Families and Springer’s correspondence

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FAMILIES AND SPRINGER’S CORRESPONDENCE

G. Lusztig

INTRODUCTION

0.1. Let $G$ be a connected reductive algebraic group over an algebraically closed field $k$ of characteristic $p$. Let $W$ be the Weyl group of $G$; let $\text{Irr} W$ be a set of representatives for the isomorphism classes of irreducible representations of $W$ over $\overline{Q}_l$, an algebraic closure of the field of $l$-adic numbers ($l$ is a fixed prime number $\neq p$).

Now $\text{Irr} W$ is partitioned into subsets called families as in [L1, Sec.9], [L3, 4.2]. Moreover to each family $F$ in $\text{Irr} W$, a certain set $X_F$, a pairing $\{,\} : X_F \times X_F \rightarrow \overline{Q}_l$, and an imbedding $F \rightarrow X_F$ was canonically attached in [L1],[L3, Ch.4]. (The set $X_F$ with the pairing $\{,\}$, which can be viewed as a nonabelian analogue of a symplectic vector space, plays a key role in the classification of unipotent representations of a finite Chevalley group [L3] and in that of unipotent character sheaves on $G$). In [L1],[L3] it is shown that $X_F = M(G_F)$ where $G_F$ is a certain finite group associated to $F$ and, for any finite group $\Gamma$, $M(\Gamma)$ is the set of all pairs $(g, \rho)$ where $g$ is an element of $\Gamma$ defined up to conjugacy and $\rho$ is an irreducible representation over $\overline{Q}_l$ (up to isomorphism) of the centralizer of $g$ in $\Gamma$; moreover $\{,\}$ is given by the “nonabelian Fourier transform matrix” of [L1, Sec.4] for $G_F$.

In the remainder of this paper we assume that $p$ is not a bad prime for $G$. In this case a uniform definition of the group $G_F$ was proposed in [L3, 13.1] in terms of special unipotent classes in $G$ and the Springer correspondence, but the fact that this leads to a group isomorphic to $G_F$ as defined in [L3, Ch.4] was stated in [L3, (13.1.3)] without proof. One of the aims of this paper is to supply the missing proof.

To state the results of this paper we need some definitions. For $E \in \text{Irr} W$ let $a_E \in \mathbb{N}, b_E \in \mathbb{N}$ be as in [L3, 4.1]. As noted in [L2], for $E \in \text{Irr} W$ we have

(a) $a_E \leq b_E$;

we say that $E$ is special if $a_E = b_E$.

For $g \in G$ let $Z_G(g)$ or $Z(g)$ be the centralizer of $g$ in $G$ and let $A_G(g)$ or $A(g)$ be the group of connected components of $Z(g)$. Let $C$ be a unipotent conjugacy class in $G$ and let $u \in C$. Let $B_u$ be the variety of Borel subgroups of $G$ that contain

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$u$; this is a nonempty variety of dimension, say, $e_C$. The conjugation action of $Z(u)$ on $B_u$ induces an action of $A(u)$ on $S_u := H^{2e_C}(B_u, Q_l)$. Now $W$ acts on $S_u$ by Springer's representation [Spr]; however here we adopt the definition of the $W$-action on $S_u$ given in [L4] which differs from Springer's original definition by tensoring by sign. The $W$-action on $S_u$ commutes with the $A(u)$-action. Hence we have canonically $S_u = \bigoplus_{E \in \Irr W} E \otimes V_E$ (as $W \times A(u)$-modules) where $V_E$ are finite dimensional $Q_l$-vector spaces with $A(u)$-action. Let $\Irr C W = \{ E \in \Irr W; V_E \not= 0 \}$; this set does not depend on the choice of $u$ in $C$. By [Spr], the sets $\Irr C W$ (for $C$ variable) form a partition of $\Irr W$; also, if $E \in \Irr C W$ then $V_E$ is an irreducible $A(u)$-module and, if $E \not= E'$ in $\Irr C W$, then the $A(u)$-modules $V_E, V_{E'}$ are not isomorphic. By [BM] we have

(b) $e_C \leq b_E$ for any $E \in \Irr C W$

and the equality $b_E = e_C$ holds for exactly one $E \in \Irr C W$ which we denote by $E_C$ (for this $E$, $V_E$ is the unit representation of $A(u)$).

Following [L3, (13.1.1)] we say that $C$ is special if $E_C$ is special. (This concept was introduced in [L2, Sec.9] although the word "special" was not used there.) From (b) we see that $C$ is special if and only if $a_{EC} = e_C$.

Now assume that $C$ is special. We denote by $F \subset \Irr W$ the family that contains $E_C$. (Note that $C \rightarrow F$ is a bijection from the set of special unipotent classes in $G$ to the set of families in $\Irr W$.) We set $\Irr C W^* = \{ E \in \Irr C W; E \in F \}$ and

\[ \mathcal{K}(u) = \{ a \in A(u); a \text{ acts trivially on } V_E \text{ for any } E \in \Irr C W^* \}. \]

This is a normal subgroup of $A(u)$. We set $\tilde{A}(u) = A(u)/\mathcal{K}(u)$, a quotient group of $A(u)$. Now, for any $E \in \Irr C W^*$, $V_E$ is naturally an (irreducible) $\tilde{A}(u)$-module.

Another definition of $\tilde{A}_u$ is given in [L3, (13.1.1)]. In that definition $\Irr C W^*$ is replaced by $\{ E \in \Irr C W; a_{E} = e_C \}$ and $\mathcal{K}(u)$, $\tilde{A}(u)$ are defined as above but in terms of this modified $\Irr C W^*$. However the two definitions are equivalent in view of the following result.

**Proposition 0.2.** Assume that $C$ is special. Let $E \in \Irr C W$.

(a) We have $a_E \leq e_C$.

(b) We have $a_E = e_C$ if and only if $E \in F$.

This follows from [L8, 10.9]. Note that (a) was stated without proof in [L3, (13.1.2)] (the proof I had in mind at the time of [L3] was combinatorial).

**0.3.** The following result is equivalent to a result stated without proof in [L3, (13.1.3)].

**Theorem 0.4.** Let $C$ be a special unipotent class of $G$, let $u \in C$ and let $F$ be the family that contains $E_C$. Then we have canonically $X_F = M(\tilde{A}(u))$ so that the pairing $\{ , \}$ on $X_F$ coincides with the pairing $\{ , \}$ on $M(\tilde{A}(u))$. Hence $G_F$ can be taken to be $\tilde{A}(u)$.

This is equivalent to the corresponding statement in the case where $G$ is adjoint, which reduces immediately to the case where $G$ is adjoint simple. It is then enough
to prove the theorem for one $G$ in each isogeny class of semisimple, almost simple algebraic groups; this will be done in §3 after some combinatorial preliminaries in §1, §2. The proof uses the explicit description of the Springer correspondence: for type $A_n, G_2$ in [Spr]; for type $B_n, C_n, D_n$ in [S1] (as an algorithm) and in [L4] (by a closed formula); for type $F_4$ in [S2]; for type $E_n$ in [AL],[Sp1].

An immediate consequence of (the proof of) Theorem 0.4 is the following result which answers a question of R. Bezrukavnikov.

Corollary 0.5. In the setup of 0.4 let $E \in \text{Irr}^\circ_\bar{C}W$ and let $V_E$ be the corresponding $\bar{A}(u)$-module viewed as an (irreducible) $\bar{A}(u)$-module. The image of $E$ under the canonical imbedding $\mathcal{F} \to X_\mathcal{F} = M(\bar{A}(u))$ is represented by the pair $(1, V_E) \in M(\bar{A}(u))$. Conversely, if $E \in \mathcal{F}$ and the image of $E$ under $\mathcal{F} \to X_\mathcal{F} = M(\bar{A}(u))$ is represented by the pair $(1, \rho) \in M(\bar{A}(u))$ where $\rho$ is an irreducible representation of $\bar{A}(u)$, then $E \in \text{Irr}^\circ_\bar{C}W$ and $\rho \cong V_E$.

0.6. Corollary 0.5 has the following interpretation. Let $Y$ be a (unipotent) character sheaf on $G$ whose restriction to the regular semisimple elements is $\neq 0$; assume that in the usual parametrization of unipotent character sheaves by $\sqcup_{x \in \mathcal{X}} X_\mathcal{X}$, $Y$ corresponds to $(1, \rho) \in M(\bar{A}(u))$ where $C$ is the special unipotent class corresponding to a family $\mathcal{F}$, $u \in C$ and $\rho$ is an irreducible representation of $\bar{A}(u)$. Then $Y|_C$ is (up to shift) the irreducible local system on $C$ defined by $\rho$.

A parametrization of unipotent character sheaves on $G$ in terms of restrictions to various conjugacy classes of $G$ is outlined in §4.

0.7. Notation. If $A, B$ are subsets of $N$ we denote by $A \cup B$ the union of $A$ and $B$ regarded as a multiset (each element of $A \cap B$ appears twice). For any set $\mathcal{X}$, we denote by $\mathcal{P}(\mathcal{X})$ the set of subsets of $\mathcal{X}$ viewed as an $F_2$-vector space with sum given by the symmetric difference. If $\mathcal{X} \neq \emptyset$ we note that $\{\emptyset, \mathcal{X}\}$ is a line in $\mathcal{P}(\mathcal{X})$ and we set $\mathcal{P}_0(\mathcal{X}) = \mathcal{P}(\mathcal{X})/\{\emptyset, \mathcal{X}\}$, $\mathcal{P}_{ev}(\mathcal{X}) = \{L \in \mathcal{P}(\mathcal{X}); |L| = 0 \mod 2\}$; let $\mathcal{P}_{ev}(\mathcal{X})$ be the image of $\mathcal{P}_{ev}(\mathcal{X})$ under the obvious map $\mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ (thus $\mathcal{P}_{ev}(\mathcal{X}) = \mathcal{P}(\mathcal{X})$ if $|\mathcal{X}|$ is odd and $\mathcal{P}_{ev}(\mathcal{X})$ is a hyperplane in $\mathcal{P}(\mathcal{X})$ if $|\mathcal{X}|$ is even). Now if $\mathcal{X} \neq \emptyset$, the assignment $L, L' \mapsto |L \cap L'| \mod 2$ defines a symplectic form on $\mathcal{P}_{ev}(\mathcal{X})$ which induces a nondegenerate symplectic form $(,)$ on $\mathcal{P}_{ev}(\mathcal{X})$ via the obvious linear map $\mathcal{P}_{ev}(\mathcal{X}) \to \mathcal{P}_{ev}(\mathcal{X})$.

For $g \in G$ let $g_s$ (resp. $g_\omega$) be the semisimple (resp. unipotent) part of $g$.

For $z \in (1/2) \mathbb{Z}$ we set $[z] = z$ if $z \in \mathbb{Z}$ and $[z] = z - (1/2)$ if $z \in \mathbb{Z} + (1/2)$.

Erratum to [L3]. On page 86, line -6 delete: ”$b' < b$” and on line -4 before ”In the language...” insert: ”The array above is regarded as identical to the array obtained by interchanging its two rows.”

On page 343, line -5, after ”respect to $M'$” insert: ”and where the group $\mathcal{G}_M$ defined in terms of $(u', M)$ is isomorphic to the group $\mathcal{G}_F$ defined in terms of $(u, G)$”.

Erratum to [L4]. In the definition of $A_\alpha, B_\alpha$ in [L4, 11.5], the condition $I \in \alpha$ should be replaced by $I \in \alpha'$ and the condition $I \in \alpha'$ should be replaced by $I \in \alpha$. 
1. Combinatorics

1.1. Let \( N \) be an even integer \( \geq 0 \). Let \( a := (a_0, a_1, a_2, \ldots, a_N) \in \mathbb{N}^{N+1} \) be such that \( a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_N \), \( a_0 < a_2 < a_4 < \cdots, a_1 < a_3 < a_5 < \cdots \). Let \( J = \{ i \in [0, N]; a_i \text{ appears exactly once in } a \} \). We have \( J = \{ i_0, i_1, \ldots, i_{2M} \} \) where \( M \in \mathbb{N} \) and \( i_0 < i_1 < \cdots < i_{2M} \) satisfy \( i_s = s \mod 2 \) for \( s \in [0, 2M] \). Hence for any \( s \in [0, 2M - 1] \) we have \( i_{s+1} = i_s + 2m_s + 1 \) for some \( m_s \in \mathbb{N} \). Let \( E \) be the set of \( b := (b_0, b_1, b_2, \ldots, b_N) \in \mathbb{N}^{N+1} \) such that \( b_0 < b_2 < b_4 < \cdots, b_1 < b_3 < b_5 < \cdots \) and such that \( [b] = [a] \) (we denote by \([b], [a]\) the multisets \( \{b_0, b_1, \ldots, b_N\}, \{a_0, a_1, \ldots, a_N\} \)). We have \( a \in E \). For \( b \in E \) we set

\[
\hat{b} = (\hat{b}_0, \hat{b}_1, \hat{b}_2, \ldots, \hat{b}_N) = (b_0, b_1, b_2, b_3, b_4 + 2, \ldots, b_N - 1 + (N/2), b_N + (N/2)).
\]

Let \([\hat{b}]\) be the multiset \( \{\hat{b}_0, \hat{b}_1, \hat{b}_2, \ldots, \hat{b}_N\} \).

For \( s \in \{1, 3, \ldots, 2M - 1\} \) we define \( a^{(s)} = (a_0^{(s)}, a_1^{(s)}, a_2^{(s)}, \ldots, a_N^{(s)}) \in E \) by

\[
(a_0^{(s)}, a_1^{(s)}, a_2^{(s)}, \ldots, a_N^{(s)}) = (a_{i_s + 1}, a_{i_s + 3}, a_{i_s + 5}, \ldots, a_{i_s + 2m_s}, a_{i_s + 2m_s + 1})
\]

and \( a_i^{(s)} = a_i \) if \( i \in [0, N] - [i_s, i_{s+1}] \). More generally for \( X \subseteq \{1, 3, \ldots, 2M - 1\} \) we define \( a^X = (a_0^X, a_1^X, a_2^X, \ldots, a_N^X) \in E \) by \( a_i^X = a_i^{(s)} \) if \( s \in X, i \in [i_s, i_{s+1}] \), and \( a_i^X = a_i \) for all other \( i \in [0, N] \). Note that \([a^X] = [\hat{a}]\). Conversely, we have the following result.

**Lemma 1.2.** Let \( b \in E \) be such that \([\hat{b}] = [\hat{a}]\). There exists \( X \subseteq \{1, 3, \ldots, 2M - 1\} \) such that \( b = a^X \).

The proof is given in 1.3-1.5.

1.3. We argue by induction on \( M \). We have

\[ a = (y_1 = y_1 < y_2 = y_2 < \cdots, y_r = y_r < a_i_0 < \cdots) \]

for some \( r \). Since \([b] = [a]\), we must have

\[ (b_0, b_2, b_4, \ldots) = (y_1, y_2, \ldots, y_r, \ldots), (b_1, b_3, b_5, \ldots) = (y_1, y_2, \ldots, y_r, \ldots). \]

Thus,

(a) \( b_i = a_i \) for \( i < i_0 \).

We have \( a = (\cdots, a_{2M} > y'_1 = y'_1 < y'_2 = y'_2 < \cdots, y'_r = y'_r) \) for some \( r' \). Since \([b] = [a]\), we must have

\[ (b_0, b_2, b_4, \ldots) = (\cdots, y'_1, y'_2, \ldots, y'_r), (b_1, b_3, b_5, \ldots) = (\cdots, y'_1, y'_2, \ldots, y'_r). \]

Thus,

(b) \( b_i = a_i \) for \( i > i_{2M} \).

If \( M = 0 \) we see that \( b = a \) and there is nothing further to prove. In the rest of the proof we assume that \( M \geq 1 \).
1.4. From 1.3 we see that

\[(a_0, a_1, a_2, \ldots, a_{i_{2M}}) = (\ldots, a_{i_{2M}-1} < x_1 = x_2 = x_2 = \cdots < x_q = x_q < a_{i_{2M}})\]

(for some \(q\)) has the same entries as \((b_0, b_1, b_2, \ldots, b_{i_{2M}})\) (in some order). Hence the pair

\[(\ldots, b_{i_{2M}-5}, b_{i_{2M}-3}, b_{i_{2M-1}}), (\ldots, b_{i_{2M}-4}, b_{i_{2M}-2}, b_{i_{2M}})\]

must have one of the following four forms.

\[(\ldots, a_{i_{2M}-1}, x_1, x_2, \ldots, x_q), (\ldots, x_1, x_2, \ldots, x_q, a_{i_{2M}}),\]

\[(\ldots, a_{i_{2M}-1}, x_1, x_2, \ldots, x_q, a_{i_{2M}}), (\ldots, x_1, x_2, \ldots, x_q, a_{i_{2M}})\]

Hence \((\ldots, b_{i_{2M}-2}, b_{i_{2M-1}}, b_{i_{2M}})\) must have one of the following four forms.

(I) \((\ldots, a_{i_{2M}-1}, x_1, x_1, x_2, \ldots, x_q, a_{i_{2M}})\),

(II) \((\ldots, x_1, a_{i_{2M-1}}, x_2, x_1, x_3, x_2, \ldots, x_q, a_{i_{2M}}, x_q)\),

(III) \((\ldots, a_{i_{2M-1}}, z, x_1, x_2, x_2, \ldots, x_q, x_q, a_{i_{2M}})\),

(IV) \((\ldots, a_{i_{2M-1}}, z', x_1, z'', x_2, x_3, x_2, \ldots, x_q, x_q, a_{i_{2M}}, x_q)\),

where \(a_{i_{2M-1}} > z, a_{i_{2M-1}} > z'' \geq z'\) and all entries in \(\ldots\) are \(< a_{i_{2M-1}}\). Correspondingly, \((\ldots, b_{i_{2M}-2}, b_{i_{2M-1}}, b_{i_{2M}})\) must have one of the following four forms.

(I) \((\ldots, a_{i_{2M}-1} + h - q, x_1 + h - q, x_1 + h - q + 1, x_2 + h - q + 1, x_2 + h - q + 2, \ldots, x_q + h - 1, x_q + h, a_{i_{2M}} + h)\),

(II) \((\ldots, x_1 + h - q, a_{i_{2M-1}} + h - q, x_2 + h - q + 1, x_1 + h - q + 1, x_3 + h - q + 2, x_2 + h - q + 1, \ldots, x_q + h - 1, x_q + h - 1, a_{i_{2M}} + h, x_q + h)\),

(III) \((\ldots, a_{i_{2M}-1} + h - q - 1, z + h - q, x_1 + h - q, x_1 + h - q + 1, x_2 + h - q + 1, x_2 + h - q + 2, \ldots, x_q + h - 1, x_q + h, a_{i_{2M}} + h)\),

(IV) \((\ldots, a_{i_{2M}-1} + h - q - 1, z' + h - q - 1, x_1 + h - q, z'' + h - q, x_2 + h - q - 1, x_1 + h - q + 1, x_3 + h - q + 2, x_2 + h - q + 1, \ldots, x_q + h - 1, x_q + h - 1, a_{i_{2M}} + h, x_q + h)\)

where \(h = \frac{i_{2M}}{2}\) and in case (III) and (IV), \(a_{i_{2M-1}} + h - q\) is not an entry of \((\ldots, b_{i_{2M}-2}, b_{i_{2M-1}}, b_{i_{2M}})\).

Since \((\ldots, \hat{a}_{i_{2M}-2}, \hat{a}_{i_{2M}-1}, \hat{a}_{i_{2M}})\) is given by (I) we see that \(a_{i_{2M-1}} + h - q\) is an entry of \((\ldots, \hat{a}_{i_{2M}-2}, \hat{a}_{i_{2M}-1}, \hat{a}_{i_{2M}})\). Using 1.3(b) we see that

\[\{\ldots, \hat{a}_{i_{2M}-2}, \hat{a}_{i_{2M}-1}, \hat{a}_{i_{2M}}\} = (\ldots, b_{i_{2M}-2}, b_{i_{2M}-1}, b_{i_{2M}})\]

as multiset. We see that cases (III) and (IV) cannot arise. Hence we must be in case (I) or (II). Thus

we have either

\[\begin{align*}
& (b_{i_{2M}-1}, b_{i_{2M}-1+1}, \ldots, b_{i_{2M}-2}, b_{i_{2M}-1}, b_{i_{2M}}) \\
\text{(a)} & = (a_{i_{2M}-1}, a_{i_{2M}-1+1}, \ldots, a_{i_{2M}-2}, a_{i_{2M}-1}, a_{i_{2M}})
\end{align*}\]

or

\[\begin{align*}
& (b_{i_{2M}-1}, b_{i_{2M}-1+1}, \ldots, b_{i_{2M}-2}, b_{i_{2M}-1}, b_{i_{2M}}) \\
\text{(b)} & = (a_{i_{2M}-1+1}, a_{i_{2M}-1}, a_{i_{2M}-1+3}, a_{i_{2M}-1+2}, \ldots, a_{i_{2M}}, a_{i_{2M}-1+1}).
\end{align*}\]
1.5. Let $a' = (a_0, a_1, a_2, \ldots, a_{i_{2M-1}-1})$, $b' = (b_0, b_1, b_2, \ldots, b_{i_{2M-1}-1})$
\begin{align*}
\hat{a}' &= (a_0, a_1 + 1, a_2 + 1, a_3 + 2, a_4 + 2, \ldots, a_{i_{2M-1}-1} + (i_{2M-1} - 1)/2), \\
\hat{b}' &= (b_0, b_1 + 1, b_2 + 1, b_3 + 2, b_4 + 2, \ldots, b_{i_{2M-1}-1} + (i_{2M-1} - 1)/2),
\end{align*}
From $\hat{a}' = [\hat{a}]$ and 1.3(b), 1.4(a), (b) we see that the multiset formed by the entries of $a'$ coincides with the multiset formed by the entries of $\hat{b}'$. Using the induction hypothesis we see that there exists $X' \subset \{1, 3, \ldots, 2M - 3\}$ such that $b' = a'X'$ where $a'X'$ is defined in terms of $a'$, $X'$ in the same way as $aX$ was defined (see 1.1) in terms of $a$, $X$. We set $X = X'$ if we are in case 1.4(a) and $X = X' \cup \{2M - 1\}$ if we are in case 1.4(b). Then we have $b = aX$ (see 1.4(a), (b)), as required. This completes the proof of Lemma 1.2.

1.6. We shall use the notation of 1.1. Let $\mathcal{I}$ be the set of all unordered pairs $(\mathcal{A}, \mathcal{B})$ where $\mathcal{A}$, $\mathcal{B}$ are subsets of $\{0, 1, 2, \ldots\}$ and $\mathcal{A} \cup \mathcal{B} = (a_0, a_1, a_2, \ldots, a_N)$ as multisets. For example, setting $\mathcal{A}_0 = (a_0, a_2, a_4, \ldots, a_N)$, $\mathcal{B}_0 = (a_1, a_3, \ldots, a_{N-1})$, we have $(\mathcal{A}_0, \mathcal{B}_0) \in \mathcal{I}$. For any subset $a$ of $\mathcal{J}$ we consider
\begin{align*}
\mathcal{A}_a &= ((\mathcal{J} - a) \cap \mathcal{A}_0) \cup (a \cap \mathcal{B}_0) \cup (\mathcal{A}_0 \cap \mathcal{B}_0), \\
\mathcal{B}_a &= ((\mathcal{J} - a) \cap \mathcal{B}_0) \cup (a \cap \mathcal{A}_0) \cup (\mathcal{A}_0 \cap \mathcal{B}_0).
\end{align*}
Then $(\mathcal{A}_a, \mathcal{B}_a) \in \mathcal{I}$ and the map $a \mapsto (\mathcal{A}_a, \mathcal{B}_a)$ induces a bijection $\mathcal{P}(\mathcal{J}) \leftrightarrow \mathcal{I}$. (Note that if $a = \emptyset$ then $(\mathcal{A}_a, \mathcal{B}_a)$ agrees with the earlier definition of $(\mathcal{A}_0, \mathcal{B}_0)$.)

Let $\mathcal{I}'$ be the set of all $(\mathcal{A}, \mathcal{B}) \in \mathcal{I}$ such that $|\mathcal{A}| = |\mathcal{A}_0|$, $|\mathcal{B}| = |\mathcal{B}_0|$. Let $\mathcal{P}(\mathcal{J})_0$ be the subspace of $\mathcal{P}_{ev}(\mathcal{J})$ spanned by the 2-element subsets
\begin{align*}
\{a_{i_0}, a_{i_1}\}, \{a_{i_2}, a_{i_3}\}, \ldots, \{a_{i_{2M-2}}, a_{i_{2M-1}}\}
\end{align*}
of $\mathcal{J}$. Let $\mathcal{P}(\mathcal{J})_1$ be the subspace of $\mathcal{P}_{ev}(\mathcal{J})$ spanned by the 2-element subsets
\begin{align*}
\{a_{i_1}, a_{i_2}\}, \{a_{i_3}, a_{i_4}\}, \ldots, \{a_{i_{2M-1}}, a_{i_{2M}}\}
\end{align*}
of $\mathcal{J}$.
Let $\mathcal{P}(\mathcal{J})_0$ (resp. $\mathcal{P}(\mathcal{J})_1$) be the image of $\mathcal{P}(\mathcal{J})_0$ (resp. $\mathcal{P}(\mathcal{J})_1$) under the obvious map $\mathcal{P}(\mathcal{J}) \to \mathcal{P}(\mathcal{J})$. Note that
\begin{enumerate}
\item $\mathcal{P}(\mathcal{J})_0$ and $\mathcal{P}(\mathcal{J})_1$ are opposed Lagrangian subspaces of the symplectic vector space $\mathcal{P}(\mathcal{J})_1$ (see 0.7); hence, (1) defines an identification $\mathcal{P}(\mathcal{J})_0 = \mathcal{P}(\mathcal{J})^*_1$
\end{enumerate}
where $\mathcal{P}(\mathcal{J})^*_1$ is the vector space dual to $\mathcal{P}(\mathcal{J})_1$.

Let $\mathcal{I}_0$ (resp. $\mathcal{I}_1$) be the subset of $\mathcal{I}$ corresponding to $\mathcal{P}(\mathcal{J})_0$ (resp. $\mathcal{P}(\mathcal{J})_1$) under the bijection $\mathcal{P}(\mathcal{J}) \leftrightarrow \mathcal{I}$. Note that $\mathcal{I}_0 \subset \mathcal{I}'$, $\mathcal{I}_1 \subset \mathcal{I}'$, $|\mathcal{I}_0| = |\mathcal{I}_1| = 2^M$.

For any $X \subset \{1, 3, \ldots, 2M - 1\}$ we set $a_X = \cup_{x \in X} \{a_i, a_{i+1}\} \in \mathcal{P}(\mathcal{J})$. Then $(\mathcal{A}_{a_X}, \mathcal{B}_{a_X}) \in \mathcal{I}_1$ is related to $a_X$ in 1.1 as follows:
\begin{align*}
\mathcal{A}_{a_X} &= \{a_X^0, a_X^2, a_X^4, \ldots, a_X^N\}, \\
\mathcal{B}_{a_X} &= \{a_X^1, a_X^3, \ldots, a_X^{N-1}\}.
\end{align*}
1.7. We shall use the notation of 1.1. Let $T$ be the set of all ordered pairs $(A, B)$ where $A$ is a subset of $\{0, 1, 2, \ldots, 1\}$, $B$ is a subset of $\{1, 2, 3, \ldots\}$, $A$ contains no consecutive integers, $B$ contains no consecutive integers, and $A \cup B = (\hat{a}_0, \hat{a}_1, \hat{a}_2, \ldots, \hat{a}_N)$ as multisets. For example, setting $A_\emptyset = (\hat{a}_0, \hat{a}_2, \hat{a}_4, \ldots, \hat{a}_N)$, $B_\emptyset = (\hat{a}_1, \hat{a}_3, \ldots, \hat{a}_{N-1})$, we have $(A_\emptyset, B_\emptyset) \in T$.

For any $(A, B) \in T$ we define $(A^-, B^-)$ as follows: $A^-$ consists of $x_0 < x_1 - 1 < x_2 - 2 < \cdots < x_p - p$ where $x_0 < x_1 < \cdots < x_p$ are the elements of $A$; $B^-$ consists of $y_1 - 1 < y_2 - 2 < \cdots < y_q - q$ where $y_1 < y_2 < \cdots < y_q$ are the elements of $B$.

We can enumerate the elements of $T$ as in [L4, 11.5]. Let $J$ be the set of all $c \in \mathbb{N}$ such that $c$ appears exactly once in the sequence

$$(\hat{a}_0, \hat{a}_1, \hat{a}_2, \ldots, \hat{a}_N) = (a_0, a_1+1, a_2+1, a_3+2, a_4+2, \ldots, a_{N-1}+(N/2), a_N+(N/2)).$$

A nonempty subset $I$ of $J$ is said to be an interval if it is of the form $\{i, i+1, i+2, \ldots, j\}$ with $i - 1 \notin J, j + 1 \notin J$ and with $i \neq 0$. Let $\mathcal{I}$ be the set of intervals of $J$. For any $s \in \{1, 3, \ldots, 2M-1\}$, the set $I_s := \{\hat{a}_{i_s}, \hat{a}_{i_s+1}, \hat{a}_{i_s+2}, \ldots, \hat{a}_{i_s+2M_s+1}\}$ is either a single interval or a union of intervals $I_s^1 \sqcup I_s^2 \sqcup \ldots \sqcup I_s^{t_s}$ ($t_s \geq 2$) where $\hat{a}_{i_s} \in I_s^1$, $\hat{a}_{i_s+2M_s+1} \in I_s^{t_s}$, $|I_s^1|, |I_s^{t_s}|$ are odd, $|I_s^h|$ are even for $h \in [2, t_s - 1]$ and any element in $I_s^e$ is less than any element in $I_s^{e'}$ for $e < e'$. Let $\mathcal{I}_s$ be the set of all $I \in \mathcal{I}$ such that $I \subset I_s$. We have a partition $\mathcal{I} = \sqcup_{s \in \{1, 3, \ldots, 2M-1\}} \mathcal{I}_s$. Let $H$ be the set of elements of $c \in J$ such that $c < a_{i_s}$ (that is such that $c$ does not belong to any interval). For any subset $\alpha \subset \mathcal{I}$ we consider

$$A_\alpha = \sqcup_{I \in \mathcal{I}-\alpha}(I \cap A_\emptyset) \cup \sqcup_{I \in \alpha}(I \cap B_\emptyset) \cup (H \cap A_\emptyset) \cup (A_\emptyset \cap B_\emptyset),$$

$$B_\alpha = \sqcup_{I \in \mathcal{I}-\alpha}(I \cap B_\emptyset) \cup \sqcup_{I \in \alpha}(I \cap A_\emptyset) \cup (H \cap B_\emptyset) \cup (A_\emptyset \cap B_\emptyset).$$

Then $(A_\alpha, B_\alpha) \in T$ and the map $\alpha \mapsto (A_\alpha, B_\alpha)$ is a bijection $\mathcal{P}(\mathcal{I}) \leftrightarrow T$. (Note that if $\alpha = \emptyset$ then $(A_\alpha, B_\alpha)$ agrees with the earlier definition of $(A_\emptyset, B_\emptyset)$.)

Let $T' = \{(A, B) \in T; |A| = |A_\emptyset|, |B| = |B_\emptyset|\}, T_1 = \{(A, B) \in T'; A^- \cup B^- = A_\emptyset^- \cup B_\emptyset^-\}$. Let $\mathcal{P}(\mathcal{I})'$ (resp. $\mathcal{P}(\mathcal{I}_1)$) be the subset of $\mathcal{P}(\mathcal{I})$ corresponding to $T'$ (resp. $T_1$) under the bijection $\mathcal{P}(\mathcal{I}) \leftrightarrow T$.

Now let $X$ be a subset of $\{1, 3, \ldots, 2M-1\}$. Let $\alpha_X = \cup_{s \in X} \mathcal{I}_s \in \mathcal{P}(\mathcal{I})$. From the definitions we see that

(a) $A_{\alpha_X}^- = A_{\alpha_X}, B_{\alpha_X}^- = B_{\alpha_X}$ (notation of 1.6). In particular we have $(A_{\alpha_X}, B_{\alpha_X}) \in T_1$. Thus $|T_1| \geq 2^M$. Using Lemma 1.2 we see that

(b) $|T_1| = 2^M$ and $T_1$ consists of the pairs $(A_{\alpha_X}, B_{\alpha_X})$ with $X \subset \{1, 3, \ldots, 2M-1\}$.

Using (a),(b) we deduce:

(c) The map $T_1 \rightarrow \mathcal{I}_1$ given by $(A, B) \mapsto (A^-, B^-)$ is a bijection.

2. COMBINATORICS (CONTINUED)

2.1. Let $N \in \mathbb{N}$. Let

$$a := (a_0, a_1, a_2, \ldots, a_N) \in \mathbb{N}^{N+1}$$
be such that \( a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_N, \ a_0 < a_2 < a_4 < \ldots, \ a_1 < a_3 < a_5 < \ldots \)
and such that the set \( J := \{ i \in [0, N]; a_i \text{ appears exactly once in } a \} \) is nonempty. Now \( J \) consists of \( \mu + 1 \) elements \( i_0 < i_1 < \cdots < i_\mu \) where \( \mu \in \mathbb{N}, \ \mu = N \mod 2 \). We have \( i_s = s \mod 2 \) for \( s \in [0, \mu] \). Hence for any \( s \in [0, \mu - 1] \) we have \( i_{s+1} = i_s + 2m_s + 1 \) for some \( m_s \in \mathbb{N} \). Let \( E \) be the set of \( a = (b_0, b_1, b_2, \ldots, b_N) \in \mathbb{N}^{N+1} \) such that \( b_0 < b_2 < b_4 < \ldots, \ b_1 < b_3 < b_5 < \ldots \) and such that \( [b] = [a] \) (we denote by \([b], [a] \) the multisets \( \{b_0, b_1, \ldots, b_N\}, \{a_0, a_1, \ldots, a_N\} \)). We have \( a \in E \).

For \( b \in E \) we set

\[
\tilde{b} = (\tilde{b}_0, \tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_N) = (b_0, b_1, b_2 + 1, b_3 + 1, b_4 + 2, b_5 + 2, \ldots) \in \mathbb{N}^{N+1}.
\]

Let \([\tilde{b}]\) be the multiset \( \{\tilde{b}_0, \tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_N\} \). For any \( s \in [0, \mu - 1] \in 2\mathbb{N} \) we define \( a^s = (a_0^s, a_1^s, a_2^s, \ldots, a_N^s) \in E \) by

\[
(a_i^s, a_{i_0+1}^s, a_{i_0+2}^s, a_{i_0+3}^s, \ldots, a_{i_s + m_s}^s, a_{i_s + m_s + 1}^s)
= (a_{i+1}^s, a_{i+2}^s, a_{i+3}^s, \ldots, a_{i_s + 2m_s}^s, a_{i_s + 2m_s + 1}^s)
\]

and \( a_i^s = a_i \) if \( i \in [0, N] - [i_s, i_{s+1}] \). More generally for a subset \( X \) of \([0, \mu - 1] \cap 2\mathbb{N} \) we define \( a^X = a^X = (a_0^X, a_1^X, a_2^X, \ldots, a_N^X) \in E \) by \( a_i^X = a_i^s \) if \( s \in X \), \( i \in [i_s, i_{s+1}] \), and \( a_i^X = a_i \) for all other \( i \in [0, N] \). Note that \([\tilde{a}^X] = [\tilde{a}] \). Conversely, we have the following result.

**Lemma 2.2.** Let \( b \in E \) be such that \( [b] = [\tilde{a}] \). Then there exists \( X \subset [0, \mu - 1] \cap 2\mathbb{N} \) such that \( b = a^X \).

The proof is given in 2.3-2.5.

2.3. We argue by induction on \( \mu \). By the argument in 1.3 we have

(a) \( b_i = a_i \) for \( i < i_0 \),

(b) \( b_i = a_i \) for \( i > i_\mu \).

If \( \mu = 0 \) we see that \( b = a \) and there is nothing further to prove. In the rest of the proof we assume that \( \mu \geq 1 \).

2.4. From 2.3 we see that \((a_{i_0}, a_{i_0+1}, \ldots, a_N) = (a_{i_0} < x_1 < x_2 < \cdots < x_p = x_p < a_{i_1} < \cdots) \) (for some \( p \)) has the same entries as \((b_{i_0}, b_{i_0+1}, \ldots, b_N) \) (in some order). Hence the pair \((b_{i_0}, b_{i_0+2}, b_{i_0+4}, \ldots), (b_{i_0+1}, b_{i_0+3}, b_{i_0+5}, \ldots) \) must have one of the following four forms.

\[
(a_{i_0}, x_1, x_2, \ldots, x_p, \ldots), (x_1, x_2, \ldots, x_p, a_{i_1}, \ldots),
\]

\[
(x_1, x_2, \ldots, x_p, a_{i_1}, \ldots), (a_{i_0}, x_1, x_2, \ldots, x_p, \ldots),
\]

Hence \((b_{i_0}, b_{i_0+1}, b_{i_0+2}, \ldots, b_N) \) must have one of the following four forms.

(I) \((a_{i_0}, x_1, x_1, x_2, \ldots, x_p, x_p, a_{i_1}, \ldots), \)
(II) \((x_1, a_{i_0}, x_2, x_1, x_3, x_2, \ldots, x_p, x_{p-1}, a_{i_1}, x_p, \ldots)\),
(III) \((a_{i_0}, x_1, x_2, x_2, \ldots, x_p, x_p, z, a_{i_1}, \ldots)\),
(IV) \((x_1, a_{i_0}, x_2, x_1, x_3, x_2, \ldots, x_p, x_{p-1}, z', x_p, z'', a_{i_1}, \ldots)\)

where \(a_{i_1} < z, a_{i_1} < z' \leq z''\) and all entries in \ldots are \(> a_{i_1}\). Correspondingly,
\((b_{i_0}, b_{i_0+1}, b_{i_0+2}, \ldots, b_N)\) must have one of the following four forms.

(I) \((a_{i_0} + h, x_1 + h, x_1 + h + 1, x_2 + h + 1, x_2 + h + 2, \ldots, x_p + h + p - 1, x_p + h + p, a_{i_1} + h + p, \ldots)\),
(II) \((x_1 + h, a_{i_0} + h, x_2 + h + 1, x_1 + h + 1, x_3 + h + 2, x_2 + h + 2, \ldots, x_p + h + p - 1, x_p + h + p, a_{i_1} + h + p, x_p + h + p, \ldots)\),
(III) \((a_{i_0} + h, x_1 + h, x_1 + h + 1, x_2 + h + 1, x_2 + h + 2, \ldots, x_p + h + p - 1, x_p + h + p, z + p, a_{i_1} + h + p + 1, \ldots)\),
(IV) \(x_{p-1} + h + p - 1, z' + h + p, x_p + h + p, z'' + h + p + 1, a_{i_1} + h + p + 1, \ldots\)

where \(h = i_0/2\) and in case (III) and (IV) \(a_{i_1} + h + p\) is not an entry of
\((\bar{o}_{i_0}, \bar{o}_{i_0+1}, \bar{o}_{i_0+2}, \ldots)\).

Since \((\bar{a}_{i_0}, \bar{a}_{i_0+1}, \bar{a}_{i_0+2}, \ldots)\) is given by (I) we see that \(a_{i_1} + h + p\) is an entry of
\((\bar{a}_{i_0}, \bar{a}_{i_0+1}, \bar{a}_{i_0+2}, \ldots)\). Using 2.3 we see that
\[
\{\bar{a}_{i_0}, \bar{a}_{i_0+1}, \bar{a}_{i_0+2}, \ldots\} = \{\bar{o}_{i_0}, \bar{o}_{i_0+1}, \bar{o}_{i_0+2}, \ldots\}
\]
as multisets. We see that cases (III) and (IV) cannot arise. Hence we must be in case (I) or (II). Thus

we have either

(a) \((b_{i_0}, b_{i_0+1}, b_{i_0+2}, \ldots, b_{i_1}) = (a_{i_0}, a_{i_0+1}, a_{i_0+2}, \ldots, a_{i_1})\)

or

(b) \((b_{i_0}, b_{i_0+1}, b_{i_0+2}, \ldots, b_{i_1}) = (a_{i_0+1}, a_{i_0}, a_{i_0+3}, a_{i_0+2}, \ldots, a_{i_1}, a_{i_1-1})\).

From 2.3 and (a),(b) we see that if \(\mu = 1\) then Lemma 2.2 holds. Thus in the rest of the proof we can assume that \(\mu \geq 2\).

2.5. Let \(a' = (a_{i_1+1}, a_{i_1+2}, \ldots, a_N), b' = (b_{i_1+1}, b_{i_1+2}, \ldots, b_N)\),

\[
\bar{a}' = (a_{i_1+1}, a_{i_1+2}, a_{i_1+3} + 1, a_{i_1+4} + 1, a_{i_1+5} + 2, a_{i_1+6} + 2, \ldots),
\]

\[
\bar{b}' = (b_{i_1+1}, b_{i_1+2}, b_{i_1+3} + 1, b_{i_1+4} + 1, b_{i_1+5} + 2, b_{i_1+6} + 2, \ldots).
\]
From $\tilde{b} = [\tilde{a}]$ and 2.3(a),2.4(a),(b) we see that the multiset formed by the entries of $\tilde{a}'$ coincides with the multiset formed by the entries of $b'$. Using the induction hypothesis we see that there exists $X' \subset [2, \mu - 1] \cap 2\mathbb{N}$ such that $b' = a'^{X'}$ where $a'^{X'}$ is defined in terms of $a', X'$ in the same way as $a^X$ (see 2.1) was defined in terms of $a, X$. We set $X = X'$ if we are in case 2.4(a) and $X = \{0\} \cup X'$ if we are in case 2.4(b). Then we have $b = a^X$ (see 2.4(a),(b)), as required. This completes the proof of Lemma 2.2.

2.6. We shall use the notation of 2.1. Let $\mathcal{F}$ be the set of all unordered pairs $(\mathcal{A}, \mathcal{B})$ where $\mathcal{A}, \mathcal{B}$ are subsets of $\{0, 1, 2, \ldots \}$ and $\mathcal{A} \cup \mathcal{B} = (a_0, a_1, a_2, \ldots, a_N)$ as multisets. For example, setting $\mathcal{A}_0 = \{a_i; i \in [0, N] \cap 2\mathbb{N}\}$, $\mathcal{B}_0 = \{a_i; i \in [0, N] \cap (2\mathbb{N} + 1)\}$, we have $(\mathcal{A}_0, \mathcal{B}_0) \in \mathcal{F}$. For any subset $a$ of $\mathcal{J}$ we consider

$$\mathcal{A}_a = ((\mathcal{J} - a) \cap \mathcal{A}_0) \cup (a \cap \mathcal{B}_0) \cup (\mathcal{A}_0 \cap \mathcal{B}_0),$$

$$\mathcal{B}_a = ((\mathcal{J} - a) \cap \mathcal{B}_0) \cup (a \cap \mathcal{A}_0) \cup (\mathcal{A}_0 \cap \mathcal{B}_0).$$

Then $(\mathcal{A}_a, \mathcal{B}_a) = (\mathcal{A}_{\mathcal{J} - a}, \mathcal{B}_{\mathcal{J} - a}) \in \mathcal{F}$ and the map $a \mapsto (\mathcal{A}_a, \mathcal{B}_a)$ induces a bijection $\mathcal{P}(\mathcal{J}) \leftrightarrow \mathcal{F}$. (Note that if $a = \emptyset$ then $(\mathcal{A}_a, \mathcal{B}_a)$ agrees with the earlier definition of $(\mathcal{A}_\emptyset, \mathcal{B}_\emptyset)$.)

Let $\mathcal{F}'$ be the set of all $(\mathcal{A}, \mathcal{B}) \in \mathcal{F}$ such that $|\mathcal{A}| = |\mathcal{A}_0|$, $|\mathcal{B}| = |\mathcal{B}_0|$. Let $\mathcal{P}(\mathcal{J})_1$ be the subspace of $\mathcal{P}(\mathcal{J})$ spanned by the following 2-element subsets of $\mathcal{J}$:

$$\{a_{i_1}, a_{i_2}\}, \{a_{i_3}, a_{i_4}\}, \ldots, \{a_{i_{\mu - 2}}, a_{i_{\mu - 1}}\}$$

if $N$ is odd)

or

$$\{a_{i_1}, a_{i_2}\}, \{a_{i_3}, a_{i_4}\}, \ldots, \{a_{i_{\mu}, a_{i_{\mu - 1}}}\}$$

if $N$ is even).

Let $\mathcal{P}(\mathcal{J})_0$ be the subspace of $\mathcal{P}(\mathcal{J})$ spanned by the following 2-element subsets of $\mathcal{J}$:

$$\{a_{i_0}, a_{i_1}\}, \{a_{i_2}, a_{i_3}\}, \ldots, \{a_{i_{\mu - 1}, a_{i_{\mu}}}\}$$

if $N$ is odd)

or

$$\{a_{i_0}, a_{i_1}\}, \{a_{i_2}, a_{i_3}\}, \ldots, \{a_{i_{\mu - 2}, a_{i_{\mu - 1}}}\}$$

if $N$ is even).

Let $\mathcal{P}(\mathcal{J})_0$ (resp. $\mathcal{P}(\mathcal{J})_1$) be the image of $\mathcal{P}(\mathcal{J})_0$ (resp. $\mathcal{P}(\mathcal{J})_1$) under the obvious map $\mathcal{P}(\mathcal{J}) \rightarrow \mathcal{P}(\mathcal{J})$.

Note that

(a) $\mathcal{P}(\mathcal{J})_0$ and $\mathcal{P}(\mathcal{J})_1$ are opposed Lagrangian subspaces of the symplectic vector space $\mathcal{P}_{ev}(\mathcal{J}), (,)$, (see 0.7); hence $(,)$ defines an identification $\mathcal{P}(\mathcal{J})_1 = \mathcal{P}(\mathcal{J})_0^*$ where $\mathcal{P}(\mathcal{J})_0^*$ is the vector space dual to $\mathcal{P}(\mathcal{J})_0$.

Let $\mathcal{F}_0$ (resp. $\mathcal{F}_1$) be the subset of $\mathcal{F}$ corresponding to $\mathcal{P}(\mathcal{J})_0$ (resp. $\mathcal{P}(\mathcal{J})_1$) under the bijection $\mathcal{P}(\mathcal{J}) \leftrightarrow \mathcal{F}$. Note that $\mathcal{F}_0 \subset \mathcal{F}'$, $\mathcal{F}_1 \subset \mathcal{F}'$, $|\mathcal{F}_0| = |\mathcal{F}_1| = 2^{|\mu/2|}$.

For any $X \subset [0, \mu - 1] \cap 2\mathbb{N}$ we set $a_X = \bigcup_{s \in X}\{a_{i_s}, a_{i_{s+1}}\} \in \mathcal{P}(\mathcal{J})$. Then $(\mathcal{A}_\alpha_X, \mathcal{B}_\alpha_X)$ is related to $a^X$ in 2.1 as follows:

$$\mathcal{A}_\alpha_X = \{a^X_i; i \in [0, N] \cap 2\mathbb{N}\}, \mathcal{B}_\alpha_X = \{a^X_i; i \in [0, N] \cap (2\mathbb{N} + 1)\}.$$
2.7. We shall use the notation of 2.1. Let \( T \) be the set of all unordered pairs \((A, B)\) where \( A \) is a subset of \([0, 1, 2, \ldots]\), \( B \) is a subset of \([1, 2, 3, \ldots]\), \( A \) contains no consecutive integers, \( B \) contains no consecutive integers, and \( A \cup B = (\bar{a}_0, \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_N) \) as multisets. For example, setting

\[
A_\emptyset = \{\bar{a}_i; i \in [0, N] \cap 2\mathbb{N}\}, B_\emptyset = (\bar{a}_i; i \in [0, N] \cap (2\mathbb{N} + 1)),
\]

we have \((A_\emptyset, B_\emptyset) \in T\).

For any \((A, B) \in T\) we define \((A^-, B^-)\) as follows: \(A^-\) consists of \(x_1 < x_2 - 1 < x_3 - 2 < \cdots < x_p - p + 1\) where \(x_1 < x_2 < \cdots < x_p\) are the elements of \(A\); \(B^-\) consists of \(y_1 < y_2 - 1 < \cdots < y_q - q + 1\) where \(y_1 < y_2 < \cdots < y_q\) are the elements of \(B\).

We can enumerate the elements of \(T\) as in [L4, 11.5]. Let \(J\) be the set of all \(c \in \mathbb{N}\) such that \(c\) appears exactly once in the sequence

\[
(\bar{a}_0, \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_N) = (a_0, a_1, a_2 + 1, a_3 + 1, a_4 + 2, a_5 + 2, \ldots).
\]

A nonempty subset \(I\) of \(J\) is said to be an interval if it is of the form \(\{i, i + 1, i + 2, \ldots, j\}\) with \(i - 1 \notin J, j + 1 \notin J\). Let \(I\) be the set of intervals of \(J\). For any \(s \in [0, \mu - 1] \cap 2\mathbb{N}\), the set \(I_s := \{\bar{a}_{i_1}, \bar{a}_{i_2+1}, \bar{a}_{i_3+2}, \ldots, \bar{a}_{i_{t_s+2m_s+1}}\}\) is either a single interval or a union of intervals \(I_{s_1} \cup I_{s_2} \cup \ldots \cup I_{s_{t_s}}\) \((t_s \geq 2)\) where \(\bar{a}_{i_s} \in I_{s_1}\), \(\bar{a}_{i_s+2m_s+1} \in I_{s_{t_s}}\), \(|I_{s_h}|\) are odd, \(|I_{s_e}|\) are even for \(h \in [2, t_s - 1]\) and any element in \(I_{s_e}\) is < than any element in \(\bar{I}_{s'}\) for \(e < e'\). Let \(I_s\) be the set of all \(I \in I\) such that \(I \subset I_s\). We have a partition \(I = \sqcup_{s \in \mathbb{N}} I_s\). For any subset \(\alpha \subset I\) we consider

\[
A_\alpha = \bigcup_{I \in I - \alpha} (I \cap A_\emptyset) \cup \bigcup_{I \in \alpha} (I \cap B_\emptyset) \cup (A_\emptyset \cap B_\emptyset),
\]

\[
B_\alpha = \bigcup_{I \in I - \alpha} (I \cap B_\emptyset) \cup \bigcup_{I \in \alpha} (I \cap A_\emptyset) \cup (A_\emptyset \cap B_\emptyset).
\]

Then \((A_\alpha, B_\alpha) \in T\) and the map \(\alpha \mapsto (A_\alpha, B_\alpha)\) is a bijection \(\mathcal{P}(I) \leftrightarrow T\). (Note that if \(\alpha = \emptyset\) then \((A_\alpha, B_\alpha)\) agrees with the earlier definition of \((A_\emptyset, B_\emptyset)\).)

Let \(T' = \{(A, B) \in T; |A| = |A_\emptyset|, |B| = |B_\emptyset|\}, T_1 = \{(A, B) \in T'; A^- \cup B^- = A_\emptyset^+ \cup B_\emptyset^+\}\) Let \(\mathcal{P}(I)'\) (resp. \(\mathcal{P}(I)_1\)) be the subset of \(\mathcal{P}(I)\) corresponding to \(T'\) (resp. \(T_1\)) under the bijection \(\mathcal{P}(I) \leftrightarrow T\).

Now let \(X\) be a subset of \([0, \mu - 1] \cap 2\mathbb{N}\). Let \(\alpha_X = \cup_{s \in X} I_s \in \mathcal{P}(I)\). From the definitions we see that

(a) \(A_{\alpha_X} = \mathfrak{A}_{\alpha_X}, B_{\alpha_X} = \mathfrak{B}_{\alpha_X}\)

(ntotation of 2.6). In particular we have \((A_{\alpha_X}, B_{\alpha_X}) \in T_1\). Thus \(|T_1| \geq 2^{\lfloor \mu/2 \rfloor}\).

Using Lemma 2.2 we see that

(b) \(|T'_1| = 2^{\lfloor \mu/2 \rfloor}\) and \(T_1\) consists of the pairs \((A_{\alpha_X}, B_{\alpha_X})\) with \(X \subset [0, \mu - 1] \cap 2\mathbb{N}\).

Using (a),(b) we deduce:

(c) The map \(T_1 \rightarrow \mathcal{P}_1\) given by \((A, B) \mapsto (A^-, B^-)\) is a bijection.
3. Proof of Theorem 0.4 and of Corollary 0.5

3.1. If $G$ is simple adjoint of type $A_n$, $n \geq 1$, then 0.4 and 0.5 are obvious: we have $A(u) = \{1\}$, $\hat{A}(u) = \{1\}$.

3.2. Assume that $G = Sp_{2n}(k)$ where $n \geq 2$. Let $N$ be a sufficiently large even integer. Now $u : k^{2n} \to k^{2n}$ has $i_e$ Jordan blocks of size $e$ ($e = 1, 2, 3, \ldots$). Here $i_1, i_3, i_5, \ldots$ are even. Let $\Delta = \{e \in \{2, 4, 6, \ldots\}; i_e \geq 1\}$. Then $A(u)$ can be identified in the standard way with $P(\Delta)$. Hence the group of characters $\hat{A}(u)$ of $A(u)$ (which may be canonically identified with the $F_2$-vector space dual to $P(\Delta)$) may be also canonically identified with $P(\Delta)$ itself (so that the basis given by the one element subsets of $\Delta$ is self-dual).

To the partition $1 \bar{1}_1 + 2 \bar{1}_2 + 3 \bar{1}_3 + \ldots$ of $2n$ we associate a pair $(A, B)$ as in [L4, 11.6] (with $N, 2m$ replaced by $2n, N$). We have $A = (\hat{a}_0, \hat{a}_2, \hat{a}_4, \ldots, \hat{a}_N)$, $B = (\hat{a}_1, \hat{a}_3, \ldots, \hat{a}_{N-1})$, where $\hat{a}_0 \leq \hat{a}_1 \leq \hat{a}_2 \leq \cdots \leq \hat{a}_N$ is obtained from a sequence $a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_N$ as in 1.1. (Here we use that $C$ is special.) Now the definitions and results in §1 are applicable. As in [L3, 4.5] the family $F$ is in canonical bijection with $\Sigma'$ in 1.6.

We arrange the intervals in $I$ in increasing order $I_{(1)}, I_{(2)}, \ldots, I_{(f)}$ (any element in $I_{(1)}$ is smaller than any element in $I_{(2)}$, etc.). We arrange the elements of $\Delta$ in increasing order $e_1 < e_2 < \cdots < e_f$; then $f = f'$ and we have a bijection $I \leftrightarrow \Delta$, $I_{(h)} \leftrightarrow e_h$; moreover we have $|I_{(h)}| = i_{e_h}$ for $h \in [1, f]$; see [L4, 11.6]. Using this bijection we see that $A(u)$ and $\hat{A}(u)$ are identified with the $F_2$-vector space $P(I)$ with basis given by the one element subsets of $I$. Let $\pi : P(I) \to P(I)^+_1$ (with $P(I)^+_1$ as in 1.7(c)) be the (surjective) $F_2$-linear map which to $X \subseteq I$ associates the linear form $L \mapsto |X \cap L| \mod 2$ on $P(I)_1$. We will show that

(a) ker $\pi = K(u)$ ($K(u)$ as in 0.1).

We identify $\text{Irr}_C^*W$ with $T'$ (see 1.7) via the restriction of the bijection in [L4, (12.2.4)] (we also use the description of the Springer correspondence in [L4, 12.3]). Under this identification the subset $\text{Irr}_C^*W$ of $\text{Irr}_C^*W$ becomes the subset $T_1$ (see 1.7) of $T'$. Via the identification $P(I)' \leftrightarrow T'$ in 1.7 and $A(u) \leftrightarrow P(I)$ (see above), the map $E \mapsto \nu_E$ from $T'$ to $\hat{A}(u)$ becomes the obvious imbedding $P(I)' \to P(I)$ (we use again [L4, 12.3]). By definition, $K(u)$ is the set of all $X \in P(I)$ such that for any $L \in P(I)_1$ we have $|X \cap L| = 0 \mod 2$. Thus, (a) holds.

Using (a) we have canonically $\hat{A}(u) = P(I)^+_1 \pi$ via $\pi$. We define an $F_2$-linear map $P(I)_1 \to \bar{P}(J)_1$ (see 1.6) by $I_s \mapsto \{a_{i_s}, a_{i_{s+1}}\}$ for $s \in \{1, 3, \ldots, 2M - 1\}$. This is an isomorphism; it corresponds to the bijection 1.7(c) under the identification $T_1 \leftrightarrow P(I)_1$ in 1.7 and the identification $\bar{X}_1 \leftrightarrow \bar{P}(J)_1$ in 1.6. Hence we can identify $P(I)^+_1$ with $\bar{P}(J)^+_1$ and with $\bar{P}(J)_0$ (see 1.6(a)). We obtain an identification $\hat{A}(u) = \bar{P}(J)_0$.

By [L3, 4.5] we have $X_{\mathcal{F}} = \bar{P}(J)$. Using 1.6(a) we see that $P(J) = M(\bar{P}(J)_0) = M(\hat{A}(u))$ canonically so that 0.4 holds in our case. From the arguments above we see that in our case 0.5 follows from 1.7(c).

3.3. Assume that $G = SO_n(k)$ where $n \geq 7$. Let $N$ be a sufficiently large integer
such that $N = n \mod 2$. Now $u : k^n \to k^n$ has $i_e$ Jordan blocks of size $e$ ($e = 1, 2, 3, \ldots$). Here $i_2, i_4, i_6, \ldots$ are even. Let $\Delta = \{e \in \{1, 3, 5, \ldots\}; i_e \geq 1\}$. If $\Delta = \emptyset$ then $A(u) = \{1\}$, $\hat{A}(u) = \{1\}$ and $G_F = \{1\}$ so that the result is trivial.

In the remainder of this subsection we assume that $\Delta \neq \emptyset$. Then $A(u)$ can be identified in the standard way with the $F_2$-subspace $P_{ev}(\Delta)$ of $P(\Delta)$ and the group of characters $\hat{A}(u)$ of $A(u)$ (which may be canonically identified with the $F_2$-vector space dual to $A(u)$) becomes $\hat{P}(\Delta)$; the obvious pairing $A(u) \times \hat{A}(u) \to F_2$ is induced by the inner product $L, L' \mapsto |L \cap L'| \mod 2$ on $\hat{P}(\Delta)$.

To the partition $1i_1 + 2i_2 + 3i_3 + \ldots$ of $n$ we associate a pair $(A, B)$ as in [L4, 11.7] (with $N, M$ replaced by $n, N$). We have $A = \{\tilde{a}_i; i \in [0, N] \cap 2\mathbb{N}\}$, $B = \{\tilde{a}_i; i \in [0, N] \cap (2\mathbb{N} + 1)\}$ where $\tilde{a}_0 \leq \tilde{a}_1 \leq \tilde{a}_2 \leq \cdots \leq \tilde{a}_N$ is obtained from a sequence $a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_N$ as in 2.1. (Here we use that $C$ is special.) Now the definitions and results in §2 are applicable. As in [L3, 4.5] (if $N$ is even) or [L3, 4.6] (if $N$ is odd) the family $\mathcal{F}$ is in canonical bijection with $\mathcal{X}$ in 2.6.

We arrange the intervals in $\mathcal{I}$ in increasing order $I_{(1)}, I_{(2)}, \ldots, I_{(f)}$ (any element in $I_{(1)}$ is smaller than any element in $I_{(2)}$, etc.). We arrange the elements of $\Delta$ in increasing order $e_1 < e_2 < \cdots < e_{f'}$; then $f = f'$ and we have a bijection $\mathcal{I} \leftrightarrow \Delta$, $I_{(h)} \leftrightarrow e_h$; moreover we have $|I_{(h)}| = i_{e_h}$ for $h \in [1, f]$; see [L4, 11.7]. Using this bijection we see that $A(u)$ is identified with $P_{ev}(\mathcal{I})$ and $\hat{A}(u)$ is identified with $\hat{P}(\mathcal{I})$. For any $X \in P_{ev}(\mathcal{I})$, the assignment $L \mapsto |X \cap L| \mod 2$ can be viewed as an element of $\hat{P}(\mathcal{I})^1_1$ (the dual space of $\hat{P}(\mathcal{I})$ in 2.7 which by 2.7(b) is an $F_2$-vector space of dimension $2^{\lceil \mu/2 \rceil}$). This induces a (surjective) $F_2$-linear map $\pi : P_{ev}(\mathcal{I}) \to \hat{P}(\mathcal{I})^1_1$. We will show that

(a) $\ker \pi = \mathcal{K}(u)$ ($\mathcal{K}(u)$ as in 0.1).

We identify $\text{Irr}_C W$ with $T'$ (see 2.7) via the restriction of the bijection in [L4, (13.2.5)] if $N$ is odd or [L4, (13.2.6)] if $N$ is even (we also use the description of the Springer correspondence in [L4, 13.3]). Under this identification the subset $\text{Irr}_C W$ of $\text{Irr}_C W$ becomes the subset $T_1$ (see 2.7) of $T'$. Via the identification $\hat{P}(\mathcal{I}') \leftrightarrow T'$ in 2.7 and $\hat{A}(u) \leftrightarrow \hat{P}(\mathcal{I})$ (see above), the map $E \mapsto \mathcal{V}_E$ from $T'$ to $\hat{A}(u)$ becomes the obvious imbedding $\hat{P}(\mathcal{I})_0 \to \hat{P}(\mathcal{I})$ (we use again [L4, 13.3]). By definition, $\mathcal{K}(u)$ is the set of all $X \in P_{ev}(\mathcal{I})$ such that for any $L \in \hat{P}(\mathcal{I})$ representing a vector in $\hat{P}(\mathcal{I})_1$ we have $|X \cap L| = 0 \mod 2$. Thus, (a) holds.

Using (a) we have canonically $\hat{A}(u) = \hat{P}(\mathcal{I})^1_1$ via $\pi$. We have an $F_2$-linear map $\hat{P}(\mathcal{I})_1 \to \hat{P}(\mathcal{J})_0$ (see 2.6) induced by $I_s \mapsto \{a_{i_s}, a_{i_{s+1}}\}$ for $s \in [0, \mu - 1] \cap 2\mathbb{N}$ ($I_s$ as in 2.7). This is an isomorphism; it corresponds to the bijection 2.7(c) under the identification $T_1 \leftrightarrow \hat{P}(\mathcal{I})_1$ in 2.7 and the identification $\mathcal{J}_1 \leftrightarrow \hat{P}(\mathcal{J})_0$ in 2.6. Hence we can identify $\hat{P}(\mathcal{I})^1_1$ with $\hat{P}(\mathcal{J})^0_0$ and with $\hat{P}(\mathcal{J})_1$ (see 2.6(a)). We obtain an identification $\hat{A}(u) = \hat{P}(\mathcal{J})_1$.

By [L3, 4.6] we have $X_F = P_{ev}(\mathcal{J})$. Using 2.6(a) we see that $\hat{P}(\mathcal{J}) = M(\hat{P}(\mathcal{J})_1) = M(\hat{A}(u))$ canonically so that 0.4 holds in our case. From the arguments above we see that in our case 0.5 follows from 2.7(c).

3.4. In 3.5-3.9 we consider the case where $G$ is simple adjoint of exceptional type.
In each case we list the elements of the set \( \text{Irr}_C W \) for each special unipotent class \( C \) of \( G \); the elements of \( \text{Irr}_C W - \text{Irr}^* C W \) are enclosed in \([\,]\). (The notation for the various \( C \) is as in [Sp2]; the notation for the objects of \( \text{Irr} W \) is as in [Sp2] (for type \( E_n \)) and as in [L3, 4.10] for type \( E_4 \).) In each case the structure of \( A(u), \bar{A}(u) \) (for \( u \in C \)) is indicated; here \( S_n \) denotes the symmetric group in \( n \) letters. The order in which we list the objects in \( \text{Irr}_C W \) corresponds to the following order of the irreducible representations of \( A(u) = S_n \):

\[
1, \epsilon \ (n = 2); \ 1, r, \epsilon \ (n = 3, G \neq G_2); \ 1, r \ (n = 3, G = G_2); \ 1, \lambda^1, \lambda^2, \sigma \ (n = 4);
\]

1, \( \nu, \lambda^1, \lambda^2, \lambda^3 \) (\( n = 5 \))

(Notation of [L3, 4.3]). Now 0.4 and 0.5 follow in our case from the tables in 3.5-3.9 and the definitions in [L3, 4.8-4.13]. (In those tables \( S_n \) is the symmetric group in \( n \) letters.)

### 3.5. Assume that \( G \) is of type \( E_8 \).

\[
\begin{align*}
\text{Irr}_{E_8} W &= \{1\}; \quad A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{E_8(a_1)} W &= \{8\}; \quad A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{E_8(a_2)} W &= \{352\}; \quad A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{E_7 A_1} W &= \{1123, 28\}; \quad A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{D_6} W &= \{2104, 160\}; \quad A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{E_7(a_1)} A_1 W &= \{560_5, [50_8]\}; \quad A(u) = S_2, \bar{A}(u) = \{1\} \\
\text{Irr}_{E_7(a_1)} W &= \{567_6\}; \quad A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{D_6(a_1)} W &= \{700_6, 300_8\}; \quad A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{E_7(a_2), A_1} W &= \{1400_7, 1008_9, 5619\}; \quad A(u) = S_3, \bar{A}(u) = S_3 \\
\text{Irr}_{A_8} W &= \{1400, 1575_10, 350_14\}; \quad A(u) = S_3, \bar{A}(u) = S_3 \\
\text{Irr}_{D_7(a_1)} W &= \{3240_9, [1050, 10]\}; \quad A(u) = S_2, \bar{A}(u) = \{1\} \\
\text{Irr}_{D_6(a_3)} W &= \{2240_{10}, [175_12], 840_{13}\}; \quad A(u) = S_3, \bar{A}(u) = S_2 \\
\text{Irr}_{D_6, A_1} W &= \{2268_{10}, 1296_{13}\}; \quad A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{E_6(a_1), A_1} W &= \{4096_{11}, 4096_{12}\}; \quad A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{E_6} W &= \{525_{12}\}; \quad A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{D_7(a_2)} W &= \{4200_{12}, 3360_{13}\}; \quad A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{E_6(a_1)} W &= \{2800_{13}, 2100_{16}\}; \quad A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{D_5, A_2} W &= \{4536_{13}, [840_{14}\}]}; \quad A(u) = S_2, \bar{A}(u) = \{1\} \\
\text{Irr}_{D_6(a_1), A_1} W &= \{6075_{14}, [700_{16}\}]}; \quad A(u) = S_2, \bar{A}(u) = \{1\} \\
\text{Irr}_{A_8, A_1} W &= \{2835_{14}\}; \quad A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{A_6} W &= \{4200_{15}\}; \quad A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{D_6(a_1)} W &= \{5600_{15}, 2400_{17}\}; \quad A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{2, A_1} W &= \{4480_{16}, 4536_{18}, 5670_{18}, 1400_{20}, 1680_{22}, 70_{32}\}; \quad A(u) = S_5, \bar{A}(u) = S_5 \\
\text{Irr}_{D_7} W &= \{2100_{20}\}; \quad A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{(A_5, A_1)^7} W &= \{5600_{21}, 2400_{23}\}; \quad A(u) = S_2, \bar{A}(u) = S_2 \\
\text{Irr}_{D_4, A_2} W &= \{4200_{15}, [168_{24}\}]}; \quad A(u) = S_2, \bar{A}(u) = \{1\} \\
\text{Irr}_{A_4, A_2, A_1} W &= \{2835_{22}\}; \quad A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{A_4, A_3} W &= \{4536_{23}\}; \quad A(u) = \{1\}, \bar{A}(u) = \{1\} \\
\text{Irr}_{D_5(a_1)} W &= \{2800_{25}, 2100_{28}\}; \quad A(u) = S_2, \bar{A}(u) = S_2
\end{align*}
\]
\[ \text{FAMILIES AND SPRINGER'S CORRESPONDENCE} \]

\[ \text{3.6. Assume that } G \text{ is adjoint of type } E_7. \]

\[ \text{Ir}_{A_4} \] \[ A_2, W = \{4200_{24}, 3360_{25}\}; \ A(u) = S_2, \bar{A}(u) = S_2 \]

\[ \text{Ir}_{D_4} \] \[ W = \{525_{36}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \]

\[ \text{Ir}_{A_4} \] \[ A_1, W = \{4096_{26}, 4096_{27}\}; \ A(u) = S_2, \bar{A}(u) = S_2 \]

\[ \text{Ir}_{A_4} \] \[ W = \{2268_{30}, 1296_{33}\}; \ A(u) = S_2, \bar{A}(u) = S_2 \]

\[ \text{Ir}_{D_4(1)} \] \[ A_2, W = \{2240_{28}, 840_{31}\}; \ A(u) = S_2, \bar{A}(u) = S_2 \]

\[ \text{Ir}_{A_4} \] \[ A_1, W = \{3240_{31}, [972_{32}]\}; \ A(u) = S_2, \bar{A}(u) = \{1\} \]

\[ \text{Ir}_{D_4(1)} \] \[ A_1, W = \{1400_{32}, 1575_{34}, 350_{38}\}; \ A(u) = S_3, \bar{A}(u) = S_3 \]

\[ \text{Ir}_{D_4(1)} \] \[ W = \{1400_{37}, 1008_{39}, 56_{49}\}; \ A(u) = S_3, \bar{A}(u) = S_3 \]

\[ \text{Ir}_{A_4} \] \[ W = \{700_{42}, 300_{44}\}; \ A(u) = S_2, \bar{A}(u) = S_2 \]

\[ \text{Ir}_{A_4} \] \[ W = \{567_{46}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \]

\[ \text{Ir}_{A_4} \] \[ W = \{560_{47}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \]

\[ \text{Ir}_{A_4} \] \[ W = \{210_{52}, 160_{55}\}; \ A(u) = S_2, \bar{A}(u) = S_2 \]

\[ \text{Ir}_{A_4} \] \[ W = \{112_{63}, 28_{68}\}; \ A(u) = S_2, \bar{A}(u) = S_2 \]

\[ \text{Ir}_{A_4} \] \[ W = \{357_{74}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \]

\[ \text{Ir}_{A_4} \] \[ W = \{8_{91}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \]

\[ \text{Ir}_0 \] \[ W = \{1_{120}\}; \ A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{2A_2} W = \{168_{21}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{A_2,3A_1} W = \{105_{21}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{A_2} W = \{210_{21}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{A_2,2A_1} W = \{189_{22}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{A_2,3A_1} W = \{120_{25}, 105_{26}\}; A(u) = S_2, \bar{A}(u) = S_2 \]
\[ \text{Irr}_{3A_1'} W = \{2136\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{A_2} W = \{56_{30}, 21_{33}\}; A(u) = S_2, \bar{A}(u) = S_2 \]
\[ \text{Irr}_{2A_1} W = \{2737\}; A(u) = \{1\}, A(u) = \{1\} \]
\[ \text{Irr}_{A_3} W = \{746\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_0 W = \{1_{63}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]

3.7. Assume that \( G \) is adjoint of type \( E_6 \).
\[ \text{Irr}_{E_6} W = \{10\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{E_6(a_1)} W = \{61\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{D_6} W = \{20_2\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{A_2} W = \{30_2, 15_3\}; A(u) = S_2, \bar{A}(u) = S_2 \]
\[ \text{Irr}_{D_5(a_1)} W = \{64_4\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{A_3} W = \{60_5\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{A_4} W = \{81_6\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{D_5} W = \{24_6\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{D_4(a_1)} W = \{80_7, 90_8, 20_{10}\}; A(u) = S_3, \bar{A}(u) = S_3 \]
\[ \text{Irr}_{2A_2} W = \{24_{12}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{A_3} W = \{81_{10}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{2A_4} W = \{60_{11}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{A_2,1} W = \{64_{13}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{A_4} W = \{30_{15}, 15_{17}\}; A(u) = S_2, \bar{A}(u) = S_2 \]
\[ \text{Irr}_{2A_1} W = \{20_{20}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{A_3} W = \{62_{25}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_0 W = \{1_{36}\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]

3.8. Assume that \( G \) is of type \( F_4 \).
\[ \text{Irr}_{F_4} W = \{1_1\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{F_4(a_1)} W = \{42, 23\}; A(u) = S_2, \bar{A}(u) = S_2 \]
\[ \text{Irr}_{F_4(a_2)} W = \{9_1\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{B_4} W = \{81\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{C_5} W = \{8_3\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{F_4(a_3)} W = \{12_1, 9_3, 6_2, 1_3\}; A(u) = S_4, \bar{A}(u) = S_4 \]
\[ \text{Irr}_{A_2} W = \{82\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{A_2} W = \{84, [1_2]\}; A(u) = S_2, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{A_1,1} W = \{9_4\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
\[ \text{Irr}_{A_2} W = \{45, 2_2\}; A(u) = S_2, \bar{A}(u) = S_2 \]
\[ \text{Irr}_0 W = \{1_4\}; A(u) = \{1\}, \bar{A}(u) = \{1\} \]
3.9. Assume that $G$ is of type $G_2$.
Irr$_{G_2}W$ is the unit representation; $A(u) = \{1\}$, $\bar{A}(u) = \{1\}$
Irr$_{G_2(a_1)}W$ consists of the reflection representation and the one dimensional representation on which the reflection with respect to a long (resp. short) simple coroot acts nontrivially (resp. trivially); $A(u) = S_3$, $\bar{A}(u) = S_3$
Irr$_0W = \{\text{sgn}\}$; $A(u) = \{1\}, \bar{A}(u) = \{1\}$

3.10. This completes the proof of Theorem 0.4 and that of Corollary 0.5.

We note that the definition of $G_{\mathcal{F}}$ given in [L3] (for type $C_n, B_n$) is $\bar{P}(\mathcal{J})_1$ (in the setup of 3.2) and $\bar{P}(\mathcal{J})_0$ (in the setup of 3.3) which is noncanonically isomorphic to $A(u)$, unlike the definition adopted here that is, $\bar{P}(\mathcal{J})_0$ (in the setup of 3.2) and $\bar{P}(\mathcal{J})_1$ (in the setup of 3.3) which makes $G_{\mathcal{F}}$ canonically isomorphic to $A(u)$.

4. Character sheaves

4.1. Let $\hat{G}$ be a set of representatives for the isomorphism classes of character sheaves on $G$. For any conjugacy class $D$ in $G$ let $D_\omega := \{g_\omega; g \in D\}$, a unipotent class in $G$. For any unipotent class $C$ in $G$ let $\mathcal{S}_C$ be the set of conjugacy classes $D$ of $G$ such that $D_\omega = C$. It is likely that the following property holds.

(a) Let $K \in \hat{G}$. There exists a unique unipotent class $C$ of $G$ such that

- for any $D \in \mathcal{S}_C$, $K|_D$ is a local system (up to shift);
- for some $D \in \mathcal{S}_C$, we have $K|_D \neq 0$;
- for any unipotent class $C'$ of $G$ such that $\dim C' \geq \dim C$, $C' \neq C$ and any $D \in \mathcal{S}_{C'}$ we have $K|_D = 0$.

We say that $C$ is the unipotent support of $K$.

(The uniqueness part is obvious.) Note that [L8, 10.7] provides some support (no pun intended) for (a).

We shall now try to make (a) more precise in the case where $K \in \hat{G}_T$, the subset of $\hat{G}$ consisting of unipotent character sheaves (that is $\hat{G}_T = \hat{G}_Q$, with the notation of [L7, 4.2]). As in [L7, 4.6] we have a partition $\hat{G}_T = \sqcup_{\mathcal{F}} \hat{G}^\text{un}_{\mathcal{F}}$ where $\mathcal{F}$ runs over the families of $W$.

In the remainder of this section we fix a family $\mathcal{F}$ of $W$ and we denote by $C$ the special unipotent class of $G$ such that $E_C \in \mathcal{F}$, see 0.1; let $u \in C$. Let $\Gamma = \hat{A}(u)$ and let $Z(u) \xrightarrow{j'} A(u) \xrightarrow{h} \Gamma$ be the obvious (surjective) homomorphisms; let $j = h j'$ : $Z(u) \rightarrow \Gamma$. Let $[\Gamma]$ be the set of conjugacy classes in $A(u)$. For $D \in \mathcal{S}_C$ let $\phi(D)$ be the conjugacy class of $j(g_s)$ in $\Gamma$ where $g \in D$ is such that $g_\omega = u$; clearly such $g$ exists and is unique up to $Z(u)$-conjugacy so that the conjugacy class of $j(g_s)$ is independent of the choice of $g$. Thus we get a (surjective) map $\phi : \mathcal{S}_C \rightarrow [\Gamma]$. For $\gamma \in [\Gamma]$ we set $\mathcal{S}_{C, \gamma} = \phi^{-1}(\gamma)$. We now select for each $\gamma \in [\Gamma]$ an element $x_\gamma \in \gamma$ and we denote by $\text{Irr} Z_{\Gamma}(x_\gamma)$ a set of representatives for the isomorphism classes of irreducible representations of $Z_{\Gamma}(x_\gamma) := \{g \in \Gamma; gx_\gamma = x_\gamma g\}$ (over $\mathbb{Q}_l$). Let $D \in \mathcal{S}_{C, \gamma}$, $\mathcal{E} \in \text{Irr} Z_{\Gamma}(x_\gamma)$. We can find $g \in D$ such that $g_\omega = u, j(g_s) = x_\gamma$ (and another choice for such $g$
must be of the form $bgb^{-1}$ where $b \in Z(u)$, $j(b) \in Z_F(x_\gamma))$. Let $E^D$ be the $G$-equivariant local system on $D$ whose stalk at $g_1 \in D$ is $\{z \in G; zg^{-1} = g_1\} \times E$ modulo the equivalence relation $(z,e) \sim (zh^{-1}, j(h)e)$ for all $h \in Z(g)$. If $g$ is changed to $g_1 = bgb^{-1}$ (as above) then $E^D$ is changed to the $G$-equivariant local system $E^D_1$ on $D$ whose stalk at $g' \in D$ is $\{z' \in G; zg^{-1} = g_1\} \times E$ modulo the equivalence relation $(z',e') \sim (z'j^{-1}, j(h)e)$ for all $h' \in Z(g_1)$. We have an isomorphism of local systems $E^D \sim E^D_1$ which for any $g' \in D$ maps the stalk of $E^D$ at $g'$ to the stalk of $E^D_1$ at $g'$ by the rule $(z,e) \mapsto (zb^{-1}, j(b)e)$. (We have $zb^{-1}g_1bz^{-1} = zg^{-1} = g'$.) This is compatible with the equivalence relations. Thus the isomorphism class of the local system $E^D$ does not depend on the choice of $g$.

4.2. Assume that $G$ is simply connected. Let $D \in S_C$. Let $s$ be a semisimple element of $G$ such that $su \in D$. Let $C_0$ be the conjugacy class of $u$ in $Z(s)$. Let $W'$ be the Weyl group of $Z(s)$ regarded as a subgroup of $W$. For any finite dimensional $W'$-module $E'$ over $Q_l$ let $E'$ be the intersection cohomology complex on $Z(s)$ defined in terms of $Z(s), E'$ in the same way as $E$ was defined in terms of $G, E$. Using $[L5, (8.8.4)]$ and the $W$-equivariance of the isomorphism in loc.cit. we see that:

(a) $E|_{sC_0} \cong (E|_{W'})|_{sC_0}[]$.

Now, if $K \in \hat{G}_{un}^F$ is of the form $E[\cdot]$ for some $E \in F$ then the computation of $K|_D$
is reduced by (a) to the computation of $E'|_{sC_0}$ for any irreducible $W'$-module $E'$ such that $(E' : E|_{W'}) > 0$ (here $(E' : E|_{W'})$ is the multiplicity of $E'$ in $E|_{W'}$). If for such $E'$ we define a unipotent class $C_E'$ of $Z(s)$ by $E' \in \text{Irr}_{C_E'} W'$ then, by a known property of $E'$, we have (with notation of 0.1 with $G$ replaced by $Z(s)$):

(b) if $C_0 = C_{E'}$ then $E'|_{sC_0}$ is the irreducible $Z(s)$-equivariant local system corresponding to $V_{E'}$;

(c) if $C_0 \neq C_{E'}$ and $\dim C_0 \geq \dim C_{E'}$ then $E'|_{sC_0} = 0$.

We say that $D$ is $E$-negligible if for any $E' \in \text{Irr} W'$ such that $(E' : E|_{W'}) > 0$ we have $\dim C_0 > \dim C_{E'}$.

(d) We say that $D$ is $E$-relevant if $\exists$ a unique $E'_0 \in \text{Irr} W'$ such that $(E'_0 : E|_{W'}) = 1$ and $C_{E'} = C_0$ (we then write $E_l = E'_0$);

for any $E' \in \text{Irr} W'$ such that $(E' : E|_{W'}) > 0, E' \neq E_l$ we have $\dim C_0 > \dim C_{E'}$

It is likely that $D$ is always $E$-negligible or $E$-relevant. If $D$ is $E$-negligible then $E|_{sC_0} = 0$ (hence $K|_D = 0$); if $D$ is $E$-relevant then $E|_{sC_0}$ (hence $K|_D$) can be explicitly computed using (b), (c).

In the remainder of this subsection we assume in addition that $G$ is almost simple of exceptional type and that $C$ is a distinguished unipotent class. In these cases one can verify that $D$ is $E$-negligible or $E$-relevant for any $E \in \mathcal{F}$ hence $K|_D$ can be explicitly computed and we can check that 4.1(b) holds. Moreover, we can compute $K|_D$ for any $K \in \mathcal{G}_{\mathcal{F}}$ (not necessarily of form $E|[]$) using an appropriate analogue of (a) (coming again from [L5, (8.8.4)]) and the appropriate analogues of (b), (c) (given in [L4]). We see that 4.1(b) holds again. Moreover we see that 4.1(c), (d) hold in these cases.

### 4.3

In this subsection we assume that $G$ is of type $E_8$ and $C$ is distinguished. In this subsection we indicate for each $D \in S_C$ the set $\mathcal{F}_D = \{ E \in \mathcal{F}; D \text{ is } E \text{-relevant} \}$ and we describe the map $E \mapsto E_l$ (see 4.2(d)). (Note that if $E \in \mathcal{F} - \mathcal{F}_D$, $D$ is $E$-negligible.) The notation is as in [Sp2]. We denote by $g_i$ an element of order $i$ of $A(u)$ (except that if $A(u) = S_5, g_2$ denotes a transposition and we denote by $g_2'$ an element of $A(u)$ whose centralizer has order 8). For each $g_i$ we denote by $\dot{g}_i$ a semisimple element of $Z(u)$ that represents $g_i$; similarly when $A(u) = S_5$, we denote by $\dot{g}_2'$ a semisimple element of $Z(u)$ that represents $g_2'$. We write $\mathcal{H}_{g_i}$ (resp. $\mathcal{F}_{g_i}$) instead of $\mathcal{F}_D$ where $D$ is the $G$-conjugacy class of $u\dot{g}_i$ (resp. of $u\dot{g}_2'$). We write $\mathcal{H}_{\dot{g}_i}$ (resp. $\mathcal{H}_{\dot{g}_2'}$) for the set of all $E_l \in \text{Irr} W'$ where $E$ runs through $\mathcal{F}_{g_i}$ (resp. $\mathcal{F}_{g_2'}$); here $W' \subset W$ is the Weyl group of $Z(\dot{g}_i)$ (resp. $Z(\dot{g}_2')$) and $E_l$ is as in 4.2(d). We write $C_{g_i}$ (resp. $C_{g_2'}$) for the conjugacy class of $u$ in $Z(\dot{g}_i)$ (resp. $Z(\dot{g}_2')$).

Assume that $C$ is the regular unipotent class. Then $A(u) = \{ 1 \}$, $\mathcal{F}_{g_1} = \mathcal{H}_{g_1} = \{ 1_{10} \}$.

Assume that $C$ is the subregular unipotent class. Then $A(u) = \{ 1 \}$, $\mathcal{F}_{g_1} = \mathcal{H}_{g_1} = \{ 8_{11} \}$.
Assume that $C = E_8(a_2)$. Then $A(u) = \{1\}, \mathcal{F}_{g_1} = \mathcal{H}_{g_1} = \{352\}$.

Assume that $C = E_7A_1$. Then $A(u) = S_2$, $Z(\dot{g}_1) = G$, $Z(\dot{g}_2)$ is of type $E_7A_1$, $\mathcal{F}_{g_1} = \mathcal{H}_{g_1} = \{1123, 282\}$, $\mathcal{F}_{g_2} = \{844\}$, $\mathcal{H}_{g_2} = \{10\}$, $C_{g_2} = \text{regular unipotent class}$.

Assume that $C = D_8$. Then $A(u) = S_2$, $Z(\dot{g}_1) = G$, $Z(\dot{g}_2)$ is of type $D_8$, $\mathcal{F}_{g_1} = \mathcal{H}_{g_1} = \{2104, 160_7\}$, $\mathcal{F}_{g_2} = \{50_8\}$, $\mathcal{H}_{g_2} = \{1\}$, $C_{g_2} = \text{regular unipotent class}$.

Assume that $C = E_7(\alpha_1)A_1$. Then $A(u) = S_2$, $Z(\dot{g}_1) = G$, $Z(\dot{g}_2)$ is of type $E_7A_1$, $\mathcal{F}_{g_1} = \mathcal{H}_{g_1} = \{560_5\}$, $\mathcal{F}_{g_2} = \{560_5\}$, $\mathcal{H}_{g_2} = \{7_1 \boxtimes 1\}$, $C_{g_2} = \text{subregular unipotent class}$ in $E_7$ factor times regular unipotent class in $A_1$ factor.

Assume that $C = D_8(\alpha_1)$. Then $A(u) = S_2$, $Z(\dot{g}_1) = G$, $Z(\dot{g}_2)$ is of type $D_8$, $\mathcal{F}_{g_1} = \mathcal{H}_{g_1} = \{700_9, 300_8\}$, $\mathcal{F}_{g_2} = \{400_7\}$, $\mathcal{H}_{g_2} = \{\text{reflection repres.}\}$, $C_{g_2} = \text{subregular unipotent class}$.

Assume that $C = E_7(\alpha_2)A_1$. Then $A(u) = S_3$, $Z(\dot{g}_1) = G$, $Z(\dot{g}_2)$ is of type $E_7A_1$, $Z(\dot{g}_3)$ is of type $E_8A_2$, $\mathcal{F}_{g_1} = \mathcal{H}_{g_1} = \{1400_7, 1008_9, 561_9\}$, $\mathcal{F}_{g_2} = \{1344_8\}$, $\mathcal{H}_{g_2} = \{27_2 \boxtimes 1\}$, $C_{g_2} = \text{subsubregular unipotent class}$ in $E_7$-factor times regular unipotent class in $A_1$ factor, $\mathcal{F}_{g_3} = \{448_9\}$, $\mathcal{H}_{g_3} = \{1\}$, $C_{g_3} = \text{regular unipotent class}$.

Assume that $C = A_8$. Then $A(u) = S_3$, $Z(\dot{g}_1) = G$, $Z(\dot{g}_2)$ is of type $D_8$, $Z(\dot{g}_3)$ is of type $A_8$, $\mathcal{F}_{g_1} = \mathcal{H}_{g_1} = \{1400_8, 1575_10, 350_14\}$, $\mathcal{F}_{g_2} = \{1050_10\}$, $\mathcal{H}_{g_2} = \{28 - \text{dimensional repres.}\}$, $C_{g_2} = \text{unipotent class}$ with Jordan blocks of size 5, 11, $\mathcal{F}_{g_3} = \{175_12\}$, $\mathcal{H}_{g_3} = \{1\}$, $C_{g_3} = \text{regular unipotent class}$.

Assume that $C = D_8(\alpha_3)$. Then $A(u) = S_3$, $Z(\dot{g}_1) = G$, $Z(\dot{g}_2)$ is of type $D_8$, $Z(\dot{g}_3)$ is of type $E_6A_2$, $\mathcal{F}_{g_1} = \mathcal{H}_{g_1} = \{2240_10, 840_13\}$, $\mathcal{F}_{g_2} = \{1400_11\}$, $\mathcal{H}_{g_2} = \{56 - \text{dimensional repres.}\}$, $C_{g_2} = \text{unipotent class}$ with Jordan blocks of size 7, 9, $\mathcal{F}_{g_3} = \{2240_10\}$, $\mathcal{H}_{g_3} = \{6_1 \boxtimes 1\}$, $C_{g_3} = \text{subregular unipotent class}$ in $E_6$-factor times regular unipotent class in $A_1$ factor.

Assume that $C = 2A_4$. Then $A(u) = S_5$, $Z(\dot{g}_1) = G$, $Z(\dot{g}_2)$ is of type $E_7A_1$, $Z(\dot{g}_3)$ is of type $D_8$, $Z(\dot{g}_4)$ is of type $E_6A_2$, $Z(\dot{g}_5)$ is of type $D_5A_3$, $Z(\dot{g}_5)$ is of type $A_4A_4$, $Z(\dot{g}_6)$ is of type $A_5A_2A_1$, $\mathcal{F}_{g_1} = \mathcal{H}_{g_1} = \{4480_{16}, 4536_{18}, 5670_{18}, 1400_{20}, 1680_{22}, 70_{32}\}$, $\mathcal{F}_{g_2} = \{7168_{17}, 5600_{19}, 448_{23}\}$, $\mathcal{H}_{g_2} = \{315_7 \boxtimes 1, 280_9 \boxtimes 1, 35_{13} \boxtimes 1\}$, $C_{g_2} = D_6(\alpha_1)A_1$ in $E_7$-factor times regular unipotent class in $A_1$-factor, $\mathcal{F}_{g_3} = \{420_20, 2688_{20} \epsilon''\}$, $\mathcal{H}_{g_3} = \{\text{repres. with symbol } (2 < 5; 0 < 3), (2 < 3; 0 < 5), (0 < 1, 4 < 5)\}$, $C_{g_3} = \text{unipotent class}$ with Jordan blocks of sizes 1, 3, 5, 7, $\mathcal{F}_{g_4} = \{3150_{18}, 1134_{20}\}$, $\mathcal{H}_{g_4} = \{30_3 \boxtimes 1, 15_5 \boxtimes 1\}$, $C_{g_4} = A_5A_1$ in $E_6$-factor times regular unipotent class in $A_2$-factor, $\mathcal{F}_{g_5} = \{1344_{19}\}$, $\mathcal{H}_{g_5} = \{5 - \text{dimensional repres.}\}$, $C_{g_5} = \text{subregular unipotent class}$ in $D_5$-factor times regular unipotent class in $A_3$-factor, $\mathcal{F}_{g_5} = \{420_{20}\}$, $\mathcal{H}_{g_5} = \{1\}$, $C_{g_5} = \text{regular unipotent class}$,
4.4. In this subsection we assume that $Irr$ class in $G$ cuspidal. They can be characterized as follows.

$L$. Then $W$ sgn image of $L$ with $\bar{\nu} \in v$. Then $W$ is a notrivial $G$-equivariant local system of rank 1 on $C$, $\mathcal{L}'$ is the inverse image of $\mathcal{L}$ under the obvious map $D \to C$. We see that 4.1(b) holds for all $K \in \hat{\mathcal{G}}_{un}^F$ and 4.1(c),(d) hold.

4.5. Assume that $\mathcal{F}$ is the family containing the unit representation of $W$. Then $C$ is the regular unipotent class of $G$ and $\hat{\mathcal{G}}_{un}^F$ consists of a single character sheaf, namely $\mathcal{Q}_1[\cdot]$. Clearly, 4.1(b),(c),(d) hold in this case.

Next we assume that $\mathcal{F}$ is the family containing the sign representation of $W$. Then $C = \{1\}$ and $\hat{\mathcal{G}}_{un}^F$ consists of a single character sheaf, namely $K = sgn[\cdot]$. Note that for any semisimple class $D$ of $G$ we have $K|_D = \mathcal{Q}_1[\cdot]$ so that 4.1(b),(c),(d) hold in this case.

References


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