UNIPOTENT ALMOST CHARACTERS OF SIMPLE p-ADIC GROUPS, II

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UNIPOTENT ALMOST CHARACTERS
OF SIMPLE $p$-ADIC GROUPS, II

G. LUSZTIG

Dedicated to Professor E. B. Dynkin for his 90th birthday

0.1. For any finite group $\Gamma$, a “nonabelian Fourier transform matrix” was introduced in [L1]. This is a square matrix whose rows and columns are indexed by pairs formed by an element of $\Gamma$ and an irreducible representation of the centralizer of that element (both defined up to conjugation). As shown in [L2], this matrix, which is unitary with square 1, enters (for suitable $\Gamma$) in the character formulas for unipotent representations of a finite reductive group.

In this paper we extend the definition of the matrix above to the case where $\Gamma$ is a reductive group over $\mathbb{C}$. We expect that this new matrix (for suitable $\Gamma$) relates the characters of unipotent representations of a simple $p$-adic group and the unipotent almost characters of that simple $p$-adic group [L6]. The new matrix is defined in §1. The definition depends on some finiteness results established in 1.2. Several examples are given in 1.4-1.6 and 1.12. In §2 a conjectural relation with unipotent characters of $p$-adic groups is stated. In §3 we consider an example arising from an odd spin group which provides some evidence for the conjecture. Since the group $\Gamma$ which enters in the conjecture is described in the literature only up to isogeny, we give a more precise description for it (or at least for the derived subgroup of its identity component) in the Appendix.

Notation. If $G$ is an affine algebraic group and $g \in G$ we denote by $g_s$ (resp. $g_u$) the semisimple (resp. unipotent) part of $g$. Let $Z_G$ be the centre of $G$ and let $G^0$ be the identity component of $G$. Let $G_{der}$ be the derived subgroup of $G$. If $g \in G$ and $G'$ is a subgroup of $G$ we set $Z_{G'}(g) = \{x \in G'; xg = gx\}$.

1. A pairing

1.1. Let $H$ be a reductive (but not necessarily connected) group over $\mathbb{C}$; let $\Sigma$ be the set of semisimple elements in $H$. Let $[H]$ be the set of irreducible components of $H$. For $x, y$ in $\Sigma$ we set $A_{x,y} = \{z \in H; zxz^{-1}y = yzxz^{-1}\}$; we have $A_{x,y} = \bigsqcup_{h \in [H]} A_{x,y}^h$ where $A_{x,y}^h = A_{x,y} \cap h$. Now $Z^0_H(x) \times Z^0_H(y)$ acts on $A_{x,y}$ by $(v, v') : z \mapsto v' zv^{-1}$, leaving stable each of the subsets $A_{x,y}^h$ ($h \in [H]$).
Lemma 1.2. The $Z^0_H(x) \times Z^0_H(y)$ action on $A_{x,y}$ in 1.1 has only finitely many orbits.

For any $c \in Z_H(y) \cap \Sigma$ we choose a maximal torus $T_c$ of $Z_{Z_H(y)}(c)$ and a maximal torus $T'_c$ of $Z_H(c)$ that contains $T_c$. (Note that $T_c \subset Z_H(c)$.) We show:

(a) If $E$ is any semisimple $H$-conjugacy class in $H$ then $E \cap cT_c$ is finite.

Since $E$ is a finite union of $H^0$-conjugacy classes, it is enough to show that, if $E'$ is any semisimple $H^0$-conjugacy class in $H$, then $E' \cap cT_c$ is finite. By [L5, I, 1.14], $E' \cap cT'_c$ is finite. Since $T_c \subset T'_c$, we see that $E' \cap cT_c$ is also finite, as required.

Let $\{c_i; i \in [1, n]\}$ be a collection of elements of $Z_H(y) \cap \Sigma$, one in each connected component of $Z_H(y)$. Let $\{c'_i; k \in [1, n']\}$ be a collection of elements of $Z_H(x) \cap \Sigma$, one in each connected component of $Z_H(x)$. Let $E$ be the $H$-conjugacy class of $x$. By (a), $E_i := E \cap c_iT_{c_i}$ is a finite set for $i \in [1, n]$. For any $i \in [1, n], e \in E_i$ we set $P_{i,e} = \{g \in H; ggxg^{-1} = e\}$. We have $P_{i,e} = p_{i,e}Z_H(x)$ for some $p_{i,e} \in H$ hence

(b) $P_{i,e} = \bigcup_{k \in [1, n']} p_{i,e}c'_k Z^0_H(x)$.

Let $z \in A_{x,y}$. We have $zzx^{-1} \in Z_H(y) \cap \Sigma$ hence $zzx^{-1} \in c_i Z^0_H(y)$ for a unique $i \in [1, n]$. Using [L5, I, 1.14(c)] (with $G$ replaced by the reductive group $Z_H(y)$) we see that $zzx^{-1}$ is $Z^0_H(y)$-conjugate to an element of $c_i T_{c_i}$. Thus, $v'zx^{-1}v^{-1} \in c_i T_{c_i}$ for some $v' \in Z^0_H(y)$. Hence $v'zx^{-1}v^{-1} = e \in E_i$ so that $v'z \in P_{i,e}$. Using (b) we see that $v'z = p_{i,e}c'_k v$ for some $v \in Z^0_H(x)$ and some $k \in [1, n']$. Hence $v'zv^{-1} = p_{i,e}c'_k$. We see that the finitely many elements $p_{i,e}c'_k (i \in [1, n], k \in [1, n'], e \in E_i)$ represent all the orbits in the lemma. This completes the proof.

Remark. The following result can be deduced from the lemma above.

(c) Let $C, C'$ be two semisimple $H^0$-conjugacy classes in $H$. Let $X = \{(g, g') \in C \times C'; gg' = g'g\}$. Then the $H^0$-action on $X$, $g_1 : (g, g') \mapsto (g_1gg_1^{-1}, g_1g'g_1^{-1})$ has only finitely many orbits.

Let $x \in C, y \in C'$. Let $\mathfrak{Z}$ be a set of representatives for the orbits of $Z_{H^0}(x) \times Z_{H^0}(y)$ on $A_{x,y}^{H^0}$. This is a finite set. (Even the orbits of the smaller group $Z^0_H(x) \times Z^0_H(y)$ form a finite set.) Let $\mathcal{O}$ be an $H^0$-orbit in $X$. In $\mathcal{O}$ we can find an element of the form $(zxz^{-1}, y)$ where $z \in H^0$, $zxz^{-1}y = yzxz^{-1}$ hence $z \in A_{x,y}^{H^0}$. Thus $z = v'z^{-1}v$ where $v' \in Z_{H^0}(y), v \in Z_{H^0}(x), z \in \mathfrak{Z}$. We have $(zxz^{-1}, y) = (v'z^{-1}v v^{-1} x v z^{-1} y) = (\zeta x^{-1} y, \zeta x^{-1} y) \in \mathcal{O}$ hence $\zeta x^{-1} y \in \mathcal{O}$. Thus the number of $H^0$-orbits in $X$ is $|\mathfrak{Z}|$, proving (c).

1.3. In the remainder of this section we fix a (not necessarily connected) reductive group $H$ over $C$ and a finite subgroup $\Lambda \subset \mathfrak{Z}_H$. Let $\Sigma$ be the set of semisimple elements in $H$.

A pair $x', y'$ of commuting elements in $\Sigma$ is said to be adapted if there exists a maximal torus $T$ of $H^0$ such that $x'Tx'^{-1} = T, y'Ty'^{-1} = T$, $(T \cap Z_H(x'))^0$ is a maximal torus of $Z_H(x')^0$ and $(T \cap Z_H(y'))^0$ is a maximal torus of $Z_H(y')^0$. For example, if $x', y'$ are contained in $H^0$ then $x', y'$ is adapted if and only there exist a maximal torus of $H^0$ that contains $x'$ and $y'$; this condition is automatically satisfied if $(H^0)_{der}$ is simply connected.
As in 1.1, let \([H]\) be the set of connected components of \(H\). Let \(x, y \in \Sigma\) and \(h \in [H]\).

For any \(x \in \Sigma\) we set \(\tilde{Z}(x) = Z_H(x)/(Z_H^0(x)\Lambda)\). For \(x \in \Sigma\) let \(I_x\) be the set of isomorphism classes of irreducible representations \(\sigma\) (over \(C\)) of \(Z_H(x)\) on which the subgroup \(Z_H(x)^0\) acts trivially. Let \(\hat{\Lambda} = \text{Hom}(\Lambda, C^*)\). If \(x \in \Sigma\) and \(\chi \in \hat{\Lambda}\) we denote by \(I_x^\chi\) the set of \(\sigma \in I_x\) such that \(\Lambda\) acts on \(\sigma\) through \(\chi\) times identity. Let \(\hat{\Sigma}\) be the set of pairs \((x, \sigma)\) where \(x \in \Sigma\) and \(\sigma \in I_x\). Now \(H\) acts on \(\hat{\Sigma}\) by \(f : (x, \sigma) \mapsto f(x, \sigma) := (fxf^{-1}, f\sigma)\) where \(f\sigma \in I_{fxf^{-1}}\) obtained from \(\sigma\) via \(\text{Ad}(f) : Z_H(x) \xrightarrow{\sim} Z_H(fxf^{-1})\).

For any \(\chi \in \hat{\Lambda}\) let \(\Sigma^\chi = \{(x, \sigma) \in \hat{\Sigma}; \sigma \in I_x^\chi\}\).

We assume given a function \(\kappa : \Sigma \times \Sigma \times [H] \to C\) such that \(\kappa(fxf^{-1}, f'yf'^{-1}, f'hf^{-1}) = \kappa(x, y, h)\) for any \((x, y, h) \in \Sigma \times \Sigma \times [H]\), \(f, f', y, \zeta, \zeta' \in \Lambda\).

We say that \(\kappa\) is a weight function. For \((x, y, h) \in \Sigma \times \Sigma \times [H]\) let \(0A^h_{x,y}\) be a set of representatives for the orbits of the action \((v, v') : z \mapsto v'zv^{-1}\) of \(Z_H^0(x) \times Z_H^0(y)\) on \(A^h_{x,y}\); this is a finite set, by 1.2. Let \(\tilde{A}^h_{x,y}\) be the set of all \(z \in 0A^h_{x,y}\) such that \(zxy^{-1}\) form an adapted pair for \(H\).

We say that \(\kappa\) is the standard weight function if \(\kappa(x, y, h) = (\tilde{A}^h_{x,y})^{-1}\) whenever \(\tilde{A}^h_{x,y} \neq \emptyset\) and \(\kappa(x, y, h) = 0\) whenever \(\tilde{A}^h_{x,y} = \emptyset\). This weight function is clearly independent of the choice of \(0A^h_{x,y}\).

We return to the general case. Let \((x, \sigma) \in \hat{\Sigma}, (y, \tau) \in \hat{\Sigma}\). We set

\[
((x, \sigma), (y, \tau)) = |\Lambda/(\Lambda \cap H^0)|^{-1}|\tilde{Z}(x)|^{-1}|\tilde{Z}(y)|^{-1} \\
\times \sum_{h \in [H]} \kappa(x, y, h) \sum_{z \in 0A^h_{x,y}} \frac{\text{tr}(zxy^{-1})\text{tr}(z^{-1}y\sigma)}{\text{tr}(z^{-1}y\sigma)}.
\]

(Clearly, this is independent of the choice of \(0A^h_{x,y}\).

In the case where \(H\) is finite, \(\Lambda = \{1\}\) and \(\kappa\) is the standard weight function (which in this case satisfies \(\kappa(x, y, h) = 1\) whenever \(A^h_{x,y} \neq 0\)) this reduces to the pairing introduced in [L1].

Note that we can take \(0A^h_{x,y}\) to be the image of \(0A^h_{x,y}^{-1}\) under \(z \mapsto z^{-1}\); it follows that \(((y, \tau), (x, \sigma)) = ((x, \sigma', (y, \tau))\). If \((x, \sigma) \in \hat{\Sigma}^\chi, (y, \tau) \in \hat{\Sigma}^{\chi'}\) with \(\chi, \chi' \in \hat{\Lambda}\) and if \(\zeta, \zeta' \in \Lambda\) then

\[
((\zeta x, \sigma), (\zeta' y, \tau)) = \chi(\zeta')\chi'(\zeta)((x, \sigma), (y, \tau)).
\]

(We can take \(0A^h_{\zeta x, \zeta' y} = 0A^h_{x, y}\). For \(f, f' \in H\) we have

\[
(f(x, \sigma), f'(y, \tau)) = ((x, \sigma), (y, \tau)).\]
(We can take $A_{x,y}^{h}f_{x,y}^{-1} = f'(0)A_{x,y}^{h'}f_{x,y}^{-1}$).

Now let $V$ be the $\mathbf{C}$-vector space with basis consisting of the elements $(x, \sigma) \in \Sigma$. Then (a) extends to a form $(,): V \times V \to \mathbf{C}$ which is linear in the first variable, antilinear in the second variable and satisfies $(v', v) = (v, v')$ for $v, v' \in V$. We have $V = \bigoplus_{\chi \in \Lambda} V^x$ where $V^x$ is the subspace of $V$ spanned by $(x, \sigma) \in \tilde{\Sigma}^x$. For any $\chi, \chi' \in \Lambda$ let

$$J_{\chi} = \{ v \in V^x; (v, V^x) = 0 \}, \quad \bar{V}_{\chi}^x = V^x/J_{\chi}^x.$$ 

Then $(,)$ induces a pairing $\bar{V}_{\chi}^x \times \bar{V}_{\chi'}^x \to \mathbf{C}$ (denoted again by $(,)$) which is linear in the first variable, antilinear in the second variable and satisfies $(v'_1, v_1) = \bar{(v_1, v'_1)}$ for $v'_1 \in \bar{V}_{\chi}^x$, $v_1 \in \bar{V}_{\chi'}^x$. Let $\mathfrak{A}_{\chi}^x$ be the image of the $\bar{\Sigma}^x$ under the obvious map $V^x \to \bar{V}_{\chi}^x$. When $\chi = 1$ we denote by $(x, \sigma)_{\chi}$ the image of $(x, \sigma) \in \tilde{\Sigma}^1$ under $V^1 \to \bar{V}^1_{\chi}$.

1.4. In 1.4-1.6 we assume that $\kappa$ is the standard weight function. In this subsection we assume that $H = H^0 \Lambda$ and that $(H^0)_{\text{der}}$ is simply connected. In this case any pair of commuting semisimple elements in $H$ is adapted. Let $x, y \in \Sigma$. We have $\bar{Z}(x) = \{1\}$, $\bar{Z}(y) = \{1\}$, $I^1_x = \{1\}$, $I^1_y = \{1\}$, $|[H]| = |\Lambda/(\Lambda \cap H^0)|$, $A_{x,y}^h \neq \emptyset$ for any $h \in [H]$. From the definitions we see that $((x, 1), (y, 1)) = 1$. Hence $(x, 1) - (x', 1) \in J^1_\chi$ for any $x, x' \in \Sigma$. We see that $\mathfrak{A}^{1}_\chi$ consists of a single element $(1, 1)_1$ which has inner product 1 with itself and dim $\bar{V}_1 = 1$. (This example applies to the situation in 1.7(a) with $u$ any unipotent element in $G = SL_N(\mathbf{C})$.)

1.5. In this subsection we assume that $H = H^0 \sqcup H^1$ and that $H^0$ is isomorphic to $\mathbf{C}^*$ (we denote by $\lambda \mapsto g_\lambda$ an isomorphism $\mathbf{C}^* \xrightarrow{\sim} H^0$); we also assume that any $r \in H^1$ satisfies $g_{\lambda}r^{-1} = g_{\lambda^{-1}}$ for all $\lambda \in \mathbf{C}^*$ and that $\Lambda = \bar{Z}_H$ that is, $\Lambda = \{1, g_{-1}\} \subset H^0$. In this case any pair of commuting semisimple elements in $H$ is adapted. We fix $r \in H^1$. We have $r^2 = 1$ or $r^2 = g_{-1}$. The case where $r^2 = 1$ (resp. $r^2 = g_{-1}$) arises in the situation in 1.7(a) with $u$ a subregular unipotent element in $G = Spin_{2n+1}(\mathbf{C})$ with $n$ even (resp. $n$ odd). We have

$$Z_H(1) = Z_H(g_{-1}) = H, Z_H(g_{\lambda}) = H^0 \text{ if } \lambda \in \mathbf{C}^* - \{1, -1\},$$

$$Z_H(r) = \{1, g_{-1}, r, rg_{-1}\}.$$ 

Hence

$$|\bar{Z}(1)| = 2, |\bar{Z}(g_{\lambda})| = 1 \text{ if } \lambda \in \mathbf{C}^* - \{1, -1\}, |\bar{Z}(r)| = 2.$$ 

We have

$$A_{g_{\lambda}, g_{\lambda'}} = H \text{ for any } \lambda, \lambda' \in \mathbf{C}^*, \quad A_{1,r} = H, \quad A_{g_{\lambda}, r} = \emptyset \text{ for any } \lambda \in \mathbf{C}^* - \{1, -1\},$$

$$A_{r, r} = \{1, g_{-1}, r, rg_{-1}, g_i, g_{-i}, rg_i, rg_{-i}\}, \text{ where } i = \sqrt{-1}.$$
Hence we can take

\[ 0 A_{g,\lambda}^{H^0} = \{ 1 \}, \quad 0 A_{g,\lambda'}^{H^1} = \{ r \} \] for any \( \lambda, \lambda' \in \mathbb{C}^* \),

\[ 0 A_{1,r}^{H^0} = \{ 1 \}, \quad 0 A_{1,r}^{H^1} = \{ r \}, \]

\[ 0 A_{g,\lambda,r}^{H^0} = \emptyset, \quad 0 A_{g,\lambda,r}^{H^1} = \emptyset \] for any \( \lambda \in \mathbb{C}^* - \{ 1, -1 \}, \)

\[ 0 A_{r,r}^{H^0} = \{ 1, g-1, g_i, g_{i-1} \}, \quad 0 A_{r,r}^{H^1} = \{ r, rg-1, rg_i, rg_{i-1} \}. \]

Let \( \epsilon \) be the non-trivial 1-dimensional representation of \( H \) which is trivial on \( H^0 \). The restriction of \( \epsilon \) to \( Z_H(r) \) is denoted again by \( \epsilon \). From the previous results we can write the 5 × 5 matrix of inner products \((x, \sigma), (y, \tau)\) with \((x, \sigma)\) running through \((1, 1), (1, \epsilon), (r, 1), (r, \epsilon), (g_\lambda, 1)\) (they index the rows), \((y, \tau)\) running through \((1, 1), (1, \epsilon), (r, 1), (r, \epsilon), (g_\lambda, 1)\) (they index the columns) and with \( \lambda, \lambda' \in \mathbb{C}^* - \{ 1, -1 \} \):

\[
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\
-\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\
1 & 1 & 0 & 0 & 2
\end{pmatrix}
\]

From the results in 1.3 we see that \((1, 1)_1, (1, \epsilon)_1, (r, 1)_1, (r, \epsilon)_1, (g_\lambda, 1)_1\) generate \( V_1^1 \) and from the matrix above we see that for any \( \lambda \in \mathbb{C}^* - \{ 1, -1 \} \) we have \((g_\lambda, 1)_1 = (1, 1)_1 + (1, \epsilon)_1\). Hence \((1, 1)_1, (1, \epsilon)_1, (r, 1)_1, (r, \epsilon)_1\) generate \( V_1^1 \) and the matrix of their inner products is

\[
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]

which is nonsingular. We see that \( \mathcal{A}_1^1 \) consists of \((1, 1)_1, (1, \epsilon)_1, (r, 1)_1, (r, \epsilon)_1, (1, 1)_1 + (1, \epsilon)_1\), and the first four of these elements form a basis of \( V_1^1 \).

1.6. Let \( E \) be a vector space of dimension \( 2n \geq 2 \) over the field \( F_2 \) with two elements with a given nondegenerate symplectic form \( \langle \cdot, \cdot \rangle : E \times E \rightarrow F_2 \). In this subsection we assume that \( H \) is a Heisenberg group attached to \( \langle \cdot, \cdot \rangle \) that is, a finite group with a given surjective homomorphism \( \psi : H \rightarrow E \) whose kernel consists of 1 and another central element \( c \) with \( c^2 = 1 \) and in which for any \( x,y \in E \) we have \( \hat{x}^2 = \hat{y}^2 = c, \hat{x}\hat{y} = c^{\langle x,y \rangle} \hat{y}\hat{x} \) for any \( \hat{x} \in \psi^{-1}(x), \hat{y} \in \psi^{-1}(y) \); we also assume that \( \Lambda = \{ 1, c \} \). (This example arises in the situation in 1.7(a) with \( u \) a suitable unipotent element in \( \mathcal{G} \) of spin type.) We have \( Z_H(1) = Z_H(c) = H \) and for \( x \in E - \{ 0 \} \), \( \hat{x} \in \psi^{-1}(x) \) we have \( Z_H(x) = \psi^{-1}(\{ x' \in E; \langle x, x' \rangle = 0 \}) \). We denote the nontrivial character of \( \Lambda \)
by \( \chi \). If \( x \in E \) and \( \dot{x} \in \psi^{-1}(x) \) then \( I_{\dot{x}}^1 \) can be identified with \( E_x := E/F_{2x} \) (to each \( \bar{y} \in E_x \) corresponds the character \( \xi \mapsto \langle \psi(\xi), y \rangle \) of \( H \) where \( y \in E \) represents \( \bar{y} \)). Now \( I_{\dot{x}}^1 \) and \( I_{\dot{x}}^1 \) consist of a single representation \( \rho \) whose character is \( 2^n \) at 1, \(-2^n \) at \( c \) and 0 at all other elements of \( H \). If \( x \in E - \{0\} \) and \( \dot{x} \in \psi^{-1}(x) \) then \( I_{\dot{x}}^1 \) consists of two representations \( \rho, \rho' \) such that the character of \( \rho \) at 1, \( c, \dot{x}, c \dot{x} \) is \( 2^{n-1}, -2^{n-1}, 2^{n-1} i, -2^{n-1} i \) (respectively), the character of \( \rho' \) at 1, \( c, \dot{x} c \dot{x} \) is \( 2^{n-1}, -2^{n-1}, 2^{n-1} i, 2^{n-1} i \) (respectively) and whose character at all other elements of \( H \) is 0; here \( i = \sqrt{-1} \). Let \( x, x', y, y' \in E \), let \( \dot{x} \in \psi^{-1}(x) \), \( \dot{x}' \in \psi^{-1}(x') \) and let \( \bar{y} \) (resp. \( \bar{y}' \)) be the image of \( y \) (resp. \( y' \)) in \( E_x \) (resp. \( E_{x'} \)). From the definitions we have

\[
((\dot{x}, \bar{y}), (\dot{x}', \bar{y}')) = 0 \text{ if } \langle x, x' \rangle \neq 0,
\]

\[
((\dot{x}, \bar{y}), (\dot{x}', \bar{y}')) = 2^{-2n+2}(-1)^{\langle x', y \rangle} + \langle x, y \rangle (\delta_{x, 0} + 1)^{-1} \delta_{x', 0} + 1)
\]

if \( \langle x, x' \rangle = 0 \).

Let \( x, x', y \in E \), let \( \dot{x} \in \psi^{-1}(x) \), \( \dot{x}' \in \psi^{-1}(x') \) and let \( \bar{y} \) be the image of \( y \) in \( E_x \); let \( \rho_1 \in I_{\dot{x}}^1 \). From the definitions we have

\[
((\dot{x}, \bar{y}), (\dot{x}', \rho_1)) = 0 \text{ if } x \neq 0,
\]

\[
((\dot{x}, \bar{y}), (\dot{x}', \rho_1)) = 2^{-n}(-1)^{\langle x', y \rangle} \text{ if } \dot{x} = 1,
\]

\[
((\dot{x}, \bar{y}), (\dot{x}', \rho_1)) = -2^{-n}(-1)^{\langle x', y \rangle} \text{ if } \dot{x} = c.
\]

Let \( x, x' \in E \), let \( \dot{x} \in \psi^{-1}(x) \), \( \dot{x}' \in \psi^{-1}(x') \) and let \( \rho_1 \in I_{\dot{x}}^1 \), \( \rho_1' \in I_{\dot{x}'}^1 \). From the definitions we see that \( ((\dot{x}, \rho_1), (\dot{x}', \rho_1')) \) is equal 0 if \( x \neq x' \), is equal to 1 if \( \dot{x} = \dot{x}', r_1 = r_1' \) or if \( \dot{x}' = c \dot{x}, r_1 \neq r_1' \) and is equal to \(-1\) if \( \dot{x}' = c \dot{x}, r_1 = r_1' \) or if \( \dot{x}' = \dot{x}, r_1 \neq r_1' \).

Let \( Z \) be the set of all pairs \((x, \bar{y})\) where \( x \in E \) and \( \bar{y} \in E_x \); let \( M \) be the matrix with rows and columns indexed by \( Z \) with entries \( M((x, \bar{y}), (x', \bar{y}')) = ((\dot{x}, \bar{y}), (\dot{x}', \bar{y}')) \) where \( \dot{x} \in \psi^{-1}(x) \), \( \dot{x}' \in \psi^{-1}(x') \). Its square \( M^2 \) has entries

\[
M^2_{(x, \bar{y}), (x', \bar{y}')} = \sum_{(x'', \bar{y}'') \in Z} ((\dot{x}, \bar{y}), (\dot{x}'', \bar{y}''))((\dot{x}'', \bar{y}''), (\dot{x}', \bar{y}'))
\]

where \( \dot{x} \in \psi^{-1}(x) \), \( \dot{x}' \in \psi^{-1}(x') \), \( \dot{x}'' \in \psi^{-1}(x'') \). Let \( x, x', y, y' \in E \) and let \( \bar{y} \) (resp.
\( \bar{y}' \) be the image of \( y \) (resp. \( y' \)) in \( E_x \) (resp. \( E_x' \)). From the definitions we have

\[
\mathcal{M}^2_{(x, \bar{y}), (x', \bar{y}')} = 2^{-4n+4} \sum_{(x'', y'') \in E \times E; (x, x'') = 0, (x'', x') = 0} (-1)^{x'' y + y'} + \langle x + x', y'' \rangle (\delta_{x,0} + 1)^{-1} (\delta_{x'',0} + 1)^{-2} (2 - \delta_{x'',0})^{-1} (\delta_{x',0} + 1)^{-1} \\
= \delta_{x,x'} 2^{-2n+4} \sum_{x'' \in E; (x,x'') = 0} (-1)^{x'' y + y'} (\delta_{x,0} + 1)^{-2} (\delta_{x'',0} + 1)^{-1} 2^{-1} \\
= \delta_{x,x'} 2^{-2n+3} \sum_{x'' \in E; (x,x'') = 0} (-1)^{x'' y + y'} (\delta_{x,0} + 1)^{-2} \\
- \delta_{x,x'} 2^{-2n+3} (\delta_{x,0} + 1)^{-2} + \delta_{x,x'} 2^{-2n+3} (\delta_{x,0} + 1)^{-2} 2^{-1} \\
= \delta_{x,x'} \delta_{\bar{y}, \bar{y}'} 2^{-2n+3} \left\{ x'' \in E; (x, x'') = 0 \right\} (\delta_{x,0} + 1)^{-2} \\
- \delta_{x,x'} 2^{-2n+2} (\delta_{x,0} + 1)^{-2} \\
= \delta_{x,x'} \delta_{\bar{y}, \bar{y}'} 2^{-2n+3} 2^{n-1} (\delta_{x,0} + 1)^{-1} - \delta_{x,x'} 2^{-2n+2} (\delta_{x,0} + 1)^{-2} \\
= \delta_{x,x'} (2 - \delta_{x,0}) (\delta_{\bar{y}, \bar{y}'} 2 - |E_x|^{-1}).
\]

We see that \( \mathcal{M}^2 \) is a direct sum over \( x \in E \) of matrices \( \mathcal{P}(x) \) indexed by \( E_x \times E_x \) where \( \mathcal{P}(x) = (2 - \delta_{x,0})(2I - |E_x|^{-1}J) \) and all entries of \( J \) are 1. Thus we have \( J^2 = |E_x|J \). Setting \( \mathcal{P}'(x) = (2 - \delta_{x,0})^{-1} \mathcal{P}(x) \) we have \( |E_x|^{-1}J = 2I - \mathcal{P}'(x) \) hence

\[
\mathcal{P}'(x)^2 - 4\mathcal{P}'(x) + 4I = (2I - \mathcal{P}'(x))^2 = |E_x|^{-2} J^2 = |E_x|^{-1} J = 2I - \mathcal{P}'(x).
\]

Thus \( \mathcal{P}'(x)^2 - 3\mathcal{P}'(x) + 2I = 0 \) that is \( (\mathcal{P}'(x) - I)(\mathcal{P}'(x) - 2I) = 0 \). This shows that \( \mathcal{P}'(x) \) is semisimple with eigenvalues 1 and 2 hence \( \mathcal{M}^2 \) is semisimple with eigenvalues \( 2 - \delta_{x,0} \) and \( 2(2 - \delta_{x,0}) \) hence \( \mathcal{M} \) is semisimple with eigenvalues \( \pm 1, \pm \sqrt{2}, \pm 2 \). In particular \( \mathcal{M} \) is invertible.

For \( z = (x, \bar{y}) \in Z \) we set \( a_z = (\hat{x}, \bar{y})_1 \in \tilde{V}_1^1 \) (where \( \hat{x} \in \psi^{-1}(x) \)); this is independent of the choice of \( \hat{x} \). Note that \( \tilde{A}_1^1 \) consists of the elements \( a_z \). From the fact that \( \mathcal{M} \) is invertible we see that \( \{a_z; z \in Z\} \) is a basis of \( \tilde{V}_1^1 \).

1.7. Let \( G \) be a connected, simply connected, almost simple reductive group over \( \mathbb{C} \). Let \( u \) be a unipotent element of \( G \). Let \( V \) be the unipotent radical of \( Z_G(u)^0 \).

(a) Until the end of 1.11 we assume that \( H = H(u) \) is a reductive subgroup of \( Z_G(u) \) such that \( Z_G(u) = HV, H \cap V = \{1\} \) and that \( \Lambda = Z_G \).

(It is well known that such \( H \) exists and that \( H \) is unique up to \( Z_G \)-conjugacy; note that \( Z_G \subset H \).) We can state the following result in which we allow \( \kappa \) to be any weight function for \( H \).

**Proposition 1.8.** Let \( \chi, \chi' \in \hat{A} \). Then \( \tilde{A}_\chi^1 \) is finite. In particular, \( \dim \tilde{V}_\chi^1 < \infty \).

The proof is given in 1.11 after some preparation in 1.9, 1.10.
1.9. For \( g \in \mathcal{G} \) we set \( T_g = \mathcal{Z}^0_{\mathcal{G}(g)} \), a torus contained in \( Z_{\mathcal{G}}(g) \). For \( g, g' \in \mathcal{G} \) we write \( g \sim g' \) if \( g'g^{-1} \in T_g = T_{g'} \). This is an equivalence relation on \( \mathcal{G} \). Note that if \( g \sim g' \) then \( x_g x^{-1} \sim x_g' x^{-1} \) for any \( x \in \mathcal{G} \). We show:

(a) Let \( g, g' \in \mathcal{G} \) be such that \( g \sim g' \). Then \( Z_{\mathcal{G}}(g) = Z_{\mathcal{G}}(g') \) and \( g_u = g'_u \).

The weaker statement that \( Z_{\mathcal{G}}(g)^0 = Z_{\mathcal{G}}(g')^0 \) is contained in the proof of [L5, I, 3.4]. Let \( s = g_s, u = g_u, s' = g'_s, u' = g'_u \). We have \( g' = tg \) where \( t \in T_g = T_{g'} \). Thus \( s' u' = tsu \). Since \( t \in Z_{\mathcal{G}}(g) \) we have \( ts = st, tu = ut \). Since \( s, t \) are commuting semisimple elements, \( ts \) is semisimple and it commutes with \( u \). By uniqueness of the Jordan decomposition of \( g' \) we have \( s' = ts, u' = u \). Let \( x \in Z_{\mathcal{G}}(g) \). Then \( xs = sx \). We have \( t \in Z_{\mathcal{G}(s)} \). Moreover, \( x \in Z_{\mathcal{G}}(s) \), hence \( xt = tx \) so that \( xs' = xt s = tsx = s'x \). Thus \( x \in Z_{\mathcal{G}}(s') \). We also have \( x \in Z_{\mathcal{G}}(u) = Z_{\mathcal{G}}(u') \). Thus, \( x \in Z_{\mathcal{G}}(s'u') \) that is, \( x \in Z_{\mathcal{G}}(g') \). Thus, \( Z_{\mathcal{G}}(g) \subset Z_{\mathcal{G}}(g') \). By symmetry we have also \( Z_{\mathcal{G}}(g') \subset Z_{\mathcal{G}}(g) \) hence \( Z_{\mathcal{G}}(g) = Z_{\mathcal{G}}(g') \). This proves (a).

1.10. In this subsection we assume that we are in the setup of Proposition 1.8. Let \( \pi : Z_{\mathcal{G}}(u) \rightarrow H \) be the homomorphism such that \( \pi(x) = x \) for \( x \in H \) and \( \pi(x) = 1 \) for \( x \in \mathcal{V} \). We show:

(a) Let \( \pi(x, \sigma), (x', \sigma'), (y, \tau) \in \tilde{\Sigma} \). Assume that for some \( v \in \mathcal{V} \) we have \( vxuv^{-1} \sim x'u \). Then \( Z_H(x) = Z_H(x') \). Assume further that \( \sigma = \sigma' \) (which makes sense by the previous sentence). Then \( ((x, \sigma), (y, \tau)) = ((x', \sigma'), (y, \tau)) \).

From \( vxuv^{-1} \sim x'u \) and 1.9(a) we see that \( Z_{\mathcal{H}}(vxuv^{-1}) = Z_{\mathcal{H}}(x'u) \). Since \( (vxuv^{-1})_s = vxv^{-1}, (vxuv^{-1})_u = u, (x'u)_s = x', (x'u)_u = u \), we deduce that \( Z_{\mathcal{H}}(vxuv^{-1}) = Z_{\mathcal{H}}(x') \). Using the semidirect product decompositions \( Z_{\mathcal{G}}(u) = (vHv^{-1})\mathcal{V}, Z_{\mathcal{G}}(u) = Hv \) we deduce

\[
Z_{vHv^{-1}}(vxuv^{-1})Z_{\mathcal{V}}(vxuv^{-1}) = Z_H(x')Z_{\mathcal{V}}(x')
\]

that is

\[
vZ_H(x)v^{-1}Z_{\mathcal{V}}(vxuv^{-1}) = Z_H(x')Z_{\mathcal{V}}(x').
\]

Applying \( \pi \), we deduce \( Z_H(x) = Z_H(x') \). This proves the first assertion of (a). From this we see that for \( z \in H \) we have \( z^{-1}yz \in Z_H(x) \) if and only if \( z^{-1}yz \in Z_H(x') \). In other words, we have \( A_{x,y} = A_{x',y} \). It follows that for any connected component \( h \) of \( H \) we have \( A^h_{x,y} = A^h_{x',y} \). Let \( z \in A_{x,y} = A_{x',y} \). Clearly, \( \text{tr}(z^{-1}yz, \sigma) = \text{tr}(z^{-1}yz, \sigma') \) (recall that \( \sigma = \sigma' \)). We claim that \( \text{tr}(zxz^{-1}, \tau) = \text{tr}(zx'z^{-1}, \tau) \). We set \( e := x'x^{-1} \in H \). From \( vxuv^{-1} \sim x'u \) we have

\[
\tilde{e} := x'uu^{-1}x^{-1}v^{-1} \in T_{x'u}.
\]

Hence

\[
z\tilde{e}z^{-1} = ze(xvx^{-1})v^{-1}z^{-1} \in zT_{x'u}z^{-1} = T'
\]

where

\[
T' := zZ_{Z_{\mathcal{G}}(x')}z^{-1} = Z_{Z_{\mathcal{G}}(zx'z^{-1})}.
\]
Now $yu \in Z_G(zx'z^{-1})$ hence $T' \subset Z_G(yu)$ that is, $T' \subset Z_{\bar{H}}(y) = Z_H(y)Z_V(y)$. It follows that $\pi(T')$ is a torus contained in $Z_H(y)$. Hence $\pi(T') \subset Z_{\bar{H}}^0(y)$. Applying $\pi$ to $ze(xux^{-1})v^{-1}z^{-1} \in T'$ we obtain $zez^{-1} \in \pi(T')$ hence $zez^{-1} \in Z_H^0(y)$. We have $zzez^{-1} = (zez^{-1})(zzz^{-1})$ where $zzz^{-1} \in Z_H^0(y)$ acts trivially on $\tau$. Hence $\text{tr}(zzez^{-1}, \tau) = \text{tr}(zzz^{-1}, \tau)$ as claimed. It follows that (a) holds.

Next we show:

(b) Let $\chi \in \Lambda$. Let $(x, \sigma) \in \bar{\Sigma}^\chi$, $x' \in \Sigma$. Assume that for some $g \in G$ we have $g x u g^{-1} \sim x' u$. Then there exists $\sigma' \in I_x^\chi$ such that for any $(y, \tau) \in \bar{\Sigma}$ we have $((x, \sigma), (y, \tau)) = ((x', \sigma'), (y, \tau))$.

From 1.9(a) we have $(gxug^{-1})_u = u$ hence $g u g^{-1} = u$ that is $g \in Z_G(u)$. We write $g = v g_1$ with $g_1 \in H$, $v \in V$. We have $g_1 x u g_1^{-1} = x'' u$ where $x'' \in \Sigma$ and $ux'' v^{-1} \sim x' u$. From 1.3 we have $((x'', \sigma'), (y, \tau)) = (x, \sigma), (y, \tau))$ where $\sigma' = g_1 g^{-1} \sigma \in I_{x''}^\chi$. From $ux'' v^{-1} \sim x' u$ and (a) we have $I_{x''}^\chi = I_{x'}^\chi$, and $(x'', \sigma'), (y, \tau)) = ((x', \sigma'), (y, \tau))$. Thus $((x, \sigma), (y, \tau)) = ((x', \sigma'), (y, \tau))$. This proves (b).

1.11. We now prove Proposition 1.8. For $g, g' \in G$ we write $g \preceq g'$ if $\gamma g g^{-1} \sim g'$ for some $\gamma \in G$. This is an equivalence relation on $G$; the equivalence classes are called the strata of $G$. According to [1.5, I, 3.7], $G$ has only finitely many strata. Let $S_1, S_2, \ldots, S_n$ be the strata of $G$ which have nonempty intersection with $u \Sigma$. For each $i \in [1, n]$ we choose $x_i \in \Sigma$ such that $ux_i \in S_i$. Now let $(x, \sigma) \in \bar{\Sigma}^\chi$. Let $S$ be the stratum of $G$ that contains $x u$. We have $S = S_i$ for some $i \in [1, n]$. Hence $gxug^{-1} \sim x_i u$ for some $g \in G$. By 1.10(b) there exists $\sigma' \in I_{x_i}^\chi$ such that $(x, \sigma) = (x_i, \sigma')$ in $V_{x_i}^\chi$. We see that $\bar{\Sigma}^\chi$ is the image under $V \rightarrow V_{x_i}^\chi$ of the finite set consisting of $(x_i, \sigma')$ with $i \in [1, n]$, $\sigma' \in I_{x_i}^\chi$. The proposition is proved.

1.12. In this subsection we assume that $H = PGL_2(C)$, $\Lambda = \{1\}$ and that $\kappa$ is the standard weight function. (This example is as in 1.7(a) for a suitable $u$ with $G$ of type $F_4$, see 1.11.)

Let $\lambda \mapsto g_{\lambda}$ be an isomorphism of $C^*$ onto a maximal torus $T$ of $H$ whose normalizer in $H$ is denoted by $N(T)$. Then any semisimple element of $H$ is conjugate to an element of the form $g_{\lambda}$, for some $\lambda \in C^*$. We have $Z_H(1) = H$, $Z_H(g_{-1}) = N(T)$, $Z_H(g_{\lambda}) = T$ if $\lambda \in C^* - \{1, -1\}$. Hence $|\bar{Z}(1)| = 1$, $|\bar{Z}(g_{-1})| = 2$, $|\bar{Z}(g_{\lambda})| = 1$ if $\lambda \in C^* - \{1, -1\}$. Let $r \in N(T) - T$. We can find $\xi \in H$ such that $\xi g_{-1} \xi^{-1} = r$. We have $A_{1, g_{\lambda}} = H$ if $\lambda \in C^*$, $A_{g_{-1}, g_{-1}} = T \cup rT \cup T \xi T \cup T \xi r T$, $A_{g_{-1}, g_{\lambda}} = T \cup rT$ if $\lambda \in C^* - \{1, -1\}$, $A_{g_{\lambda}, g_{\lambda'}} = T \cup rT$ if $\lambda, \lambda' \in C^* - \{1, -1\}$. Hence we can take

$$0 A^H_{1, g_{\lambda}} = \{1\}, \quad 0 A^H_{g_{-1}, g_{-1}} = \{1, r, \xi, \xi r\},$$

$$0 A^H_{g_{-1}, g_{-1}} = \{1, r\} \text{ if } \lambda \in C^* - \{1, -1\},$$

$$0 A^H_{g_{\lambda}, g_{\lambda'}} = \{1, r\} \text{ if } \lambda \in C^* - \{1, -1\}. $$

In our case any pair of commuting semisimple elements is adapted except for a pair $H$-conjugate to the pair $g_{-1}, r$. We have:

$$\kappa(1, g_{\lambda}, H) = 1, \quad \kappa(g_{-1}, g_{-1}) = 1/2,$$
Recall that \( \kappa(g_-, g_\lambda) = 1/2 \) if \( \lambda \in \mathbb{C}^* - \{1, -1\} \),
\[ \kappa(g_\lambda, g_{\lambda'}) = 1/2 \) if \( \lambda \in \mathbb{C}^* - \{1, -1\} \).

Let \( \epsilon \) be the non-trivial 1-dimensional representation of \( N(T) \) which is trivial on \( T \). From the previous results we can write the \( 4 \times 4 \) matrix of inner products \( ((x, \sigma), (y, \tau)) \) with \( (x, \sigma) \) running through \((1,1), (g_-, 1), (g_-, \epsilon), (g_\lambda, 1) \) (they index the rows), \( (y, \tau) \) running through \((1,1), (g_-, 1), (g_-, \epsilon), (g_{\lambda'}, 1) \) (they index the columns) and with \( \lambda, \lambda' \in \mathbb{C}^* - \{1, -1\} \):
\[
\begin{pmatrix}
1 & \frac{1}{2} & \frac{1}{2} & 1 \\
\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\
1 & \frac{1}{2} & \frac{1}{2} & 1
\end{pmatrix}
\]

We see that for any \( \lambda \in \mathbb{C}^* - \{1, -1\} \) we have \((g_\lambda, 1) = (1,1) \) in \( \mathcal{V}_1 \). Also, \((1,1) = (g_-, 1) + (g_-, \epsilon) \) in \( \mathcal{V}_1 \). Hence \((g_-, 1), (g_-, \epsilon) \) generate \( \mathcal{V}_1 \) and the matrix of their inner products is
\[
\begin{pmatrix}
\frac{1}{2} & 0 \\
0 & 1
\end{pmatrix}
\]
which is nonsingular, so that \( \mathcal{A}_1 \) is equal to \( \{(g_-, 1), (g_-, \epsilon)\} \) and is a basis of \( \mathcal{V}_1 \).

2. A conjecture

2.1. Let \( k \) be an algebraic closure of a finite \( F_q \). Let \( K = k((t)) \) where \( t \) is an indeterminate. Let \( K_0 = F_q((t)) \), a subfield of \( K \). Let \( G \) be a simple adjoint algebraic group defined and split over \( F_q \). Let \( \mathcal{G} \) be a connected, simply connected, almost simple reductive group over \( \mathbb{C} \) of type dual to that of \( G \). Let \( P_0 \) be an Iwahori subgroup of \( G := G(K) \) such that \( P_0 \cap G(K_0) \) is an Iwahori subgroup of \( G(K_0) \). Let \( \mathcal{W} \) be an indexing set for the set of \((P_0, P_0)\)-double cosets in \( G \). We regard \( \mathcal{W} \) as a group as in [L4]. As in loc.cit., \( \mathcal{W} \) is the semidirect product of a (normal) subgroup \( \mathcal{W}' \) (an affine Weyl group) and a finite abelian subgroup \( \Omega \). For any \( \xi \in \Omega \) let \( \xi G \) be the union of the \((P_0, P_0)\) double cosets in \( G \) indexed by elements in \( \mathcal{W}' \xi \). We have \( G = \cup_{\xi \in \Omega} \xi G \). Let \( G_{\text{rsc}} \) be the set of regular semisimple compact elements of \( G \), see [L6, 3.1]. We have \( G_{\text{rsc}} = \cup_{\xi \in \Omega} \xi G_{\text{rsc}} \) where \( \xi G_{\text{rsc}} = \xi G \cap G_{\text{rsc}} \). We set \( \xi G(K_0)_{\text{rsc}} = \xi G_{\text{rsc}} \cap G(K_0) \). Let \( \mathcal{U} \) be the set of isomorphism classes of unipotent representations of \( G(K_0) \), see [L4]. To each \( \rho \in \mathcal{U} \) we attach a sign \( \Delta(\rho) \in \{1, -1\} \) as follows: from the definition, to \( \rho \) corresponds a unipotent cuspidal representation \( \rho_0 \) of a finite reductive group and \( \Delta(\rho) \) is the invariant \( \Delta(\rho_0) \) defined in [L2, 4.23, 4.21]. (We almost always have \( \Delta(\rho) = 1 \).) Recall that [L4] gives a bijection \( \mathcal{Z}^1 \leftrightarrow \mathcal{U} \) where for \( \xi \in \Omega \), \( \mathcal{Z}^\xi \) is the set of all pairs \((g, \sigma)\) (up to \( \mathcal{G}\)-conjugation) where \( g \in \mathcal{G} \) and \( \sigma \) is an irreducible representation of
For $\zeta \in \mathfrak{Z}^1$ let $R_\zeta$ be the corresponding unipotent representation, let $\tilde{R}_\zeta$ be the corresponding standard representation of $G(K_0)$ (which has $R_\zeta$ as a quotient) and let $\phi^{\tilde{\zeta}}_\zeta$ be $\Delta(R_\zeta)$ times the restriction of the character of $\tilde{R}_\zeta$ to $\xi G(K_0)_{rsc}$. For any $\zeta \in \mathfrak{Z}^\xi$ let $t_\zeta : \xi G(K_0)_{rsc} \to \mathbb{C}$ be the unipotent almost character defined as in [L6, 3.9] (we assume that [L6, 3.8(f)] holds).

We now fix a unipotent element $u$ of $G$ and a reductive subgroup $H$ of $Z_G(u)$ as in 1.7. Then the definitions in §1 can be applied to this $H$ and to $\Lambda = Z_G$ (a subgroup of $H$). In particular $\Sigma, \tilde{\Sigma}, V, V^\chi_{\chi}', A^1_{\chi}'$ and the pairing 1.3(b) are defined. If $(x, \sigma) \in \tilde{\Sigma}^1$ then $\sigma$ can be viewed as an irreducible representation of $Z_G(g)/Z_G(g)^0$ on which $Z_G$ acts according to $\xi$ (we can identify $\Omega = \text{Hom}(Z_G, \mathbb{C}^*)$).

We preserve the setup of 2.1 and we assume that $\Lambda = G$ and any $\xi \in \Omega$ is the standard weight function for $H$.

Conjecture 2.2. We assume that $\kappa$ is the standard weight function for $H$.

(a) For any $\xi \in \Omega$ there is a unique isomorphism of vector spaces $\Theta_\xi : V^1_\xi \to V_\xi(u)$ such that for any $(x, \sigma) \in \tilde{\Sigma}^1$ we have $\Theta_\xi((x, \sigma)_{\xi}) = \phi^{\tilde{\xi}}_{x, \sigma}$. In particular $\Theta_\xi$ defines a bijection $A^1_{\xi} \to A^1_\xi$.

(b) There is a unique subset $B^1_1$ of $A^1_1$ such that $B^1_1$ is a basis of $V^1_1$ and any element of $A^1_{\xi}$ is a $\mathbb{R}_{\geq 0}$-linear combination of elements in $B^1_{\xi}$.

(c) Let $(x, \sigma) \in \tilde{\Sigma}^1$. We set $(x, \sigma)^*_1 = \sum_{b \in B^1_1} (b, (x, \sigma)) b \in V^1_1$. We have $\Theta_1((x, \sigma)^*_1) = t_{x, \sigma}$ up to a nonzero scalar factor.

3. An example

3.1. We preserve the setup of 2.1 and we assume that $G$ is of type $C_n$ with $n \geq 2$; then we can take $G = \text{Spin}_{2n+1}(\mathbb{C})$. We take $u$ in 2.1 to be a subregular unipotent element in $G$ so that $H$ in 2.1 is as in 1.5 (with $r^2 = g^{n+1}$). We take $\kappa$ to be the standard weight function for $H$.

The simple reflections of the affine Weyl group $W'$ in 2.1 are denoted by $s_0, s_1, \ldots, s_n$ where $s_is_{i+1}$ has order 4 if $i = 0$ or $i = n - 1$ and order 3 if $0 < i < n - 1$; $s_i, s_j$ commute if $|i - j| \geq 2$. For any $i \in [0, n]$ let $W'_{\xi_i}$ be the subgroup of $W'$ generated by $\{s_j; j \in [0, n] - \{i\}\}$; this can be viewed as the Weyl group of the reductive quotient $G'$ of a maximal parahoric subgroup of $G$ containing $P_0$. Note that $G'$ is defined over $F_q$.

Let $\hat{H}_q$ be the extended affine Hecke algebra (over $\mathbb{C}$) with parameter $q$ corresponding to $W$ and let $H_q$ be unextended affine Hecke algebra with parameter $q$ corresponding to $W'$ (a subalgebra of $\hat{H}_q$); note that $\hat{H}_q$ contains an element $\omega$ such that $\omega^2 = 1$, $\hat{H}_q = H_q \oplus H_q \omega$ and such that $x \mapsto \omega x \omega$ is the algebra automorphism of $H_q$ induced by the automorphism of $W'$ given by $s_i \mapsto s_{n-i}$ for $i \in [0, n]$. 


For any $i \in [0, n]$ let $\mathcal{H}_q^i$ be the Hecke algebra with parameter $q$ corresponding to $W^i$, viewed naturally as a subalgebra of $\mathcal{H}_q$.

We consider the following $W'$-graphs (in the sense of [KL] but with the $\mu$-function allowed to take complex values):

(a) $v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{n-1} \rightarrow v_n,$

(b) $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{n-1},$

(c) $v_0 \rightarrow v_n.$

($d_\lambda$) $v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{n-1} \rightarrow v_n \rightarrow v_{n-1}' \rightarrow v_{n-2}' \rightarrow \cdots \rightarrow v_1'.$

(In ($d_\lambda$), $v_1'$ is also joined with $v_0$.)

In each case the vertex $v_j$ or $v_j'$ has associated set of simple reflections $\{s_j\}$. For (a) we have $\mu(v_0, v_1) = 2$, $\mu(v_n, v_{n-1}) = 2$, all other $\mu$ are equal to 1. For (b) all $\mu$ are equal to 1. For (c) there are no edges hence no $\mu$-function. For ($d_\lambda$) with $\lambda \in C^*$ we have $\mu(v_0, v_1) = \lambda', \mu(v_1, v_0) = \lambda^{-1}, \mu(v_n, v_n') = \lambda', \mu(v_n', v_n) = \lambda^{-1}$, all other $\mu$ are equal to 1. For any $i \in [0, n]$, the $W'$-graphs (a),(b),(c) give rise to a $W'^i$-graph with the same vertices (except the one marked with $s_i$), same $\mu$-function and same associated set of simple reflections (at any non-removed vertex). This $W'^i$-graph is denoted by $(a^i),(b^i),(c^i)$ respectively. Let $n_a^i, n_b^i, n_c^i$ be the number of vertices marked with $s_i$ in (a),(b),(c) respectively (this number is 1 or 0).

For each of the $W'$-graphs (a),(b),(c),($d_\lambda$) we consider the $C$-vector space $E^a, E^b, E^c, E^{ds}$ spanned by the vertices of the graph, viewed as a $\mathcal{H}_q$-module in the standard way defined by the $W'$-graph structure. This $\mathcal{H}_q$-module extends to a $\tilde{\mathcal{H}}_q$-module in which $\omega$ acts by permuting the bases elements according to: $v_i \leftrightarrow v_{n-i}$ (for (a),(b),(c)) and (in case ($d_\lambda$)), $v_i \leftrightarrow v_{n-i}'$ if $i \in [1, n-1]$, $v_0 \leftrightarrow v_n$. Let $\hat{E}^a, \hat{E}^b, \hat{E}^c, \hat{E}^{ds}$ be the admissible representations of $G(K_0)$ generated by their subspace of $P_0 \cap G(K_0)$-invariant vectors such that the natural $\mathcal{H}_q$-module structure on this subspace is $E^a, E^b, E^c, E^{ds}$ respectively. Now the $\mathcal{H}_q$-modules $E^a, E^b, E^c, E^{ds}$ can be specialized to $q = 1$ (using the $W'$-graph structure) and yield $W$-modules $E^a_1, E^b_1, E^c_1, E^{ds}_1$ respectively.

If $i \in [0, n]$ for each of the $W'^i$-graphs ($a^i$),($b^i$),($c^i$) we consider the $C$-vector space $E^{a^i}, E^{b^i}, E^{c^i}$ spanned by the vertices of the graph, viewed as a $\mathcal{H}_q^i$-module in the standard way defined by the $W'^i$-graph structure. The restriction of the $\mathcal{H}_q^i$-module $E^a, E^b, E^c$ to $\mathcal{H}_q^i$-module is isomorphic to $E^{a^i} \oplus C^{n_{a^i}}, E^{b^i} \oplus C^{n_{b^i}}, E^{c^i} \oplus C^{n_{c^i}}$ respectively. Here $C^m$ is 0 if $m = 0$ while if $m = 1$ it is the $\mathcal{H}_q^i$-module defined by the $W'^i$ graph with a single vertex without any associated simple reflection.
Let $\hat{E}^a_i, \hat{E}^b_i, \hat{E}^c_i$ be the representations of $G^i(F_q)$ generated by their subspace of vectors invariant under a Borel subgroup such that the natural $\mathcal{H}^i_q$-module structure on this subspace is $E^a_i, E^b_i, E^c_i$ respectively. Now the $\mathcal{H}^i_q$-modules $E^a_i, E^b_i, E^c_i$ can be specialized to $q = 1$ (using the $\mathcal{W}^i$-graph structure) and yield $\mathcal{W}^i$-modules $E^a_i, E^b_i, E^c_i$ respectively.

With notation in 2.1 and 1.5, $\hat{E}^a_i, \hat{E}^b_i, \hat{E}^c_i$ can be identified with the standard representations $\hat{R}_{1,1}, \hat{R}_{1,\epsilon}, \hat{R}_{r,1}$ of $G(K_0)$. On the other hand the standard representation $\hat{R}_{1,g_0}$ (where $\nu \in \mathbb{C}^* - \{1, -1\}$) is among the representations $\hat{E}^d_{\lambda}$. Let $\phi^*_{1,1}, \phi^*_{1,\epsilon}, \phi^*_q$ be the restriction of the character of $\hat{R}_{1,1}, \hat{R}_{1,\epsilon}, \hat{R}_{r,1}, \hat{R}_{1,g_0}$ to the set $G(K_0)_{\text{vrc}}$ of very regular compact elements (see [KmL]) in $G(K_0)$. According to [KmL], $G(K_0)_{\text{vrc}}$ has a canonical partition $\sqcup G(K_0)^\theta_{\text{vrc}}$ into pieces invariant under conjugation by $G(K_0)$, where $\theta$ runs over the set of $\mathcal{W}$-conjugacy classes of elements of finite order in $\mathcal{W}$. Any finite dimensional $\mathcal{W}$-module $\rho$ gives rise to a class function $e_{\rho} : G(K_0)_{\text{vrc}} \to \mathbb{C}$ where $e_{\rho}|_{G(K_0)^\theta_{\text{vrc}}}$ is the constant equal to the value of $\rho$ at any element of $\theta$. In particular the class functions $e_{E^a_1}, e_{E^b_1}, e_{E^c_1}$ on $G(K_0)_{\text{vrc}}$ are well defined; we denote them by $e_{1,1}, e_{1,\epsilon}, e_{r,1}$. We set $e_{r,\epsilon} = 0$ (a function on $G(K_0)_{\text{vrc}}$).

We have:

(e) \[ \phi^*_{1,1,\nu} = \phi^*_{1,1} + \phi^*_{1,\epsilon}. \]

To prove (e) it is enough to show that $\hat{E}^d_{\lambda}$ and $\hat{E}^a_i \oplus \hat{E}^b_i$ have the same character on $G(K_0)_{\text{vrc}}$ for any $\lambda \in \mathbb{C}^*$. By the results of [KmL], the character of $\hat{E}^d_{\lambda}$ on $G(K_0)_{\text{vrc}}$ depends only on the restrictions of $E^d_{\lambda}$ to the various $\mathcal{H}^i_q$ and in particular is independent of $\lambda$. Hence we can assume that $\lambda = 1$. Thus it is enough to show that $E^d_{1} \cong E^a _1 \oplus E^b _1$ as a $\mathcal{H}^i_q$-module. The $\mathcal{W}$-graph $(d_1)$ admits an involution $v_i \leftrightarrow v_i'$ ($i \in [1, n-1]$), $v_0 \mapsto v_0$, $v_n \mapsto v_n$. Its eigenspaces give the required direct sum decomposition of the $\mathcal{H}^i_q$-module $E^d_{1}$.

We have the following equalities.

(f) \[ \phi^*_{1,1} = 1/2(e_{1,1} + e_{1,\epsilon} + e_{r,1} + e_{r,\epsilon}), \]

(g) \[ \phi^*_{1,\epsilon} = 1/2(e_{1,1} + e_{1,\epsilon} - e_{r,1} - e_{r,\epsilon}), \]

(h) \[ \phi^*_{r,1} = 1/2(e_{1,1} - e_{1,\epsilon} + e_{r,1} - e_{r,\epsilon}). \]

To prove (f) it is enough to show that for any $i \in [0, n]$ and any conjugacy class $\theta_0$ of $\mathcal{W}$, the two sides of (f) take the same value on $G(K_0)^\theta_{\text{vrc}}$ where $\theta$ is the $\mathcal{W}$-conjugacy class in $\mathcal{W}$ that contains $\theta_0$. By [KmL], $\phi^*_{1,1}|_{G(K_0)^\theta_{\text{vrc}}}$ is the constant equal to $n_a^i$ plus the value of the character of $\hat{E}^a_i$ at a regular semisimple element.
of $G^i(F_v)$ of type $\theta_0$. By the results of [L2], this is equal to $n^i_a$ plus the character of $1/2(E^a_i + E^b_i + E^c_i)$ at $\theta_0$ hence it is equal to $n^i_a$ plus the character of $1/2(e_{1,1} - n^i_a + e_{1,\epsilon} - n^i_b + e_{r,1} - n^i_c + e_{r,\epsilon})$ at $\theta$. It remains to note that $n^i_a = 1/2(n^i_a + n^i_b + n^i_c)$.

The proof of (g) and (h) is entirely similar; it is based on [KmL], [L2] and the equalities $n^i_b = 1/2(n^i_a + n^i_b - n^i_c)$, $n^i_c = 1/2(n^i_a - n^i_b + n^i_c)$.

Now $\check{R}_{r,\epsilon}$ is an irreducible representation of $G(K_0)$ which is not Iwahori-spherical. Let $\phi^*_{r,\epsilon}$ be the restriction of its character to $G(K_{0,v})$. Using again [KmL], [L2], we see that

$$\phi^*_{r,\epsilon} = 1/2(e_{1,1} - e_{1,\epsilon} - e_{r,1} + e_{r,\epsilon}).$$

From (f)-(i) we deduce:

$$1/2(\phi^*_{1,1} + \phi^*_{1,\epsilon} + \phi^*_{r,1} + \phi^*_{r,\epsilon}) = e_{1,1},$$
$$1/2(\phi^*_{1,1} - \phi^*_{r,1} + \phi^*_{1,\epsilon} - \phi^*_{r,\epsilon}) = e_{1,\epsilon},$$
$$1/2(\phi^*_{1,1} - \phi^*_{1,\epsilon} + \phi^*_{r,1} - \phi^*_{r,\epsilon}) = e_{r,1},$$
$$1/2(\phi^*_{1,1} - \phi^*_{1,\epsilon} - \phi^*_{r,1} + \phi^*_{r,\epsilon}) = e_{r,\epsilon}.$$

which, together with (e), confirm the Conjecture 2.2 in a very special case.

**Appendix**

**A.0.** Let $G, u, V, H = H(u)$ be as in 1.7. In this appendix we give some information on the structure of $H$.

**A.1.** Assume first that $G = SL(V)$ where $V$ is a $\C$-vector space of dimension $N \geq 2$. We assume that $V = \bigoplus_{i \geq 1} V_i$ where $V_i = W_i \otimes E_i$ and $W_i, E_i$ are $\C$-vector spaces of dimension $i, m_i$ respectively. Let $I = \{i \geq 1; m_i \geq 1\}$. For $i \in I$ we have an imbedding $\tau_i : GL(E_i) \to GL(V_i)$, $g \mapsto 1_{W_i} \otimes g$. Let $\Phi : \prod_{i \in I} GL(E_i) \to GL(V)$ be the homomorphism $(g_i) \mapsto \prod_{i \in I} \tau_i(g_i)$ (an imbedding). Let $H = SL(V) \cap$ image of $\Phi$. Then $H$ is of the form $H(u)$ for some $u \in G$ (as in 1.7) and all $H(u)$ as in 1.7 are obtained up to conjugacy. Note that $\Phi^{-1}(H)$ is the set of all $(x_i)_{i \in I}$ with $x_i \in GL(E_i)$, $\prod_{i \in I} \det(x_i)^i = 1$ and $\Phi^{-1}((H^0)_{der})$ is $\prod_{i \in I} SL(E_i)$. Thus $(H^0)_{der}$ is simply connected.

**A.2.** For any $\C$-vector space $V$ with a fixed symmetric or symplectic nondegenerate bilinear form $(,) : V \times V \to \C$ we denote by $Is_V$ be the group of isometries of $(,)$.

Until the end of A.8 we assume that $V$ is a $\C$-vector space of dimension $N \geq 1$ with a fixed symmetric nondegenerate bilinear form $(,) : V \times V \to \C$; we write $O_V$ instead of $Is_V$. Let $V_* = \{v \in V; (v, v) = 1\}$. For any $v \in V_*$ define $r_v \in O_V$ by $r_v(v) = -v$, $r_v(v') = v'$ if $v' \in V, (v, v') = 0$ (a reflection).
Let $C(V)$ be the Clifford algebra of $(,)$ that is, the quotient of the tensor algebra of $V$ by the two-sided ideal generated by the elements $v \otimes v' + v' \otimes v - 2(v, v')$ with $v, v' \in V$. As a vector space we have $C(V) = C(V)^+ \oplus C(V)^-$ where $C(V)^+$ (resp. $C(V)^-$) is spanned by the products $v_1 v_2 \ldots v_n$ with $v_i \in V$ and $n$ even (resp. $n$ odd). Note that for $v \in V$, we have $v^2 = 1$ in $C(V)$. Let $\tilde{O}_V$ be the subgroup of the group of invertible elements of $C(V)$ consisting of the elements $v_1 v_2 \ldots v_n$ with $v_i \in V$, $n \geq 0$.

Let $^0\tilde{O}_V = \tilde{O}(V) \cap C(V)^+$, $^1\tilde{O}_V = \tilde{O}_V \cap C(V)^- = \tilde{O}_V - ^0\tilde{O}_V$. Then $^0\tilde{O}_V$ is a subgroup of index $2$ of $\tilde{O}_V$. We set $c = -1 \in C(V)$. If $v \in V$, $(v, v) = -1$, we have $v^2 = -1$ in $C(V)$. Hence $c \in ^0\tilde{O}_V$. If $\xi \in \tilde{O}_V, v \in V$, we have $\xi v \xi^{-1} \in V$. (Indeed, for $\xi = v_1 v_2 \ldots v_n$ as in (a), we have $\xi v \xi^{-1} = (-1)^n r v_1 r v_2 \ldots r v_n$.) Hence $\xi \mapsto [v \mapsto \xi v \xi^{-1}]$ is a homomorphism $\bar{\beta} : \tilde{O}_V \to O_V$ with image $O_V$, if $N$ is even, and $O_V^0$ if $N$ is odd. The kernel of $\beta : ^0\tilde{O}_V \to O_V^0$ is $\{1, c\}$. If $N \geq 2$ we have $^0\tilde{O}_V = O_V^0$. If $N = 1$, $^0\tilde{O}_V$ is of order $2$.

A.3. In the setup of A.2 we assume $G = ^0\tilde{O}_V$. We also assume (until the end of A.8) that $V = \bigoplus_{i \geq 1} V_i$ where $V_i = W_i \otimes E_i$ and $W_i, E_i$ are $C$-vector spaces of dimension $i, m_i$ respectively with given nondegenerate bilinear forms $(,)$ (both symmetric if $i$ is odd, both symplectic if $i$ is even) such that $(w \otimes e, w' \otimes e') = (w, w')(e, e')$ for $w, w' \in W_i, e, e' \in E_i$, and $(V_i, V_j) = 0$ for $i \neq j$.

Let $I = \{i \geq 1; m_i \geq 1\}, I_{odd} = I \cap (2\mathbb{Z} + 1), I_{even} = I \cap (2\mathbb{Z})$. For any $t \geq 2$ let $I_{odd}^t = \{i \in I_{odd}; m_i \geq t\}$. For any $i \in I$ let $\beta_i : \tilde{O}_{V_i} \to O_{V_i}$ be the homomorphism defined like $\beta$ with $V$ replaced by $V_i$. For any $i \in I_{odd}$ let $w_1, w_2, \ldots, w_t$ be an orthonormal basis of $W_i$ with respect to $(,)$.

In this appendix, any product over $I$ or $I_{odd}$ is taken using the order of $I$ or $I_{odd}$ induced from the obvious order on $\mathbb{N}$. The imbedding $\tau_i : I S_{E_i} \to O_{V_i}$, $g \mapsto 1_{V_i} \otimes g$, restricts to an imbedding $I S_{E_i}^0 \to O_{V_i}^0$. Let $\Phi : \prod_{i \in I} I S_{E_i} \to O_V$ be the homomorphism $g \mapsto \bigoplus_{i \in I} \tau_i(g_i)$ (an imbedding); it restricts to an imbedding $\Phi^0 : \prod_{i \in I} I S_{E_i}^0 \to O_V^0$. Let $\bar{R} = O_V^0 \cap$ image of $\Phi$. We have $\text{image of } \Phi^0 = \bar{R}^0$.

For $i \in I_{odd}$ we choose $e_i \in E_i$, and we define $y_i \in O_V$ by $y_i = 1$ on $W_i \otimes E_i$ and $y_i = -1$ on the perpendicular to $W_i \otimes E_i$ in $V$. Thus $y_i = \tau_i(-r_{e_i})$ on $V_i$, $y_i = -1$ on $V_j$ for $j \neq i$. (Here $r_{e_i}$ is a reflection in $E_i$.) We see that $y_i \in$ image of $\Phi$, $y_i^2 = 1$.

Let $g = \bigoplus_{j \in I} \tau_j(t_j) \in O_V$ where $t_j \in I S_{E_j}$. From the definitions we see that for any $i \in I_{odd}$ we have

(a) $y_i g y_i^{-1} = \bigoplus_{j \in I} \tau_j(t_j') \in O_V$ where $t_j' \in I S_{E_j}$ is given by $t_j' = r_{e_i} t_i r_{e_i}$ and $t_j' = t_j$ if $j \neq i$. In particular if $g \in \bar{R}^0$ then $y_i g y_i^{-1} \in \bar{R}_0$. Let $\bar{\Gamma}$ be the (finite abelian) subgroup of $O_V$ generated by $\{y_i; i \in I_{odd}\}$. Let $\bar{\Gamma}^*$ be the subgroup of $\bar{\Gamma}$ consisting of elements which are products of an even number of generators $y_i$. If $I_{odd} = \emptyset$ we have $\bar{\Gamma} = \bar{\Gamma}^* = \{1\}$. If $I_{odd} \neq \emptyset$ then $\bar{\Gamma}^+$ is a subgroup of index $2$ of $\bar{\Gamma}$. We have $\bar{R} = \bar{R}^0 \bar{\Gamma}^+$ and $\bar{R}_0^0 \cap \bar{\Gamma}^+ = \{1\}$. Let
H = \{g \in \mathcal{G}; \beta(x) \in R\}. Then H is of the form H(u) for some u \in \mathcal{G} (as in 1.7) and all H(u) as in 1.7 are obtained up to conjugacy.

A.4. Our next objective is to describe the structure of H, see A.6(e), A.7.

For i \in I we set S_i = 0\tilde{O}_{E_i} if i \in I_{odd} and S_i = I_sE_i if i \in I_{even}. For i \in I let R'_i = \{\xi \in 0\tilde{O}_{V_i}; \beta_i(\xi) \in \tau_i(I_s^0_{E_i})\}. We set R_i = R'_i if i \in I_{odd} and R_i = R'_i if i \in I_{even}.

For i \in I_{odd} let c'_i be -1, viewed as an element of S_i. For i \in I let c_i be -1, viewed as an element of 0\tilde{O}(V_i); note that, if i \in I_{odd}, we have c_i \in R_i.

If i \in I_{odd} then R_i contains

\[x_{i;e,f} = (w^1_i \otimes e) \ldots (w^1_i \otimes e)(w^1_i \otimes f) \ldots (w^1_i \otimes f) = -x_{i;e,f}\]

for any e, f \in E_i. (Indeed, x_{i;e,f} projects to (1_{W_i} \otimes e)(1_{W_i} \otimes f), where r_e, r_f are reflections in E_i.) Since r_e r_f generate \tau_i(O^0_{E_i}), we see that the elements x_{i;e,f} generate R_i.

If i \in I_{odd} and m_i = 1 we have \tau_i(I_s^0_{E_i}) = \{1\} and R_i = \{1, c_i\}; we see that there is a unique isomorphism S_i \simto R_i. Thus \tau_i^0 lifts uniquely to an isomorphism \tilde{t}_i : S_i \simto R_i which carries c'_i to c_i.

If i \in I_{even}^2, from an argument in [L3, 14.3], we see that R_i is connected; being a double covering of O^0_{E_i}, there is a unique isomorphism 0\tilde{O}_{E_i} \simto R_i which induces the identity on O^0_{E_i}. Thus \tau_i^0 lifts uniquely to an isomorphism \tilde{t}_i : S_i \simto R_i which carries c'_i to c_i.

If i \in I_{even} then, since R'_i is a double covering (with kernel \{1, c_i\}) of S_i which is simply connected, we see that R'_i = R_i \cup c_i R_i and that \tau_i lifts uniquely to an isomorphism \tilde{t}_i : S_i \simto R_i.

We set S = \prod_{i \in I} S_i; for any s \in S we write s_i for the i-component of s. Putting together the isomorphisms \tilde{t}_i we get

\[S \simto \prod_{i \in I} R_i \subset \prod_{i \in I} 0\tilde{O}_{V_i},\]

Composing with the homomorphism

(a)

\[\prod_{i \in I} 0\tilde{O}_{V_i} \to \mathcal{G}\]

induced by \prod_{i \in I} C(V_i) \to C(V) (multiplication in C(V)) we obtain the homomorphism \Phi' in the commutative diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\Phi'} & \mathcal{G} \\
\alpha \downarrow & & \beta \downarrow \\
\prod_{i \in I} I_s^0_{E_i} & \xrightarrow{\Phi^0} & O^0_V
\end{array}
\]
where $\alpha$ is the obvious homomorphism.

Let $R$ be the image of $\Phi'$. Clearly, $R$ is a closed subgroup of $G$. We show that $\beta(R) = R^0$. Since $\alpha$ is surjective, this follows from (b) and the fact that the image of $\Phi^0$ is $R^0$. Let

$$A = \{s \in S; s_i = 1 \text{ if } i \in I_{\text{even}}, s_i = c_i^{n_i} \text{ if } i \in I_{\text{odd}}, n_i \in \mathbb{Z}/2, \sum_{i \in I_{\text{odd}}} n_i = 0\}.$$

We show:

\begin{equation}
\text{(c)} \quad \text{Ker}(\Phi') = A.
\end{equation}

From the definitions, if $s \in S$ is given by $s_i = 1$ if $i \in I_{\text{even}}, s_i = c_i^{n_i}$ if $i \in I_{\text{odd}}$ where $n_i \in \mathbb{Z}/2$, then $\Phi'(s) = c_i^{m_i}$ where $m = \sum_{i \in I_{\text{odd}}} n_i$. In particular, we have $A \subset \text{Ker}(\Phi')$. Conversely, let $(\xi_i)_{i \in I} \in S$ be an element in $\text{Ker}(\Phi')$. We set $\tilde{\xi}_i = \tilde{x}_i(\xi_i) \in R_i$. Viewing $\tilde{\xi}_i$ as an element of $C(V_i) \subset C(V)$, we have $\prod_{i \in I} \tilde{\xi}_i = 1$ (product in $C(V)$). Let $I_0$ be the set of all $i \in I$ such that $\tilde{\xi}_i$ is a (nonzero) scalar. From the definition of $C(V)$ we see that, if $b_i^j$ ($j \in [1, 2^{\dim V_i}]$ is a basis of $C(V_i)$, then the products $b^f := \prod_{i \in I} b_i^{f(i)}$ (for various functions $f$ which to each $i \in I$ associate $f(i) \in [1, \dim C(V_i)]$) form a basis of $C(V)$. We can assume that $b_1^1 = \tilde{\xi}_i$ for all $i \in I$ and $b_2^2 = 1$ for all $i \in I - I_0$. Let $f_1, f_2$ be such that $f_1(i) = 1$ for all $i \in I, f_2(i) = 2$ for $i \in I - I_0, f_2(i) = 1$ for $i \in I_0$. We have $b_1^{f_1} = 1, b_2^{f_1} = \lambda$ with $\lambda \in C^*$. Thus $b_2^{f_2} = \lambda b_2^{f_1}$. Since $b_2^{f_2}, b_2^{f_1}$ are part of a basis of $C(V)$, we see that $f_1 = f_2$ hence $I - I_0 = \emptyset$. Thus $\tilde{\xi}_i$ is a nonzero scalar for any $i \in I$. It follows that the element $\tilde{\xi}_i \in \tilde{O}_{V_i}$ satisfies $\beta_i(\tilde{\xi}_i) = 1$ hence $\tilde{\xi}_i = c_i^{n_i}$ (hence $\xi_i = c_i^{n_i}$) for some $n_i \in \mathbb{Z}/2$. If in addition, $i \in I_{\text{even}}$, then $\tilde{\xi}_i \in R_i$ and $c_i \notin R_i$ implies that $n_i = 0$. Now if $i \in I_{\text{odd}}$, then $c_i$ viewed as an element of $C(V)$ is equal to $c$. Hence from $\prod_{i \in I} \tilde{\xi}_i = 1$ (in $C(V)$) it follows that $\sum_{i \in I_{\text{odd}}} n_i = 0$. Thus $\text{Ker}(\Phi') \subset A$. The opposite containment is obvious. This proves (c).

We show:

\begin{itemize}
  \item [(d)] If $I_{\text{odd}}^{\geq 2} \neq \emptyset$ then $R = R^0 = H^0$ and $c \in H^0$.
\end{itemize}

If $i \in I$ and $m_i \geq 2$, then $R_i = R_i^0$. Hence

$$\prod_{i \in I} R_i = \prod_{i \in I; m_i = 1} R_i \times \prod_{i \in I; m_i \geq 2} R_i^0.$$

Thus $R$ is generated by $R^0$ and by $\Phi'(c_i)$ for various $i \in I$ such that $m_i = 1$. To show that $R = R^0$ it is enough to show that if $i \in I$ is such that $m_i = 1$ (hence $i$ is odd) then $\Phi'(c_i) \in R^0$. By assumption we can find $i' \in I_{\text{odd}}^{\geq 2}$. From (c) we see that $\Phi'(c_i^{c_i^{i'}}) = 1$. Hence

$$\Phi'(c_i^{i'}) = \Phi'(c_i^{i'}) \in \Phi'(R_{i'}) = \Phi'(R_{i'}^0) \subset R^0.$$
Thus we have $R = R^0$. Since $\beta(R) = R^0$, $H = \beta^{-1}(R) \cap G$, we have $\dim R = \dim H = \dim \bar{R}$ and $R \subset H$ so that $R^0 = H^0$. From the proof of (c) we see that, if $s \in S$ is given by $s_i = c_i'$ for $i = i'$ (as above), $s_i = 1$ for $i \in I - \{i', i\}$, then $\Phi'(s) = c$. Thus, $c \in R$. Since $R = H^0$ we see that $c \in H^0$; (d) is proved.

We show:

(e) If $I_{odd}^2 \neq \emptyset$ then $\beta$ induces an isomorphism $H/H^0 \sim R/\bar{R}^0 = baG^+$. Hence $H/H^0$ is a finite abelian group of order $2^{I_{odd}^2-1}$.

From A.3 we have $\bar{R}/\bar{R}^0 = \Gamma^+$. From the definitions, $\beta$ induces a surjective homomorphism $H \to \bar{R}$ with kernel $\{1, c\}$. It is then enough to note that $\{1, c\} \subset H^0$ (see (d)).

A.5. Let $i \in I_{odd}$. We define an automorphism $a_i : S \to S$ by $s \mapsto s'$ where $s' = e_i s_i e_i^{-1}$ (product in $O_{E_i}$), $s_j = s_j$ for $j \neq i$. Since $a_i$ induces the identity map on $\text{Ker}(\Phi')$ (see A.4(c)), it follows that $a_i$ induces an automorphism $a_i' : R \to R$ (recall that $R$ is the image of $\Phi'$). We set

$$\tilde{y}_i = (w_i^1 \otimes e_i) \ldots (w_i^k \otimes e_i) \in \tilde{O}_V.$$  

We show:

(a) For any $\tilde{g} \in R$ we have $\tilde{y}_i \tilde{g} \tilde{y}_i^{-1} = a_i'(\tilde{g}) \in R$.

Assume first that $m_i = 1$. We have $a_i = 1$ (in this case, $e_i$ is in the centre of $O_{E_i}$). Now $R_i$ is generated by $x_{i; e_i, e_i} = \tilde{y}_i^2$ (see A.4) and by $c_i$ hence $\tilde{y}_i$ commutes with any element of $R_i$; it also commutes with any element of $R_j, j \neq i$ hence $\tilde{y}_i \tilde{g} \tilde{y}_i^{-1} = \tilde{g}$. Thus (a) is proved in this case. Next we assume that $m_i \geq 2$. We have $\beta(\tilde{y}_i) = y_i$ hence

$$\beta(\tilde{y}_i \tilde{g} \tilde{y}_i^{-1}) = y_i \beta(\tilde{g}) y_i^{-1} \in y_i \bar{R}^0 y_i^{-1} = \bar{R}^0$$

(see (4.3(a))). Thus, $\tilde{y}_i \bar{R} \tilde{y}_i^{-1} \in \beta^{-1}(\bar{R}^0) \cap G \subset H$. Since $R = H^0$, see A.4(d), we see that $\tilde{y}_i \bar{R} \tilde{y}_i^{-1} = \tilde{y}_i H^0 \tilde{y}_i^{-1}$ is a connected subgroup of $H$. It follows that $\tilde{y}_i \bar{R} \tilde{y}_i^{-1} = H^0 = R$. Thus $Ad(\tilde{y}_i)$ is an automorphism of $R$. From the definition and A.3(a) we have $\beta(a_i'(\tilde{g})) = y_i \beta(\tilde{g}) y_i^{-1} = \beta(\tilde{y}_i \tilde{y}_i^{-1})$. Thus $\tilde{y}_i \tilde{y}_i^{-1} = a_i'(\tilde{g}) f(\tilde{g})$ where $f : R \to \ker \beta$ is a morphism. Now $R$ is connected (see A.4(d)) and $\ker \beta$ is finite hence $f$ is constant. Clearly, $f(1) = 1$. It follows that $f(\tilde{g}) = 1$ for any $\tilde{g} \in R$, proving (a).

A.6. Let $\Delta$ be the group defined by the generators $z_i (i \in I_{odd})$ and $c_*$ with relations: $c_* = 1$, $c_* z_i = z_i c_*$, $z_i^2 = c_*^{(1-1)/2}$ for $i \in I_{odd}$ and $z_i z_i' = c_* z_i' z_i$ for $i \neq i'$ in $I_{odd}$. (If $I_{odd} = \emptyset$, $\Delta$ is the group of order 2 with generator $c_*$.). From the definition we see that any element of $\Delta$ can be written in the form $c_* \prod_{i \in I_{odd}} z_i^n$ where $n_i \in \mathbb{Z}/2, n \in \mathbb{Z}/2$. It follows that $|\Delta| \leq 2^{I_{odd}^2+1}$. The assignment $c_* \mapsto c$, $z_i \mapsto \tilde{y}_i = (w_i^1 \otimes e_i) \ldots (w_i^k \otimes e_i) \in \tilde{O}_V$ defines a homomorphism $\zeta : \Delta \to \tilde{O}_V$. For any $i \in I_{odd}$ we have $\beta(\tilde{y}_i) = y_i$. Thus $\beta(\zeta(\Delta))$ is the subgroup of $O_V$ generated by $y_i (i \in I_{odd})$, an (abelian) group of order $2^{I_{odd}^2+1}$. Since $\zeta(\Delta)$ contains $c$ which is in
Let $s$ be such that $s \delta \in \ker(\zeta)$. We write $\delta = c^n \prod_{i \in I_{\text{odd}}} z_i^{n_i}$ where $u_i \in \mathbb{Z}/2, n \in \mathbb{Z}/2$. We set $\xi_i = \tilde{t}_i(s_i) \in R_i$.

Assume first that $I_{\text{odd}} \neq \emptyset$. We have $1 = c^n \prod_{i \in I} \xi_i \prod_{i \in I_{\text{odd}}} \tilde{y}_i^{n_i}$ hence $\prod_{i \in I} f_i = \text{power of } c$, where $f_i = \tilde{t}_i \tilde{y}_i^{n_i}$ for $i \in I_{\text{odd}}, f_i = \xi_i$ for $i \in I_{\text{even}}$. We have $f_i \in C(V_i)^{u_i}$ if $i \in I_{\text{odd}}$ and $f_i \in C(V_i)^0$ if $i \in I_{\text{even}}$. Now the subspaces $\prod_{i \in I} C(V_i)^{u_i}$ of $C(V)$ (with $r_i \in \{0, 1\}$) form a direct sum decomposition of $C(V)$. It follows that $u_i = 0$ for any $i \in I_{\text{odd}}$. Thus we have $c^n \prod_{i \in I} \xi_i = 1$. If $n = 0$ then from the previous equality we deduce as in the proof of A.5(c) that $s_i = c_i^{n_i}$ if $i \in I_{\text{odd}}, s_i = 1$ if $i \in I_{\text{even}}$, where $n_i \in \mathbb{Z}/2, \sum_{i \in I_{\text{odd}}} n_i = 0$. If $n = 1$ we pick $i_0 \in I_{\text{odd}}$ and we define $s' \in S$ by $s'_{i_0} = c_i^n s_{i_0}$, $s'_i = s_i$ if $i \in I \setminus \{i_0\}$. Then $\zeta(s') = 1$ from which we deduce as above that $s'_i = c_i^{n_i}$ if $i \in I_{\text{odd}}$ with $n_i \in \mathbb{Z}/2, \sum_{i \in I_{\text{odd}}} n_i = 0$. Hence $s'_i = c_i^{n_i}$ for $i \in I_{\text{odd}}, s_i = 1$ for $i \in I_{\text{even}}$ with $n'_i \in \mathbb{Z}/2, \sum_{i \in I_{\text{odd}}} n'_i = 1$. We see that $\ker \zeta \subset A'$. The opposite containment is obvious. This completes the proof when $I_{\text{odd}} \neq \emptyset$.

Assume next that $I_{\text{odd}} = \emptyset$. We have $\Delta = \{1, c_+\}$. In our case the homomorphism $\alpha$ in A.4(b) defines a connected covering of a simply connected group hence is an isomorphism; since $\Phi^0$ in A.4(b) is injective it follows that $\beta$ in A.4(b) is injective when restricted to $R$, hence $c \notin R$. We have $1 = c^n \prod_{i \in I} \xi_i$. Hence $c^n \in R$. Since $c \notin R$ it follows that $n = 0$. Thus, $1 = \prod_{i \in I} \xi_i$. From this we deduce as in the proof of A.4(c) that $s = 1$. This completes the proof of (a).

Let $\Delta^+$ be the subgroup of $\Delta$ consisting of the elements of the form $c^n \prod_{i \in I_{\text{odd}}} z_i^{n_i}$ where $n_i \in \mathbb{Z}/2, n \in \mathbb{Z}/2, \sum_{i \in I_{\text{odd}}} n_i = 0$. We have $|\Delta^+| = 2^{\left| I_{\text{odd}} \right|}$ if $I_{\text{odd}} \neq \emptyset$, $|\Delta^+| = 2$ if $I_{\text{odd}} = \emptyset$. We regard the semidirect product $S \cdot \Delta^+$ as a subgroup of $S \cdot \Delta$ in an obvious way. Let $\zeta_0 : S \cdot \Delta^+ \rightarrow \mathcal{G}$ be the restriction of $\zeta : S \cdot \Delta \rightarrow \tilde{O}_V$. From (a) we deduce:
(b) The kernel of \( \zeta_0 : S \cdot \Delta^+ \to \mathcal{G} \) is equal to \( A' \). The image of \( \zeta_0 \) is the subgroup of \( \mathcal{G} \) generated by \( R \), by \( \tilde{y}_i \tilde{y}_i' \) \( (i, i' \in I_{odd}) \) and by \( c \). Hence it is equal to \( H \). We see that \( \zeta_0 \) defines an isomorphism

\[
(S \cdot \Delta^+)/A' \overset{\sim}{\to} H.
\]

If \( I_{odd} \neq \emptyset \), we can identify

\[
(S \cdot \Delta^+)/A' = (S^0 \cdot \Delta^+)/A'_1
\]

where

\[
A'_1 = \{sc^n_i; s \in S, n \in \mathbb{Z}/2, s_i = c^n_i, \text{ if } i \in I_{odd}^{\geq 2}, s_i = 1 \text{ if } i \in I_{even} \cup (I_{odd} - I_{odd}^{\geq 2}), n_i \in \mathbb{Z}/2, n = \sum_{i \in I_{odd}^{\geq 2}} n_i \}.
\]

We show:

(e) If \( I_{odd}^{\geq 2} = \emptyset \) then \( H = S^0 \cdot \Delta^+ \); hence \( H/H^0 = \Delta^+ \).

Assume first that \( I_{odd} = \emptyset \). Then \( A' = \{1\}, \Delta^+ = \{1, c_*\}, S = S^0 \) and (c) becomes \( S^0 \cup S^0 c_* \overset{\sim}{\to} H \); thus (e) holds. If \( I_{odd} \neq \emptyset \) and \( I_{odd}^{\geq 2} = \emptyset \) then we have \( A'_1 = \{1\} \) so that (e) follows from (d).

A.7. In this subsection we assume that \( I_{odd}^{\geq 2} \neq \emptyset \). Using A.6(c),(d) we see that in this case we have \( H^0 = S^0/A'_2 \) where

\[
A'_2 = \{s \in S; s_i = c^n_i, \text{ if } i \in I_{odd}^{\geq 2}, s_i = 1 \text{ if } i \in I_{even} \cup (I_{odd} - I_{odd}^{\geq 2}), n_i \in \mathbb{Z}/2, \sum_{i \in I_{odd}^{\geq 2}} n_i = 0 \},
\]

so that for \( i \in I_{odd}^{\geq 2} \), the image of \( c_i' \) in \( H^0 \) is independent of \( i \); we denote it by \( c' \). Note that the \( \Delta^+ \) action on \( S^0 \) induces an action of \( \Delta^+ \) on \( S^0/A'_2 \) and we can form the semidirect product \( (S^0/A'_2) \cdot \Delta^+ \). From A.6(c),(d) we see that \( H = ((S^0/A'_2) \cdot \Delta^+)/\{1, c'c_*\} \). Now \( (S^0/A'_2)_{der} \) is the image of \( (S^0)_{der} = \prod_{i \in I_{even} \cup I_{odd}^{\geq 3}} S_i \) under \( p : S^0 \to (S^0/A'_2)_{der} \). The condition that \( (S^0/A'_2)_{der} \) is simply connected is that the restriction of \( p \) to \( (S^0)_{der} \) has trivial kernel; that kernel is

\[
A'_2 \cap (S^0)_{der} = \{s \in S; s_i = c^n_i, \text{ if } i \in I_{odd}^{\geq 3}, s_i = 1 \text{ if } i \in I_{even} \cup (I_{odd} - I_{odd}^{\geq 3}), n_i \in \mathbb{Z}/2, \sum_{i \in I_{odd}^{\geq 2}} n_i = 0 \},
\]

and this is trivial precisely when \( |I_{odd}^{\geq 3}| \leq 1 \).
A.8. From the results in A.6, A.7 we see that \((H^0)_{\text{der}}\) is simply connected if and only if \(|I_{\text{odd}}| \leq 1\).

A.9. Let \(V\) be a \(\mathbb{C}\)-vector space of dimension \(N \geq 4\) with a fixed symmetric nondegenerate symplectic form \((,): V \times V \to \mathbb{C}\). Let \(\mathcal{G} = Is_V\). We assume that \(V = \bigoplus_{i \geq 1} V_i\) where \(V_i = W_i \otimes E_i\) and \(W_i, E_i\) are \(\mathbb{C}\)-vector spaces of dimension \(i, m_i\) respectively with given nondegenerate bilinear forms \((,)\) (so that if \(i\) is odd, \((,)\) is symmetric for \(W_i\) and symplectic for \(E_i\); if \(i\) is even, \((,)\) is symmetric for \(E_i\) and symplectic for \(W_i\)) such that \((w \otimes e, w' \otimes e') = (w, w')(e, e')\) for \(w, w' \in W_i, e, e' \in E_i\) and \((V_i, V_j) = 0\) for \(i \neq j\). Let \(I = \{i \geq 1; m_i \geq 1\}\), \(I_{\text{odd}} = I \cap (2\mathbb{Z} + 1)\), \(I_{\text{even}} = I \cap (2\mathbb{Z})\).

We have an imbedding \(\tau_i : Is_{E_i} \to Is_{V_i}, g \mapsto 1_{W_i} \otimes g\). Let \(\Phi : \prod_{i \in I} Is_{E_i} \to \mathcal{G}\) be the homomorphism \((g_i) \mapsto \bigoplus_{i \in I} \tau_i(g_i)\) (an imbedding); it restricts to an imbedding \(\Phi^0 : \prod_{i \in I} Is_{E_i}^0 \to \mathcal{G}\). Let \(H = \text{image of } \Phi\). Then \(H\) is of the form \(H(u)\) for some \(u \in \mathcal{G}\) (as in 1.7) and all \(H(u)\) as in 1.7 are obtained up to conjugacy. We have image of \(\Phi^0 = H^0\).

We see that \((H^0)_{\text{der}}\) is simply connected if and only if the number of \(i \in I_{\text{even}}\) such that \(m_i \geq 3\) is 0.

A.10. In this subsection we assume that \(L\) is the centralizer of a torus \(S\) in \(\mathcal{G}\), that \(u \in L\) and that \((H^0)_{\text{der}}\) is simply connected. Let \(\mathcal{V}'\) be the unipotent radical of \(Z_L(u)^0\); let \(\mathcal{V}''\) be the unipotent radical of \(Z_{L_{\text{der}}}(u)^0\). We show:

(a) \(Z_L(u) = H' \mathcal{V}'\) (semidirect product) where \(H'\) is reductive and \((H^{0'})_{\text{der}}\) is simply connected;
(b) \(Z_{L_{\text{der}}}(u) = H'' \mathcal{V}''\) (semidirect product) where \(H''\) is reductive and \((H^{0''})_{\text{der}}\) is simply connected.

We can assume that \(S \subset H\). We have \(Z_L(u) = Z_\mathcal{G}(u) \cap Z_\mathcal{G}(S) = HV \cap Z_\mathcal{G}(S) = Z_H(S)Z_\mathcal{V}(S)\). Thus we can assume that \(H' = Z_H(S)\) so that \(H^{0'} = Z_{H^0}(S)\).

Since \((H^0)_{\text{der}}\) is simply connected, it follows that \((Z_{H^0}(S))_{\text{der}}\) is simply connected, proving (a). Let \(T\) be the connected centre of \(L\). We have \(L = L_{\text{der}}T\). Clearly, \(Z_L(u) = Z_{L_{\text{der}}}(u)T\). Thus we can assume that \(H' = H''T\) so that \(H^{0'}, H^{0''}\) have the same derived group, proving (b).

A.11. In the remainder of this appendix we assume that \(\mathcal{G}\) (in 1.7) is of exceptional type and \(u, H(u)\) are as in 1.7. As G. Seitz pointed out to the author, the structure of \(H = H(u)\) can in principle be extracted from [LS]. More precisely, the tables [LS, 22.3.1-22.3.5] contain information on the structure of the Lie algebra of \(H^0\), on the structure of \(H/H^0\) and the action of \(H^0\) on the Lie algebra of \(\mathcal{G}\). From this one can recover the precise structure of \(H^0\); using in addition the tables [LS, 22.2.1-22.2.6] one can recover the structure of the extension \(1 \to H^0 \to H \to H/H^0 \to 1\). (When \(\mathcal{G}\) is of type \(E_7\) or \(E_6\) then, in addition to the corresponding tables in [LS] for the corresponding adjoint groups, we must use the tables for \(E_8\) by regarding \(\mathcal{G}\) as a derived subgroup of a Levi subgroup of a parabolic subgroup of a group of type \(E_8\); we also use A.10.) In this way we find the following results. If \(\mathcal{G}\) is of type \(E_6\) or \(G_2\) then \((H^0)_{\text{der}}\) is simply connected.
Assume now that \( G \) is of type \( F_4 \). If \( u \) is of type \( A_1 \tilde{A}_1 \) (notation of \([LS]\)) then \( H = H^0 = PGL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \). If \( u \) is of type \( B_3 \) (notation of \([LS]\)) then \( H = H^0 = PGL_2(\mathbb{C}) \). For all other \( u \), \((H^0)_{der}\) is simply connected.

Assume now that \( G \) is of type \( E_7 \). If \( u \) is of type \( A_2A_1 \tilde{A}_1 \) (notation of \([LS]\)) then \( H = H^0 = SL_2(\mathbb{C})^3/\{\pm 1\} \) with \( \{\pm 1\} \) imbedded diagonally in the centre of \( SL_2(\mathbb{C})^3 \). For all other \( u \), \((H^0)_{der}\) is simply connected.

Assume now that \( G \) is of type \( E_8 \). If \( u \) is of type \( A_2A_1 \tilde{A}_1 \) (notation of \([LS]\)) then \( H = H^0 = SL_2(\mathbb{C})^2/\{\pm 1\} \) with \( \{\pm 1\} \) imbedded diagonally in the centre of \( SL_2(\mathbb{C})^2 \). If \( u \) is of type \( D_4(a_1)A_2 \) (notation of \([LS]\)) then \( H = PGL_3(\mathbb{C}) \cdot \mathbb{Z}/2 \) (semidirect product with the generator of \( \mathbb{Z}/2 \) acting on \( PGL_3(\mathbb{C}) \) by an outer involution). If \( u \) is of type \( D_5(a_1)A_1 \) (notation of \([LS]\)) then \( H = H^0 = PGL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \). If \( u \) is of type \( A_6 \) (notation of \([LS]\)) then \( H = H^0 = SL_2(\mathbb{C})^2/\{\pm 1\} \) with \( \{\pm 1\} \) imbedded diagonally in the centre of \( SL_2(\mathbb{C})^2 \). For all other \( u \), \((H^0)_{der}\) is simply connected.

References


