### Geometric Interpretation of Half-Plane Capacity

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GEOMETRIC INTERPRETATION OF HALF-PLANE CAPACITY

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Abstract
Schramm-Loewner Evolution describes the scaling limits of interfaces in certain statistical mechanical systems. These interfaces are geometric objects that are not equipped with a canonical parametrization. The standard parametrization of SLE is via half-plane capacity, which is a conformal measure of the size of a set in the reference upper half-plane. This has useful harmonic and complex analytic properties and makes SLE a time-homogeneous Markov process on conformal maps. In this note, we show that the half-plane capacity of a hull $A$ is comparable up to multiplicative constants to more geometric quantities, namely the area of the union of all balls centered in $A$, tangent to $\mathbb{R}$, and the (Euclidean) area of a 1-neighborhood of $A$ with respect to the hyperbolic metric.

1 Introduction

Suppose $A$ is a bounded, relatively closed subset of the upper half plane $\mathbb{H}$. We call $A$ a compact $\mathbb{H}$-hull if $A$ is bounded and $\mathbb{H} \setminus A$ is simply connected. The half-plane capacity of $A$, $\text{hcap}(A)$, is defined in a number of equivalent ways (see [1], especially Chapter 3). If $g_A$ denotes the unique conformal
transformation of \( \mathbb{H} \setminus A \) onto \( \mathbb{H} \) with \( g_A(z) = z + o(1) \) as \( z \to \infty \), then \( g_A \) has the expansion
\[
g_A(z) = z + \frac{\text{hcap}(A)}{z} + O(|z|^{-2}), \quad z \to \infty.
\]
Equivalently, if \( B_t \) is a standard complex Brownian motion and \( \tau_A = \inf\{t \geq 0 : B_t \notin \mathbb{H} \setminus A\} \),
\[
\text{hcap}(A) = \lim_{y \to \infty} y \mathbb{E}[\text{Im}(B_{\tau_A})].
\]
Let \( \text{Im}[A] = \sup\{\text{Im}(z) : z \in A\} \). Then if \( y \geq \text{Im}[A] \), we can also write
\[
\text{hcap}(A) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathbb{E}[x+iy] \text{Im}(B_{\tau_A}) \, dx.
\]
These last two definitions do not require \( \mathbb{H} \setminus A \) to be simply connected, and the latter definition does not require \( A \) to be bounded but only that \( \text{Im}[A] < \infty \).
For \( \mathbb{H} \)-hulls (that is, for relatively closed \( A \) for which \( \mathbb{H} \setminus A \) is simply connected), the half-plane capacity is comparable to a more geometric quantity that we define. This is not new (the second author learned it from Oded Schramm in oral communication), but we do not know of a proof in the literature\(^3\). In this note, we prove the fact giving (nonoptimal) bounds on the constant. We start with the definition of the geometric quantity.

**Definition 1.** For an \( \mathbb{H} \)-hull \( A \), let \( \text{hsiz}(A) \) be the 2-dimensional Lebesgue measure of the union of all balls centered at points in \( A \) that are tangent to the real line. In other words
\[
\text{hsiz}(A) = \text{area} \left( \bigcup_{x+iy \in A} B(x+iy, R) \right),
\]
where \( B(z, \epsilon) \) denotes the disk of radius \( \epsilon \) about \( z \).

In this paper, we prove the following.

**Theorem 1.** For every \( \mathbb{H} \)-hull \( A \),
\[
\frac{1}{66} \text{hsiz}(A) < \text{hcap}(A) < \frac{7}{2\pi} \text{hsiz}(A).
\]

## 2 Proof of Theorem 1

It suffices to prove this for weakly bounded \( \mathbb{H} \)-hulls, by which we mean \( \mathbb{H} \)-hulls \( A \) with \( \text{Im}(A) < \infty \) and such that for each \( \epsilon > 0 \), the set \( \{x+iy : y > \epsilon\} \) is bounded. Indeed, for \( \mathbb{H} \)-hulls that are not weakly bounded, it is easy to verify that \( \text{hsiz}(A) = \text{hcap}(A) = \infty \).

We start with a simple inequality that is implied but not explicitly stated in [H]. Equality is achieved when \( A \) is a vertical line segment.

**Lemma 1.** If \( A \) is an \( \mathbb{H} \)-hull, then
\[
\text{hcap}(A) \geq \frac{\text{Im}[A]^2}{2}.
\]

\(^3\)After submitting this article, we learned that a similar result was recently proved by Carto Wong as part of his Ph.D. research.
Lemma 3. For every $c > 0$, let

$$
\rho_c := \frac{2\sqrt{2}}{\pi} \arctan \left( e^{-\theta} \right), \quad \theta = \theta_c = \frac{\pi}{4c}.
$$

Then, for any $c > 0$, if $A$ is a weakly bounded $\mathbb{H}$-hull and $x_0 + iy_0 \in A$ with $y_0 = \text{Im}(A)$, then

$$
\text{hcap}(A) \geq \rho_c^2 y_0^2 + \text{hcap} \left( A \setminus \mathcal{R}(z, 2c) \right).
$$

Proof. Due to the continuity of hcap with respect to the Hausdorff metric on $\mathbb{H}$-hulls, it suffices to prove the result for $\mathbb{H}$-hulls that are path-connected. For two $\mathbb{H}$-hulls $A_1 \subseteq A_2$, it can be seen using the Optional stopping theorem that $\text{hcap}(A_1) \leq \text{hcap}(A_2)$. Therefore without loss of generality, $A$ can be assumed to be of the form $\eta(0, T]$ where $\eta$ is a simple curve with $\eta(0+) \in \mathbb{R}$, parameterized so that $\text{hcap}[\eta(0, t)] = 2t$. In particular, $T = \text{hcap}(A)/2$. If $g_t = g_{\eta(0, t)}$, then $g_t$ satisfies the Loewner equation

$$
\partial_t g_t(z) = \frac{2}{g_t(z) - g_t(0)} - \frac{g_t(z) - z}{|g_t(z) - z|^2}, \quad g_0(z) = z,
$$

which implies

$$
\partial_t \eta_t^2 \leq \frac{4\eta_t}{|g_t(z) - g_t(0)|^2} \leq 4,
$$

where $U : [0, T] \to \mathbb{R}$ is continuous. Suppose $\text{Im}(z) > 2 \text{hcap}(A)$ and let $Y_t = \text{Im}[g_t(z)]$. Then (2) gives

$$
-\partial_t \eta_t^2 \leq \frac{4\eta_t}{|g_t(z) - g_t(0)|^2} \leq 4,
$$

which implies

$$
\eta_t^2 \geq \frac{1}{4} \eta_0^2 - 4T > 0.
$$

This implies that $z \not\in A$, and hence $\text{Im}[A] \leq 2 \text{hcap}(A)$.

The next lemma is a variant of the Vitali covering lemma. If $c > 0$ and $z = x + iy \in \mathbb{H}$, let

$$
\mathcal{J}(z, c) = (x - cy, x + cy),
$$

$$
\mathcal{R}(z, c) = \mathcal{J}(z, c) \times (0, y) = \{x' + iy' : |x' - x| < cy, 0 < y' \leq y\}.
$$

Lemma 2. Suppose $A$ is a weakly bounded $\mathbb{H}$-hull and $c > 0$. Then there exists a finite or countably infinite sequence of points $\{z_1 = x_1 + iy_1, z_2 = x_2 + iy_2, \ldots \} \subseteq A$ such that:

- $y_1 \geq y_2 \geq y_3 \geq \cdots$;
- the intervals $\mathcal{J}(x_1, c), \mathcal{J}(x_2, c), \ldots$ are disjoint;
- $A \subseteq \bigcup_{j=1}^{\infty} \mathcal{R}(z_j, 2c)$. (3)

Proof. We define the points recursively. Let $A_0 = A$ and given $\{z_1, \ldots, z_j\}$, let

$$
A_j = A \setminus \left[ \bigcup_{k=1}^{j} \mathcal{R}(z_j, 2c) \right].
$$

If $A_j = \emptyset$ we stop, and if $A_j \neq \emptyset$, we choose $z_{j+1} = x_{j+1} + iy_{j+1} \in A$ with $y_{j+1} = \text{Im}[A_j]$. Note that if $k \leq j$, then $|x_{j+1} - x_k| \geq 2c y_k \geq c(y_k + y_{j+1})$ and hence $\mathcal{J}(z_{j+1}, c) \cap \mathcal{J}(z_k, c) = \emptyset$. Using the weak boundedness of $A$, we can see that $y_j \to 0$ and hence (3) holds.

Theorem. Let $A$ be a weakly bounded $\mathbb{H}$-hull, let $z_0 \in A$, and let $\gamma \subseteq A$. Then

$$
\text{hcap}(A) \geq \rho_c^2 y_0^2 + \text{hcap}(A \setminus \mathcal{R}(z, 2c)).
$$

Proof. We define the points recursively. Let $A_0 = A$ and given $\{z_1, \ldots, z_j\}$, let

$$
A_j = A \setminus \left[ \bigcup_{k=1}^{j} \mathcal{R}(z_j, 2c) \right].
$$

If $A_j = \emptyset$ we stop, and if $A_j \neq \emptyset$, we choose $z_{j+1} = x_{j+1} + iy_{j+1} \in A$ with $y_{j+1} = \text{Im}[A_j]$. Note that if $k \leq j$, then $|x_{j+1} - x_k| \geq 2c y_k \geq c(y_k + y_{j+1})$ and hence $\mathcal{J}(z_{j+1}, c) \cap \mathcal{J}(z_k, c) = \emptyset$. Using the weak boundedness of $A$, we can see that $y_j \to 0$ and hence (3) holds.

Lemma 3. For every $c > 0$, let

$$
\rho_c := \frac{2\sqrt{2}}{\pi} \arctan \left( e^{-\theta} \right), \quad \theta = \theta_c = \frac{\pi}{4c}.
$$

Then, for any $c > 0$, if $A$ is a weakly bounded $\mathbb{H}$-hull and $x_0 + iy_0 \in A$ with $y_0 = \text{Im}(A)$, then

$$
\text{hcap}(A) \geq \rho_c^2 y_0^2 + \text{hcap}(A \setminus \mathcal{R}(z, 2c)).
$$
Proof. By scaling and invariance under real translation, we may assume that \( \text{Im}[A] = y_0 = 1 \) and \( x_0 = 0 \). Let \( S = S_r \) be defined to be the set of all points \( z \) of the form \( x + iy \) where \( x + iy \in A \setminus \mathcal{R}(i, 2c) \) and \( 0 < u \leq 1 \). Clearly, \( S \cap A = A \setminus \mathcal{R}(i, 2c) \).

Using the capacity inequality [1] (3.10)\
\[
\text{hcap}(A_1 \cup A_2) - \text{hcap}(A_2) \leq \text{hcap}(A_1) - \text{hcap}(A_1 \cap A_2),
\]
we see that\
\[
\text{hcap}(S \cup A) - \text{hcap}(S) \leq \text{hcap}(A) - \text{hcap}(S \cap A).
\]

Hence, it suffices to show that\
\[
\text{hcap}(S \cup A) - \text{hcap}(S) \geq \rho^2_x.
\]

Let \( f \) be the conformal map of \( \mathbb{H} \setminus S \) onto \( \mathbb{H} \) such that \( \varepsilon - f(z) = O(1) \) as \( z \to \infty \). Let \( S' := S \cup A \).

By properties of halfplane capacity [1] (3.8) and (1),\
\[
\text{hcap}(S') - \text{hcap}(S) = \text{hcap}(f(S' \setminus S)) \geq \frac{\text{Im}[f(i)]^2}{2}.
\]

Hence, it suffices to prove that\
\[
\text{Im}[f(i)] \geq \sqrt{2} \rho = \frac{4}{\pi} \arctan \left( e^{-\theta} \right).
\]

By construction, \( S \cap \mathcal{R}(z_r, 2c) = \emptyset \). Let \( V = (-2c, 2c) \times (0, \infty) = \{ x + iy : |x| < 2c, y > 0 \} \) and let \( \tau_V \) be the first time that a Brownian motion leaves the domain. Then [1] (3.5),\
\[
\text{Im}[f(i)] = 1 - \mathbb{P} \left[ \text{Im}(B_{\tau_V}) \right] \geq \mathbb{P} \left\{ B_{\tau_V} \in [-2c, 2c] \right\} \geq \mathbb{P} \left\{ B_{\tau_V} \in [-2c, 2c] \right\}.
\]

The map \( \Phi(z) = \sin(\theta z) \) maps \( V \) onto \( \mathbb{H} \) sending \( [-2c, 2c] \) to \([-1, 1]\) and \( \Phi(i) = i \sin \theta \). Using conformal invariance of Brownian motion and the Poisson kernel in \( \mathbb{H} \), we see that\
\[
\mathbb{P} \left\{ B_{\tau_V} \in [-2c, 2c] \right\} = \frac{2}{\pi} \arctan \left( \frac{1}{\sinh \theta} \right) = \frac{4}{\pi} \arctan \left( e^{-\theta} \right).
\]

The second equality uses the double angle formula for the tangent. \( \square \)

Lemma 4. Suppose \( c > 0 \) and \( x_1 + iy_1, x_2 + iy_2, \ldots \) are as in Lemma 2. Then\
\[
\text{hsiz}(A) \leq [\pi + 8c] \sum_{j=1}^{\infty} y_j^2.
\]

If \( c \geq 1 \), then\
\[
\pi \sum_{j=1}^{\infty} y_j^2 \leq \text{hsiz}(A).
\]

Proof. A simple geometry exercise shows that\
\[
\text{area} \left[ \bigcup_{x + iy \in \mathcal{R}(z_r, 2c)} \mathcal{R}(x + iy, y) \right] = [\pi + 8c] y_j^2.
\]
Since
\[ A \subset \bigcup_{j=1}^{\infty} \mathcal{A}(z_j, 2c), \]
the upper bound in (6) follows. Since \( c \geq 1 \), and the intervals \( \mathcal{A}(z_j, c) \) are disjoint, so are the disks \( \mathcal{B}(z_j, y_j) \). Hence,
\[
\text{area} \left[ \bigcup_{x+i \in A} \mathcal{B}(x+i y, y) \right] \geq \text{area} \left[ \bigcup_{j=1}^{\infty} \mathcal{A}(z_j, y_j) \right] = \pi \sum_{j=1}^{\infty} y_j^2.
\]

Proof of Theorem 1. Let \( V_j = A \cap \mathcal{A}(z_j, c) \). Lemma 3 tells us that
\[
\text{hcap} \left[ \bigcup_{k=j}^{\infty} V_j \right] \geq \rho_c^2 y_j^2 + \text{hcap} \left[ \bigcup_{k=j+1}^{\infty} V_j \right],
\]
and hence
\[
\text{hcap}(A) \geq \rho_c^2 \sum_{j=1}^{\infty} y_j^2. \tag{8}
\]
Combining this with the upper bound in (6) with any \( c > 0 \) gives
\[
\frac{\text{hcap}(A)}{\text{hsiz}(A)} \geq \frac{\rho_c^2}{\pi + 8c}.
\]
Choosing \( c = \frac{8}{5} \) gives us
\[
\frac{\text{hcap}(A)}{\text{hsiz}(A)} > \frac{1}{66}.
\]
For the upper bound, choose a covering as in Lemma 2. Subadditivity and scaling give
\[
\text{hcap}(A) \leq \sum_{j=1}^{\infty} \text{hcap} \left[ \mathcal{A}(z_j, 2c y_j) \right] = \text{hcap} \left[ \mathcal{A}(i, 2c) \right] \sum_{j=1}^{\infty} y_j^2. \tag{9}
\]
Combining this with the lower bound in (6) with \( c = 1 \) gives
\[
\frac{\text{hcap}(A)}{\text{hsiz}(A)} \leq \frac{\text{hcap} \left[ \mathcal{A}(i, 2) \right]}{\pi}.
\]
Note that \( \mathcal{A}(i, 2) \) is the union of two real translates of \( \mathcal{A}(i, 1) \), \( \text{hcap} \left[ \mathcal{A}(i, 2) \right] \leq 2 \text{hcap} \left[ \mathcal{A}(i, 1) \right] \) whose intersection is the interval \( (0, i] \). Using (4), we see that
\[
\text{hcap} \left[ \mathcal{A}(i, 2) \right] \leq 2 \text{hcap} \left[ \mathcal{A}(i, 1) \right] - \text{hcap}((0, i]) = 2 \text{hcap} \left[ \mathcal{A}(i, 1) \right] - \frac{1}{2}.
\]
But \( \mathcal{A}(i, 1) \) is strictly contained in \( A' := \{ z \in \mathbb{H} : |z| \leq \sqrt{2} \} \), and hence
\[
\text{hcap} \left[ \mathcal{A}(i, 1) \right] < \text{hcap}(A') = 2.
\]
The last equality can be seen by considering $h(z) = z + 2z^{-1}$ which maps $\mathbb{H} \setminus A'$ onto $\mathbb{H}$. Therefore,

$$\text{hc} \{\mathcal{R}(i, 2)\} < \frac{7}{2},$$

and hence

$$\frac{\text{hc}(A)}{\text{hsiz}(A)} < \frac{7}{2\pi}.$$ 

An equivalent form of this result can be stated\footnote{This formulation was suggested to us by Scott Sheffield and the anonymous referee.} in terms of the area of the 1-neighborhood of $A$ (denoted $\text{hyp}(A)$) in the hyperbolic metric. The unit hyperbolic ball centered at a point $x + iy$ is the Euclidean ball with respect to which $x + iy/e$ and $x = iy e$ are diametrically opposite boundary points. For any $c$, choosing a covering as in Lemma 2,

$$\text{hyp}(A) < \left( \left( \frac{e}{2} \right)^2 \pi + 4ec \right) \sum_{j=1}^{\infty} y_j^2.$$ 

So by \footnote{This formulation was suggested to us by Scott Sheffield and the anonymous referee.},

$$\frac{\text{hc}(A)}{\text{hyp}(A)} > \rho_c^2 \left( \left( \frac{e}{2} \right)^2 \pi + 4ec \right)^{-1}.$$ 

Setting $c$ to $\frac{8}{5}$,

$$\frac{\text{hc}(A)}{\text{hyp}(A)} > \frac{1}{100}.$$ 

For any $c > \frac{e-e^{-1}}{2}$,

$$\text{hyp}(A) \geq \pi \left( \frac{e-e^{-1}}{2} \right)^2 \sum_{j=1}^{\infty} y_j^2.$$ 

So by \footnote{This formulation was suggested to us by Scott Sheffield and the anonymous referee.},

$$\frac{\text{hc}(A)}{\text{hyp}(A)} \leq \frac{\text{hc} \{\mathcal{R}(i, 2)\}}{\pi \left( \frac{e-e^{-1}}{2} \right)^2}.$$ 

Thereby,

$$\text{hc} \{\mathcal{R}(i, 3)\} \leq \text{hc} \{\mathcal{R}(i, 1)\} + \text{hc} \{\mathcal{R}(i, 2)\} - \text{hc} \{(0, i)\} \leq 5.$$ 

Therefore,

$$\frac{1}{100} < \frac{\text{hc}(A)}{\text{hyp}(A)} < \frac{20}{\pi(e-e^{-1})^2}.$$ 

References