L[subscript 2]-gain optimization for robust bipedal walking on unknown terrain

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Abstract—In this paper we seek to quantify and explicitly optimize the robustness of a control system for a robot walking on terrain with uncertain geometry. Geometric perturbations to the terrain enter the equations of motion through a relocation of the hybrid event “guards” which trigger an impact event; these perturbations can have a large effect on the stability of the robot and do not fit into the traditional robust control analysis and design methodologies without additional machinery. We attempt to provide that machinery here. In particular, we quantify the robustness of the system to terrain perturbations by defining an \( L_2 \) gain from terrain perturbations to deviations from the nominal limit cycle. We show that the solution to a periodic dissipation inequality provides a sufficient upper bound on this gain for a linear approximation of the dynamics around the limit cycle, and we formulate a semidefinite programming problem to compute the \( L_2 \) gain for the system with a fixed linear controller. We then use either binary search or an iterative optimization method to construct a linear robust controller and to minimize the \( L_2 \) gain. The simulation results on canonical robots suggest that the \( L_2 \) gain is closely correlated to the actual number of steps traversed on the rough terrain, and our controller can improve the robot’s robustness to terrain disturbances.

I. INTRODUCTION

Bipedal robots are subject to many sources of uncertainty during walking; these could include a push from human, an unexpected gust of wind, or parametric uncertainties of unmodeled friction forces. Among all of these uncertainties, we focus in this paper on geometric perturbations to the terrain height. Unlike uncertainties which affect the continuous dynamics of the system, which can be accommodated with traditional approaches to robust control analysis and synthesis, terrain uncertainty manifests itself directly in the hybrid dynamical systems nature of a walking robot. A perturbation in terrain height appears as changes in the timing and dynamics of a ground contact event. Changes in the ground contact events can have a major stabilizing or de-stabilizing effect on legged robots. Although it is natural to apply robust control analysis and/or synthesis to a (typically numerical) approximation of walking dynamics on the Poincaré map, or even to apply time-domain methods from robust control to the continuous phases of the dynamics, applying robust control to the hybrid systems uncertainty requires additional care.

In this paper, we define an \( L_2 \) gain to quantify the robustness of bipedal robots to terrain disturbances. Moreover, we present a semidefinite programming formulation for computing an upper bound of the \( L_2 \) gain based on dissipation inequality. We further demonstrate that through binary search or iterative optimization, that upper bound can be optimized by constructing robust linear controllers. We validate our paradigm on canonical robots.

II. RELATED WORK

Extensive research has been performed on dealing with uncertainties for a continuous linear system [24], [16]. A common approach is to quantify the robustness of the system by its \( L_2 \) gain. By searching over the storage function that satisfies the dissipation inequality, an upper bound of the \( L_2 \) gain can be determined for a closed-loop system, and an \( H_{\infty} \) controller can be constructed for a given \( L_2 \) gain upper bound [12]. In this paper, we extend this approach to hybrid dynamical systems, with uncertainty existing in the guard function.

For limit cycle walkers, the robotics community has realized that careful local linearization of the dynamics around a limit cycle can provide powerful tools for orbital stability analysis[7] and control design[17]. This analysis can also lead to regional stability analysis[10] and can be lead to receding-horizon control strategies for dealing with terrain sensed at runtime[9]. These results suggest that linear controllers in the transversal coordinate can be used for nonlinear hybrid systems like bipedal robots. In this paper, we will show that robust linear controller even in the original (non-reduced) coordinates can improve the stability of a robot walking over unknown terrain.

Morimoto employs DDP method to optimize an \( H_{\infty} \) cost function, so as to improve the robustness of a bipedal
walker[13]. His simulation results demonstrate that this controller enables the robot traverse longer distance under joint disturbances. This result indicates that $H_\infty$ norm, and thus the $L_2$ gain is a good robust measure of the nonlinear system like bipedal robots.

One common approach in analyzing periodic legged locomotion is to construct a discrete step-to-step function, namely the Poincaré map, and analyze the properties of this discrete map [11], [14], [21]. Based on the Poincaré map, Hobbel define the gain sensitivity norm to measure the robustness of limit cycle walkers [8]. Byl uses mean first-passage time to measure the robustness to unknown terrain given that the terrain height is drawn from a known distribution [4]. Moreover, Park designs an $H_\infty$ controller for the discrete Poincaré map [15] for a bipedal walker. However, there some important limitations to Poincaré map analysis and control. In Poincaré analysis, it is difficult to include continuous dynamics uncertainty, and in Poincaré synthesis, control decisions can be made only once per one cycle, so opportunities for mid-step corrections are missed. For most systems, the Poincaré map does not have a closed form representation; it can only be numerically approximated instead of being exactly computed. In our approach, instead of relying on the Poincaré map, we study the continuous formulation of the hybrid dynamical system directly.

This paper is organized as follows: In Section III we present the definition of the $L_2$ gain for a limit cycle walker (III-A); a semidefinite programming formulation to compute an upper bound of the $L_2$ gain (III-B); control synthesis for a given $L_2$ gain upper-bound (III-C); and a paradigm to optimize such an upper bound (III-D). In section IV, we validate our approach on canonical robots. We then conclude our work in the last section.

III. APPROACH

Bipedal walking robots with pin feet are commonly modeled as hybrid systems with continuous modes interconnected by transition functions [22], [5]. Suppose the robot foot hits the ground with inelastic impact. The ground transition is then modeled as an impulsive mapping. For simplicity, we consider the system with only one continuous mode and one transition function

$$\dot{x} = f(x, u) \text{ if } \phi(x, u) > 0$$

$$x_h^t = \Delta(x_h^t) \text{ if } \phi(x_h^t, h) = 0$$

(1)

(2)

Where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, $h$ is the height of the terrain at the incoming ground impact. $\phi$ represents the distance between the foot and the terrain. When the distance decreases to zero, the ground impact occurs and the transition function maps the pre-impact state $x_h^t$ to the post-impact state $x_h^{t+1}$. We suppose that we have access to perfect state information. Notice that unlike disturbances in the continuous dynamical system, in the rough terrain walking case, the terrain disturbance exists only in the guard function, the hybrid part of the system.

Terrain uncertainty can arise from many situations, like terrain perception error. Also, when we have planned a nominal gait, and do not want to re-plan for every small difference in the perceived height, that small difference can also be regarded as terrain uncertainty. Suppose the nominal terrain is $h^n$. On such terrain $h^n$, we have planned a limit cycle $(x^*, u^*)$ as the desired walking pattern. Denote the terrain disturbance as $\tilde{h} = h - h^n$. The goal is to design a phase tracking controller such that the error between the actual trajectory and the desired one $(x^*, u^*)$ is small, while the unknown terrain disturbance is present. The magnitude of the error signal for an infinite time horizon is defined as

$$\int_{0}^{\infty} |e(t)|^2 dt = \int_{0}^{\infty} |x(t) - x^*(t)|_Q^2 + |u(t) - u^*(t)|_R^2$$

(3)

where $Q, R$ are given positive definite matrices for state error and control error respectively; $\tau$ is the phase being tracked on the nominal trajectory. For simplicity we suppose the clock of the tracked phase elapses at the same speed as the world clock in the continuous mode, (more advanced phase tracking, like the transversal coordinate, is also possible.) For the ground impact, since it does not make sense for the perturbed trajectory after the impact to track the nominal trajectory prior to the impact, the phase is reset by a function $\Pi$ to the post-impact phase $\tau_h^+$ every time the impact takes place. Namely

$$\tau = 1 \text{ if } \phi(x(t), h) > 0$$

$$\Pi(x_h^+, x^*(\tau_h^+)) = 0 \text{ if } \phi(x_h^+, h) = 0$$

(4)

(5)

Notice that the projection function $\Pi$ can be an implicit function of $\tau_h^+$. For example, $\tau_h^+$ may be chosen such that the stance leg angle of the post-impact state $x_h^+$ is the same as the stance leg angle of the nominal state $x^*(\tau_h^+)$ [1].

We measure the influence of terrain disturbance $\bar{h}$ on the error signal $e$ by its $L_2$ gain, which is defined as the infimum of $\gamma$ such that the equation below is lower bounded.

$$-\int_{0}^{\infty} |e(t)|^2 dt + \gamma^2 \sum_{n=1}^{\infty} \bar{h}[n]^2 > -\infty$$

(6)

where $\bar{h}[n]$ is the terrain height disturbance at the $n^{th}$ ground impact.

We can not use Equation 6 directly to compute the $L_2$ gain of the system. The challenge lies in the following two aspects:

1) The time horizon stretches to infinity, while numerically determining the boundedness of the summation of an infinite time signal is not straightforward. The effects of terrain disturbances accumulate over time, and such effect is prohibitively difficult to explicitly compute.

2) The disturbance is a discrete signal, while the error is continuous. We lack the tools to tackle such a “hybrid” condition.

To overcome the above difficulties, rather than relying on Poincaré map, which is the traditional approach, we analyze the hybrid dynamical system (Equation 1, 2) directly in the following sections. We rely on the fact that the nominal trajectory $(x^*, u^*)$ is periodic.
when the perturbed trajectory hits the impact surface at time \( t_\text{c} \). The infinite horizon is split every time the tracked phase \( \tau \) jumps from \( n \) to 0 at the end of each step. The propagation of the perturbed trajectory \( \bar{x}(\tau) \) is mapped by the state-transition function \( \Pi \) from \( x_\text{h} \). The post-impact phase \( \tau^+_0 \) is determined by the phase reset function \( \Phi \). The world clock does not change at the impact, namely \( t_\text{c}' = t_\text{c} \).

A. \( L_2 \) gain for a periodic phase tracking system

To get rid of the infinite time horizon problem, our solution is to break it into small steps, and only analyze the continuous state propagation within each step independently. We define a step based on the nominal trajectory being tracked. If we fix a certain event \( E \) on the nominal trajectory (such an event should be distinct from the ground impact, for example, the robot reaches its apex with that nominal state configuration), a step starts at \( \tau = 0 \) right after event \( E \) happens on the nominal trajectory, and ends when the same event \( E \) takes place again on the nominal trajectory \( (x^*(\tau), u^*(\tau)) \). Since the nominal trajectory is periodic, the state/control on the nominal trajectory that triggers event \( E \) at the end of the step exactly equals the state/control at the beginning of the step. We denote the time length in between the two events as \( T \), and we reset the tracked phase from \( T \) to 0 at the end of each step. The propagation of the perturbed and nominal trajectories within one step is illustrated in Fig.2. The step interval is \( \tau \in [0, T] \), with only one ground impact taking place in the middle of the step. Note that when the perturbed trajectory hits the impact surface \( \phi = 0 \) at time \( t_\text{c} \) with state \( x_\text{h} \), the nominal trajectory does not necessarily hit the same impact surface. To summarize, when the robot walks on rough terrain, there are two clocks running simultaneously, the world clock and the tracked phase clock. The infinite horizon is split every time the tracked phase clock jumps from \( T \) to 0.

Suppose that at the start of the \( n^{th} \) step, the world clock of the perturbed trajectory is at time \( t_\text{E}[n] \), and without loss of generality, we can assume \( t_\text{E}[\ell] = 0 \). Equation 6 can be reformulated as the summation of disturbance and error signal within each step in Equation 7.

\[
\sum_{n=1}^{\infty} \left( \gamma^2 \tilde{h}[n] - \int_{t_\text{E}[n]}^{t_\text{E}[n+1]} |x(t) - x^*(t)|^2_Q + |u(t) - u^*(t)|^2_R dt \right) > -\infty
\]

(7)

For the given step \( n \), we define the state error within that step as \( \bar{x}^*(\tau) = x(t) - x^*(t) \), the control error as \( \bar{u}^*(\tau) = u(t) - u^*(t) \), with the initial condition \( \bar{x}^*(0) = x(t_\text{E}[n]) - x^*(0) \) and \( \bar{u}^*(0) = u(t_\text{E}[n]) - u^*(0) \). We get the following lemma on the \( L_2 \) gain of the system with an infinite time horizon:

**Lemma 3.1:** For the hybrid dynamical system described by Equation 1 and 2, a sufficient condition for the \( L_2 \) gain no larger than a constant \( \gamma \), is that there exists a storage function \( V : [0, T] \times \mathbb{R}^n \to \mathbb{R} \), such that the following conditions 8-10 hold

\[
\gamma^2 \tilde{h}^2 - \int_0^{t_\text{E}[n]} |\bar{x}^*(\tau)|^2_Q + |\bar{u}^*(\tau)|^2_R \, d\tau \\
- \int_{t_\text{E}[n]}^{t_\text{E}[n+1]} |\bar{x}^*(\tau)|^2_Q + |\bar{u}^*(\tau)|^2_R \, d\tau \\
\geq V(T, \bar{x}^*(T)) - V(0, \bar{x}^*(0)) \quad \forall \bar{h} \in \mathbb{R}, \bar{x}^*(0) \in \mathbb{R}^n
\]

(8)

And the constraints on the two ends of the step

\[
V(T, z) \geq V(0, z) \quad \forall z \in \mathbb{R}^n
\]

(9)

**Proof:**

\[
\sum_{n=1}^{\infty} \left( \gamma^2 \tilde{h}[n] - \int_{t_\text{E}[n]}^{t_\text{E}[n+1]} |x(t) - x^*(t)|^2_Q + |u(t) - u^*(t)|^2_R dt \right) \\
= \sum_{n=1}^{\infty} \left( \gamma^2 \tilde{h}[n] - \int_0^{t_\text{E}[n]} |x(t) - x^*(t)|^2_Q + |u(t) - u^*(t)|^2_R \, d\tau \\
- \int_{t_\text{E}[n]}^{t_\text{E}[n+1]} |x(t) - x^*(t)|^2_Q + |u(t) - u^*(t)|^2_R \, d\tau \right) \\
\geq \sum_{n=1}^{\infty} V(T, x(t_\text{E}[n+1]) - x^*(T)) - V(0, x(t_\text{E}[n]) - x^*(0)) \\
= \sum_{n=2}^{\infty} \left( V(T, x(t_\text{E}[n+1]) - x^*(T)) - V(0, x(t_\text{E}[n]) - x^*(0)) \right) \\
+ \lim_{m \to \infty} V(T, x(t_\text{E}[m]) - x^*(T)) - V(0, x(t_\text{E}[1]) - x^*(0)) > -\infty
\]

Note that due to the periodicity of the nominal trajectory, \( x^*(T) = x^*(0) \).

**Lemma 3.1** enables us to get rid of the infinite time horizon problem and only analyze conditions 8-10 within one step. It is conservative as it does not capture the recovery motions that take one bad step before taking a different good step. In the sequel, we will drop the superscript “\( n \)” in \( \bar{x} \) and \( \bar{u} \), as each step is analyzed independent of the other steps.

Still, within one step, we have the continuous error signal \( e \) and the discrete terrain disturbance \( \tilde{h} \) in condition (8). To separate them, we consider the following sufficient conditions for equation (8)

\[
V(\tau, \bar{x}(\tau)) \leq -|\bar{x}(\tau)|^2_Q - |\bar{u}(\tau)|^2_R \\
\text{if } \phi(x^*(\tau) + \bar{x}(\tau), h^* + \tilde{h}) > 0
\]

(11)

\[
V(\tau^+_h, \bar{x}^+) - V(\tau^-_h, \bar{x}^-) \leq \gamma^2 \tilde{h}^2 \\
\text{if } \phi(x^*(\tau^-_h) + \bar{x}^-, h^* + \tilde{h}) > 0
\]

(12)

where \( \bar{x}^+ = \Delta(x^*(\tau^-_h) + \bar{x}^-) - x^*(\tau^-_h) \) is the post-impact state error. Notice that we isolate the mode transition out and treat both pre-impact state error \( \bar{x}^- \) and post-impact state error \( \bar{x}^+ \) as time-independent variables.
For simplicity, we restrict the storage function \( V \) to a quadratic form,
\[
V(\tau, \tilde{x}) = \tilde{x}'S(\tau)\tilde{x}
\]
where \( S \in \mathbb{R}^{n \times n} \) satisfies the following conditions
\[
\begin{align*}
\dot{V} &= \tilde{x}'S(\tau)\tilde{x} + 2\tilde{x}'S(\tau)\tilde{x} \leq -|\tilde{x}|^2 - |\tilde{u}|_Q^2 \\
&\text{if } \phi(x'(\tau) + \bar{x}(\tau), h^* + \bar{h}) > 0 \\
\bar{x}'S(\bar{\tau}^+\tau)\bar{x} - \bar{x}'S(\bar{\tau}^-\tau)\bar{x} &\leq \gamma^2 h^2 \\
&\text{if } \phi(x'(\bar{\tau}^+\tau) + \bar{x}^-, h^* + \bar{h}) = 0 \\
S(T) &\geq S(0) \\
S(T) &\geq 0
\end{align*}
\] (13)
\( \Theta \) and consider a small perturbation of the terrain height \( \bar{x}^+ \) together; moreover, there exists a constraint that the guard variable state error \( \bar{x}^- \) and \( \bar{h} \) is a matrix defined in Equation 18. We suppose
\[
\begin{align*}
V(x', \bar{\tau}^+\tau) &= f(x'(\bar{\tau}^+\tau), u'(\bar{\tau}^+\tau)) \\
&= \phi(x'(\tau), h^* + \bar{h}) > 0 \\
\bar{x}'S(\bar{\tau}^+\tau)\bar{x} - \bar{x}'S(\bar{\tau}^+\tau)\bar{x} &\leq \gamma^2 h^2 \\
&\text{if } \phi(x'(\bar{\tau}^+\tau) + \bar{x}^-, h^* + \bar{h}) = 0 \\
S(T) &\geq S(0) \\
S(T) &\geq 0
\end{align*}
\] (14)
Equations 14c and 14d are obtained by substituting Equation 13 into Equation 9 and 10 respectively.

We aim to design a linear time-varying controller \( \bar{u}(\tau) = K(\tau)\bar{x}(\tau) \) and search for the storage matrix \( S \) satisfying Conditions 14a-14d, which guarantee that the closed-loop system has an \( L_2 \) gain no larger than \( \gamma \). The condition 14b for the mode transition is tricky, as the post-impact phase \( \bar{\tau}^+ \) is determined by the phase reset function \( \Pi \), thus we cannot compute \( \bar{x}^- \) from the state transition function \( \Delta \) only, it has to be computed with the phase reset and guard function together; moreover, there exists a constraint that the guard function should be zero, we also wish to get rid of this constraint. For simplicity, we focus on the linearized case, and consider a small perturbation of the terrain height \( \bar{h} \), and a small drift of the pre-impact state \( \tilde{x}^- \). Note that when there is no terrain disturbance and pre-impact state error is zero, the ground impact happens at the nominal impact time \( \tau_{0} \), which is the phase that the nominal trajectory hits the nominal terrain guard function. Moreover, the post-impact state error \( \tilde{x}^+ \) will also be zero when \( \bar{h} = 0 \) and \( \tilde{x}^- = 0 \). To analyze the situation when both \( \tilde{x}^- \) and \( \bar{h} \) are small, by taking full derivative of the guard function, the following condition should hold.
\[
\phi_3(f(x'(\tau_{0})), u'(\tau_{0}))d\tau_{0} + \phi_3dx^- + \phi_0\bar{h}d\bar{h} = 0
\]
where we denote \( dq \) as the infinitesimal variation of some variable \( q \). We suppose that the robot does not graze on the ground when impact happens, thus \( \phi_3(f(x'(\tau_{0})), u'(\tau_{0})) \neq 0 \), and the variation of the pre-impact time \( d\tau_{0} \) can be uniquely determined by the pre-impact state variation \( dx^- \) and terrain variation \( d\bar{h} \).
\[
d\tau_{0} = -\frac{\phi_0\bar{h}d\bar{h} - \phi_3dx^-}{\phi_3(f(x'(\tau_{0})), u'(\tau_{0}))}
\]
Likewise, we take full differentiation on the state transition function \( \Delta \) and phase reset function \( \Pi \) also. Combining these two full differentiations with Equation 15, we get the following matrix equation.
\[
L\begin{bmatrix}
\begin{array}{cc}
\begin{bmatrix}
d\tau_{0}^- \\
d\tau_{0}^+
\end{bmatrix} & \begin{bmatrix}
d\tau_{0}^- \\
d\tau_{0}^+
\end{bmatrix} =
\begin{bmatrix}
\phi_0 & -\phi_3 \\
0 & \Delta_\rho
\end{bmatrix}
\end{array}
\begin{bmatrix}
d\bar{h} \\
dx^-
\end{bmatrix}
\end{bmatrix}
\]
Where \( L \) is a matrix defined in Equation 18. We suppose the phase reset function uniquely determines the post-impact phase, hence the term \( \Pi, f(x'(\tau_{0}^+)), u'(\tau_{0}^+)) \neq 0 \). So the matrix \( L \) is nonsingular, and we can solve \( \begin{bmatrix} d\tau_{0}^-, d\tau_{0}^+, dx^- \end{bmatrix} \) from \( \begin{bmatrix} d\bar{h}, dx^- \end{bmatrix} \) as a linear mapping. Suppose that
\[
dx^+ = T_1d\bar{h} + T_2dx^-
\]
And we do the first order expansion of the term \( \tilde{x}^+, S(\tau_{0}) \) and \( S(\tau_{0}) \) as follows:
\[
\tilde{x}^+ = T_1\tilde{h} + T_2\tilde{x}^+ + o(\tilde{h}) + o(\tilde{x}^-)
\]
\[
S(\tau_{0}) = S(\tau_{0}^+) + S(\tau_{0}^-) + o(\tau_{0}^+) + o(\tau_{0}^-)
\]
where \( \tilde{x}_0^+, \tau_0^- \) are variations of pre- and post-impact phases respectively. We use a second order approximation of Condition 14b around the nominal terrain height \( \tilde{h} = 0 \) and nominal trajectory \( \tilde{x}^- = 0 \). By substituting 20a-20c into 14b, the second order approximation is
\[
(T_1\tilde{h} + T_2\tilde{x}^+)'S(\tau_{0}^-)(T_1\tilde{h} + T_2\tilde{x}^-) - \tilde{x}^-S(\tau_{0}^-)\tilde{x}^- \leq \gamma^2 h^2
\]
Notice that we will then only need to verify the storage matrix \( S \) at the nominal pre-impact time \( \tau_{0}^- \) and post-impact time \( \tau_{0}^+ \). Equation 21 is equivalent to the following linear matrix inequality (LMI) below
\[
\begin{bmatrix}
T_2^T & T_1^T
\end{bmatrix}
\begin{bmatrix}
S(\tau_{0}^+) & 0 \\
0 & \gamma^2
\end{bmatrix}
\geq 0
\]
For the continuous mode condition 14a, we linearize the state dynamics Equation 1. Denote \( A = \frac{df}{dx}, B = \frac{df}{dx}, \) and we require the following condition to hold
\[
-S \succeq (A + BK)'S + S(A + BK) + Q + K'RK
\]
Notice that by dropping the constraint \( \phi(x'(\tau), \bar{x}^-), h^* + \bar{h}) > 0 \), equation 23 is more conservative than equation 14a, as we require the inequality holds for any state error \( \tilde{x} \). To summarize the discussions above, we have the following theorem

**Theorem 3.2:** For a hybrid system with dynamics defined as Equation 1 and 2, a sufficient condition for the controller \( \bar{u} = K\xi \) making the feedback system with \( L_2 \) gain no larger than \( \gamma \), is that there exists a matrix \( S: [0, T] \rightarrow \mathbb{R}^{n \times n} \) and satisfying the following conditions
\[
\begin{align*}
-S \succeq (A + BK)'S + S(A + BK) + Q + K'RK &\quad (24a) \\
\begin{bmatrix}
T_2^T & T_1^T
\end{bmatrix}
\begin{bmatrix}
S(\tau_{0}^+) & 0 \\
0 & \gamma^2
\end{bmatrix}
\leq 0 &\quad (24b) \\
S(T) \geq S(0) &\quad (24c) \\
S(T) \geq 0 &\quad (24d)
\end{align*}
\]
**B. Computing \( L_2 \) gain**

For a given linear controller \( \bar{u}(\tau) = K(\tau)\xi(\tau) \), our goal is determine an upper bound of its \( L_2 \) gain based on Theorem
3.2, the problem is formulated as

$$L = \begin{bmatrix} 
\phi_z f(x^*(\tau^+), u^*(\tau^+)) & 0 \\
-\Delta_z f(x^*(\tau^+), u^*(\tau^+)) & 0 \\
0 & \Pi_{\tau^+}+\Pi_{\tau^+} f(x^*(\tau^+), u^*(\tau^+)) \end{bmatrix}$$

(18)

Based on Corollary 3.4, if we denote $\hat{F} = A + BK$, $\hat{M} = Q + K'RK$, condition 25b can be reformulated as

$$S(\tau^+) \succeq \Phi_{\hat{F}}(T, \tau^+) P S(T) P^T \Phi_{\hat{F}}(T, \tau^+) + \int_{\tau^+}^T \Phi_{\hat{F}}(\sigma, \tau^+) (Q + K'RK) \Phi_{\hat{F}}(\sigma, \tau^+) d\sigma$$

(32a)

$$S(0) \preceq \Phi_{\hat{F}}(\tau^+, 0) S(\tau^+) \Phi_{\hat{F}}(\tau^+, 0) + \int_{0}^{\tau^+} \Phi_{\hat{F}}(\sigma, 0) (Q + K'RK) \Phi_{\hat{F}}(\sigma, 0) d\sigma$$

(32b)

The state transition matrix and the integral term can be numerically computed. The decision variables are reduced to the storage matrix $S$ at time 0, $\tau^+$, $\tau^+$, $T$ and a scalar $\gamma$. We can further reduce the number of decision variables by checking the special properties of the optimal solution. Notice that if the tuple $(\gamma, S_1)$ satisfies Condition 25b-25e, then we can construct a new tuple $(\gamma, S_2)$ satisfying

$$S_2(0) = S_2(T)$$

(33a)

$$S_2(\tau^+) = S_1(\tau^+) + \frac{\lambda_{\min}(S_2(0) - S_1(0))}{\lambda_{\max}(\Phi_F(\tau^+, 0) \Phi_F(\tau^+, 0))} I$$

(33b)

$$S_2(\tau^+) = S_1(\tau^+)$$

(33c)

$$S_2(T) = S_1(T)$$

(33d)

It can be easily verified that $S_2$ satisfies constraints 32a, 32b, 25d and 25e. Moreover, since $S_2(\tau^+) \geq S_1(\tau^+)$, $S_2(\tau^+) = S_1(\tau^+)$, $(\gamma, S_2)$ also satisfies constraint 25c, hence $(\gamma, S_2)$ is a feasible solution to the program 25a-25e. Namely, in the optimization program 25a-25e, by replacing the constraint 25d with strict equality 33a, the optimal value does not change. Thus the decision variable $S(0)$ can be dropped, as it can be replaced by $S(T)$.

We further show that the optimization program can be simplified by dropping $S(T)$. Condition 25e is the only constraint involving $S(T)$. But if $S(T) \geq 0$, then inequality 32a implies $S(\tau^+) \geq 0$. On the other hand, if $S(\tau^+) \geq 0$, then inequality 25c implies $S(S(\tau^+)) \geq T_0 S(\tau^+) S(T) \geq 0$, and by inequality 32b, we have $S(0) \geq \Phi_F(T, \tau^+) S(\tau^+) \Phi_F(T, \tau^+) \geq 0$, thus $S(T) = S(0) \geq 0$. So $S(T) \geq 0$ iff $S(\tau^+) \geq 0$. Constraint 25e can be replaced by $S(\tau^+) \geq 0$, and the decision variable $S(T)$ can be dropped.

In all, with powerful conic programming solver like SeDuMi [18], we can solve the following semidefinite programming problem (SDP) to determine an upper bound of the $L_2$ gain, given a fixed linear controller $\bar{u} = K\bar{x}$.

$$\min_{\xi \in \mathcal{S}} \zeta$$

s.t. $S(\tau^+) \succeq \Psi^T \Psi + \tilde{N}$

$$\begin{bmatrix} T_2 \bar{T} \\ \bar{T} \end{bmatrix} S \begin{bmatrix} T_2 \\ T_1 \end{bmatrix} - \begin{bmatrix} S \xi \\ 0 \end{bmatrix} \preceq 0$$

(34c)

$$S(\tau^+) \succeq 0$$

(34d)
Where
\[ S^+ = S(\tau^+_{i}), S^- = S(\tau^-_{i}), \zeta = \gamma^2 \]
\[ \Psi = \Phi_F(\tau^+_{i}, 0) \Phi_F(T, \tau^-_{i}) \]
\[ \hat{N} = \int^{T_{i}}_{\tau^+_{i}} \Phi_F(\sigma, \tau^+_{i})'(Q + K'R)K \Phi_F(\sigma, \tau^+_{i})d\sigma + \Phi_F(T, \tau^-_{i})' \]
\[ \left( \int^{T_{i}}_{0} \Phi_F(\sigma, 0)'(Q + K'R)K \Phi_F(\sigma, 0)d\sigma \right) \Phi_F(T, \tau^-_{i}) \]

where \( \Psi, \hat{N} \) can be numerically computed.

### 4. Robust control synthesis

Given a constant \( \gamma \) and a hybrid system defined in Equations 1, 2, we want to construct a linear controller \( \hat{u}(\tau) = K(\tau) \hat{x}(\tau) \), such that the closed loop system has an \( L_2 \) gain no larger than \( \gamma \). Based on Theorem 3.2, it is equivalent to computing the storage matrix \( S \) and the control gain \( K \) satisfying constraints 24a-24d. When \( \gamma^2 > T_i^0 S(\tau^+_{i}) T_i \), \( \gamma \) is a non-decreasing sequence in Algorithm 1, as in each iteration, the tuple \( (\gamma, P_{i}) \) which solves the Riccati equation is a feasible solution to the semidefinite programming problem in the next iteration.

#### Algorithm 1 Iterative Optimization

1. At iteration \( i \), given a linear controller gain \( K_{i} \), solve the following semidefinite programming problem to determine \( \gamma \) as an upper bound of the \( L_2 \) gain of the closed-loop system

\[
\begin{align*}
\min_{S^+_i, S^-_i, \zeta_i} & \quad \zeta_i \\
\text{s.t.} & \quad S^+_i, S^-_i \geq 0 \\
& \quad S^+_i \succeq \Psi F S^-_i \Psi F + \hat{N}_i \\
& \quad S^+_i \succeq 0 \\
& \quad \zeta_i > T_i^0 S^+_i T_i \\
\end{align*}
\]

Where
\[ F_i = A + B K_{i} \]
\[ \hat{N}_i = \int^{T_i}_{\tau^+_{i}} \Phi_F(\sigma, \tau^+_{i})'(Q + K'R)K \Phi_F(\sigma, \tau^+_{i})d\sigma + \Phi_F(T, \tau^-_{i})' \left( \int^{T_i}_{T_{i}} \Phi_F(\sigma, 0)'(Q + K'R)K \Phi_F(\sigma, 0)d\sigma \right) \Phi_F(T, \tau^-_{i}) \]

\[ \gamma = \sqrt{\zeta_i} \]

2. For the upper bound \( \gamma \), construct a \( \gamma \)-sub-optimal controller \( K_{i} \) by computing a periodic solution to the following Riccati equation

\[
\begin{align*}
P_i(\tau^+_{i}) &= T_{i}^0 P(\tau^+_{i}) T_i (\gamma^2 - T_{i}^0 P(\tau^+_{i}) T_i)^{-1}T_{i}^0 P(\tau^+_{i}) T_i + T_{i}^0 S(\tau^+_{i}) T_i \\
- \dot{P}_i &= A' P_i + P_i A + Q - P_i B R^{-1} B' P_i \forall \tau \neq \tau_{h} \\
P(T) &= P(0) \\
\end{align*}
\]

Find the \( \gamma \)-suboptimal controller \( K_{i} \)
\[ K_{i} = -R^{-1} B' P_i \]

3. \( i \leftarrow i + 1 \), with \( K_{i+1} = K_{i} \)

The binary search approach scales better for systems with high degrees of freedom (DOF). But the initial guess of
\(\gamma\) is unclear, and sometimes suffers the numeric tolerance issue of determining convergence. We can combine these two approaches, given an initial linear controller (not necessarily robust), we can compute the \(L_2\) gain of the closed-loop system associated with that controller, and such \(L_2\) gain can be an initial guess of the binary search. When convergence is hard to be decided by numeric integrating Riccati equation, we can switch to the iterative optimization to get a good estimate of the optimal \(\gamma\).

IV. RESULTS

A. Compass Gait

A compass gait robot is the simplest dynamic walking model (Fig 3). It is well known that the compass gait robot has a very narrow region of attraction and can easily fall down over rough terrain. We compare two limit cycles, the passive one and the robust one [6]. We run simulations of the robot walking over unknown terrain with virtual slope drawn from \([2^\circ, 8^\circ]\). The comparison is summarized in Table I. \(\gamma\) is computed through the semidefinite programming formulation in Section III-D. The limit cycle that has smaller \(\gamma\) can traverse much longer distance than the one with large \(\gamma\). This good agreement between the big gap of \(\gamma\) and the distinction of their actual performance on the rough terrain suggests that the \(L_2\) gain is a good indicator for the capability of traversing unknown terrain.

<table>
<thead>
<tr>
<th>Limit Cycle</th>
<th>Passive</th>
<th>Robust</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\gamma)</td>
<td>292.6646</td>
<td>21.2043</td>
</tr>
<tr>
<td>Average number of steps before falling down</td>
<td>&lt; 10</td>
<td>(\infty)</td>
</tr>
</tbody>
</table>

For the passive limit cycle, we construct two robust controllers based on the algorithm in Section III-D, and compute their \(L_2\) gain by solving the the semidefinite programming problem in Section III-B. We run 100 simulations of the compass gait robot walking on a rough terrain with the LQR controller and the two robust controllers. The virtual slope of the terrain changes from step to step, and the angle is drawn uniformly from \([-2^\circ, 2^\circ]\).

B. RABBIT

RABBIT is a five-link planar bipedal walker constructed jointly by several French laboratories. Grizzle’s group in University of Michigan have done extensive research into modeling and control of this robot [20], [5]. We apply our approach to the rigid-body model of RABBIT created by Westervelt (Fig.6).

We consider two different walking gaits for RABBIT. For gait 1 we construct an LQR controller, the SDP program computes that for the closed-loop system of RABBIT following gait 1, its \(\gamma = 522.0822\); for gait 2 we use binary search to find a controller, which makes \(\gamma \in [5000, 6000]\). We then take 40 simulations of the robot model walking on a rough terrain. The virtual slope of the terrain changes from step to step, and the angle is drawn uniformly from \([-2^\circ, 2^\circ]\). The comparison of their average number of steps traversed also close. The comparison between those three controllers are shown in Fig 4 and 5, the straight line is \(y = x\). Each dot represents a simulation of the compass gait on the same terrain for three controllers. In Fig 4, most points are above the \(y = x\) line, indicating that Robust1 controller enables the robot to traverse more steps than LQR does. In Fig.5, most points are along the \(y = x\) line, indicating the performance of the two robust controllers is close.

<table>
<thead>
<tr>
<th>controller</th>
<th>LQR</th>
<th>Robust2</th>
<th>Robust1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\gamma)</td>
<td>292.6646</td>
<td>274.2038</td>
<td>269.9908</td>
</tr>
<tr>
<td>Average number of steps before falling down</td>
<td>40.61</td>
<td>89.27</td>
<td>96.21</td>
</tr>
</tbody>
</table>

Fig. 4: LQR vs Robust1

Fig. 5: Robust2 vs Robust1
is shown in Table III. This again shows that the upper bound of $L_2$ gain is a good indicator of the capability of traversing rough terrain. Moreover, it suggests that our method scales well to a system of RABBIT’s complexity.

**TABLE III**: Comparison between two control strategies of RABBIT

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>522.0822</th>
<th>5000, 6000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average number of steps</td>
<td>20.325</td>
<td>1.675</td>
</tr>
</tbody>
</table>

V. CONCLUSIONS AND FUTURE WORKS

In this paper, we study the robustness of a bipedal robot to unknown terrain elevations. Given a desired walking pattern, we define a continuous error signal for the hybrid dynamical system. We quantify the robustness to terrain disturbance by the $L_2$ gain of the closed-loop system. Given a fixed linear controller, we present a semidefinite programming approach to compute an upper bound of the $L_2$ gain, and a control synthesis scheme to design a robust controller so as to bring down the $L_2$ gain. The simulation results validate that the $L_2$ gain is a good indicator of the capability to traverse unknown terrain. And our robust controller can improve such capability.

It will be easy to extend our scheme to include disturbances in the continuous time, since the analysis is performed on the continuous signal in our paper.

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