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Reconstructing permutations from cycle minors

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Abstract

The $i$th cycle minor of a permutation $p$ of the set $\{1, 2, \ldots, n\}$ is the permutation formed by deleting an entry $i$ from the decomposition of $p$ into disjoint cycles and reducing each remaining entry larger than $i$ by 1. In this paper, we show that any permutation of $\{1, 2, \ldots, n\}$ can be reconstructed from its set of cycle minors if and only if $n \geq 6$. We then use this to provide an alternate proof of a known result on a related reconstruction problem.

1 Background and Notation

For any positive integer $n$, let $[n]$ denote the set $\{1, 2, 3, \ldots, n\}$. Let $S_n$ be the set of all permutations of $[n]$. Consider a permutation $p \in S_n$ and the corresponding sequence $p(1), p(2), \ldots, p(n)$, which we abbreviate $p_1p_2\ldots p_n$.

Definition. Let $n \geq 2$, $p \in S_n$ and $i \in [n]$. The $i$th sequence reduction of $p$, denoted $p \downarrow i$, is the permutation of $[n-1]$ formed by first deleting $p_k = i$ from the $p_1p_2\ldots p_n$ and then decreasing any number greater than $i$ in the resulting sequence by 1.

For instance, $13425 \downarrow 3 = 1324$, because we first remove the 3 from 13425, leaving 1425, after which we decrease the 4 and the 5 by 1. We denote by $R(p)$ the set of all sequence reductions of $p$ and by $M(p)$ the multiset of all sequence reductions of $p$. For example, $R(13425) = \{2314, 1234, 1324, 1342\}$ and $M(13425) = \{2314, 1234, 1324, 1324, 1342\}$.

Several reconstruction problems related to sequence reductions have been formulated. One such problem asks for which $n$ can any permutation of length $n$ be uniquely reconstructed from its set of sequence reductions. The analogous reconstruction problem for multisets of sequence reductions has also been investigated. Formally, these problems are equivalent to asking for which $n$ is the restriction of $R$ (or $M$, respectively) to $S_n$ an injective map.
These problems are motivated by the famous Ulam Conjecture [7], which states that a graph with \( n \geq 3 \) vertices can be reconstructed from its multiset of induced \((n-1)\)-vertex subgraphs. Harary [2] conjectured further that if \( n \geq 4 \), we can reconstruct a graph with \( n \) nodes from its set (ignoring multiplicity) of induced \((n-1)\)-vertex subgraphs. The problem of reconstructing from sequence reductions is a natural analogue of the Ulam conjecture for permutations in the following sense. The inversion graph of a permutation \( p \in S_n \) is the graph with vertices labeled 1, 2, \ldots, \( n \) and with an edge between vertices \( i \) and \( j \) with \( i < j \) if and only if \( i \) is to the right of \( j \) in the sequence \( p_1p_2\ldots p_n \). The \((n-1)\)-vertex subgraphs of the inversion graph of a permutation \( p \) are isomorphic (ignoring the labels) to the inversion graphs of the corresponding sequence reductions of \( p \).

The following theorem has been proven independently by Ginsburg [1], Ince [3], Raykova [5], and Smith [6].

**Theorem 1.1.** Let \( n \geq 5 \) be a positive integer, and let \( p \) and \( q \) be two permutations in \( S_n \). Then \( R(p) = R(q) \) implies that \( p = q \) (and thus \( M(p) = M(q) \) implies that \( p = q \)).

In addition, there are counterexamples for \( n = 2, 3, \) and \( 4 \). We have \( M(3142) = M(2413), M(312) = M(231), \) and \( M(12) = M(21), \) and the same counterexamples hold for \( R \).

In this paper we solve a natural variant on the problem of reconstructing permutations from their sequence reductions. We also use this variant to provide an alternate proof of Theorem 1.1.

### 1.1 Cycle Minors

Rather than considering a permutation \( p \in S_n \) as a sequence consisting of the numbers in \([n]\), we consider the decomposition of \( p \) into disjoint cycles, i.e. a composition of disjoint cycles of the form \((i, p(i), p(p(i)), \ldots)\).

**Definition.** Let \( n \geq 2, p \in S_n \) and \( i \in [n] \). Then the \( i \)th cycle minor of \( p \), denoted \( p \downarrow i \), is the permutation of \([n-1]\) formed by first deleting the entry \( i \) from the decomposition of \( p \) into disjoint cycles, and then subtracting 1 from any number greater than \( i \).

For example, suppose \( n = 9 \) and \( p = (1546)(279)(3)(8) \). The permutation \( p \) can be represented by the directed graph shown in Figure 1.

![Figure 1](image)

Figure 1: The directed graph associated with the permutation \((1546)(279)(3)(8)\).

Then \( p \downarrow 5 \) is the permutation \((145)(268)(3)(7)\), which has the directed graph shown in Figure 2, where new edges and labels are shown in red.
considering the representation of a permutation as a directed graph, it is clear that the definition of cycle minor is independent of such choices.

**Definition.** We denote the set of cycle minors of \( p \) by \( C(p) \) and the multiset of cycle minors of \( p \) by \( \mathcal{M}(p) \).

We will also use the following conventions throughout the next section. If \( p(a) = b \), we write \( a \mapsto b \) in \( p \). We say that \( a \) and \( b \) are adjacent in \( p \) if either \( a \mapsto b \) or \( b \mapsto a \) in \( p \). We also say that a permutation \( p \) “contains” the \( k \)-cycle \( (a_1a_2\ldots a_k) \), or the \( k \)-cycle is “in” \( p \), if it appears in the decomposition of \( p \) into disjoint cycles.

## 2 Reconstruction from Cycle Minors

We now state our main result.

**Theorem 2.1.** Suppose \( n \geq 6 \) and \( p, q \in S_n \) such that \( C(p) = C(q) \). Then \( p = q \).

Furthermore, there are counterexamples for \( n = 2, 3, 4, \) and \( 5 \):

\[
C((12)) = C((1)(2)) = \{(1)\},
\]
\[
C((123)) = C((132)) = \{(12)\},
\]
\[
C((13)(24)) = C((14)(23)) = \{(2)(13), (1)(23), (3)(12)\},
\]
\[
C((14253)) = C((13524)) = \{(1423), (1342), (1324), (1243)\}.
\]

The counterexample for \( n = 5 \) is the only pair of permutations in \( S_5 \) with the same set of cycle minors.

To prove Theorem 2.1, we first provide several preliminary lemmas.

**Lemma 2.2.** Let \( n \geq 3 \). Suppose \( p, q \in S_n \) and \( C(p) = C(q) \). Then \( C(p \downarrow n) = C(q \downarrow n) \) and \( C(p \downarrow 1) = C(q \downarrow 1) \).

**Proof.** First, notice that for any \( p \in S_n \) and any \( 1 \leq i < j \leq n \), the permutations \( p \downarrow j \downarrow i \) and \( p \downarrow i \downarrow j - 1 \) are each formed by deleting \( i \) and \( j \) simultaneously from the
cycle representation of \( p \) and subsequently subtracting 1 from any number that is between \( i \) and \( j \) and subtracting 2 from any number that is greater than \( j \). Hence, we obtain

\[
p \downarrow i \downarrow (j - 1) = p \downarrow j \downarrow i
\]

for any \( 1 \leq i < j \leq n \).

From the definition of \( \mathcal{C}(p \downarrow n) \) and using (2.1) repeatedly, we find

\[
\mathcal{C}(p \downarrow n) = \{ p \downarrow n \downarrow i \mid 1 \leq i \leq n - 1 \}
= \{ p \downarrow i \downarrow (n - 1) \mid 1 \leq i \leq n - 1 \}
= \{ p \downarrow i \downarrow (n - 1) \mid 1 \leq i \leq n \}
= \{ r \downarrow (n - 1) \mid r \in \mathcal{C}(p) \}
\]

Similarly, we have

\[
\mathcal{C}(q \downarrow n) = \{ t \downarrow (n - 1) \mid t \in \mathcal{C}(q) \}
\]

Since \( \mathcal{C}(p) = \mathcal{C}(q) \) by assumption, it follows that \( \mathcal{C}(p \downarrow n) = \mathcal{C}(q \downarrow n) \). A similar argument shows that \( \mathcal{C}(p \downarrow 1) = \mathcal{C}(q \downarrow 1) \).

\[\Box\]

Lemma 2.3. Suppose \( n \geq 2 \) and \( p, q \in S_n \) such that \( p \downarrow 1 = q \downarrow 1 \) and \( p \downarrow n = q \downarrow n \). Then one of the following is true:

(i) \( p = q \).

(ii) \( p = (1n) \circ q \circ (1n) \) and either \( p(1) = n \) or \( p(n) = 1 \); in other words, 1 and \( n \) are adjacent in \( p \) and interchanging them results in \( q \).

(iii) 1 and \( n \) are fixed points of \( p \), \((1n) \) is a 2-cycle in \( q \) and \( p(i) = q(i) \) for all \( i \neq 1, n \).

(iv) 1 and \( n \) are fixed points of \( q \), \((1n) \) is a 2-cycle in \( p \) and \( p(i) = q(i) \) for all \( i \neq 1, n \).

Proof. Let \( s = p \downarrow 1 = q \downarrow 1 \) and \( t = p \downarrow n = q \downarrow n \). We consider several cases.

Case 1. Suppose that \( n - 1 \) is a fixed point of \( s \).

Let \( m = p(n) \), so that \( n \mapsto m \) in \( p \), and assume \( m \not\in \{1, n\} \). Then \((n - 1) \mapsto (m - 1) \) in \( p \downarrow 1 = s \), which is a contradiction. Thus either \( n \mapsto 1 \) or \( n \mapsto n \) in \( p \). Similarly, if \( k \mapsto n \) in \( p \) then \( k = 1 \) or \( k = n \). Thus either \( n \) is a fixed point of \( p \) or \((1n) \) is a 2-cycle in \( p \), and by an analogous argument, the same holds for \( q \).

If \( n \) is a fixed point of both \( p \) and \( q \), we see that \( p \) and \( q \) are otherwise identical since \( p \downarrow n = q \downarrow n \). Thus (i) is satisfied.

If \( n \) is a fixed point of one of the permutations, say \( p \), and \((1n) \) is a 2-cycle in \( q \), then since 1 is a fixed point of \( q \downarrow n = p \downarrow n \), the element 1 must be a fixed point of \( p \) as well. The remaining cycles are unchanged by removing \( n \) from \( p \) or \( q \) to form \( t \), and thus either (iii) or (iv) is satisfied.

Finally, if both \( p \) and \( q \) contain the 2-cycle \((1n) \), then again the remaining cycles are unchanged by removing \( n \) from \( p \) or \( q \) to form \( t \), and so \( p \) and \( q \) satisfy (i).

Therefore, if \( n - 1 \) is a fixed point of \( s \), \( p \) and \( q \) satisfy one of (i), (iii), or (iv).
Case 2. Suppose 1 is a fixed point of \( t \).

Let \( m = p(1) \), so that \( 1 \mapsto m \) in \( p \), and assume \( m \not\in \{1, n\} \). Then \( 1 \mapsto m \) in \( p \downarrow n = t \), which is a contradiction. Thus either \( 1 \mapsto 1 \) or \( 1 \mapsto n \) in \( p \). Similarly, if \( k \mapsto 1 \) in \( p \) then \( k = 1 \) or \( k = n \). Thus either 1 is a fixed point of \( p \) or \((1n)\) is a 2-cycle in \( p \), and by an analogous argument, the same holds for \( q \).

If 1 is a fixed point of both \( p \) and \( q \), we see that \( p \) and \( q \) are otherwise identical since \( p \downarrow 1 = q \downarrow 1 \). Thus (i) is satisfied.

If 1 is a fixed point of one of the permutations, say \( p \), and \((1n)\) is a 2-cycle in the other, \( q \), then since \( n - 1 \) is a fixed point of \( q \downarrow 1 = p \downarrow 1 \), the element \( n \) must be a fixed point of \( p \) as well. The remaining cycles are unchanged by deleting \( n \) from \( p \) or from \( q \) to form \( t \), and thus either (iii) or (iv) is satisfied.

Finally, if both \( p \) and \( q \) contain the 2-cycle \((1n)\), then again the remaining cycles are unchanged by removing \( n \) from \( p \) or \( q \) to form \( t \), and so \( p \) and \( q \) satisfy (i).

Therefore, if 1 is a fixed point of \( t \), \( p \) and \( q \) satisfy one of (i), (iii), or (iv).

Case 3. Suppose \( n - 1 \) is not a fixed point of \( s \) and 1 is not a fixed point of \( t \).

Let \( a = t^{-1}(1) \) and \( b = t(1) \), so that \( a \mapsto 1 \mapsto b \) in \( t \). Since 1 is not a fixed point of \( t \), neither \( a \) nor \( b \) can be equal to 1. Recall that \( t \) can be formed by deleting \( n \) from \( p \) or from \( q \) in cycle notation. It follows that either \( a \mapsto 1 \mapsto b \), \( a \mapsto 1 \mapsto n \mapsto b \), or \( a \mapsto n \mapsto 1 \mapsto b \) in \( p \), and the same is true of \( q \).

Let \( c = s^{-1}(n - 1) + 1 \) and \( d = s(n - 1) + 1 \), so that \( (c - 1) \mapsto (n - 1) \mapsto (d - 1) \) in \( t \). Since \( n - 1 \) is not a fixed point of \( s \), neither \( c \) nor \( d \) can be equal to \( n \). Recall that \( s \) can be formed by deleting \( 1 \) from \( p \) or from \( q \) in cycle notation. It follows that either \( c \mapsto n \mapsto d \), \( c \mapsto 1 \mapsto n \mapsto d \), or \( c \mapsto n \mapsto 1 \mapsto d \) in \( p \), and the same is true of \( q \).

Notice that \( a, b \leq n - 1 \) and \( c, d \geq 2 \) since \( s, t \in S_{n-1} \). Recall that neither \( a \) nor \( b \) is equal to 1 and neither \( c \) nor \( d \) is equal to \( n \). Thus \( a, b, c, d \not\in \{1, n\} \).

Suppose 1 and \( n \) are not adjacent in \( p \). Then \( c \mapsto n \mapsto d \) in \( p \). If 1 and \( n \) are adjacent in \( q \), then \( c \mapsto 1 \) in \( q \downarrow n = t \), whereas \( c \mapsto d \) in \( p \downarrow n = t \), which is a contradiction since \( d \neq 1 \). Thus 1 and \( n \) are not adjacent in \( q \). It follows that \( c \mapsto n \mapsto d \) in \( q \). Since \( p \downarrow n = q \downarrow n \), it follows that \( p = q \), and we are in case (i).

If instead 1 and \( n \) are adjacent in \( p \), then since 1 and \( n \) have distinct images under \( p \) and \( p^{-1} \), it follows that \( a = c \) and \( b = d \). Thus 1 and \( n \) are adjacent in \( q \) as well, for otherwise \( a \mapsto 1 \mapsto b \) and \( a \mapsto n \mapsto b \) in \( q \), which is impossible since \( a, b \not\in \{1, n\} \).

Thus 1 and \( n \) are adjacent in \( p \) and \( q \), and since \( p \downarrow n = q \downarrow n \), either \( p = q \) or \( p \) can be formed by reversing the positions of 1 and \( n \) in \( q \). Hence, either (i) or (ii) is satisfied, which completes the proof.

We now have the tools required to prove Theorem 2.1.

**Proof.** We use induction on \( n \geq 6 \). The base case, \( n = 6 \), has been verified by computer, by checking that the 720 permutations in \( S_6 \) have distinct sets of cycle minors.

Let \( n \geq 7 \) and assume that for any \( r, s \in S_{n-1} \), \( C(r) = C(s) \) implies that \( r = s \). Let \( p, q \in S_n \), and assume \( C(p) = C(q) \). Then \( C(p \downarrow n) = C(q \downarrow n) \) and \( C(p \downarrow 1) = C(q \downarrow 1) \) by Lemma 2.2. It follows from the inductive hypothesis that \( p \downarrow 1 = q \downarrow 1 \) and \( p \downarrow n = q \downarrow n \).
Now, assume \( p \neq q \). Then by Lemma 2.3, either \((1n)\) is a 2-cycle of \( p \) and 1 and \( n \) are fixed points of \( q \) (or vice versa) and \( p \) and \( q \) are otherwise identical, or 1 and \( n \) are adjacent in \( p \) and interchanging them results in \( q \).

Suppose that the former is true, and without loss of generality suppose \((1n)\) is a 2-cycle of \( p \) and 1 and \( n \) are fixed points of \( q \). Let \( a \) be an integer between 2 and \( n - 1 \) inclusive. Then in \( p \upharpoonright a \), the elements 1 and \( n - 1 \) form a 2-cycle. However, for any \( b \in [n] \), either 1 or \( n - 1 \) (possibly both) is a fixed point of \( q \upharpoonright b \). Thus \( p \upharpoonright a \) is an element of \( \mathcal{C}(p) \) that is not an element of \( \mathcal{C}(q) \), which is a contradiction.

Suppose instead that 1 and \( n \) are adjacent in \( p \) and interchanging them results in \( q \). Without loss of generality, suppose further that \( 1 \mapsto n \) in \( p \) and \( n \mapsto 1 \) in \( q \). If 1 and \( n \) form a 2-cycle in \( p \) then \( p = q \), which is a contradiction. Therefore 1 and \( n \) are in a cycle of length at least 3.

Suppose one of \( p \) and \( q \), say \( q \), is an \( n \)-cycle. Then \( p = (1n) \circ q \circ (1n) \) is an \( n \)-cycle as well. Thus either \( p \) and \( q \) are both \( n \)-cycles or neither is an \( n \)-cycle. We consider these two cases separately.

**Case 1.** Suppose neither \( p \) nor \( q \) is an \( n \)-cycle. Let \( k \) be the length of the cycle in \( p \) (and hence the length of the cycle in \( q \)) containing 1 and \( n \). Let \( i \) be an element of \( p \) that is not in the cycle containing 1 and \( n \), and let \( r = p \upharpoonright i \). We show that \( r \) is not a cycle minor of \( q \).

Notice that \( 1 \mapsto (n - 1) \) in a cycle of length \( k \) of \( r \). Suppose \( q \upharpoonright i \) is a cycle minor of \( q \) for which 1 and \( n - 1 \) occur in a cycle of length \( k \). If \( i \) is a member of the \( k \)-cycle in \( q \) containing 1 and \( n \), then either 1 or \( n - 1 \) occurs in a cycle of length \( k - 1 \) in \( q \upharpoonright i \), which is a contradiction. Hence \( i \) is not a member of the \( k \)-cycle of \( q \) containing 1 and \( n \). Since \( n \mapsto 1 \) in \( q \), it follows that \( (n - 1) \mapsto 1 \) in \( q \upharpoonright i \).

Thus, any cycle minor of \( q \) having 1 and \( n - 1 \) in a cycle of length \( k \) has \( n - 1 \mapsto 1 \). Thus \( r \) cannot be a cycle minor of \( q \), and we have a contradiction.

**Case 2.** Suppose \( p \) and \( q \) are both \( n \)-cycles. Then for any \( i \notin \{1, n\} \), we have \( 1 \mapsto (n - 1) \) in \( p \upharpoonright i \), but \( (n - 1) \mapsto 1 \) in \( q \upharpoonright i \). Since \( \mathcal{C}(p) = \mathcal{C}(q) \), either \( q \upharpoonright 1 \) or \( q \upharpoonright n \) must have \( 1 \mapsto (n - 1) \).

Suppose that \( 1 \mapsto (n - 1) \) in \( q \upharpoonright 1 = p \upharpoonright 1 \). Then since \( n \mapsto 1 \) in \( q \) and \( 1 \mapsto n \) in \( p \), it follows that \( 2 \mapsto n \) in \( q \) and \( 2 \mapsto 1 \mapsto n \) in \( p \). Thus \( 1 \mapsto (n - 1) \) in \( p \upharpoonright i \) for any \( i \neq n \). Since \( \mathcal{C}(p) = \mathcal{C}(q) \) and there are elements of \( \mathcal{C}(q) \) in which \( (n - 1) \mapsto 1 \), we must have \( (n - 1) \mapsto 1 \) in \( p \upharpoonright n \). This is impossible because \( 2 \mapsto 1 \) in \( p \upharpoonright n \), and \( n - 1 > 2 \). Thus, we have a contradiction.

Suppose instead that \( 1 \mapsto (n - 1) \) in \( q \upharpoonright n = p \upharpoonright n \). Then since \( n \mapsto 1 \) in \( q \) and \( 1 \mapsto n \) in \( p \), it follows that \( 1 \mapsto n - 1 \) in \( q \) and \( 1 \mapsto n \mapsto n - 1 \) in \( p \). Thus \( 1 \mapsto (n - 1) \) in \( p \upharpoonright i \) for any \( i \neq 1 \). Since \( \mathcal{C}(p) = \mathcal{C}(q) \) and there are elements of \( \mathcal{C}(q) \) in which \( (n - 1) \mapsto 1 \), we must have \( (n - 1) \mapsto 1 \) in \( p \upharpoonright 1 \). But \( n - 1 \mapsto n - 2 \) in \( p \upharpoonright 1 \), and \( n - 2 > 1 \). Thus, we have a contradiction.

Having found a contradiction in all cases, we conclude that \( p = q \). It follows by induction that for all \( n \geq 6 \), if \( p, q \in S_n \) and \( \mathcal{C}(p) = \mathcal{C}(q) \), then \( p = q \). \(\square\)
Corollary 2.4. Suppose \( n \geq 5 \) and \( p, q \in S_n \) such that \( \mathcal{M}(p) = \mathcal{M}(q) \). Then \( p = q \).

**Proof.** If \( \mathcal{M}(p) = \mathcal{M}(q) \) then \( \mathcal{C}(p) = \mathcal{C}(q) \), so if \( n \geq 6 \) then \( p = q \) by Theorem 2.1. For \( n = 5 \), we only need to verify that \( \mathcal{M}((14253)) \neq \mathcal{M}((13524)) \) since this is the only pair of permutations in \( S_5 \) for which reconstruction from the set of cycle minors fails, and indeed this is the case.

The permutations given above for \( n = 2 \), 3, and 4 that have the same set of cycle minors also have the same multisets of cycle minors, so 5 is the smallest value of \( n \) greater than 1 for which permutations can be reconstructed from their multisets of cycle minors.

### 3 Application to Sequence Reductions

Theorem 2.1 can be applied to the problem of reconstructing permutations from their sequence reductions.

**Definition.** Let \( p = p_1p_2 \ldots p_n \) be a permutation, written in sequence form. Then the associated cyclic permutation of \( p \), denoted \( \hat{p} \), is the permutation having cycle representation \( (p_1p_2 \ldots p_n) \).

For example, if \( p = 31524 \) (as a sequence), then \( \hat{p} = (31524) \), the cyclic permutation which sends 3 to 1, 1 to 5, etc.

The following corollary to Theorem 2.1 provides a natural link between cycle minors and sequence reductions.

**Corollary 3.1.** Suppose \( n \geq 6 \), \( p, q \in S_n \) and \( \mathcal{C}(\hat{p}) = \mathcal{C}(\hat{q}) \). If there exists an index \( i \) such that \( p(i) = q(i) \), then \( p = q \).

**Proof.** If \( \mathcal{C}(\hat{p}) = \mathcal{C}(\hat{q}) \) then \( \hat{p} = \hat{q} \), meaning \( p \) and \( q \) are cyclic permutations of one another. In other words, there exists a positive integer \( k \) such that \( p = q \circ (123 \ldots n)^k \).

Suppose \( i \) is such that \( p(i) = q(i) \). Note that \( p(i) = q \circ (123 \ldots n)^k(i) = q(i+k) = q(i) \), where the index \( i + k \) is taken modulo \( n \). Since \( q \) is a bijection, it follows that \( k \) is divisible by \( n \), and so \( p = q \).

Recall that \( R(p) \) denotes the set of all sequence reductions of \( p \). We now provide an alternate proof of Theorem 1.1 using Corollary 3.1, and the fact, shown by Ginsburg [1] and Ince [3], that the position of \( n \) in \( p \) is uniquely determined by \( R(p) \) for \( n \geq 5 \).

**Theorem 1.1.** Let \( n \geq 5 \) and \( p, q \in S_n \). If \( R(p) = R(q) \), then \( p = q \).

**Proof.** The case \( n = 5 \) has been verified by a computer search. Suppose \( n \geq 6 \) and \( p, q \in S_n \) are such that \( R(p) = R(q) \). Notice that \( \hat{p} \downarrow i = p \downarrow \hat{i} \), so we have

\[
\begin{align*}
\mathcal{C}(\hat{p}) &= \{ \hat{p} \downarrow i \mid 1 \leq i \leq n \} \\
&= \{ p \downarrow \hat{i} \mid 1 \leq i \leq n \} \\
&= \{ \hat{s} \mid s \in R(p) \} \\
&= \{ \hat{s} \mid s \in R(q) \} \\
&= \mathcal{C}(\hat{q}).
\end{align*}
\]
Since \( n \geq 5 \), the position of \( n \) is the same in \( p \) and \( q \) by the result of Ginsburg and Ince, so by Corollary 3.1, we have \( p = q \) as desired.

## 4 Cycle \( k \)-minors

We have shown that a permutation in \( S_n \) can be reconstructed from its set of cycle minors for \( n \geq 6 \). The analogous result for sequence reductions has been extended to the case of \( k \)-reductions, where a \( k \)-reduction of a permutation \( p \) is a permutation that is the result of \( k \) successive sequence reductions starting from \( p \). Reconstruction of permutations from their \( k \)-reductions has been studied in [3], [5], and [6].

Ince [3] and Raykova [5] have shown that for any \( k \), permutations of \( n \) can be reconstructed from their sets of \( k \)-reductions whenever \( n \) is sufficiently large, and we ask if a similar theorem holds for cycle minors. We define a cycle \( k \)-minor as follows.

**Definition.** Let \( k \) and \( n \) be positive integers with \( k < n \) and let \( p \in S_n \). A cycle \( k \)-minor of \( p \) is any permutation in \( S_n \) formed by taking \( k \) successive cycle minors of \( p \).

Alternatively, to form a cycle \( k \)-minor we delete some \( k \) numbers of the cycle representation of \( p \) and then replace the remaining numbers with the numbers 1; 2; \ldots; \( n - k \) so as to preserve the ordering. Let \( C_k(p) \) and \( M_k(p) \) be the set and multiset of cycle \( k \)-minors of \( p \). The following conjecture is a natural generalization of Theorem 2.1.

**Conjecture 4.1.** For any positive integer \( k \), there exists a positive integer \( N_k \) such that for all \( n \geq N_k \), permutations in \( S_n \) can be reconstructed from their sets of cycle \( k \)-minors.

From computer calculations, it seems that we can reconstruct permutations from their set of cycle 2-minors if and only if \( n \geq 7 \), and from their set of cycle 3-minors if and only if \( n \geq 10 \).

For sufficiently large \( n \), we can uniquely determine the sizes of the cycles of a permutation in \( S_n \) from its set of cycle \( k \)-minors as follows. A partition \( \lambda \) of a positive integer \( n \) is sequence \( \lambda_1, \lambda_2, \ldots, \lambda_m \) of positive integers which satisfy \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \) and \( \sum_{i=1}^{m} \lambda_i = n \).

We can associate each permutation \( p \in S_n \) having cycles of length \( \lambda_1, \lambda_2, \ldots, \lambda_m \) in non-increasing order with this partition \( \lambda \) of \( n \). For example, permutations in \( S_8 \) having decomposition into disjoint cycles of the form \((abc)(def)(gh)\) are associated with the partition 3,3,2 of 8.

Recall that the conjugacy classes of \( S_n \) consist of all permutations having the same associated partition. Let \( \lambda \) be a partition of \( n \), and let \( \mu \) be a partition of \( n - k \). We say that \( \mu \) is a \( k \)-minor of \( \lambda \) if \( \mu_i \leq \lambda_i \) for all \( i \).

For any positive integer \( m \), let \( \rho(m) \) denote the smallest divisor of \( m \) that is greater than or equal to \( \sqrt{m} \). In [4], we show the following.

**Theorem 4.2.** Let \( n \) and \( k \) be positive integers with \( k < n \). For \( n \not\in \{5, 12, 21, 32\} \), define

\[
g(n) = \min_{0 \leq t \leq n} \rho(n + 2 - t) - 2 + t
\]
and also define $g(5) = 1$, $g(12) = 3$, $g(21) = 5$, $g(32) = 7$. Then partitions of $n$ can be reconstructed from their sets of $k$-minors if and only if $k \leq g(n)$.

Clearly, the partition associated with a cycle $k$-minor of a permutation $p$ is a $k$-minor of the partition associated with $p$. Thus we have the following corollary to Theorem 4.2.

**Corollary 4.3.** The conjugacy class of a permutation can be reconstructed from its set of cycle $k$-minors whenever $k \leq g(n)$.

In the case $k = 1$, this is not sufficient to reconstruct the permutation as well, since we can reconstruct partitions of $n \geq 3$ from their 1-minors, whereas we require $n \geq 6$ in order to reconstruct permutations from their cycle 1-minors. Nevertheless, this may be a useful intermediate step in settling Conjecture 4.1 and finding the values $N_k$.

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**References**


