when $n = 0$, there will be (at least) $n$-fold degenerate edge states, and when $n = \pm 1$, there will be (at least) $n/2$-fold degenerate edge states.

So the edge states of the holes must be gapless or degenerate, at least when $n > 2$. Also the gapless low-energy excitations or the degenerate states must transform nontrivially under the the $Z_n$ symmetry transformations. In Refs. [73, 76], using the nontrivial statistics of the monodromy defects, one can argue more generally that edge states of the holes must be gapless or degenerate even for the $n = 2$ case.

B. Bosonic $Z_n$ SPT state in 4 + 1D

Next, we consider bosonic $Z_n$ SPT state in 4 + 1D, labeled by $m \in \mathcal{H}^3(Z_n,\mathbb{R}/\mathbb{Z})$, $m \neq 0$. We assume the space to have a topology $M_2 \times M_2$. We have shown that, n identical $Z_n$ monodromy defects in $M_2$ will induce a $2+1D$ $Z_n$ SPT state on $M_2$, labeled by $3m \in \mathcal{H}^3(Z_n,\mathbb{R}/\mathbb{Z})$ (see Sec. III A 6). Again, we can realize the n identical $Z_n$ monodromy defects through $n$ large holes on $M_2$ and assume that the $Z_n$ symmetry is not spontaneously broken at the edge of the holes. Then depending on if each hole contains a $Z_n$ monodromy defect or not, the $2+1D$ $Z_n$ SPT state on $M_2$ will be labeled by $3m$ or $0$ in $\mathcal{H}^3(Z_n,\mathbb{R}/\mathbb{Z})$. We see that twisting the boundary condition around the edges of the $n$ holes changes the $2+1D$ $Z_n$ SPT state on $M_2$. Since each twist costs no energy in the large hole limit, the edge states of a hole must be gapless or degenerate, at least when mod$(3m,n) \neq 0$.

C. $U(1)$ SPT state in 2 + 1D and beyond

We have discussed bosonic and fermionic $U(1)$ SPT states in 2 + 1D. Those $U(1)$ SPT states are characterized by a nonzero Hall conductance. In Refs. [105, 106], it was shown that a nonzero Hall conductance implies gapless edge excitations. Here we review the argument.

We consider a 2D space with a hole and $2\pi n$ flux far away from the hole (see Fig. 5). We assume that there is no $U(1)$ symmetry breaking. The $2\pi n$ flux will induce a nonzero charge $Q = nm$, $m \in \mathbb{Z}$. As we move the $2\pi n$ flux into the hole, the induced $U(1)$ charge will become the charge on the edge. Since $2\pi n$ flux in the hole does not change the boundary condition, the induced $U(1)$ charge is an excitation of the edge.

If the $2\pi n$ flux is generated by a weak field, moving the $2\pi n$ flux into the hole represents a weak perturbation. Since the weak perturbation causes a finite change in the induced charge and also since the fact that there are infinite many weak perturbations causes infinite many different changes in the induced charges, the excitations on the edge of the hole is gapless.

We can also use a similar argument to show the following.

$SPT$ invariant 32. Nontrivial bosonic and fermionic $U(1)$ SPT states have gapless boundary excitations in any dimensions.

VI. SPT INVARIANTS OF SPT STATES WITH SYMMETRY $G = GG \times SG$

In this section, we discuss some examples of SPT states where the symmetry group has a form $G = GG \times SG$.

A. Bosonic $U(1) \times Z_2$ SPT phases

Let us first consider bosonic $U(1) \times Z_2$ SPT phases. We note that $U(1) \times Z_2$ is a subgroup of SO(3). Therefore, the $U(1) \times Z_2$ SPT phases can be realized by spin systems where the spin rotation symmetry is partially broken.

1. 1 + 1D

The SPT states with a non-Abelian symmetry $U(1) \times Z_2$ in 1 + 1 dimensions are described by $\mathcal{H}^2(U(1) \times Z_2,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}$, whose elements can be labeled by a subset of $\{m_0, m_1, m_2\}$, according to the result in Appendix D:

$$m_0 \in \mathcal{H}^2(\mathbb{Z}_2,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_1,$$

$$m_1 \in \mathcal{H}^1(\mathbb{Z}_2,\mathcal{H}^1(U(1),\mathbb{R}/\mathbb{Z})_{\mathbb{Z}_2}) = \mathcal{H}^1(\mathbb{Z}_2,\mathbb{Z}_2) = \mathbb{Z}_2,$$

$$m_2 \in \mathcal{H}^2(\mathbb{Z}_2,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_1.$$ (108)

The second equation in the above is obtained by noting that the nonhomogenous cocycle $\omega_1(\theta) \in \mathcal{H}^1(U(1),\mathbb{R}/\mathbb{Z}) \equiv \mathbb{Z}$ has a form $\omega_1(\theta) = m_2 \frac{\theta}{2\pi}$, $m_2 \in \mathbb{Z}$ [i.e., $e^{i2\pi \omega_1(\theta)}$ forms a 1D representation of the $U(1)$]. Under the $Z_2$ transformation $g$, $\omega_1(\theta)$ transforms as $\omega_1(\theta) \mapsto \omega_1(g\theta g^{-1}) = -\omega_1(\theta)$ or $m \mapsto -m$, since $g^2 \equiv -1$. Therefore, $Z_2$ has a nontrivial action on $\mathcal{H}^1(U(1),\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$. We rewrite $\mathbb{Z}_2$ as $\mathbb{Z}_2$, and $\mathcal{H}^1(U(1),\mathbb{R}/\mathbb{Z})$ as $\mathcal{H}^1(U(1),\mathbb{R}/\mathbb{Z})_{\mathbb{Z}_2}$, to indicate such a nontrivial action.

Note that $\mathcal{H}^1(\mathbb{Z}_2,\mathbb{Z}_2) = \mathbb{Z}_1$, while $\mathcal{H}^1(\mathbb{Z}_2,\mathbb{Z}_2) = \mathbb{Z}_2$. This is because the cocycle condition for $\mathcal{H}^1(\mathbb{Z}_2,\mathbb{Z}_2)$ is

$$(d\omega_1)(g_0,g_1) = g_0 \cdot \omega_1(g_1) - \omega_1(g_0 g_1) + \omega_1(g_0) = 0,$$

$$g_0,g_1 \in \mathbb{Z}_2 = \{1, -1\}.$$ (109)
Using \( g_0 \cdot \omega_1(g_1) = \pm \omega_1(g_1) \) when \( g_0 = \pm 1 \), we can reduce the above to
\[
\omega_1(1) = 0, \quad -\omega_1(-1) - \omega_1(1) + \omega_1(-1) = 0. \tag{110}
\]
So the cocycles are given by
\[
\omega_1(1) = 0, \quad \omega_1(-1) = \text{integer}. \tag{111}
\]
The 1-coboundaries are given by
\[
(d\omega_0)(g_0) = g_0 \cdot \omega_0 - \omega_0 \tag{112}
\]
or
\[
(d\omega_0)(1) = 0, \quad (d\omega_0)(-1) = \text{even integer}. \tag{113}
\]
We see that \( \mathcal{H}^1(Z_2, \mathbb{Z}_2) = Z_2 \).

We also note that every element in \( \mathcal{H}^2(U(1) \times Z_2, \mathbb{R}/Z) \) can be labeled by at least one \((m_0, m_1, m_2)\), but it is possible that not every \((m_0, m_1, m_2)\) labels an element in \( \mathcal{H}^2(U(1) \times Z_2, \mathbb{R}/Z) \). In other words, the two sets, \((m_0, m_1, m_2)\) and \( \mathcal{H}^2(U(1) \times Z_2, \mathbb{R}/Z) \), are related by a sequence
\[
\{(m_0, m_1, m_2)\} \to \mathcal{H}^2(U(1) \times Z_2, \mathbb{R}/Z) \to 0. \tag{114}
\]
In this particular case, since \([m_0] = [m_2] = Z_1\), we know that \([m_1] = Z_2\) and \( \mathcal{H}^2(U(1) \times Z_2, \mathbb{R}/Z) \) has a one-to-one correspondence.

To measure \( m_1 \), we put the system on a finite line \( l_1 \). At an end of the line, we get degenerate states that form a projective representation of \( U(1) \times Z_2 \) [44–46], if \( m_1 \neq 0 \). If we view \( U(1) \times Z_2 \) as a subgroup of \( \text{SO}(3) \), the projective representations of \( U(1) \times Z_2 \) are simply half-integer spin representations of \( \text{SO}(3) \).

One way to understand such a result is to gauge the \( U(1) \times Z_2 \) symmetry, the \( U(1) \times Z_2 \) SPT states are described by a gauge topological term (induced by integrating out the matter fields),
\[
\mathcal{L}_{\text{top}} = \frac{m_1}{2} F, \tag{115}
\]
where \( F \) is the field strength two form for the \( U(1) \)-gauge field. Under \( Z_2 \) transformation, \( F \to -F \). Since \( \int_{\Sigma_2} \frac{d^2 F}{2} = m_1 \pi \) on any closed 1 + 1D space-time manifold \( M_2 \), \( \mathcal{L}_{\text{top}} \) respects the \( Z_2 \) symmetry, since \( m_1 \) is an integer.

If the space-time \( M_2 \) has a boundary, the above topological term naively reduce to an effective Lagrangian on the boundary
\[
\mathcal{L}_{0+1D} = \frac{m_1}{2} A, \tag{116}
\]
where \( A \) is the gauge potential one form. This is nothing but a 1D \( U(1) \) Chern-Simons term with a fractional coefficient. However, such a 1D \( U(1) \) Chern-Simons term breaks the \( Z_2 \) symmetry. So only if the \( Z_2 \) symmetry is broken at the boundary can the topological term reduce to the 1D Chern-Simons term on the boundary. If the \( Z_2 \) symmetry is not broken, we have an effective boundary theory,
\[
\mathcal{L}_{0+1D} = \frac{m_1 \sigma}{2} A + \mathcal{L}(\sigma), \tag{117}
\]
where the \( \sigma(x) \) field only takes two values \( \sigma = \pm 1 \). We see that if \( m_1 = 0 \), the ground state of the 0 + 1D system is not degenerate (ground) = \([\sigma = 1] + [\sigma = -1]\). If \( m_1 = 1 \), the ground states of the 0 + 1D system is degenerate, which are described by \([\sigma = \pm 1]\) states carrying fractional \( \pm 1/2 \) \( U(1) \) charges. Such states form a projective representation of \( U(1) \times Z_2 \).

We can also view the \( U(1) \times Z_2 \) SPT states as \( Z_2 \times Z_2 \) SPT states. Using the results in SPT invariant 16, we find the following.

SPT invariant 33. Consider a 1 + 1D bosonic \( U(1) \times Z_2 \) SPT state labeled by \( m_1 \in \mathcal{H}^2(U(1) \times Z_2, \mathbb{R}/Z) \). If we put the SPT state on a circle \( S^1 \), adding \( \pi \) flux of \( U(1) \) through \( S^1 \) will induce a \( Z_2 \) charge \( m_1 \), and adding \( \pi \) flux of \( Z_2 \) through \( S^1 \) will induce a \( U(1) \) charge \( m_1 \) even integers in the ground state.

2. 2 + 1D

The \( U(1) \times Z_2 \) SPT states in 2 + 1D dimensions are described by \( \mathcal{H}^4(U(1) \times Z_2, \mathbb{R}/Z) = Z_2 \), whose elements can be labeled by a subset of \([m_0, m_1, m_2, m_3]\) (see Appendix D), where
\[
m_0 \in \mathcal{H}^4(U(1), \mathbb{R}/Z) = Z, \quad m_1 \in \mathcal{H}^4(Z_2, \mathcal{H}^2(U(1), \mathbb{R}/Z)) = \mathcal{H}^2(Z_2, Z), \quad m_2 \in \mathcal{H}^4(Z_2, \mathcal{H}^4(U(1), \mathbb{R}/Z)) = \mathcal{H}^2(Z_2, Z_2), \quad m_3 \in \mathcal{H}^4(Z_2, \mathbb{R}/Z) = Z_2. \tag{118}
\]
We see that the 2 + 1D \( U(1) \times Z_2 \) SPT states can be viewed as 2 + 1D \( U(1) \) SPT states [described by \( \mathcal{H}^4(U(1), \mathbb{R}/Z) = Z \)] or 2 + 1D \( Z_2 \) SPT states [described by \( \mathcal{H}^4(Z_2, \mathbb{R}/Z) = Z_2 \)]. Their SPT invariants have been discussed before.

3. 3 + 1D

The 3 + 1D \( U(1) \times Z_2 \) SPT states are described by \( \mathcal{H}^4(U(1) \times Z_2, \mathbb{R}/Z) = Z_2 \), whose elements can be labeled by a subset of \([m_0, m_1, m_2, Z_3, m_4]\) (see Appendix D), where
\[
m_0 \in \mathcal{H}^4(U(1), \mathbb{R}/Z) = Z, \quad m_1 \in \mathcal{H}^4(Z_2, \mathcal{H}^4(U(1), \mathbb{R}/Z)) = \mathcal{H}^2(Z_2, Z), \quad m_2 \in \mathcal{H}^4(Z_2, \mathcal{H}^4(U(1), \mathbb{R}/Z)) = \mathcal{H}^2(Z_2, Z_2), \quad m_3 \in \mathcal{H}^4(Z_2, \mathbb{R}/Z) = Z_2, \quad m_4 \in \mathcal{H}^4(Z_2, \mathbb{R}/Z) = Z_2. \tag{119}
\]
Note that \( Z_2 \) has a trivial action on \( \mathcal{H}^4(U(1), \mathbb{R}/Z) \). To construct the SPT invariants that probe \( m_4 \), we can view the \( U(1) \times Z_2 \) SPT states as \( Z_2 \times Z_2 \) SPT states and use the result in Sec. IV E 3. This is because, as we replace \( U(1) \) with \( Z_2 \), \( \mathcal{H}^4(Z_2, \mathcal{H}^4(U(1), \mathbb{R}/Z)) \) becomes \( \mathcal{H}^4(Z_2, \mathcal{H}^4(Z_2, \mathbb{R}/Z)) = Z_2 \). In Sec. IV E 3, we have discussed how to measure \( \mathcal{H}^4(Z_2, \mathcal{H}^4(Z_2, \mathbb{R}/Z)) \). The same setup also measures \( \mathcal{H}^4(Z_2, \mathcal{H}^4(U(1), \mathbb{R}/Z)) \). This allows us to obtain the following result.

SPT invariant 34. Consider a 3 + 1D bosonic \( U(1) \times Z_2 \) SPT state labeled by \( m_3 \in \mathcal{H}^4(U(1) \times Z_2, \mathbb{R}/Z) \). If we put the SPT state on a space with topology \( S^1 \times M_2 \), adding \( \pi \) flux of \( U(1) \) through \( S^1 \) will induce a bosonic \( Z_2 \) SPT state in the 2D space labeled by \( m_3 \) in \( \mathcal{H}^4(Z_2, \mathbb{R}/Z) \). This also implies that a \( \pi \) flux vortex line in \( U(1) \) will carry the gapless/degenerate edge states [53] of the 2 + 1D bosonic \( Z_2 \) SPT state labeled by \( m_3 \) in \( \mathcal{H}^4(Z_2, \mathbb{R}/Z) \).
B. Bosonic $U_c(1) \times [U_r(1) \times Z_3]$ SPT states

After the preparation of the last section, in this section, we use the tools (i.e., the SPT invariants) developed so far to study a more complicated example: bosonic $U_c(1) \times [U_r(1) \times Z_3]$ SPT states in various dimensions. We note that $U_r(1) \times Z_2$ is a subgroup of SO(3). So the results obtained here apply to integer-spin boson gas with boson number conservation. For this reason, we call $U_r(1)$ the charge $U(1)$ and $U_c(1)$ the spin $U(1)$.

1. 1 + 1D

The different $U_c(1) \times [U_r(1) \times Z_3]$ bosonic SPT states in 1 + 1D are described by $\mathcal{H}^2(U_c(1) \times [U_r(1) \times Z_3], \mathbb{R}/\mathbb{Z})$. According to the K"unneth formula (see Appendix C),

$$\mathcal{H}^2(U_c(1) \times [U_r(1) \times Z_3], \mathbb{R}/\mathbb{Z})$$

= $\mathcal{H}^2(U_c(1), \mathbb{R}/\mathbb{Z}) \oplus \mathcal{H}^2(U_r(1) \times Z_3, \mathbb{R}/\mathbb{Z})$.

We see that there are two $U_c(1) \times [U_r(1) \times Z_3]$ bosonic SPT states in 1 + 1D (including the trivial one), labeled by $m_0 = 0, 1$. The SPT states involve only the $U_r(1) \times Z_3$ symmetry. The nontrivial 1D SPT state carries a projective representation of $U_r(1) \times Z_3$ at each end if the 1D SPT state form an open chain [44–46]. This state was discussed in the last section.

2. 2 + 1D

a. Group cohomology description. The different $U_c(1) \times [U_r(1) \times Z_3]$ bosonic SPT states in 1 + 1D are described by $\mathcal{H}^3(U_c(1) \times [U_r(1) \times Z_3], \mathbb{R}/\mathbb{Z})$:

$$\mathcal{H}^3(U_c(1) \times [U_r(1) \times Z_3], \mathbb{R}/\mathbb{Z})$$

= $\mathcal{H}^3(U_r(1), \mathbb{R}/\mathbb{Z}) \oplus \mathcal{H}^3(U_r(1) \times Z_2, \mathbb{R}/\mathbb{Z}) \oplus \mathcal{H}^3(U_r(1), \mathbb{R}/\mathbb{Z})$.

We see that $U_c(1) \times [U_r(1) \times Z_3]$ bosonic SPT states in 2 + 1D are labeled by $m_0, m_3 \in \mathbb{Z}$.

b. The $(m_0, 0, 0, 0)$ SPT states. We note that a $(m_0, 0, 0, 0)$ SPT state is still nontrivial if we break the $Z_2$ symmetry and the charge $U(1)$ symmetry since $\mathcal{H}^3(U_c(1), \mathbb{R}/\mathbb{Z}) = \mathbb{Z}$ for the spin $U(1)$ symmetry. Thus, if we probe the $(m_0, 0, 0, 0)$ SPT state by a nondynamical $U(1)$-gauge field $A_\mu$, after we integrate out the matter fields, we obtain the following quantized gauge topological term in 3 + 1D [54],

$$\mathcal{L}_{2+1D} = \frac{2m_0}{4\pi} A_\mu \partial_\nu A_\lambda \epsilon^{\mu\nu\lambda},$$

which characterizes the $(m_0, 0, 0, 0)$ SPT state. The Hall conduce for the charge $U(1)$ symmetry is quantized as an even integer $\sigma_{xy} = \frac{2eB}{\pi}$, which is the SPT invariant that fully characterizes the $(m_0, 0, 0, 0)$ SPT states.

c. The $(0, m_0, 0, 0)$ SPT states. Again, the $(0, m_0, 0, 0)$ SPT states only involve the $U_c(1) \times Z_2$ symmetry. The charge $U(1)$ is not relevant here. So we drop it in the following discussion. To probe the $(0, m_0, 0, 0)$ SPT states, we create two identical monodromy defects of the spin $U(1)$ symmetry, each with a $\pi$ twist. Such monodromy defects do not break the $U_r(1) \times Z_3$ symmetry. The SPT invariant for the $(0, m_0, 0, 0)$ SPT states is the total $Z_2$ charge of the two monodromy defects, which is given by $m_0$. Such a SPT invariant fully characterizes the $(0, m_0, 0, 0)$ SPT states.

In fact, we can view the $2 + 1D$ $U_c(1) \times Z_3$ SPT states as $Z_2 \times Z_3$ SPT states. Then the above SPT invariant is one of those discussed in Sec. IV E 1.

3. 3 + 1D

a. Group cohomology description. The different $U_c(1) \times [U_r(1) \times Z_3]$ bosonic SPT states in 3 + 1D are described by $\mathcal{H}^4(U_c(1) \times [U_r(1) \times Z_3], \mathbb{R}/\mathbb{Z})$. According to the K"unneth formula (see Appendix C),

$$\mathcal{H}^4(U_c(1) \times [U_r(1) \times Z_3], \mathbb{R}/\mathbb{Z})$$

= $\mathcal{H}^4(U_r(1), \mathbb{R}/\mathbb{Z}) \oplus \mathcal{H}^4(U_r(1) \times Z_2, \mathbb{R}/\mathbb{Z}) \oplus \mathcal{H}^4(U_r(1), \mathbb{R}/\mathbb{Z})$.

where we have only kept the nonzero terms, and

$$\mathcal{H}^4(U_c(1), \mathbb{R}/\mathbb{Z})$$

= $\mathbb{Z} \oplus \mathbb{Z} = \{m_0, m_0\}$.

We see that there are four $U_c(1) \times [U_r(1) \times Z_3]$ bosonic SPT states in 3 + 1D (including the trivial one), labeled by $m_0 = 0, 1$ and $m_2 = 0, 1$. The SPT state $(m_0, m_2) = (1, 0)$ involves only the $U_r(1) \times Z_2$, which is discussed in Sec. V A 3. On the other hand, the $(m_0, m_2) = (0, 1)$ SPT state involves the full $U_c(1) \times [U_r(1) \times Z_3]$ symmetry and is new.

b. The $(m_0, m_2) = (0, 1)$ SPT state. One way to probe the $(m_0, m_2) = (0, 1)$ SPT state is to couple the the $U_c(1)$ and $U_r(1)$ charges to nondynamical gauge fields $A_\mu$ and $A_\lambda$. After we integrate out the matter fields, we obtain the following
quantized gauge topological term in 3 + 1D [54]:
\[ \mathcal{L}_{3+1D} = \frac{\pi}{(2\pi)^2} \partial_\mu A_{\nu\lambda} \partial_\lambda A_{\nu\mu} e^{\mu \nu \lambda}. \] (130)

The structure of the above quantized gauge topological term is consistent with corresponding group cohomology class \( H^2(\mathbb{U}(1) \times \mathbb{Z}_2, \mathbb{R}/\mathbb{Z}) \).

To understand the physical properties (i.e., the SPT invariants) of the \((m_0,m_2) = (0,1)\) SPT state, let us assume that the 3 + 1D space-time has a topology \( M_2 \times M_2 \). We also assume that the \( A_{\mu} \) gauge field has \( 2\pi \) flux on \( M_2 \). In the large \( M_2 \) limit, the Lagrangian (130) reduces to an effective Lagrangian on \( M_2 \) which has a form
\[ \mathcal{L}_{M_2} = \frac{\pi}{2\pi} \partial_\mu A_{\nu\mu} e^{\mu \nu}. \] (131)

We note that the \( A_{\mu} \) gauge configuration preserves the \( U(1) \times \{ U(1) \times \mathbb{Z}_2 \} \) symmetry. The above Lagrangian is the effective Lagrangian of the \( U(1) \times \{ U(1) \times \mathbb{Z}_2 \} \) symmetric theory on \( M_2 \) probed by the \( A_{\mu} \) gauge field. Such an effective Lagrangian implies that the \( U(1) \times \mathbb{Z}_2 \) symmetric theory on \( M_2 \) describes a nontrivial \( U(1) \times \mathbb{Z}_2 \) SPT state labeled by the nontrivial element \( m_2 = 1 \) in \( H^2(\mathbb{U}(1) \times \mathbb{Z}_2, \mathbb{R}/\mathbb{Z}) \).

We can prove the basic conjecture regarding to the boundary of SPT phases, for the case of \( U(1) \times \{ U(1) \times \mathbb{Z}_2 \} \) SPT state.

C. Bosonic \( U(1) \times \mathbb{Z}_2^2 \) SPT phases

In this section, we study bosonic \( U(1) \times \mathbb{Z}_2^2 \) SPT phases. Those SPT phases can be realized by charged bosons with time-reversal symmetry.

I. 1 + 1D

Let us first consider 1 + 1D SPT states with symmetry \( U(1) \times \mathbb{Z}_2^2 \), which are described by \( H^2(\mathbb{U}(1) \times \mathbb{Z}_2^2, (\mathbb{R}/\mathbb{Z})_T) = \mathbb{Z}_2 \). According to the result in Appendix D, the elements in \( H^2(\mathbb{U}(1) \times \mathbb{Z}_2^2, (\mathbb{R}/\mathbb{Z})_T) \) can be labeled by a subset of \( \{ (m_0, m_1, m_2) \} \), where
\[ m_0 \in H^0(\mathbb{Z}_2^2, H^2(\mathbb{U}(1), \mathbb{R}/\mathbb{Z})) = \mathbb{Z}_1, \]
\[ m_1 \in H^1(\mathbb{Z}_2^2, H^2(\mathbb{U}(1), \mathbb{R}/\mathbb{Z})) = H^2(\mathbb{Z}_2^2, \mathbb{Z}) = \mathbb{Z}_1, \] (132)
\[ m_2 \in H^2(\mathbb{Z}_2^2, H^0(\mathbb{U}(1), \mathbb{R}/\mathbb{Z})) = H^2(\mathbb{Z}_2^2, (\mathbb{R}/\mathbb{Z})_T) = \mathbb{Z}_2. \]

where we have used the fact that \( \mathbb{Z}_2^2 \) has a trivial action on \( H^0(\mathbb{U}(1), \mathbb{R}/\mathbb{Z}) \). We see that \( m_2 = 0, 1 \) describes the two 1 + 1D \( U(1) \times \mathbb{Z}_2^2 \) SPT states. The \( U(1) \) symmetry is irrelevant here. Therefore, we get the following.

SPT invariant 36. A 1 + 1D bosonic \( U(1) \times \mathbb{Z}_2^2 \) SPT state labeled by \( m_2 = 1 \) has a degenerate Kramer doublet at an open boundary.

2. 2 + 1D

Next, we consider the \( U(1) \times \mathbb{Z}_2^2 \) SPT states in 2 + 1 dimensions, which are described by \( H^2(\mathbb{U}(1) \times \mathbb{Z}_2^2, (\mathbb{R}/\mathbb{Z})_T) = \mathbb{Z} \oplus \mathbb{Z}_2 \). The elements in \( H^2(\mathbb{U}(1) \times \mathbb{Z}_2^2, (\mathbb{R}/\mathbb{Z})_T) \) can be labeled by a subset of \( \{ (m_0, m_1, m_2, m_3) \} \) (see Appendix D), where
\[ m_0 \in H^0(\mathbb{Z}_2^2, H^2(\mathbb{U}(1), \mathbb{R}/\mathbb{Z})) = H^0(\mathbb{Z}_2^2, \mathbb{Z}_T) = \mathbb{Z}_1, \]
\[ m_1 \in H^1(\mathbb{Z}_2^2, H^2(\mathbb{U}(1), \mathbb{R}/\mathbb{Z})) = \mathbb{Z}_1, \]
\[ m_2 \in H^2(\mathbb{Z}_2^2, H^0(\mathbb{U}(1), \mathbb{R}/\mathbb{Z})) = H^2(\mathbb{Z}_2^2, (\mathbb{R}/\mathbb{Z})_T) = \mathbb{Z}_2, \] (133)
\[ m_3 \in H^3(\mathbb{Z}_2^2, H^0(\mathbb{U}(1), \mathbb{R}/\mathbb{Z})) = H^3(\mathbb{Z}_2^2, (\mathbb{R}/\mathbb{Z})_T) = \mathbb{Z}_1, \]

where we have used the fact that \( \mathbb{Z}_2^2 \) has a trivial action on \( H^0(\mathbb{U}(1), \mathbb{R}/\mathbb{Z}) \) and a nontrivial action on \( H^2(\mathbb{U}(1), \mathbb{R}/\mathbb{Z}) \). We see that the \( U(1) \times \mathbb{Z}_2^2 \) SPT states are described by \( m_3 = 0, 1 \).

To measure \( m_2 \), we note that if we break the \( U(1) \) symmetry down to \( \mathbb{Z}_2 \), we get the nontrivial SPT state with \( m_2 = 1 \) is still nontrivial, since \( H^2(\mathbb{Z}_2^2, H^0(\mathbb{U}(1), \mathbb{R}/\mathbb{Z})) = H^2(\mathbb{Z}_2^2, H^0(\mathbb{U}(1), \mathbb{R}/\math{Z}_2)) = H^2(\mathbb{Z}_2, \mathbb{Z}_2^2) \). If we break the \( \mathbb{Z}_2^2 \) symmetry at the same time, the \( U(1) \times \mathbb{Z}_2^2 \) SPT state described by \( m_2 = 1 \) will become a trivial \( \mathbb{Z}_2 \) SPT state. So the \( U(1) \times \mathbb{Z}_2^2 \) SPT state described by \( m_2 = 1 \) is also a nontrivial \( \mathbb{Z}_2 \times \mathbb{Z}_2^2 \) SPT state. The above consideration suggest that such a \( \mathbb{Z}_2 \times \mathbb{Z}_2^2 \) SPT state is described by a nontrivial \( m_3 \) in Eq. (97).

So we can use the SPT invariant that detects \( m_1 \) of the \( \mathbb{Z}_2 \times \mathbb{Z}_2^2 \) SPT states to detect \( m_2 \) of the \( U(1) \times \mathbb{Z}_2^2 \) SPT states. Thus, we get the following.

SPT invariant 37. Consider a 2 + 1D bosonic \( U(1) \times \mathbb{Z}_2^2 \) SPT state labeled by \( m_2 = 1 \) in \( H^2(\mathbb{U}(1) \times \mathbb{Z}_2^2, (\mathbb{R}/\mathbb{Z})_T) \). If we put the state on a cylinder \( I \times S_1 \), then the states on one boundary will form Kramer doublets if we twist the boundary condition around \( S_1 \) by the \( \pi \) rotation in \( U(1) \). This also implies that a \( U(1) \) monodromy defect generated by \( \pi \) rotation carries a degenerate Kramer doublet.
3. 3 + 1D

Last, we consider the $U(1) \times Z_2^T$ SPT states in 3 + 1 dimensions. Several SPT invariants for such states were discussed in Refs. [56,70]. The $U(1) \times Z_2^T$ SPT states are described by $H^4(U(1) \times Z_2^T, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The elements in $H^4(U(1) \times Z_2^T, \mathbb{R}/\mathbb{Z})$ can be labeled by a subset of $\{(m_0, m_1, m_2, m_3, m_4)\}$ (see Appendix D), where

$$m_0 \in H^4(Z_2^{T}, H^4(U(1), \mathbb{R}/\mathbb{Z})) = \mathbb{Z}_1,$$
$$m_1 \in H^4(Z_2^{T}, H^4(U(1), \mathbb{R}/\mathbb{Z})) = H^4(Z_2^T, \mathbb{Z}_2),$$
$$m_2 \in H^4(Z_2^{T}, H^4(U(1), \mathbb{R}/\mathbb{Z})) = \mathbb{Z}_1,$$
$$m_3 \in H^4(Z_2^{T}, H^4(U(1), \mathbb{R}/\mathbb{Z})) = H^4(Z_2^T, \mathbb{Z}_2),$$
$$m_4 \in H^4(Z_2^{T}, H^4(U(1), \mathbb{R}/\mathbb{Z})) = H^4(Z_2^T, \mathbb{R}/\mathbb{Z}).$$

We see that the 3 + 1D $U(1) \times Z_2^T$ SPT states are labeled by $m_1 = 0.1$ and $m_4 = 0.1$. $m_4$ labels different 3 + 1D $Z_2^T$ SPT states where the $U(1)$ symmetry is irrelevant.

To probe $m_1$, we may gauge the $U(1)$ symmetry. We believe that the $U(1) \times Z_2^T$ SPT states labeled by $(m_1, m_4) = (m_1, 0)$ are described by the following $U(1)$-gauge topological term:

$$\mathcal{L}_{\text{top}} = \frac{m_1}{(2\pi)^2} F^2. \quad (135)$$

Under the $Z_2^T$ time-reversal transformation, $F^2 \rightarrow -F^2$ and $e^{i\int_{M_4} \frac{\alpha_{14}}{2\pi \alpha_{14}}} \rightarrow e^{-i\int_{M_4} \frac{\alpha_{14}}{2\pi \alpha_{14}}} F^2$. Since $\int_{M_4} \frac{\alpha_{14}}{2\pi \alpha_{14}} F^2 = \pi m_1 \times \text{integers}$, on any closed 3 + 1D orientable space-time manifold $M_4$, the $Z_2^T$ symmetry is preserved since $m_1$ is an integer. $m_1 = \text{odd}$ describes the nontrivial 3 + 1D $U(1) \times Z_2^T$ SPT state, while $m_1 = \text{even}$ describes the trivial SPT state. Now we see that $m_1$ can be measured by the statistical effect discussed in Refs. [70,100,101].

**SPT invariant 38.** In a 3 + 1D bosonic $U(1) \times Z_2^T$ SPT state labeled by $(m_1, m_4) = (m_1, 0)$, a dyon of the $U(1)$ gauge field with $[U(1) \text{ charge}, \text{magnetic charge}] = (q, m)$ has a statistics $(-\eta_{m_1-q})$, where $\eta_{m_1} = \text{boson}$ and $\eta_{m_1} = \text{fermion}$. If the space-time $M_4$ has a boundary, the topological term (135) reduces to an effective Lagrangian on the boundary,

$$\mathcal{L}_{2+1D} = \frac{m_1}{(2\pi)^2} A F, \quad (136)$$

if the $Z_2^T$ time-reversal symmetry is broken on the boundary. The above is nothing but a 2 + 1D $U(1)$ Chern-Simons term with a quantized Hall conductance $\sigma_{xy} = m_1/2\pi$ [56]. Thus, we get the following.

**SPT invariant 39.** In a 3 + 1D bosonic $U(1) \times Z_2^T$ SPT state labeled by $(m_1, m_4)$, the gapped time-reversal symmetry-breaking boundary has a Hall conductance $\sigma_{xy} = m_1/2\pi$ + $\sigma_{xy}^{\text{even}}/2\pi$.

If the $Z_2^T$ symmetry is not broken, we actually have an effective boundary theory,

$$\mathcal{L}_{2+1D} = \frac{m_1 \sigma}{4\pi} A F + \xi(\sigma), \quad (137)$$

where the $\sigma(x)$ field only takes two values $\sigma = \pm 1$. The gapless edge states on the domain wall between $\sigma = 1$ and $\sigma = -1$ regions may give rise to the gapless boundary excitations on the 2 + 1D surface.

### VII. SUMMARY

It has been shown that the SPT states and some of the SET states can be described by the cocycles in the group cohomology class $H^4(G, \mathbb{R}/\mathbb{Z})$ [48,77]. In this paper, we construct many SPT invariants which allow us to physically measure the cocycles in $H^4(G, \mathbb{R}/\mathbb{Z})$ fully. The constructed SPT invariants allow us to physically or numerically detect and characterize the SPT states and some of the SET states.

The SPT invariants are constructed by putting the SPT states on a space-time with a topology $M_k \times M_{d-1-k}$ and gauging a subgroup $GG$ of the symmetry group $G$. We then put a nontrivial $GG$ gauge configuration on the closed manifold $M_k$. When $k = 1$, the gauge configuration can be a gauge flux through the ring. When $k = 2$, the gauge configuration can be a gauge flux through $M_2$ if $GG$ is continuous or a few identical gauge fluxes through $M_2$ (if $GG$ is discrete), etc.

When $M_{d-1-k}$ is large, the SPT states on $M_k \times M_{d-1-k}$ can be viewed as a SPT state on $M_{d-1-k}$ with a symmetry $SG$, where $SG$ is a subgroup of $G$ that commutes with $GG$. The $SG$ SPT state on $M_{d-1-k}$ is described by $H^4(G, \mathbb{R}/\mathbb{Z})$ by measuring the the cocycles in $H^4(G, \mathbb{R}/\mathbb{Z})$. This way, we can measure the the cocycles in $H^4(G, \mathbb{R}/\mathbb{Z})$ by measuring the cocycles in $H^4(SG, \mathbb{R}/\mathbb{Z})$. When $d - k = 1$, the cocycles in $H^4(SG, \mathbb{R}/\mathbb{Z})$ can be measured by measuring the $SG$ quantum number of the ground state. When $d - k = 2$, we can choose the space-time $M_{d-1-k}$ to have a space described by a finite line. Then the cocycles in $H^4(SG, \mathbb{R}/\mathbb{Z})$ can be measured by measuring the projective representation of $SG$ at one end of the line.

In Table I, we list the SPT invariants for some simple bosonic SPT phases. In Table II, we list the SPT invariants for a few fermionic SPT phases. More SPT invariants are described by the SPT invariant statements in the paper. Those SPT invariants also allow us to understand some of the SPT states for interacting fermions. We list those results in Table III [71,102].

### ACKNOWLEDGMENTS

I like to thank Xie Chen, Zheng-Cheng Gu, Max Metlitski, and Juven Wang for many helpful discussions. This research is supported by NSF Grants No. DMR-1005541, No. NSFC 11074140, and No. NSFC 11274192. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research.

---

**Table III.** The fermionic SPT phases with the full symmetry $G_f$. Here 0 means that there is only a trivial SPT phase. $Z_a$ means that the nontrivial SPT phases plus the trivial phase are labeled by the elements in $Z_a$.

<table>
<thead>
<tr>
<th>$G_f$</th>
<th>0 + 1D</th>
<th>1 + 1D</th>
<th>2 + 1D</th>
<th>3 + 1D</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_2^T$</td>
<td>$Z_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$U(1)$</td>
<td>$Z$</td>
<td>0</td>
<td>$Z$</td>
<td>0</td>
</tr>
<tr>
<td>$U(1) \times U(1)$</td>
<td>$Z \oplus Z$</td>
<td>0</td>
<td>$Z \oplus Z \oplus Z$</td>
<td>0</td>
</tr>
<tr>
<td>$U(1) \times Z_2^T$</td>
<td>$Z \oplus Z_2$</td>
<td>0</td>
<td>$Z$</td>
<td>0</td>
</tr>
<tr>
<td>$Z_2 \times Z_2^T$</td>
<td>$Z_2 \oplus Z_2$</td>
<td>$Z_2$</td>
<td>$Z_8$</td>
<td>?</td>
</tr>
</tbody>
</table>
The group cohomology class \( a \) is called a 

\[ a \in H^d(G, \mathbb{M}) \]

is a subgroup of \( H^d(G, \mathbb{M}) \) also induces a mapping \( \tilde{G} \) to \( \mathbb{M} \)

\[ \tilde{G} = \tilde{G} \cdot \mathbb{M} \]

with \( \mathbb{M} \) being a group homomorphism.

A homogeneous \( d \) cocycle is a function \( \nu_d : G^{d+1} \to \mathbb{M} \), which satisfies

\[ \nu_d(g_0, \ldots, g_d) = \nu_d(gg_0, \ldots, gg_d), \ g, g_i \in G. \quad (A3) \]

We denote the set of \( d \) cocycles as \( C^d(G, \mathbb{M}) \). Clearly, \( C^d(G, \mathbb{M}) \) is an Abelian group. homogeneous group cocycle

Let us define a mapping \( d \) (group homomorphism) from \( C^d(G, \mathbb{M}) \) to \( C^{d+1}(G, \mathbb{M}) \),

\[ (d
\]

where \( g_0, \ldots, \hat{g}_i, \ldots, g_{d+1} \) is the sequence \( g_0, \ldots, g_i, \ldots, g_{d+1} \) with \( g_i \) removed. One can check that \( d^2 = 0 \). The homogeneous \( d \) cocycles are then the homogeneous \( d \) cocycles that also satisfy the cocycle condition

\[ d\nu_d = 0. \quad (A5) \]

We denote the set of \( d \) cocycles as \( Z^d(G, \mathbb{M}) \). Clearly, \( Z^d(G, \mathbb{M}) \) is an Abelian subgroup of \( C^d(G, \mathbb{M}) \).

Let us denote \( B^d(G, \mathbb{M}) \) as the image of the map \( d \) : \( C^{d-1}(G, \mathbb{M}) \to C^d(G, \mathbb{M}) \) and \( B^0(G, \mathbb{M}) = \{0\} \). The elements in \( B^d(G, \mathbb{M}) \) are called \( d \) coboundary. Since \( d^2 = 0 \), \( B^d(G, \mathbb{M}) \) is a subgroup of \( Z^d(G, \mathbb{M}) \):

\[ B^d(G, \mathbb{M}) \subset Z^d(G, \mathbb{M}). \quad (A6) \]

The group cohomology class \( H^d(G, \mathbb{M}) \) is then defined as

\[ H^d(G, \mathbb{M}) = Z^d(G, \mathbb{M})/B^d(G, \mathbb{M}). \quad (A7) \]

We note that the \( d \) operator and the cocycles \( C^d(G, \mathbb{M}) \) (for all values of \( d \)) form a so-called cochain complex,

\[ \cdots \to C^0(G, \mathbb{M}) \xrightarrow{d} C^1(G, \mathbb{M}) \xrightarrow{d} C^2(G, \mathbb{M}) \xrightarrow{d} \cdots, \quad (A8) \]

which is denoted as \( C(G, \mathbb{M}) \). So we may also write the group cohomology \( H^d(G, \mathbb{M}) \) as the standard cohomology of the cochain complex \( H^d[C(G, \mathbb{M})] \).

2. Nonhomogeneous group cocycle

The above definition of group cohomology class can be rewritten in terms of nonhomogeneous group cocycles. An nonhomogeneous group \( d \) cocycle is a function \( \omega_d : G^d \to M \). All \( \omega_d(g_1, \ldots, g_d) \) form \( C^d(G, \mathbb{M}) \).

The nonhomogeneous group cocycles and the homogeneous group cocycles are related as

\[ \nu_d(g_0, g_1, \ldots, g_d) = \omega_d(g_0, \ldots, g_{d-1}, d). \quad (A9) \]

with

\[ g_0 = 1, \ \ g_1 = g_0g_1, \ \ g_2 = g_1g_2, \ldots. \quad (A10) \]

Now the \( d \) map has a form on \( \omega_d \):

\[ (d\omega_d)(g_0, \ldots, g_d,d+1) = g_0 \cdot \omega_d(g_1, \ldots, g_d,d+1) \]

\[ + \sum_{i=1}^d (-)^i \omega_d(g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_d,d+1) \]

\[ + (-)^{d+1} \omega_d(g_0, \ldots, g_{d-1}). \quad (A11) \]

This allows us to define the nonhomogeneous group \( d \) cocycles which satisfy \( d\omega_d = 0 \) and the nonhomogeneous group \( d \) coboundaries which have a form \( \omega_d = d\mu_{d-1} \). In the following, we use nonhomogeneous group cocycles to study group cohomology. Geometrically, we may view \( g_i \) as living on the vertex \( i \) and \( g_{ij} \) as living on the edge connecting the two vertices \( i \) to \( j \).

3. “Normalized” cocycles

We know that each elements in \( H^d(G, \mathbb{R}/\mathbb{Z}) \) can be represented by many cocycles. In the following, we are going describe a way to simplify the cocycles so that the simplified cocycles can still represent all the elements in \( H^d(G, \mathbb{R}/\mathbb{Z}) \).

The simplification is obtained by considering normalized cochains \([108]\), which satisfy

\[ \omega_d(g_1, \ldots, g_d) = 0, \ \text{if one of } \ g_i = 1. \quad (A12) \]

One can check that the \( d \) operator maps a normalized cochain to a normalized cochain. The group cohomology classes obtained from the ordinary cochains is isomorphic to the group cohomology classes obtained from the normalized cochains. Let us use \( \tilde{C}^d(G, \mathbb{M}) \), \( \tilde{Z}^d(G, \mathbb{M}) \), and \( \tilde{B}^d(G, \mathbb{M}) \) to denote the normalized cochains, cocycles, and coboundaries, respectively. We have \( H^d(G, \mathbb{M}) = \tilde{Z}^d(G, \mathbb{M})/\tilde{B}^d(G, \mathbb{M}) \).

4. A “differential form” notion for group cocycles

We know that a cocycle \( \omega_d \) in \( H^d(G, \mathbb{R}/\mathbb{Z}) \) is a linear map that maps a \( d \)-dimensional complex \( M \), with \( g_i \) on the vertices or \( g_{ij} \) on the edges, to a mod-1 number in \( \mathbb{R}/\mathbb{Z} \). Let us use a differential form notation to denote such a map:

\[ \int_M \omega_d(g_{ij}) \in \mathbb{R}/\mathbb{Z}. \quad (A13) \]

In the above, we have regarded \( \omega_d(g_{ij}) \) as a function of \( g_{ij} \) on the edges. We may also view \( \omega_d \) as a function of \( g_i \) on the vertices by replacing \( g_{ij} \) with \( g_{ij}g_{ij}^{-1} : \omega_d(g_{ij}g_{ij}^{-1}) \). A differential form \( F \) is a linear map from a complex (or a manifold) to a real number:

\[ \int_M F \in \mathbb{R}. \quad (A14) \]
In fact, we can use a differential form $F_d(g_{ij})$ (that depends on $g_{ij}$'s on the edges) to represent $\omega_d(g_{ij})$:

$$\int_M \omega_d(g_{ij}) = \int_M F_d(g_{ij}) \mod 1. \quad (A15)$$

So we can treat $\omega_d(g_{ij})$ as a differential form, or more precisely, a **discretized differential form**. In fact, the cocycle is an analog of closed form.

In this paper, we use such a notation to describe the fixed-point (or the ideal) Lagrangian for the SPT states. The ideal fixed-point actions for SPT states contain only a pure topological term which always has a form

$$S_{\text{top}} = 2\pi \int_M \omega_d(g_{ij}), \quad (A16)$$

where $\omega_M$ is a cocycle in $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ and $M$ is the space-time complex. The factor $2\pi$ is needed to make the action amplitude $e^{2\pi i \int_M \omega_d(g_{ij})}$ well defined. The expression (A16) reflects the direct connection between the SPT phases and cocycles in $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$.

**APPENDIX B: RELATION BETWEEN $H^{d+1}(BG, \mathbb{Z})$ AND $\mathcal{H}^d_B(G, \mathbb{R}/\mathbb{Z})$**

We can show that the topological cohomology of the classifying space, $H^{d+1}(BG, \mathbb{Z})$, and the Borel-group cohomology, $\mathcal{H}^d_B(G, \mathbb{R}/\mathbb{Z})$, are directly related,

$$H^{d+1}(BG, \mathbb{Z}) \simeq \mathcal{H}^d_B(G, \mathbb{R}/\mathbb{Z}). \quad (B1)$$

This result is obtained from Ref. [109]. On page 16 of Ref. [109], it is mentioned in Remark IV.16(3) that $\mathcal{H}^d_B(G, \mathbb{R}) \simeq Z_1$ [there, $\mathcal{H}^d_B(G, M)$ is denoted as $\mathcal{H}^d_{\text{Borel}}(G, M)$, which is equal to $\mathcal{H}^d_{\text{SM}}(G, M)$]. It is also shown in Remark IV.16(1) and in Remark IV.16(3) that $\mathcal{H}^d_{\text{SM}}(G, \mathbb{Z}) = H^d(BG, \mathbb{Z})$ and $\mathcal{H}^d_{\text{SM}}(G, \mathbb{R}/\mathbb{Z}) = H^{d+1}(BG, \mathbb{Z})$ [where $G$ can have a nontrivial action on $\mathbb{R}/\mathbb{Z}$ and $\mathbb{Z}$, and $H^{d+1}(BG, \mathbb{Z})$ is the usual topological cohomology on the classifying space $BG$ of $G$].

Therefore, we have

$$\mathcal{H}^d_B(G, \mathbb{R}/\mathbb{Z}) = \mathcal{H}^d_{\text{SM}}(G, \mathbb{Z}) = H^{d+1}(BG, \mathbb{Z}),$$

$$\mathcal{H}^d_B(G, \mathbb{R}) = Z_1, \quad d > 0. \quad (B2)$$

These results are valid for both continuous groups and discrete groups, as well as for $G$ having a nontrivial action on the modules $\mathbb{R}/\mathbb{Z}$ and $\mathbb{Z}$.

**APPENDIX C: THE KÜNNETH FORMULA**

The Künneth formula is a very helpful formula that allows us to calculate the cohomology of chain complex $X \times X'$ in terms of the cohomology of chain complex $X$ and chain complex $X'$. The Künneth formula is expressed in terms of the tensor-product operation $\otimes_R$ and the torsion-product operation $\boxtimes_R \equiv \text{Tor}_R$, which have the following properties:

$$M \otimes_R M' \simeq M' \otimes_R M,$$

$$Z \otimes_R M \simeq M \otimes_R Z = M,$$

$$Z_n \otimes_R M \simeq M \otimes_R Z_n = M/nM,$$

$$Z_n \otimes_R \mathbb{R}/\mathbb{Z} \simeq \mathbb{R}/\mathbb{Z} \otimes_R Z_n = 0,$$

$$Z_m \otimes \mathbb{Z} Z_n = Z_{(m,n)},$$

$$(M' \oplus M'') \otimes_R M \simeq (M' \otimes_R M) \oplus (M'' \otimes_R M),$$

$$M \otimes_R (M' \oplus M'') = (M \otimes_R M') \oplus (M \otimes_R M''); \quad (C1)$$

and

$$\text{Tor}_R^d(M, M') \equiv M \boxtimes_R M',$$

$$M \boxtimes_R M' \simeq M' \boxtimes_R M,$$

$$Z \boxtimes \mathbb{Z} M = M \boxtimes \mathbb{Z} Z = 0,$$

$$Z_n \boxtimes \mathbb{Z} M = \{m \in M \mid nm = 0\},$$

$$Z_n \boxtimes \mathbb{Z} \mathbb{R}/\mathbb{Z} = Z_n,$$

$$Z_{mn} \boxtimes \mathbb{Z} Z_{(m,n)},$$

$$M' \otimes M'' \boxtimes_R M \simeq M' \boxtimes_R M \oplus M'' \boxtimes_R M,$$

$$M \boxtimes_R M' \oplus M'' \simeq M \boxtimes_R M' \oplus M \boxtimes_R M.$$

Here $R$ is a ring and $M', M'', M'$ are $R$ modules. A $R$ module is like a vector space over $R$ (i.e., we can "multiply" a vector by an element of $R$).

The Künneth formula itself is given by (see Ref. [110] page 247)

$$H^d(X \times X', M \otimes_R M')$$

$$\simeq \bigoplus_{k=0}^d H^k(X, M) \otimes_R H^{d-k}(X', M')$$

$$\oplus \bigoplus_{k=0}^{d+1} H^k(X, M) \boxtimes_R H^{d-k+1}(X', M'). \quad (C3)$$

Here $R$ is a principle ideal domain and $M, M', M''$ are $R$ modules such that $M \boxtimes_R M = 0$. We also require that $M'$ and $H^d(X', Z)$ are finitely generated, such as $M' = Z \oplus \cdots \oplus Z \oplus Z_n \oplus Z_{mn} \oplus \cdots$.

For more details on principal ideal domain and $R$ module, see the corresponding Wiki articles. Note that $Z$ and $R$ are principal ideal domains, while $\mathbb{R}/\mathbb{Z}$ is not. Also, $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ are not finitely generated $R$ modules if $R = \mathbb{Z}$. The Künneth formula works for topological cohomology where $X$ and $X'$ are treated as topological spaces. The Künneth formula also works for group cohomology, where $X$ and $X'$ are treated as groups, $X = G$ and $X' = G'$, provided that $G'$ is a finite group. However, the above Künneth formula does not apply for Borel-group cohomology where $X = G'$ is a continuous group, since in that case $\mathcal{H}^d_B(G', \mathbb{Z})$ is not finitely generated.

As the first application of Künneth formula, we like to use it to calculate $H^*(X, M)$ from $H^*(X', Z)$, by choosing $R = M' = Z$. In this case, the condition $M \boxtimes_R M' = M \boxtimes_R Z = 0$ is always satisfied. So we have

$$H^d(X \times X', M)$$

$$\simeq \bigoplus_{k=0}^d H^k(X, M) \otimes \mathbb{Z} H^{d-k}(X', \mathbb{Z})$$

$$\oplus \bigoplus_{k=0}^{d+1} H^k(X, M) \boxtimes \mathbb{Z} H^{d-k+1}(X', \mathbb{Z}). \quad (C4)$$
The above is valid for topological cohomology. It is also valid for group cohomology:

\[
\mathcal{H}^d(G \times G', \mathbb{M}) \\
\simeq \left[ \oplus_{k=0}^{d} \mathcal{H}^d(G, \mathbb{M}) \otimes_{\mathbb{Z}} \mathcal{H}^{d-k}(G', \mathbb{Z}) \right] \\
\oplus \left[ \oplus_{k=0}^{d-1} \mathcal{H}^d(G, \mathbb{M}) \otimes_{\mathbb{Z}} \mathcal{H}^{d-k-1}(G', \mathbb{Z}) \right],
\]

where we have used

\[
\mathcal{H}^1(G', \mathbb{Z}) = 0.
\]

If we further choose \( \mathbb{M} = \mathbb{R}/\mathbb{Z} \), we obtain

\[
\mathcal{H}^d(G \times G', \mathbb{R}/\mathbb{Z}) \\
\simeq \mathcal{H}^d(G, \mathbb{R}/\mathbb{Z}) \oplus \mathcal{H}^d(G', \mathbb{R}/\mathbb{Z}) \\
\oplus \left[ \oplus_{k=0}^{d-1} \mathcal{H}^d(G, \mathbb{R}/\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{H}^{d-k-1}(G', \mathbb{R}/\mathbb{Z}) \right],
\]

where \( G' \) is a finite group.

We can further choose \( X \) to be the space of one point (or the trivial group of one element) in Eq. (C4) or Eq. (C5) and use

\[
\mathcal{H}^d(X, \mathbb{M}) = \begin{cases} 
\mathbb{M}, & \text{if } d = 0, \\
0, & \text{if } d > 0,
\end{cases}
\]

to reduce Eq. (C4) to

\[
\mathcal{H}^d(X, \mathbb{M}) \simeq \mathbb{M} \otimes_{\mathbb{Z}} \mathcal{H}^d(X, \mathbb{Z}) \oplus \mathbb{M} \otimes_{\mathbb{Z}} \mathcal{H}^{d+1}(X, \mathbb{Z}),
\]

where \( X' \) is renamed as \( X \). The above is a form of the universal coefficient theorem which can be used to calculate \( H^*(X, \mathbb{M}) \) from \( H^*(X, \mathbb{Z}) \) and the module \( \mathbb{M} \). The universal coefficient theorem works for topological cohomology where \( X \) is a topological space. The universal coefficient theorem also works for group cohomology where \( X \) is a finite group.

Using the universal coefficient theorem, we can rewrite Eq. (C4) as

\[
\mathcal{H}^d(X \times X', \mathbb{M}) \simeq \oplus_{k=0}^{d} \mathcal{H}^k(X, \mathbb{H}^{d-k}(X', \mathbb{M})).
\]

The above is valid for topological cohomology. It is also valid for group cohomology,

\[
\mathcal{H}^d(G \times G', \mathbb{M}) \simeq \oplus_{k=0}^{d} \mathcal{H}^d(G, \mathbb{M}) \otimes_{\mathbb{Z}} \mathcal{H}^{d-k}(G', \mathbb{M}),
\]

provided that both \( G \) and \( G' \) are finite groups.

We may apply the above to the classifying spaces of group \( G \) and \( G' \). Using \( B(G \times G') = BG \times BG' \), we find

\[
\mathcal{H}^d(BG \times BG', \mathbb{M}) \simeq \oplus_{k=0}^{d} \mathcal{H}^d(BG, \mathbb{H}^{d-k}(BG', \mathbb{M})).
\]

Choosing \( \mathbb{M} = \mathbb{R}/\mathbb{Z} \) and using Eq. (B2), we have

\[
\mathcal{H}^d_B(G \times G', \mathbb{R}/\mathbb{Z}) = \mathcal{H}^{d+1}(BG \times BG', \mathbb{Z}) \\
= \oplus_{k=0}^{d+1} \mathcal{H}^k(BG, \mathbb{H}^{d+1-k}(BG', \mathbb{Z})).
\]

Equation (C15) is valid for any groups \( GG \) and \( SG \). Choosing \( X = BG, \mathbb{M} = \mathbb{Z}_n \), Eq. (C10) becomes

\[
\mathcal{H}^d(G, \mathbb{Z}_n) \simeq \mathcal{H}^d(G, \mathbb{Z}) \oplus \mathcal{H}^d(G, \mathbb{Z}) \oplus \mathcal{H}^{d+1}(G, \mathbb{Z}),
\]

where we have used Eq. (C14). Equation (C16) is valid for any compact group \( G \). Using Eq. (C16), we find that

\[
\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z}) = \mathcal{H}^d(G, \mathbb{R}/\mathbb{Z}) \\
\oplus \mathcal{H}^d(G, \mathbb{R}/\mathbb{Z}) \oplus \mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z}),
\]

Using Eq. (C16), we also find that

\[
\mathcal{H}^d(\mathbb{Z}_n, \mathbb{Z}_n) = \begin{cases} 
\mathbb{Z}_n, & \text{if } d = 0, \\
0, & \text{if } d > 0,
\end{cases}
\]

and

\[
\mathcal{H}^d(U(1), \mathbb{Z}_n) = \begin{cases} 
\mathbb{Z}_n, & \text{if } d = \text{even}, \\
0, & \text{if } d = \text{odd}.
\end{cases}
\]

**APPENDIX D: LYNDON-HOCHSCHILD-SERRE SPECTRAL SEQUENCE**

The Lyndon-Hochschild-Serre spectral sequence (see Ref. [111], pages 280 and 291, and Ref. [108]) allows us to understand the structure of \( \mathcal{H}^d(GG \times SG, \mathbb{R}/\mathbb{Z}) \) to a certain degree. (Here \( GG \times SG \equiv PSG \) is a group extension of \( GG \) by \( SG \).) We find that \( \mathcal{H}^d(GG \times SG, \mathbb{R}/\mathbb{Z}) \), when viewed as an Abelian group, contains a chain of subgroups,

\[
[0] = H_{d+1} \subset H_d \subset \cdots \subset H_0 = \mathcal{H}^d(GG \times SG, \mathbb{R}/\mathbb{Z}),
\]

such that \( H_k/H_{k+1} \) is a subgroup of a factor group of \( \mathcal{H}^d(SG, \mathbb{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})) \), i.e., \( \mathcal{H}^d(SG, \mathbb{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})_{SG}) \) contains a subgroup \( \Gamma_k \), such that

\[
H_k/H_{k+1} \subset \mathcal{H}^d(SG, \mathbb{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})_{SG})/\Gamma_k,
\]

\[
k = 0, \ldots, d.
\]
stress this point. We also have
\[ H_0/H_1 \subset \mathcal{H}^d(G,G,\mathbb{R}/\mathbb{Z})_{SG}, \]
\[ H_d/H_{d+1} = H_d = \mathcal{H}^d(SG,\mathbb{R}/\mathbb{Z})/\Gamma^d. \] (D3)

In other words, all the elements in \( \mathcal{H}^d(G,G,\mathbb{R}/\mathbb{Z}) \) can be one-to-one labeled by \( (x_0,x_1,\ldots,x_d) \), with
\[ x_k \in H_k/H_{k+1} \subset \mathcal{H}^d(SG,\mathcal{H}^{d-k}(GG,\mathbb{R}/\mathbb{Z})_{SG})/\Gamma^k. \] (D4)
The above discussion implies that we can also use \((m_0,m_1,\ldots,m_d)\) with
\[ m_k \in \mathcal{H}^k(SG,\mathcal{H}^{d-k}(GG,\mathbb{R}/\mathbb{Z})_{SG}) \] (D5)
to label all the elements in \( \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) \). However, such a labeling scheme may not be one to one, and it may happen that only some of \((m_0,m_1,\ldots,m_d)\) correspond to the elements in \( \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) \). However, on the other hand, for every element in \( \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) \), we can find a \((m_0,m_1,\ldots,m_d)\) that corresponds to it.

APPENDIX E: A DUALITY RELATION BETWEEN THE SPT AND THE SET PHASES

There is a duality relation between the SPT and the SET phases described by a weak-coupling gauge field \([59,73,82]\). We first review a simple formal description of such a duality relation. Then we review an exact description for finite gauge groups.

1. A simple formal description

To understand the duality between the SPT and the SET phases, we note that a SPT state with symmetry \( G \) in \( d \)-dimensional space-time \( M \) can be described by a nonlinear \( \sigma \) model with \( G \) as the target space,
\[ S = \int_M d^dx \left\{ \frac{1}{\lambda_s} [\partial g(x^\mu)]^2 + i W_{\text{top}}(g) \right\}, \] (E1)
in the large \( \lambda_s \) limit. Here we triangulate the \( d \)-dimensional space-time manifold \( M \) to make it a lattice or a \( d \)-dimensional complex, and \( g(x^\mu) \) lies on the vertices of the complex: \( g(x^\mu) = [g_i] \), where \( i \) labels the vertices (the lattice sites). So \( \int d^dx \) is, in fact, a sum over lattice sites and \( \partial \) is the lattice difference operator. The above action \( S \) actually defines a lattice theory. \( W_{\text{top}}(g(x^\mu)) \) is a lattice topological term which satisfies
\[ \int_M d^dx W_{\text{top}}([g_i]) = \int_M d^dx W_{\text{top}}([gg_i]) \in \mathbb{R}, \quad g, g_i \in G, \]
\[ \int_M d^dx W_{\text{top}}([g(x^\mu)]) = 0 \mod 2\pi, \quad \text{if } M \text{ has no boundary}. \] (E2)

We have rewritten \( W_{\text{top}}(g(x^\mu)) \) as \( W_{\text{top}}([g_i]) \) to stress that the topological term is defined on lattice. \( W_{\text{top}}([g_i]) \) satisfying (E2) are the group cocycles. Thus, the lattice topological term \( W_{\text{top}}([g_i]) \) is defined and described by the elements (the cocycles) in \( \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) \). [48,49] This is why the bosonic SPT states are described by \( \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) \).

If \( G \) contains a normal subgroup \( GG \subset G \), we can “gauge” \( GG \) to obtain a gauge theory in the bulk,
\[ S = \int d^dx \left\{ \frac{[\partial - iA]g \cdot A}{\lambda_s} + \frac{\text{Tr}(F_{\mu\nu})^2}{\lambda} + i W_{\text{gauge}}(g,A) \right\}, \] (E3)
where \( A \) is the \( GG \) gauge potential. When \( \lambda \) is small the above theory is a weak-coupling gauge theory with a gauge group \( GG \) and a global symmetry group \( SG = G/GG \).

The topological term \( W_{\text{gauge}}(g,A) \) in the gauge theory is a generalization of the Chern-Simons term \([54,86,87]\), which is obtained by gauging the topological term \( W_{\text{top}}(g) \) in the nonlinear \( \sigma \) model. The two topological terms \( W_{\text{gauge}}(g,A) \) and \( W_{\text{top}}(g) \) are directly related when \( A \) is a pure gauge:
\[ W_{\text{gauge}}(g,A) = W_{\text{top}}[h(x)g(x)], \]
where \( A = h^{-1}\partial h, \quad h \in GG \). (E4)

[A more detailed description of the two topological terms \( W_{\text{top}}(g) \) and \( W_{\text{gauge}}(g,A) \) on lattice can be found in Refs. [54,87]. See also the next section.] So the topological term \( W_{\text{gauge}}(g,A) \) in the gauge theory is also classified by the same \( \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) \) that classifies \( W_{\text{top}}(g) \). (We would like to remark that although both topological terms \( W_{\text{top}}(g) \) and \( W_{\text{gauge}}(g,A) \) are classified by the same \( \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) \), when \( \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) = \mathbb{Z} \), the correspondence can be tricky: For a topological term \( W_{\text{gauge}}(g) \) that corresponds to an integer \( k \) in \( \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) \), its corresponding topological term \( W_{\text{gauge}}(g,A) \) may correspond to an integer \( nk \) in \( \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) \). However, for finite group \( G \), the correspondence is one to one.)

When the space-time dimensions \( d = 3 \) or when \( d > 3 \) and \( GG \) is a finite group, the theory (E3) is gapped in \( \lambda_s \to \infty \) and \( \lambda \to 0 \) limit, which describe a SET phase with symmetry group \( SG \) and gauge group \( GG \). Such SET phase is described by \( \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}) \).

2. Exactly soluble gauge theory with a finite gauge group \( GG \) and a global symmetry group \( SG \)

To understand the above formal results more rigorously, we would like to review the exactly soluble models of weak-coupling gauge theories with a finite gauge group \( GG \) and a global symmetry group \( SG \). The exactly soluble models were introduced in Refs. [21,59,73,112]. The exactly soluble models is defined on a space-time lattice, or more precisely, a triangulation of the space-time. So we start by describing such a triangulation.

a. Discretize space-time

Let \( M_{\text{tri}} \) be a triangulation of the \( d \)-dimensional space-time. We call the triangulation \( M_{\text{tri}} \) as a space-time complex, and a cell in the complex as a simplex. In order to define a generic lattice theory on the space-time complex \( M_{\text{tri}} \), it is important to give the vertices of each simplex a local order. A nice local scheme to order the vertices is given by a branching structure \([48,49,113]\). A branching structure is a choice of orientation of each edge in the \( d \)-dimensional complex so that there is no oriented loop on any triangle (see Fig. 6).
FIG. 6. (Color online) Two branched simplices with opposite orientations. (a) A branched simplex with positive orientation and (b) a branched simplex with negative orientation.

The branching structure induces a local order of the vertices on each simplex. The first vertex of a simplex is the vertex with no incoming edges, and the second vertex is the vertex with only one incoming edge, etc. So the simplex in Fig. 6(a) has the following vertex ordering: 0, 1, 2, 3.

The branching structure also gives the simplex (and its subsimplices) an orientation denoted by $s_{ij-k} = \pm 1$. Figure 6 illustrates two 3-simplices with opposite orientations $s_{0123} = 1$ and $s_{0123} = -1$. The red arrows indicate the orientations of the 2-simplices which are the subsimplices of the 3-simplices. The black arrows on the edges indicate the orientations of the 1-simplices.

### b. Lattice gauge theory with a global symmetry

To define a lattice gauge theory with a gauge group $GG$ and a global symmetry group $SG$, let $G$ be an extension of $SG$ by $GG$: $G = GG \ltimes SG$. Here we assume $GG$ to be a finite group.

In our lattice gauge theory, the degrees of freedom on the vertices of the space-time complex, is described by $g_i \in G$ where $i$ labels the vertices. The gauge degrees of freedom are on the edges $ij$ which are described by $h_{ij} \in GG$.

The action amplitude $e^{-S_{\text{lat}}}$ for a $d$ cell $(ij \cdots k)$ is complex function of $g_i$ and $h_{ij}$: $V_{ij \cdots k}(\{g_i\},\{h_{ij}\})$. The total action amplitude $e^{-S}$ for configuration (or a path) is given by

$$e^{-S} = \prod_{(ij-k)} [V_{ij \cdots k}(\{g_i\},\{h_{ij}\})]^{\nu_{ij-k}},$$

(E5)

where $\prod_{(ij-k)}$ is the product over all the $d$ cells $(ij \cdots k)$. Note that the contribution from a $d$ cell $(ij \cdots k)$ is $V_{ij \cdots k}(\{g_i\},\{h_{ij}\})$ or $V_{ij \cdots k}^*(\{g_i\},\{h_{ij}\})$ depending on the orientation $s_{ij-k}$ of the cell. Our lattice theory is defined by following imaginary-time path integral (or partition function)

$$Z_{\text{gauge}} = \sum_{\{h_{ij}\},\{g_i\}} \prod_{(ij-k)} [V_{ij \cdots k}(\{g_i\},\{h_{ij}\})]^{\nu_{ij-k}}.$$ 

(E6)

If the above action amplitude $\prod_{(ij-k)} [V_{ij \cdots k}(\{g_i\},\{h_{ij}\})]^{\nu_{ij-k}}$ on closed space-time complex ($\partial M_{\text{int}} = \emptyset$) is invariant under the gauge transformation

$$h_{ij} \rightarrow g_i h_{ij} h_{ij}^{-1}, \quad g_i \rightarrow g_i^*,$$

(E7)

then the action amplitude $V_{ij \cdots k}(\{h_{ij}\},\{g_i\})$ defines a gauge theory of gauge group $GG$. If the action amplitude is invariant under the global transformation

$$h_{ij} \rightarrow h_{ij}^* = g_i h_{ij} h_{ij}^{-1}, \quad g_i \rightarrow g_i^*,$$

(E8)

then the action amplitude $V_{ij \cdots k}(\{h_{ij}\},\{g_i\})$ defines a $GG$ lattice gauge theory with a global symmetry $SG = G/GG$. We need to mod out $GG$ since when $h \in GG$, it is a part of a gauge transformation which does not change the physical states, instead of a global symmetry transformation which changes a physical state to another.)

However, in this paper, we are mainly considering a system with a global symmetry $G$, where we gauged a subgroup $GG \subset G$. The resulting gauge connection $h_{ij}$ is treated as nondynamical probe fields.

Using a cocycle $v_d(g_0, g_1, \ldots, g_d) \in H^d(G, \mathbb{R}/\mathbb{Z})$, $g_i \in G$ [where $v_d(g_0, g_1, \ldots, g_d)$ is a real function over $G^{d+1}$], we can construct an action amplitude $V_{ij \cdots k}^d(h_{ij},\{g_i\})$ that defines a gauge theory with gauge group $SG$ and global symmetry $SG$. The gauge theory action amplitude is obtained from $v_d(g_0, g_1, \ldots, g_d)$ as

$$V_{01-\cdots d}(\{h_{ij}\},\{g_i\}) = 0, \quad h_{ij} h_{jk} \neq h_{ik},$$

$$V_{01-\cdots d}(\{h_{ij}\},\{g_i\}) = e^{2\pi i v_d(h_{00}, h_{01}, \ldots, h_{d0})}$$

$$= e^{2\pi i w_d(h_{01}, \cdots, h_{d-1,d})},$$

(E9)

where $h_i$ are given by

$$h_0 = 1, \quad h_1 = h_0 h_{001}, \quad h_2 = h_1 h_{112}, \quad h_3 = h_2 h_{23}, \ldots,$$

(E10)

and $w_d$ is the nonhomogenous cocycle that corresponds to $v_d$, $w_d(h_{01}, h_{12}, \ldots, h_{d-1,d}) = v_d(h_0 h_1, \ldots, h_d)$. (E11)

To see that the above action amplitude defines a $GG$ lattice gauge theory with a global symmetry $SG$, we note that the cocycle satisfies the cocycle condition

$$v_d(g_0, g_1, \ldots, g_d) = v_d(g_0, g_1, \ldots, g_d) \mod 1, \quad g \in G,$$

$$\sum_{d} v_d(g_0, g_1, \ldots, g_{d+1}) = 0 \mod 1,$$

(E12)

where $g_0, \ldots, g_d$ is the sequence $g_0, \ldots, g_d, \ldots, g_{d+1}$ with $g_d$ removed. Using such a property, one can check that the above action amplitude $V_{01-\cdots d}(\{h_{ij}\},\{g_i\})$ is invariant under the global symmetry transformation (E8). We can also rewrite the partition function as [see Eq. (E9)]

$$Z = \sum_{\{h_{ij}\},\{g_i\}} \prod_{(ij-k)} [V_{ij \cdots k}(\{g_i^{-1} h_{ij} g_j\},\{1\})]^{\nu_{ij-k}},$$

(E13)

which is explicitly gauge invariant. Thus, it defines a symmetric gauge theory with a gauge group $GG$ and a global symmetry group $SG$.

We note that the action amplitude is nonzero only when $h_i h_{jk} = h_{ik}$ or $h_i h_{jk} h_{ik}^{-1} = 1$. The condition $h_i h_{jk} h_{ik}^{-1} = 0$, i.e., gauge flux = 1, is the zero-flux condition on the triangle $(ijk)$ or the flat connection condition. The corresponding gauge theory is in the weak-coupling limit (actually is at the zero coupling). This condition can be implemented precisely only when $GG$ is finite. With the flat connection condition $h_{ij} h_{jk} = h_{ik}$, $h_i$’s and the gauge equivalent sets of $h_{ij}$ have a one-to-one correspondence.

Since the total action amplitude $\prod_{(ij-k)}[V_{ij \cdots k}(\{h_{ij}\},\{g_i\})]^{\nu_{ij-k}}$ on a sphere is always equal to 1 if the gauge flux vanishes, $V_{ij \cdots k}(\{h_{ij}\},\{g_i\})$ describes a quantized topological term in weak-coupling gauge theory (or
In SET states, the gauge connection $h_{ij}$ on the links is a nondynamical probe field. In this case, the gauge connection $h_{ij}$ on a particular form of branched graph. Here is an example in 1 + 1D example.

c. From path integral to Hamiltonian

A path integral can give us an amplitude $Z[[g_i', h_{ij}', g_i, h_{ij}]]$ for a configuration $[g_i, h_{ij}]$ at $t$ to another configuration $[g_i', h_{ij}']$ at $t'$. We like to interpret $Z[[g_i', h_{ij}', g_i, h_{ij}]]$ as the amplitude of an evolution in imaginary time by a Hamiltonian:

$$Z[[g_i', h_{ij}', g_i, h_{ij}]] = \langle g_i', h_{ij}' | e^{-(t'-t)H} | g_i, h_{ij} \rangle.$$  \hspace{1cm} (E14)

However, such an interpretation may not be valid since $Z[[g_i', h_{ij}', g_i, h_{ij}]]$ may not give rise to a Hermitian matrix. It is a worrisome realization that path integral and Hamiltonian evolution may not be directly related.

Here we would like to use the fact that the path integral that we are considering is defined on the branched graphs with a “reflection” property [see Eq. (E5)]. We like to show that such a path integral is better related to Hamiltonian evolution. The key is to require that each time step of evolution is given by branched graphs of the form in Fig. 7. One can show that $Z[[g_i', h_{ij}', g_i, h_{ij}]]$ obtained by summing over all the internal indices in the branched graphs Fig. 7 has a form

$$Z[[g_i', h_{ij}', g_i, h_{ij}]] = \sum_{[g_i, h_{ij}]} U^*[[g_i', h_{ij}'], [g_i', h_{ij}]] U[[g_i', h_{ij}'], [g_i, h_{ij}]]$$ \hspace{1cm} (E15)

and represents a positive-definite Hermitian matrix. Thus, the path integral of the form (E5) always corresponds to a Hamiltonian evolution in imaginary time. In fact, the above $Z[[g_i', h_{ij}', g_i, h_{ij}]]$ can be viewed as an imaginary-time evolution $T = e^{-\Delta tH}$ for a single time step.

For most cases studied in this paper, $h_{ij}$ is a static probe field. In those cases, $h_{ij}$ is the same on all the time slices and $h_{ij} = 1$ on the vertical time links. In this case, $Z[[g_i', h_{ij}', g_i, h_{ij}]]$ (with fixed $h_{ij}$) can still be viewed as an imaginary-time evolution $T = e^{-\Delta tH}$ for a single time step, where only $g_i$’s are dynamical.

For the ideal path integrals with the action amplitudes described by the cocycles, we can reduce the double-layer time step to a single-layer time step, using the retriangulation invariance of the action amplitudes if the space has no boundary. If the space does have boundary, we can still reduce the double-layer time step to a single-layer time step, but with some extra terms on the boundary (see Fig. 8).

APPENDIX F: PHYSICAL PROPERTIES OF DEFECTS

If we view $h_{ij}$ in the last section as a static probe field, then the formalism developed in the last section can be viewed as the path-integral description of SPT states with possible monodromy defects or other possible twists described as the “gauge configuration” $h_{ij}$ on the links. In this section, we use such a formalism to study the physical properties of defects in SPT states.

1. Symmetry transformations and their nonfactorization

First let us examine how symmetry transformations act on the defects. Consider a system with symmetry $G$. The evolution operator $T = e^{-iH}$ satisfies

$$W_g T W_g^{-1} = T, \quad g \in G,$$

where $W_g$ represents the symmetry. We would like to examine the amplitude of the evolution from a configuration $[g_i, h_{ij}]$ to its symmetry $g$ transformed configuration $[gg_i, h_{ij}]$ (where we have assumed that $gh_{ij}g^{-1} = h_{ij}$). More precisely, we want to examine the trace $\text{Tr}(T^N W_g)$. Such a trace can be expressed as a graph which is periodic in the time direction, with one layer of vertical time links given by $h_{ij} = g$ and other layers of vertical time links given by $h_{ij} = 1$ (see Fig. 9).

FIG. 7. Each time step of evolution is given by the path integral on a particular form of branched graph. Here is an example in 1 + 1D.

FIG. 8. The reduction of double-layer time step to single-layer time step on space with boundary in a 1 + 1D example.

FIG. 9. (Color online) (a) The trace $\text{Tr}(T^N W_g)$ can be represented by a graph which is periodic in the time direction, with one layer of vertical time links with $h_{ij} = g$ in a 1 + 1D example. Those vertical time links are marked by red lines crossing them. (a) $\rightarrow$ (b) We can use the retriangulation invariance of the action amplitudes to set all the internal $g_i$’s to a fixed $g^*$ without changing the action amplitude. (b) For fixed $g^*$, we can rewrite one graph as three graphs, where the middle graph just represents a phase factor.
For the ideal path integrals with the action amplitudes described by the cocycles, the action amplitudes only depend on the $g_i$'s on the boundary. (Here we assume that $h_{ij}$'s are fixed nondynamical probe fields. We can use the retriangulation factor: $T_{\text{bulk}}^{g_{n}}$ whose matrix elements are given by $T_{\text{bulk}}^{g_{n}}(g_{k,1}g_{k,2}g_{k,3}g_{k,4}; h_{01},h_{02},h_{1})$. The complex is formed by three tetrahedrons: (0122), (0012), and (0112). The triangles (012) and (012') are on two time slices. A defect is described by the dynamical variables $g_i,g_j$. When $h_{ij}$ on the links crossed by the red lines are nontrivial ($h_{ij}=h_{ij} 
eq 1$), the defect is a monodromy defect. When $h_i=g$ on the vertical links (the time links), it describes the insertion $W_{g}$ in the path integral. Note that $g_i,g_j$ are transformed by $g^*$'s and the complex (b) is formed by four complexes of the type in (a), represented by the four colors of the base triangles.

2. The low-energy effective theory and low-energy effective symmetry at the monodromy defects

In this section, we are going to apply the formalism developed in the last section to study the low-energy effective theory and low-energy effective symmetry at the monodromy defects in 2+1D $Z_n$ SPT states. The monodromy defects are created by a $h_d$ twist ($h_d \in Z_n$).

A $Z_n$ monodromy defect is described by Fig. 10(b). The low-energy degrees of freedom in the defect are described by $g \in Z_n$. Let us use $g^{(k)} = e^{\frac{2\pi ik}{n}}$, $k = 0, \ldots, n-1$, to describe the $Z_n$ group elements. The states on a defect are described by $|g^{(k)}\rangle$. To construct the path integral Eq. (F2) that describes low-energy dynamics of the defects, let us first introduce

$$C(g_0,g_1,g_2;g_0',g_1,g_2';h_{01},h_{02},h_1) = e^{2\pi i\omega_0(g_0,g_1,g_2,g_0',g_1,g_2';h_{01},h_{02},h_1)} \times e^{2\pi i\omega_{01}(g_0,g_1,g_2,h_{01},h_{02},h_1)} \times e^{2\pi i\omega_{12}(g_0,g_1,g_2,h_{01},h_{02},h_1)}.$$  

(F4)

Physically, the above is the action amplitude for the ideal fixed-point system described by (E9) on the complex in Fig. 10(a). Using $C(g_0,g_1,g_2,g_0',g_1,g_2';h_{01},h_{02},h_1)$, we can construct a $|G| \times |G|$ matrix $U_{\text{def}}(g^*,h_d,h)$ whose matrix elements are given by

$$(U_{\text{def}}(g^*,h_d,h))_{g_0,g_0'} = C(g_0,g_1,g_2;g_0',g_1,g_2';1,1,h) \times C(g^*,g_1,g_2;g^*,g_1,g_2;1,1,h) \times C(g^*,g^*,g_2;g^*,g^*,g_2;h_d,h) \times C(g^*,g^*,g^*;g^*,g^*,g^*;h_d,1).$$

(F5)

Then the $|G| \times |G|$ matrix $T_{\text{def}}^{\Delta \tau}(g^*,h_d)$,

$$T_{\text{def}}^{\Delta \tau}(g^*,h_d) = [U_{\text{def}}(g^*,h_d,1)]^T U_{\text{def}}(g^*,h_d,1),$$

will generate the imaginary-time evolution for a single defect. We have (for two defects)

$$(U_{\text{def}}(g^*,h_d,h))^N = U_{\text{def}}^{\Delta \tau} T_{\text{def}}^{\Delta \tau}(g^*,h_d)^N T_{\text{def}}^{\Delta \tau}(g^*,h_d)^N, \quad (F7)$$

where $T$ is the imaginary-time evolution operator $e^{-\Delta \tau H}$ of the whole system for a single time step, $T_{\text{def}}^{\Delta \tau}$ is the imaginary-time evolution operator for a single defect, and the bulk contribution $U_{\text{def}}^{\Delta \tau} = 1$.

Let us calculate $T_{\text{def}}^{\Delta \tau}(g^*,h_d)$ for the monodromy defects in the 2+1D $Z_n$ SPT state. We always choose $g^* = 1$. The cocycles in $H^2(Z_n,\mathbb{R}/\mathbb{Z})$ are labeled by $m = 0,1,\ldots,n-1$, and are given by

$$\omega_0(g^{(k)}_1,g^{(k)}_2,g^{(k)}_3) = e^{\frac{2\pi i}{n} m(k_1+k_2-k_3)}.$$

(F8)

where $[k]_n$ is a shorthand notation for $[k]_n \equiv \text{mod}(k,n)$.  

(F9)

In the following, we only consider the $Z_n$ SPT phases described by $m = 1$.

Let us first concentrate on $2+1D Z_2$ SPT states. Using the cocycles, we find that, for a 2+1D $Z_2$ SPT state,

$$U_{\text{def}}(g^*,h_d = 1,h = 1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (F10)$$

$$T_{\text{def}}^{\Delta \tau}(g^*,h_d = 1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}. \quad (F10)$$
We find that for a trivial monodromy defect, the ground state on a defect is given by $|g_1 = 1\rangle + |g = -1\rangle$, which is an expected result. We also find that

$$U_{\text{def}}(g^*, h_d = -1, h = 1) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

\[(\text{F11})\]

$$T_{\Delta r}^{\Delta r}(g^*, h_d = -1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}. $$

This means that the nontrivial monodromy defect carries two degenerate states $g = \pm 1$. However, the degeneracy can be lifted by perturbations that respect the symmetry.

To study the $Z_2$ symmetry of the defects, let us consider the path integral

$$\text{Tr}(W_g T^N) = U_g^{\text{bulk}} U_0^{\text{bulk}} \text{Tr}(W_g^{\text{def}} T_{\Delta r}^{\Delta r}(g^*, h_d))^{N} \times \text{Tr}(W_g^{\text{def}} T_{\Delta r}^{\Delta r}(g^*, h_d))^{N},$$

\[(\text{F12})\]

where $W_g$, $g \in Z_2$ is a representation of $Z_2$ acting on the total system: $|\{g_1\}\rangle \rightarrow |\{g g_1\}\rangle$, and $W_g^{\text{def}}$ describes how $Z_2$ symmetry transformation act on the low-energy degrees of freedom on the defect. We note that now the phase factor contribution from the bulk $U_g^{\text{bulk}}$ has a $g$ dependence, and thus becomes nontrivial.

Let us first calculate $W_g^{\text{def}}$. Note that $\text{Tr}(T_{\Delta r}^{\Delta r}(g^*, h_d))^{N}$ is a trace of a product of many $U_{\text{def}}(g^*, h_d, h = 1)$ operators. To calculate $\text{Tr}W_g^{\text{def}} T_{\Delta r}^{\Delta r}(g^*, h_d))^{N}$, we just need to replace one of the $U_{\text{def}}(g^*, h_d, h = 1)'s$ with $U_{\text{def}}(g^*, h_d, h = g)$. Therefore, we have

$$[U_{\text{def}}(g^*, h_d, 1)]^i [U_{\text{def}}(g^*, h_d, g)] = [U_{\text{def}}(g^*, h_d, 1)]^i U_{\text{def}}(g^*, h_d, 1) W_g^{\text{def}}.$$  

\[(\text{F13})\]

For the $Z_2$ SPT state, we find

$$U_{\text{def}}(g^*, h_d = -1, h = 1) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

\[(\text{F14})\]

$$U_{\text{def}}(g^*, h_d = -1, h = -1) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. $$

Equation (F13) becomes (for $h_d = -1$)

$$\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} W_{g}^{\text{def}}.$$  

\[(\text{F15})\]

We find that

$$W_{g}^{\text{def}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = icos^2$$ 

\[(\text{F16})\]

for a nontrivial monodromy defect.

Next, let us calculate the phase factor from the bulk, $U_g^{\text{bulk}}$. For this purpose, we introduce

$$U(g, h_{01}, h_{12}) = \frac{e^{i2\pi g_1 h_1} g_1}{e^{i2\pi g_1 h_1}} g_1^{i2\pi g_1 h_1} g_2^{i2\pi g_1 h_1} (\text{F17})$$

which is the action amplitude on a single space-time complex in Fig. 10(a) with $g_1 = g_2 = g^* = 1$. We find that (see Fig. 11)

$$U(-1, -1, -1) = -1, $$

$$U(g, h_{01}, h_{12}) = 1\text{ otherwise.}$$

\[(\text{F18})\]

![FIG. 11.](image) A graphic representation of $U(-1, h_{01}, h_{12})$. The edges crossed by the red lines have $h_{ij} = -1$. The edges not crossed by the red lines have $h_{ij} = 1$. The gauge configurations in (a) and (b) have $U(-1, h_{01}, h_{12}) = -1$. The gauge configuration in (c) and other configurations have $U(-1, h_{01}, h_{12}) = 1$.

The total action amplitude for the bulk is given by

$$U_g^{\text{bulk}} = \prod_{(ij)} U_{ij}^{\text{def}}(g, h_{01}, h_{12})/ U_{ij}^{\text{def}}(1, h_{01}, h_{12}).$$  

\[(\text{F19})\]

where $s_{ijk}$ describes the orientation of the triangle $(ijk)$, and $\prod_{(i,j,k)}$ is a product over all the triangles that are not monodromy defects (i.e., contain no $Z_2$ flux). From Fig. 3, we see that $U_{g1} = -1$ for two identical monodromy defects. Therefore, the low-energy effective $Z_2$ symmetry transformation $W_g$ is given by

$$W_g = U_g^{\text{bulk}} W_g^{\text{def}} \otimes W_g^{\text{def}}.$$  

\[(\text{F20})\]

For $g = -1$, we have

$$W_{-1} = U_{g1}^{\text{bulk}} W_{-1}^{\text{def}} \otimes W_{-1}^{\text{def}} = -i\sigma^2 \otimes i\sigma^2,$$ 

\[(\text{F21})\]

where the first $i\sigma^2$ acts on the states on the first monodromy defect and the second $i\sigma^2$ acts on the second monodromy defect.

The above calculation can be generalized to $n$ identical monodromy defects in a $2+1D$ $Z_n$ SPT state, described by the cocycle Eq. (F8). We find that the low-energy effective $Z_n$ symmetry transformation $W_g$ is given by

$$W_{g}^{(n)} = U_g^{(n)} W_g^{(0)} \otimes \cdots \otimes W_g^{(0)},$$  

\[(\text{F22})\]

for $n$ terms

$$g^{(k)} = e^{2\pi i k/n}.$$  

\[(\text{F23})\]

Here $W_g^{(k)}$ is a $n \times n$ matrix acting on the states on one $Z_n$ monodromy defect. If we choose $|g^{(k)}\rangle$ to be the basis of the states on one $Z_n$ monodromy defect, the action of $W_g^{(0)}$ is given by

$$W_g^{(0)}|g^{(k)}\rangle = f_k|g^{(k)}\rangle, \quad k = 0, 1, \ldots, n - 1,$$ 

\[(\text{F24})\]

which is a pure phase factor which is given by Eq. (F19). For the $Z_n$ SPT state described by the cocycle Eq. (F8), we find that

$$U_g^{(1)}(h_{01}, h_{12})^{(k,k')} = e^{2\pi i(k-k'-k'k)/n^2},$$  

\[(\text{F25})\]

for $k, k' = 0, 1, \ldots, n - 1$. 

035147-34
identical Z₃ monodromy defects are given in Fig. 10(b). The yellow triangle contributes a phase factor \( e^{2\pi i/3} \) to \( U^{\text{bulk}}_{g^{(1)}} \).

This gives us (see Fig. 12)

\[
U^{\text{bulk}}_{g^{(1)}} = \prod_{k=0}^{n-1} e^{2\pi i(k+1)/n} = e^{2\pi i/n}.
\]  

We note that \( (W^{\text{def}}_{g^{(1)}})^n = e^{2\pi i/n} \). So we may say that each monodromy defect carries \( \frac{1}{n} + \text{integer } Z_n \) charges. The fact that \( U^{\text{bulk}}_{g^{(1)}} = e^{2\pi i/n} \) implies that the bulk also carries an \( Z_n \) charge 1. So, we get the following.

**SPT invariant 40.** \( n \) identical elementary monodromy defects (i.e., generated by the twist \( h_d = g^{(1)} \)) in \( 2 + 1DZ_n \) SPT states on a torus always carry a total \( Z_n \) charge 2 if the \( Z_n \) SPT states are described by the \( m = 1 \) cocycle in \( \hat{H}^1(Z_n, \mathbb{R}/\mathbb{Z}) \) [see Eq. (F8)].

Although we only present the derivation of the above result for a particular choice of cocycles as in Eq. (F8), we have checked that the result remains valid for any choices of cocycles. In other words, the above result does not change if we add a coboundary to the cocycle that describes the SPT state. There is a simple way to understand SPT invariant 40. We may view the \( Z_n \) SPT state as a \( U(1) \) SPT state with Hall conductance \( \sigma_{xy} = 2\pi/n \) U(1) flux which will carry a nuclear \( 2\pi/n \) U(1) charge. A \( 2\pi/n \) U(1) charge corresponds to a \( 2\pi \) \( Z_n \) charge. Thus, \( n \) identical \( Z_n \) monodromy defects carry a total \( Z_n \) charge 2.