Lagrangian caps

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Lagrangian caps

Yakov Eliashberg∗  Emmy Murphy
Stanford University  MIT

Abstract

We establish an $h$-principle for exact Lagrangian embeddings with concave Legendrian boundary. We prove, in particular, that in the complement of the unit ball $B$ in the standard symplectic $\mathbb{R}^{2n}$, $2n \geq 6$, there exists an embedded Lagrangian $n$-disc transversely attached to $B$ along its Legendrian boundary.

1 Introduction

Question. Let $B$ be the round ball in the standard symplectic $\mathbb{R}^{2n}$. Is there an embedded Lagrangian disc $\Delta \subset \mathbb{R}^{2n} \setminus \text{Int } B$ with $\partial \Delta \subset \partial B$ such that $\partial \Delta$ is a Legendrian submanifold and $\Delta$ transversely intersects $\partial B$ along its boundary?

If $n = 2$ then such a Lagrangian disc does not exist. Indeed, it is easy to check that the existence of such a Lagrangian disc implies that the Thurston-Bennequin invariant $\text{tb}(\partial \Delta)$ of the Legendrian knot $\partial \Delta \subset S^3$ is equal to $+1$. On the other hand, the knot $\partial \Delta$ is sliced, i.e its 4-dimensional genus is equal to 0. But then according to Lee Rudolph’s slice Bennequin inequality \cite{8} we should have $\text{tb}(\partial \Delta) \leq -1$, which is a contradiction.

As far as we know no such Lagrangian discs have been previously constructed in higher dimensions either. We prove in this paper that if $n > 2$ such discs exist in abundance. In particular, we prove

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Theorem 1.1. Let $L$ be a smooth manifold of dimension $n > 2$ with non-empty boundary such that its complexified tangent bundle $T(L) \otimes \mathbb{C}$ is trivial. Then there exists an exact Lagrangian embedding $f : (L, \partial L) \to (\mathbb{R}^{2n} \setminus \text{Int } B, \partial B)$ with $f(\partial \Delta) \subset \partial B$ such that $f(\partial \Delta) \subset \partial B$ is a Legendrian submanifold and $f$ transverse to $\partial B$ along the boundary $\partial L$.

Note that the triviality of the bundle $T(L) \otimes \mathbb{C}$ is a necessary (and according to Gromov’s $h$-principle for Lagrangian immersions, \cite{suff} sufficient) condition for existence of any Lagrangian immersion $L \to \mathbb{C}^n$.

In fact, we prove a very general $h$-principle type result for Lagrangian embeddings generalizing this claim, see Theorem 2.2 below. As corollaries of this theorem we get

- an $h$-principle for Lagrangian embeddings in any symplectic manifold with a unique conical singular point, see Corollary 6.1
- a general $h$-principle for embeddings of flexible Weinstein domains, see Corollary 6.3
- construction of Lagrangian immersions with minimal number of self-intersection points; this is explored in a joint paper of the authors with T. Ekholm and I. Smith, \cite{joint}.

Theorem 2.2 together with the results from the book \cite{book} yield new examples of rationally convex domains in $\mathbb{C}^n$, which will be discussed elsewhere. The authors are thankful to Stefan Nemirovski, whose questions concerning this circle of questions motivated the results of the current paper.

2 Main Theorem

Loose Legendrian submanifolds

Let $(Y, \xi)$ be a $(2n-1)$-dimensional contact manifold. Let us recall that each contact plane $\xi_y, y \in Y$, carries a canonical linear symplectic structure defined up to a scaling factor. Thus, there is a well defined class of isotropic and, in particular, Lagrangian linear subspaces of $\xi_y$. Given a $k$-dimensional, $k \leq n - 1$, manifold $\Lambda$, an injective homomorphism $\Phi : T\Lambda \to TY$ covering a map $\phi : \Lambda \to Y$ is called isotropic (or if $k = n - 1$ Legendrian) if $\Phi(T\Lambda) \subset \xi$ and $\Phi(Tx\Lambda) \subset \xi_{\phi(x)}$ is isotropic for each $x \in \Lambda$.

Given a $(2n-1)$-dimensional contact manifold $(Y, \xi)$, an embedding $f : \Lambda \to Y$ is called isotropic if it is tangent to $\xi$; if in addition $\dim \Lambda = n - 1$ then it is called
Legendrian. The differential of an isotropic (resp. Legendrian) embedding is an isotropic (resp. Legendrian) homomorphism.

Two Legendrian embeddings $f_0, f_1 : \Lambda \to Y$ are called formally Legendrian isotopic if there exists a smooth isotopy $f_t : \Lambda \to Y$ connecting $f_0$ and $f_1$ and a 2-parametric family of injective homomorphisms $\Phi^s : T\Lambda \to TY$, such that $\Phi^0 = df_0, \Phi^1 = df_1$, and $\Phi^s_t$ is a Legendrian homomorphism ($s, t \in [0, 1]$).

The results of this paper essentially depend on the theory of loose Legendrian embeddings developed in [7]. This is a class of Legendrian embeddings into contact manifolds of dimension $> 3$ which satisfy a certain form of an $h$-principle. For the purposes of this paper we will not need a formal definition of loose Legendrian embeddings, but instead just describe their properties.

Let $\mathbb{R}^{2n-1}_{\text{std}} := (\mathbb{R}^{2n-1}, \xi_{\text{std}} = \{dz - \sum_{i=1}^{n-1} y_i dx_i = 0\})$ be the standard contact $\mathbb{R}^{2n-1}$, $n > 2$, and $\Lambda_0 \subset \mathbb{R}^{2n-1}_{\text{std}}$ be the Legendrian $\{z = 0, y_i = 0\}$. Note that a small neighborhood of any point on a Legendrian in a contact manifold is contactomorphic to the pair $(\mathbb{R}^{2n-1}_{\text{std}}, \Lambda_0)$. There is another Legendrian $\tilde{\Lambda}$, called the universal loose Legendrian, which is equal to $\Lambda_0$ outside of a compact subset, and formally Legendrian isotopic to it. A picture of $\tilde{\Lambda}$ is given in Figure 2.1 though we do not use any properties of $\Lambda$ besides those stated above. A connected Legendrian submanifold $\Lambda \subset Y$ is called loose, if there is a contact embedding $(\mathbb{R}^{2n-1}_{\text{std}}, \tilde{\Lambda}) \to (Y, \Lambda)$. We refer the interested readers to the paper [7] and the book [1] for more information. The following proposition summarizes the properties of loose Legendrian embeddings.

**Proposition 2.1.** For any contact manifold $(Y, \xi)$ of dimension $2n - 1 > 3$ the set of connected loose Legendrians have the following properties:

(i) For any Legendrian embedding $f : \Lambda \to Y$ there is a loose Legendrian embedding $\tilde{f} : \Lambda \to Y$ which coincides with $f$ outside an arbitrarily small neighborhood of a point $p \in \Lambda$ and which is formally isotopic to $f$ via a formal Legendrian isotopy supported in this neighborhood.

(ii) Let $f_0, f_1 : \Lambda \to Y$ be two loose Legendrian embeddings of a connected $\Lambda$ which coincide outside a compact set and which are formally Legendrian isotopic via a compactly supported isotopy. Then $f_0, f_1$ are Legendrian isotopic via a compactly supported Legendrian isotopy.

(iii) Let $f_1 : \Lambda \to Y$, $t \in [0, 1]$, be a smooth isotopy which begins with a lose Legendrian embedding $f_0$. Then it can be $C^0$-approximated by a Legendrian isotopy $\tilde{f}_t : \Lambda \to Y$, $t \in [0, 1]$, beginning with $\tilde{f}_0 = f_0$. 
Statement (i) is the **Legendrian stabilization** construction which replaces a small neighborhood of a point on a Legendrian submanifold by the model \((\mathbb{R}^{2n-1}, \~\Lambda)\). It was first described for \(n > 2\) in [3]. The main part of Proposition 2.1, parts (ii) and (iii), are proven in [7]. Notice that (ii) implies that if a Legendrian is already loose that any further stabilizations do not change its Legendrian isotopy class.

**Symplectic manifolds with negative Liouville ends**

Throughout the paper we use the terms **closed submanifold** and **properly embedded submanifold** as synonyms, meaning a submanifold which is a closed subset, but not necessarily a closed manifold itself.

Let \(L\) be an \(n\)-dimensional smooth manifold. A **negative end** structure on \(L\) is a choice of

- a codimension 1 submanifold \(\Lambda \subset L\) which divides \(L\) into two parts: \(L = L_- \cup L_+\), \(L_- \cap L_+ = \Lambda\), and

- a non-vanishing vector field \(S\) on \(\mathcal{O}p \ L_- \subset L\) which is outward transverse to
the boundary $\Lambda = \partial \mathcal{L}^-$, and such that the negative flow $S^{-t} : \mathcal{L}^- \to \mathcal{L}^-$ is defined for all $t$ and all its trajectories intersect $\Lambda$.

In other words, there is a canonical diffeomorphism $\mathcal{L}^- \to (-\infty, 0] \times \Lambda$ which is defined by sending the ray $(-\infty, 0] \times x$, $x \in \Lambda$, onto the trajectory of $-S$ originated at $x \in \Lambda$.

Alternatively, the negative end structure can be viewed as a negative completion of the manifold $\mathcal{L}^+$ with boundary $\Lambda$:

$$L = \mathcal{L}^+ \cup \bigcup_{0 \times \lambda \in \Lambda \setminus (0, x) \sim x \in \Lambda} (-\infty, 0] \times \Lambda.$$  

Negative end structures which differ by a choice of the cross-section $\Lambda$ transversely intersecting all the negative trajectories of $L$ will be viewed as equivalent.

Let $(X, \omega)$ be a $2n$-dimensional symplectic manifold. A properly embedded co-oriented hypersurface $Y \subset X$ is called a contact slice if it divides $X$ into two domains $X = X_- \cup X_+$, $X_- \cap X_+ = Y$, and there exists a Liouville vector field $Z$ in a neighborhood of $Y$ which is transverse to $Y$, defines its given co-orientation and points into $X_+$. Such hypersurfaces are also called symplectically convex [4], or of contact type [9].

If the Liouville field extends to $X_-$ as a non-vanishing Liouville field such that the negative flow $Z^{-t}$ is defined for all $t \geq 0$ and all its trajectories in $X_-$ intersect $Y$ then $X_-$ with a choice of such $Z$ is called a negative Liouville end structure of the symplectic manifold $(X, \omega)$.

The restriction $\alpha$ of the Liouville form $\lambda = i(Z) \omega$ to $Y$ is a contact form on $Y$ and the diffeomorphism $(-\infty, 0] \times Y \to X_-$ which sends each ray $(-\infty, 0] \times x$ onto the trajectory of $-Z$ originated at $x \in \Lambda$ is a Liouville isomorphism between the negative symplectization $((-\infty, 0] \times Y, d(\alpha))$ of the contact manifold $(Y, \{\alpha = 0\})$ and $(X_-, \lambda)$. Hence alternatively the negative Liouville end structure can be viewed as a negative completion of the manifold $X_+$ with the negative contact boundary $Y$, i.e. as an attaching the negative symplectization $((-\infty, 0] \times Y, d(\alpha))$ of the contact manifold $(Y, \{\alpha = 0\})$ to $X_+$ along $Y$.

A negative Liouville end structure which differs by another choice of the cross-section $Y$ transversely intersecting all negative trajectories of $X$ will be viewed as an equivalent one. Note that the holonomy along trajectories of $X$ provides a contactomorphism between any two transverse sections. Any such transverse section will be called a contact slice.

If the symplectic form $\omega$ is exact and the Liouville form $\lambda$ is extended as a Liouville
form, still denoted by $\lambda$, to the whole manifold $X$, then we will call $(X, \lambda)$ a Legendrian embedding.

Let $L$ be an $n$-dimensional manifold with a negative end, and $X$ a symplectic $2n$-manifold with a negative Liouville end. A proper Lagrangian immersion $f : L \to X$ is called cylindrical at $-\infty$ if it maps the negative end $L_-$ of $L$ into a negative end $X_-$ of $X$, the restriction $f|_{L_-}$ is an embedding, and the differential $df|_{T_{L_-}}$ sends the vector field $S$ to $Z$. Composing the restriction of $f$ to a transverse slice $\Lambda$ with the projection of the negative Liouville end of $X$ to $Y$ along trajectories of $Z$ we get a Legendrian embedding $f_{-\infty} : \Lambda \to Y$, which will be called the asymptotic negative boundary of the Lagrangian immersion $f$.

The action class

Given a proper Lagrangian immersion $f : L \to X$, we consider its mapping cylinder $C_f = L \times [0, 1] \cup_{(x, 1) \sim f(x)} X$, which is homotopy equivalent to $X$, and denote respectively by $H^2(X, f)$ and $H^2(X, f)$ the 2-dimensional cohomology groups $H^2(C_f, L \times 0)$ and $H^2(C_f, L \times 0) := \lim_{K \subset C_f} H^2(C_f \setminus K, (L \times 0) \setminus K)$, where the direct limit is taken over all compact subsets $K \subset C_f$. We denote by $r_{-\infty}$ the restriction homomorphism $r_{-\infty} : H^2(X, f) \to H^2(X, f)$. If $f$ is an embedding then $H^2(X, f)$ and $H^2(X, f)$ are canonically isomorphic to $H^2(X, f(L))$ and $H^2(X, f(L)) := \lim_{K \subset X} H^2(X \setminus K, f(L) \setminus K)$, respectively. We define the relative action class $A(f) \in H^2(X, f)$ of a proper Lagrangian immersion $f : L \to X$ as the class defined by the closed 2-form which is equal $\omega$ on $X$ and to 0 on $L \times 0$. We say that $f$ is weakly exact if $A(f) = 0$. The relative action class at infinity $A_{-\infty}(f) \in H^2_{-\infty}(X, f)$ is defined as $A_{-\infty}(f) := r_{-\infty}(A_{-\infty})$. We note we have $A_{-\infty}(f) = A_{-\infty}(g)$ if Lagrangian immersions $f, g$ coincide outside a compact set.

Consider next a compactly supported Lagrangian regular homotopy, $f_t : L \to X$, $0 \leq t \leq 1$, and write $F : L \times [0, 1] \to X$, for $F(x, t) = f_t(x)$. Let $\alpha$ denote the 1-form on $L \times [0, 1]$ defined by the equation $\alpha := \frac{\partial}{\partial t}(F^*\omega)$, where $t$ is the coordinate on the second factor of $L \times [0, 1]$. Then the restrictions $\alpha_t := \alpha|_{L \times \{t\}}$ are closed for all $t \in [0, 1]$. We call the Lagrangian regular homotopy $f_t$ a Hamiltonian regular homotopy if the cohomology class $[\alpha_t] \in H^1(L)$ is independent of $t$. It is straightforward to verify that for a Hamiltonian regular homotopy $f_t$ the action class $A(f_t)$ remains constant. Note, however, that the converse is not necessarily true.

If $X$ is a Liouville manifold, then we define the absolute action class $a(f) \in H^1(L)$
as the class of the closed form $f^*\lambda$, and call a Lagrangian immersion $f$ \textit{exact} if $a(f) = 0$. Note that in that case we have $\delta(a(f)) = A(f)$, where $\delta$ is the boundary homomorphism $H^1(L) \to H^2(X, f)$ from the exact sequence of the pair $(C_f, L \times 0)$. We will also use the notation

$$H^1_\infty(L) := \lim_{K \subset L} H_1(L \setminus K), \quad r_\infty : H^1(L) \to H^1_\infty(L), \quad a_\infty(f) = r_\infty(a(f)).$$

If the the immersion $f$ is cylindrical at $-\infty$ then the class $a_\infty(f)$ vanishes on $L_-$. 

\textbf{Statement of main theorems}

We say that a symplectic manifold $X$ has infinite Gromov width if an arbitrarily large ball in $\mathbb{R}^{2n}$ admits a symplectic embedding into $X$. For instance, a complete Liouville manifold have infinite Gromov width.

\textbf{Theorem 2.2.} Let $f : L \to X$ be a cylindrical at $-\infty$ proper embedding of an $n$-dimensional, $n \geq 3$, connected manifold $L$, such that its asymptotic negative Legendrian boundary has a component which is loose in the complement of the other components. Suppose that there exists a compactly supported homotopy of injective homomorphisms $\Psi_t : TL \to TX$ covering $f$ and such that $\Psi_0 = df$ and $\Psi_1$ is a Lagrangian homomorphism. If $n = 3$ assume, in addition, that the manifold $X \setminus f(L)$ has infinite Gromov width. Then given a cohomology class $A \in H^2(X, f(L))$ with $r_\infty(A) = A_\infty(f)$, there exists a compactly supported isotopy $f_t : L \to X$ such that

- $f_0 = f$;
- $f_1$ is Lagrangian;
- $A(f_1) = A$ and
- $df_1 : TL \to TX$ is homotopic to $\Phi_1$ through Lagrangian homomorphisms.

If $X$ is a Liouville manifold with a negative contact end, then one can in addition prescribe any value $a \in H^1(L)$ to the absolute action class $a(f_1)$ provided that $r_\infty(a) = a_\infty$, and in particular make the Lagrangian embedding $f_1$ exact.

We do not know whether the infinite width condition when $n = 3$ is really necessary, or it is just a result of deficiency of our method.
Suppose we are given a smooth proper immersion \( f : L^n \to X^{2n} \) with only transverse double points and which is an embedding outside of a compact subset. If \( L \) is connected, \( L \) is orientable and \( X \) is oriented and \( n \) is even, we define the relative self-intersection index of \( f \), denoted \( I(f) \), to be the signed count of intersection points, where the sign of an intersection \( f(p^0) = f(p^1) \) is +1 or −1 depending on whether the orientation defined by \( (df_{p^0}(L), df_{p^1}(L)) \) agrees or disagrees with the orientation on \( X \). Because \( n \) is even, this sign does not depend on the ordering \( (p^0, p^1) \); if \( n \) is odd or \( L \) is non-orientable we instead define \( I(f) \) as an element of \( \mathbb{Z}_2 \). If \( X \) is simply connected a theorem of Whitney [10] implies that \( f \) is regularly homotopic with compact support to an embedding if and only if \( I(f) = 0 \).

Theorem 2.2 will be deduced in Section 5 from the following

**Theorem 2.3.** Let \((X, \lambda)\) be a simply connected Liouville manifold with a negative end \( X_- \), and \( f : L \to X \) a cylindrical at \(-\infty\) exact self-transverse Lagrangian immersion with finitely many self intersections. Suppose that \( I(f) = 0 \), and the asymptotic negative boundary \( \Lambda \) of \( f \) has a component which is loose in the complement of the others. If \( n = 3 \) suppose, in addition, that \( X \setminus f(L) \) has infinite Gromov width. Then there exists a compactly supported Hamiltonian regular homotopy \( f_t \), connecting \( f_0 = f \) with an embedding \( f_1 \).

**Remark.** If \( X \) is not simply connected the statement remains true if the self-intersection index \( I(f) \) is understood as an element of the group ring of \( \pi_1(X) \).

## 3 Weinstein recollections and other preliminaries

### Weinstein cobordisms

We define below a slightly more general notion of a Weinstein cobordism than is usually done (comp. [1]), by allowing cobordisms between non-compact manifolds. Let \( W \) be a \( 2n \)-dimensional smooth manifold with boundary. We allow \( W \), as well as its boundary components to be non-compact. Suppose that the boundary \( \partial W \) is presented as the union of two disjoint subsets \( \partial_\pm W \) which are open and closed in \( \partial W \). A **Weinstein cobordism** structure on \( W \) is a triple \((\omega, Z, \phi)\), where \( \omega \) is a symplectic form on \( W \), \( Z \) is a Liouville vector field, and \( \phi : W \to [m, M] \) a Morse function with finitely many critical points, such that

- \( \partial_- W = \{\phi = m\} \) and \( \partial_+ W = \{\phi = M\} \) are regular level sets;
- the vector field \( Z \) is gradient like for \( \phi \), see [1], Section 9.3;
• outside a compact subset of $W$ every trajectory of $Z$ intersects both $\partial_-W$ and $\partial_+W$.

The function $\phi$ is called a Lyapunov function for $Z$. The Liouville form $\lambda = i(Z)\omega$ induces contact structure on all regular levels of the function $\phi$. All $Z$-stable manifolds of critical points of the function $\phi$ are isotropic for $\omega$ and, in particular, indices of all critical points are $\leq n = \frac{\dim W}{2}$. A Weinstein cobordism $(W,\omega, X, \phi)$ is called subcritical if indices of all critical points are $< n$.

Extension of Weinstein structure

The following lemma is the standard handle attaching statement in the Weinstein category (see [9] and [1]). We provide a proof here because we need it in a slightly different than it is presented in [9] and [1].

**Lemma 3.1.** Let $(X,\lambda)$ be a Liouville manifold with boundary, $Z$ the Liouville field corresponding to $\lambda$ (i.e. $i_Z\omega = \lambda$ where $\omega = d\lambda$) and $Y \subset \partial X$ a (union of) boundary component(s) of $X$ such that $Z$ is inward transverse to $Y$. Let $(\Delta, \partial \Delta) \subset (X, Y)$ be a $k$-dimensional ($k \leq n$) isotropic disc, which is tangent to $Z$ near $\partial \Delta$. If $k = 1$ suppose, in addition, that $\int_\Delta \lambda = 0$, and if $k < n$ suppose, in addition, that $\Delta$ is extended to (a germ of) a Lagrangian submanifold $(L, \partial L) \subset (X, Y)$ which is also tangent to $Z$ near $\partial L$. Then for any neighborhoods $U \supset \Delta$ and $\Omega \supset Y$ there exists a Weinstein cobordism $(W,\omega, Z, \phi)$ with the following properties:

• $Y \cup \Delta \subset W \subset \Omega \cup U$;

• $\partial_-W = Y$;

• the function $\phi$ has a unique critical point $p$ of index $k$ at the center of the disc $\Delta$;

• the disc $\Delta$ is contained in the $\tilde{Z}$-stable manifold of the point $p$;

• the field $\tilde{Z}|_{L \cap W}$ is tangent to $L$;

• the Liouville form $\tilde{\lambda} = i(\tilde{Z})\omega$ can be written as $\lambda + dH$ for a function $H$ compactly supported in $U \setminus Y$. 
Proof. Let us set \( L = \Delta \) if \( k = n \). For a general case we can assume that \( L = \Delta \times \mathbb{R}^{n-k} \). Let \( \omega_{st} \) denote the symplectic form on \( T^*(L) = T^*L \times T^*\mathbb{R}^k = \Delta^k \times \mathbb{R}^k \times \mathbb{R}^{n-k} \times \mathbb{R}^{n-k} \) given by the formula

\[
\omega_{st} = \sum_{i=1}^{k} dp_i \wedge dq_i + \sum_{j=1}^{n-k} du_j \wedge dv_j
\]

with respect to the coordinates \((q, p, v, u) \in \Delta^k \times \mathbb{R}^k \times \mathbb{R}^{n-k} \times \mathbb{R}^{n-k}\) which correspond to this splitting. Denote by \( \lambda_k \) the Liouville form

\[
\lambda_k = \sum_{i=1}^{k} (2p_i dq_i + q_i dp_i) + \frac{1}{2} \sum_{i=1}^{n-k} (v_i du_j - u_j dv_j),
\]

d\( \lambda_k = \omega_{st} \). Note that the Liouville field

\[
Z_k := \sum_{i=1}^{k} \left(-q_i \frac{\partial}{\partial q_i} + 2p_i \frac{\partial}{\partial p_i}\right) + \frac{1}{2} \sum_{i=1}^{n-k} \left(v_i \frac{\partial}{\partial v_i} + u_j \frac{\partial}{\partial u_j}\right)
\]

corresponding to the form \( \lambda_k \) is gradient like for the quadratic function

\[
Q := \sum_{i=1}^{k} (p_i^2 - q_i^2) + \sum_{i=1}^{n-k} (u_j^2 + v_j^2),
\]
tangent to \( L \), and the disc \( \Delta \) serves as the \( Z_k \)-stable manifold of its critical point. Using the normal form for the Liouville form \( \lambda \) near \( \partial L \) (see [9], and also [1], Proposition 6.6) and the Weinstein symplectic normal form along the Lagrangian \( L \) we can find, possibly decreasing the neighborhoods \( \Omega \) and \( U \), a symplectomorphism \( \Phi : U \to U' \), where \( U' \) is a neighborhood of \( \Delta \) in \( T^*L \), such that

- \( \Phi(L \cap U) = L \cap U' \), \( \Phi(\Delta \cap U) = \Delta \cap U' \);
- \( \Phi^* \omega_{st} = \omega \);
- \( \Phi^* \lambda_k = \lambda \) on \( \Omega \cap U \);
- \( \Phi(Y \cap U) = \{Q = -1\} \cap U' \).

Thus the closed, and hence exact 1-form \( \Phi^* \lambda - \lambda_k \) vanishes on \( \Omega' := \Phi(\Omega \cap U) \), and therefore, using the condition \( \int_{\Delta} \lambda = 0 \) when \( k = 1 \), we can conclude that

\( \lambda_k = \Phi^* \lambda + dH \) for a function \( H : G \to \mathbb{R} \) vanishing on \( \Omega' \supset \partial \Delta \). Let \( \theta : U' \to [0, 1] \) be a \( C^\infty \)-cut-off function equal to 0 outside a neighborhood \( U_1' \supset \Delta, U_1' \Subset U' \), and
equal to 1 on a smaller neighborhood \( U_2' \supset \Delta, U_2' \subseteq U_1' \). Denote \( \tilde{H} := \theta H \). Then the form \( \tilde{\lambda} := \Phi^* \lambda + d\tilde{H} \) coincides with \( \Phi^* \lambda \) on \( \Omega' \cup (U' \setminus U_1') \), and equal to \( \lambda_k \) on \( U_2' \). Then, according to Corollary 9.21 from [1], for any sufficiently small \( \varepsilon > 0 \) and a neighborhood \( U_3' \supset \Delta, U_3' \subseteq U_2' \), there exists a Morse function \( \hat{Q} : U' \to \mathbb{R} \) such that

- \( \hat{Q} \) coincides with \( Q \) on \( \{ Q \leq -1 \} \cup \{ \{ Q \leq -1 + \varepsilon \} \setminus U_2' \}; 
- \( \hat{Q} \) and \( Q \) are target equivalent over \( U_3' \), i.e. there exists a diffeomorphism \( \sigma : \mathbb{R} \to \mathbb{R} \) such that over \( U_3' \) we have \( \hat{Q} = \sigma \circ Q \); 
- \( -1 + \varepsilon \) is a regular value of \( \hat{Q} \) and \( \{ \hat{Q} \leq -1 + \varepsilon \} \subseteq \Omega' \cup U_2' \); 
- inside \( \hat{W} := \{ -1 \leq \hat{Q} \leq -1 + \varepsilon \} \subseteq U' \) the function \( \hat{Q} \) has a unique critical point.

Denote \( \tilde{Q} := \hat{Q} \circ \Phi : U \to \mathbb{R} \). Let us extend the function \( \tilde{Q} \) to the whole manifold \( X \) in such a way that

- \( \{ \tilde{Q} = -1 \} \setminus U = Y \setminus U \), 
- \( \{ -1 \leq \tilde{Q} \leq -1 + \varepsilon \} \setminus U \subset \Omega \setminus U \), 
- the function \( \tilde{Q}|_{X \setminus U} \) has no critical values in \([-1, -1 + \varepsilon]\) and 
- the Liouville vector field \( Z \) is gradient like for \( \tilde{Q} \) on \( \{ -1 \leq \tilde{Q} \leq -1 + \varepsilon \} \setminus U \).

Let us define \( W := \{ -1 \leq \tilde{Q} \leq -1 + \varepsilon \} \subset X \),

\[
\tilde{\lambda} = \begin{cases} 
\Phi^* \lambda + d\tilde{H} \circ \Phi, & \text{on } U, \\
\lambda, & \text{on } X \setminus U.
\end{cases}
\]

Let \( \tilde{Z} \) be the Liouville field \( \omega \)-dual to the Liouville form \( \tilde{\lambda} \). Then the Weinstein cobordism \( (W, \omega, \tilde{Z}, \phi := \hat{H} \circ \Phi) \) has the required properties.

We will also need the following simple

**Lemma 3.2.** Let \((X, \lambda)\) be a Liouville manifold and \( f : L \to X \) a Lagrangian immersion. Let \( p \in X \) be a transverse self-intersection point. Then there exists a symplectic embedding \( h : B \to X \) of a sufficiently small ball in \( \mathbb{R}_{st}^{2n} \) into \( X \) such that \( h(0) = p \) and \( h^{-1}(f(L)) = B \cap (\{ x = 0 \} \cup \{ y = 0 \}) \).
Proof. By the Weinstein neighborhood theorem, there exist coordinates in a symplectic ball near \( p \) so that \( f(L) \) is given by \( \{ x = 0 \} \cup \{ y = dg(x) \} \) for some function \( g : \mathbb{R}^n \to \mathbb{R} \) so that \( dg(0) = 0 \) (here we use natural coordinates on \( T^*\mathbb{R}^n \)). By transversality the critical point of \( g \) at 0 is non-degenerate. Composing with the symplectomorphism \( (x, y) \mapsto (x, y - dg(x)) \) gives the desired coordinates. \( \Box \)

Cancellation of critical points in a Weinstein cobordism

The following proposition concerning cancellations of critical points in a Weinstein cobordism is proven in [1], see there Proposition 12.22.

**Proposition 3.3.** Let \((W, \omega, Z_0, \phi_0)\) be a Weinstein cobordism with exactly two critical points \( p, q \) of index \( k \) and \( k - 1 \), respectively, which are connected by a unique \( Z \)-trajectory along which the stable and unstable manifolds intersect transversely. Let \( \Delta \) be the closure of the stable manifold of the critical point \( p \). Then there exists a Weinstein cobordism structure \((\omega, Z_1, \phi_1)\) with the following properties:

\begin{itemize}
  \item\( (Z_1, \phi_1) = (Z_0, \phi_0) \) near \( \partial W \) and outside a neighborhood of \( \Delta \);
  \item\( \phi_1 \) has no critical points.
\end{itemize}

From Legendrian isotopy to Lagrangian concordance

The following Lemma about Lagrangian realization of a Legendrain isotopy is proven in [5], see there Lemma 4.2.5.

**Lemma 3.4.** Let \( f_t : \Lambda \to (Y, \xi = \{ \alpha = 0 \}) \), \( t \in [0, 1] \), be a Legendrian isotopy connecting \( f_0, f_1 \). Let us extend it to \( t \in \mathbb{R} \) as independent of \( t \) for \( t \notin [0, 1] \). Then there exists a Lagrangian embedding

\[ F : \mathbb{R} \times \Lambda \to \mathbb{R} \times Y, d(e^{t\alpha}) \],

of the form \( F(t, x) = (\tilde{f}_t(x), h(t, x)) \) such that

- \( F(t, x) = (f_1(x), t) \) and \( F(x, -t) = f_0(x) \) for \( t > C \), for a sufficiently large constant \( C \);
- \( \tilde{f}_t(x) \) \( C^\infty \)-approximate \( f_t(x) \).
4 Action-balanced Lagrangian immersions

Suppose we are given an exact proper Lagrangian immersion $f : L \to X$ of an orientable manifold $L$ into a simply connected Liouville manifold $(X, \lambda)$ with finitely many transverse self-intersection points. For each self-intersection point $p \in X$ we denote by $p^0, p^1 \in L$ its pre-images in $L$. The integral $a_{SI}(p, f) = \int_{\gamma} f^* \lambda$, where $\gamma : [0, 1] \to L$ is any path connecting the points $\gamma(0) = p^0$ and $\gamma(1) = p^1$, will be called the action of the self-intersection point $p$. Of course, the sign of the action depends on the ordering of the pre-images $p^0$ and $p^1$. We will fix this ambiguity by requiring that $a_{SI}(p, f) > 0$ (by a generic perturbation of $f$ we can assume there are no points $p$ with $a_{SI}(p, f) = 0$).

A pair of self-intersection points $(p, q)$ is called a balanced Whitney pair if $a_{SI}(p, f) = a_{SI}(q, f)$ and the intersection indices of $df(T_{p_0}L)$ with $df(T_{p_1}L)$ and of $df(T_{q_0}L)$ with $df(T_{q_1}L)$ have opposite signs. In the case where $L$ is non-orientable we only require that $p$ and $q$ have the same action. A Lagrangian immersion $f$ is called balanced if the set of its self-intersection points can be presented as the union of disjoint balanced Whitney pairs.

The goal of this section is the following

**Proposition 4.1.** Let $(X, \lambda)$ be a simply connected Liouville manifold with a negative end and $f : L \to X$ a proper exact and cylindrical at $-\infty$ Lagrangian immersion with finitely many transverse double points. If $n = 3$ suppose, in addition, that $X \setminus f(L)$ has infinite Gromov width. Then there exists an exact cylindrical at $-\infty$ Lagrangian regular homotopy $f_t : L \to X$, $t \in [0, 1]$, which is compactly supported away from the negative end, and such that $f_0 = f$ and $f_1$ is balanced.

If the asymptotic negative boundary of $f$ has a component which is loose in the complement of the other components and $I(f) = 0$ then the Lagrangian regular homotopy $f_t$ can be made fixed at $-\infty$.

Note that Proposition 4.1 is the only step in the proof of the main results of this paper where one need the infinite Gromov width condition when $n = 3$.

The following two lemmas will be used to reduce the action of our intersection points in the case where we only have a finite amount of space to work with, for example when $X_+$ is compact. In the case where $X_+$ contains a symplectic ball $B_R$ of arbitrarily large radius, e.g. in the situation of Theorem 1.1 these lemmas are not needed.
Lemma 4.2. Consider an annulus $A := [0, 1] \times S^{n-1}$. Let $x, z$ be coordinates corresponding to the splitting, and $y, u$ the dual coordinates in the cotangent bundle $T^*A$, so that the canonical Liouville form $\lambda$ on $T^*A$ is equal to $ydx + udz$. Then for any integer $N > 0$ there exists a Lagrangian immersion $\Delta : A \rightarrow T^*A$ with the following properties:

- $\Delta(A) \subset \{|y| \leq \frac{5}{N}, ||u|| \leq \frac{5}{N}\}$;
- $\Delta$ coincides with the inclusion of the zero section $j_A : A \hookrightarrow T^*A$ near $\partial A$;
- there exists a fixed near $\partial A$ Lagrangian regular homotopy connecting $j_A$ and $\Delta$;
- $\int \lambda = 1$, where $\zeta$ is the $\Delta$-image of any path connecting $S^{n-1} \times 0$ and $S^{n-1} \times 1$ in $A$;
- action of any self-intersection point of $\Delta$ is $< \frac{1}{N}$;
- the number of self-intersection points is $< 8N^3$.

Proof. Consider in $\mathbb{R}^2$ with coordinates $(x, y)$ the rectangulars

$$I_{j,N} = \left\{ \frac{j}{5N^2} \leq x \leq \frac{j}{5N^2} + \frac{1}{5N}, 0 \leq y \leq \frac{5}{N} \right\}, j = 0, \ldots (N - 1)N.$$ 

Consider a path $\gamma$ in $\mathbb{R}^2$ which begins at the origin, travels counter-clockwise along the boundary of $I_{0,N}$, then moves along the $x$-axis to the point $(\frac{1}{5N^2}, 0)$, travels counter-clockwise along the boundary of $I_{1,N}$ etc., and ends at the point $(1, 0)$. Note that $\int ydx = \frac{N-1}{N}$. We also observe that squares $I_{j,N}$ and $I_{i,N}$ intersect only when $|i - j| \leq N$, and hence for any self-intersection point $p$ of $\gamma$ its action is bounded by $N \frac{1}{N^2} = \frac{1}{N}$. Let us $C^\infty$-approximate $\gamma$ by an immersed curve $\gamma_1$ with transverse self-intersections and which coincides with $\gamma$ near its end points. We can arrange that

- $\left| \int_{\gamma_1} ydx - 1 \right| < \frac{2}{N}$;
- action of any self-intersection point of $\gamma_1$ is $< \frac{1}{N}$;
- the number of self-intersection points is $< 2N^3$.
• the curve $\gamma_1$ is contained in the rectangular $\{0 \leq x \leq \frac{1}{5}, 0 \leq y \leq \frac{N}{5}\}$.

See Figure 4.1. The only non-trivial statement is the upper bound on the number of self-intersections. Notice that there are less than $N^2$ loops, and each loop intersects at most $2N$ other loops, in 2 points each. Thus the number of self intersections, double counted, is less than $4N^3$.

We will assume that $\gamma_1$ is parameterized by the interval $[0, \frac{1}{5}]$. Let $r_N$ denote the affine map $(x, y) \mapsto (x + \frac{1}{5}, -\frac{N}{5}y)$. We define a path $\gamma_2 : [\frac{1}{5}, \frac{2}{5}] \to \mathbb{R}^2$ by the formula

$$\gamma_2(t) = r_N(\gamma_1(t - \frac{1}{5})).$$

Note that the immersion $\gamma_{12} : [0, \frac{2}{5}] \to \mathbb{R}^2$ which coincides with $\gamma_1$ on $[0, \frac{1}{5}]$ and with $\gamma_2$ on $[\frac{1}{5}, \frac{2}{5}]$ is regularly homotopic to the straight interval embedding via a homotopy which is fixed near the end of the interval, and which is inside $\{0 \leq x \leq \frac{2}{5}, -\frac{5}{N^2} \leq y \leq \frac{N}{5}\}$. We also note that $\left| \int_{\gamma_{12}} y dx - 1 \right| < \frac{3}{N}$. See Figure 4.2.

We further extend $\gamma_{12}$ to an immersion $\gamma_{123} : [0, 1] \to \mathbb{R}^2$ by extending it to $[\frac{2}{5}, 1]$ as a graph of function $\theta : [\frac{2}{5}, 1] \to [-\frac{5}{N}, \frac{N}{5}]$ with

$$\int_{\gamma_{12}}^1 \theta(x) dx = 1 - \int_{\gamma_{12}} y dx,$$

which implies $\int_{\gamma_{123}} y dx = 1$. 

---

Fig. 4.1: The curve $\gamma_1$ when $N = 3$. 

---
Let $j_{S^{n-1}}$ denote the inclusion $S^{n-1} \to T^*S^{n-1}$ as the 0-section. Consider a Lagrangian immersion $\Gamma : A \to T^*A$ given by the formula
\[ \Gamma(x, z) = (\gamma_{123}(x), j_{S^{n-1}}(z)) \in T^*[0, 1] \times T^*S^{n-1} = T^*A. \]

The Lagrangian immersion $\Gamma$ self-intersects along spheres of the form $p \times S^{n-1}$ where $p$ is a self-intersection point of $\tilde{\gamma}$. By a $C^\infty$-perturbation of $\Gamma$ we can construct a Lagrangian immersion $\Delta : A \to T^*A$ with transverse self-intersection points which have all the properties listed in Lemma 4.2. Indeed, for each of the $4N^3$ intersection points $p$ of $\gamma_{123}$, the sphere $p \times S^{n-1}$ can be perturbed to have two self-intersections. The other required properties are straightforward from the construction. \hfill \Box

**Remark 4.3.** Given any $a > 0$ we get, by scaling the Lagrangian immersion $\Delta$ with the dilatation $(y, u) \mapsto (ay, au)$, a Lagrangian immersion $\Delta_a : A \to T^*A$ which satisfy

- $\int \lambda = a$, where $\zeta$ is the $\Delta_a$-image of any path connecting the boundary $S^{n-1} \times 0$ and $S^{n-1} \times 1$ of $A$;
- action of any self-intersection point of $\Delta_a$ is $< \frac{a}{N}$;
- the number of self-intersection points is $< 8N^3$;
- $\Delta_a(A) \subset \{|y|, ||u|| \leq \frac{5a}{N}\}$;
- the immersion $\Delta_a$ is regularly homotopic relative its boundary to the inclusion $A \hookrightarrow T^*A$. 

**Fig. 4.2:** The curve $\gamma_{12}$. 

Let $j_{S^{n-1}}$ denote the inclusion $S^{n-1} \to T^*S^{n-1}$ as the 0-section. Consider a Lagrangian immersion $\Gamma : A \to T^*A$ given by the formula
\[ \Gamma(x, z) = (\gamma_{123}(x), j_{S^{n-1}}(z)) \in T^*[0, 1] \times T^*S^{n-1} = T^*A. \]
Given a proper Lagrangian immersion $f : L \to X$ with finitely many transverse self-intersection points, we denote the number of self-intersection points by $\text{SI}(f)$. The action of a self-intersection point $p$ of $f$ is denoted by $a_{\text{SI}}(p, f)$. We set $a_{\text{SI}}(f) := \max_p |a_{\text{SI}}(p, f)|$, where the maximum is taken over all self-intersection points of $f$.

**Lemma 4.4.** Let $f_0 : L \to (X, \lambda)$ be a proper exact Lagrangian immersion into a simply connected Liouville manifold with finitely many transverse self-intersection points. Then for any sufficiently large integer $N > 0$ there exists a fixed at infinity $C^0$-small exact Lagrangian regular homotopy $f_1 : L \to X$, $t \in [0, 1]$, such that $f_1$ has transverse self-intersections,

$$a_{\text{SI}}(f_1) \leq \frac{a_{\text{SI}}(f)}{N}, \quad \text{SI}(f_1) \leq 9N^3\text{SI}(f_0).$$

**Proof.** Let $p_1, \ldots, p_k$ be the self-intersection points of $f_0$ and $p_1^0, p_1^1, \ldots, p_k^0, p_k^1$ their pre-images, $k = \text{SI}(f_0)$. Let us recall that we order the pre-images in such a way that $a_{\text{SI}}(p_i, f_0) > 0$, $i = 1, \ldots, k$. Choose

- disjoint embedded $n$-discs $D_i \ni p_i^1$, $i = 1, \ldots, k$, which do not contain any other pre-images of double points, and
- annuli $A_i \subset D_i$ bounded by two concentric spheres in $D_i$.

For a sufficiently large $N > 0$ there exist disjoint symplectic embeddings $h_i$ of the domains $U_i := \{ |y|, |u| \leq \frac{5a_{\text{SI}}(p_i, f_0)}{N} \} \subset T^* A$ in $X$, $i = 1, \ldots, k$, such that $h_i^{-1}(f_0(L)) = h_i^{-1}(A_i) = A$. Then, using Remark 4.3, we find a Lagrangian regular homotopy $f_1$ supported in $\bigcup h_i(U_i)$ which annihilates the action of points $p_i$, i.e. $a_{\text{SI}}(p_i, f_1) = 0$, $i = 1, \ldots, k$, and which creates no more than $8kN^3$ new self-intersection points of action $< \frac{a_{\text{SI}}(f_0)}{N}$. Hence, the total number of self-intersection points of $f_1$ satisfies the inequality $\text{SI}(f_1) < 9\text{SI}(f_0)N^3$.

The next lemma is a local model which will allow us to match the action of a given intersection point, during our balancing process. For a positive $C$ we denote by $Q_C$ the parallelepiped

$$\{ |z| \leq C, |x_i| \leq 1, |y_i| \leq C, \ i = 1, \ldots, n-1 \}$$
in the standard contact space $\mathbb{R}^{2n-1}_{st} = (\mathbb{R}^{2n-1}, \xi = \{ \alpha_{st} := dz - \sum_{i=1}^{n-1} y_i dx_i = 0 \})$. Let $SQ_C$ denote the domain $[\frac{1}{2}, 1] \times Q_C$ in the symplectization $(0, \infty) \times Q_C$ of $Q_C$ endowed with the Liouville form $\lambda_0 := s \alpha_{st}$. We furthermore denote by $L'$ the Lagrangian rectangular $\{ z = t, y = 0; j = 1, \ldots, n - 1 \} \cap SQ_C \subset SQ_C, t \in [-C, C]$.

Lemma 4.5. For any positive $b_0, b_1, \ldots, b_k \in (0, \infty), k \geq 0$, such that

$$\frac{C}{4k + 4} > b_0 > \max(b_1, \ldots, b_k),$$

and a sufficiently small $\varepsilon > 0$ there exists a Lagrangian isotopy which starts at $L^{-\varepsilon}$, fixed near $1 \times Q_C$ and $[\frac{1}{2}, 1] \times \partial Q_C$, cylindrical near $\frac{1}{2} \times Q_C$, and which ends at a Lagrangian submanifold $\tilde{L}^{-\varepsilon}$ with the following properties:

- $\tilde{L}^{-\varepsilon}$ intersects $L^0$ transversely at $k + 1$ points $B_0, B_1, \ldots, B_k$;
- if $\gamma_{B_j}, j = 0, \ldots, k$, is a path in $\tilde{L}^{-\varepsilon}$ connecting the point $B_j$ with a point on the boundary $\partial Q_C$, then
  $$\int_{\gamma_{B_j}} \lambda_0 = b_j, j = 0, \ldots, k;$$
- the intersection indices of $L^0$ and $\tilde{L}^{-\varepsilon}$ at the points $B_0, B_1, \ldots, B_k$ are equal to $1, -1, \ldots, -1$, respectively.
- $\tilde{L}^{-\varepsilon} \cap \{ s = \frac{1}{2} \}$ is a Legendrian submanifold in $Q_C$ defined by a generating function which is equal to $-\varepsilon$ near $\partial Q_C$ and positive over a domain in $Q_C$ of Euler characteristic $1 - k$.

Proof. We have

$$\omega := d\lambda_0 = ds \wedge dz - \sum_{i=1}^{n-1} dx_i \wedge d(s y_i) = -d(z ds + \sum_{i=1}^{n-1} v_i dq_i),$$

we denoted $v_i := sy_i, i = 1, \ldots, n - 1$. Let $I^{n-1} \subset \mathbb{R}^{n-1}$ be the cube $\{ \max_{i=1, \ldots, n-1} |q_i| \leq 1 \}$. Choose a smooth non-negative function $\theta : [\frac{1}{2}, 1] \rightarrow \mathbb{R}$ such that

- $\theta(s) = s$ for $s \in [\frac{1}{2}, \frac{5}{8}]$;
• $\theta$ has a unique local maximum at a point $\frac{3}{4}$;
• $\theta(s) = 0$ for $s$ near 1;
• the derivative $\theta'$ is monotone decreasing on $[\frac{5}{8}, \frac{3}{4}]$.

For any $\tilde{b}_0, \ldots, \tilde{b}_k \in (0, \frac{C}{2k+2})$ which satisfy $\tilde{b}_0 > \max(\tilde{b}_1, \ldots, \tilde{b}_k)$ one can construct a smooth non-negative function $\phi : I^{n-1} \to \mathbb{R}$ with the following properties:

• $\phi = 0$ near $\partial I^{n-1}$;
• $\max_{i=1, \ldots, n-1} \left| \frac{\partial \phi}{\partial q_i} \right| < \frac{C}{2}$;
• besides degenerate critical points corresponding to the critical value 0, the function $\phi$ has $k+1$ positive non-degenerate critical points: 1 local maximum $\tilde{B}_0$ and $k$ critical points $\tilde{B}_1, \ldots, \tilde{B}_k$ of index $n-2$ with critical values $\tilde{b}_0, \tilde{b}_1, \ldots, \tilde{b}_k$ respectively.

Take a positive $\varepsilon < \min(\tilde{b}_0, \ldots, \tilde{b}_k, \frac{C}{2k+8})$ and define a function $h : [\frac{1}{2}, 1] \times I^{n-1} \to \mathbb{R}$ by the formula

$$h(s, q) = -\varepsilon s + \theta(s)\phi(q), \ s \in \left[\frac{1}{2}, 1\right], q \in I^{n-1}.$$ 

Thus the function $h$ is equal to $s(-\varepsilon + \phi(q))$ for $s \in [\frac{1}{2}, \frac{5}{8}]$ and equal to $-\varepsilon s$ near the rest of the boundary of $[\frac{1}{2}, 1] \times I^{n-1}$. The function $h$ has one local maximum at a point $(s_0, \tilde{B}_0)$ and $k$ index $n-1$ critical points with coordinates $(s_j, \tilde{B}_j), \ j = 1, \ldots, k$. Here the values $s_j \in \left[\frac{5}{8}, \frac{3}{4}\right]$ are determined from the equations $\tilde{b}_j\theta'(s_j) = \varepsilon, \ j = 0, \ldots, k$. Respectively, the critical values are equal to $\tilde{b}_j := -\varepsilon s_j + \theta(s_j)\tilde{b}_j$. For $\tilde{b}_j$ near $\varepsilon$ we have $\tilde{b}_j < \varepsilon$, while for $\tilde{b}_j$ close to $\frac{C}{2k+2}$ we have $\tilde{b}_j > \frac{C}{4k+4}$. Hence, by continuity, any critical values $b_0, b_1, \ldots b_k \in (\varepsilon, \frac{C}{2k+4})$ which satisfy the inequality $b_0 > \max(b_1, \ldots, b_k)$ can be realized.

The required Lagrangian manifold $\tilde{L}^{-\varepsilon}$ can be now defined by the equations

$$z = \frac{\partial h}{\partial s}, \ x_j = q_j, \ v_j = \frac{\partial h}{\partial p_j}, \ j = 1, \ldots, n-1, \ s \in \left[\frac{1}{2}, 1\right], q \in I^{n-1},$$

or returning to $x, y, z, s$ coordinates by the equations

$$\tilde{L}^{-\varepsilon} = \left\{ z = \frac{\partial h}{\partial s}, y_j = \frac{1}{s} \frac{\partial h}{\partial q_j} \right\}.$$ 

It is straightforward to check that $\tilde{L}^{-\varepsilon}$ has the required properties.
After using Lemma 4.4 to shrink the action of an intersection point, Lemma 4.5 applied with $k = 0$, will allow us to balance any negative intersection point. Positive intersection points still provide a challenge though, because the intersection point with the largest action created by Lemma 4.5 is always positive. The following lemma solves this issue.

**Lemma 4.6.** Let $f : L \rightarrow (X, \lambda)$ be a proper exact Lagrangian immersion into a simply connected $X$ and $D \subset L$ an $n$-disc which contains no double points of the immersion $f$. Then for any $A > 0$ and a sufficiently small $\sigma > 0$ there exists a supported in $D$ Hamiltonian regular homotopy of $f$ to $\widetilde{f}$ which creates a pair $p_+, p_-$ of additional self-intersection points such that $a_{SI}(p_\pm, \widetilde{f}) = A \pm \sigma$, the self-intersection indices of $p_\pm$ have opposite signs and can be chosen at our will.

Let us introduce some notation. Consider a domain $U_\varepsilon := \{ -2\varepsilon < p_1 < 1 + 2\varepsilon, \max_{1 \leq i \leq n} |q_i| < 2\varepsilon, \max_{1 \leq j \leq n} |p_j| < 2\varepsilon \}$ in the standard symplectic $\mathbb{R}^{2n}_{st} = (\mathbb{R}^{2n}, \sum_1^n dp_i \wedge dq_i)$. Let $L^t$ be the Lagrangian plane $\{ p_1 = t, p_j = 0 \text{ for } j = 2, \ldots, n \} \cap U_\varepsilon \subset U_\varepsilon$, $t \in \{0, 1\}$. Note that $pdq|_{L^1} = tdq_1$. We will also use the following notation associated with $U_\varepsilon$:

- $u_\pm \in L^1$ denote the points with coordinates $p = (1, 0, \ldots, 0), q = (\pm \delta_1, 0, \ldots, 0)$;
- $z_\pm \in L^0$ denote the points with coordinates $p = (0, 0, \ldots, 0), q = (\pm \delta_1, 0, \ldots, 0)$
- $c_0$ denote the point with coordinates $p = (0, 0, \ldots, 0), q = (-\varepsilon, 0, \ldots, 0)$;
- $c_1$ denote the point with coordinates $p = (1, 0, \ldots, 0), q = (-\varepsilon, 0, \ldots, 0)$;
- $J^1_\pm$ denote the intervals connecting $c_1$ and $u_\pm$;
- $J^0_\pm$ denote the intervals connecting $c_0$ and $z_\pm$.

We will use in the proof of 4.6 the following

**Lemma 4.7.** There exists a Lagrangian isotopy $f_t : L^1 \rightarrow U_\varepsilon$ fixed near $\partial L^1$ and starting at the inclusion $f_0 : L^1 \hookrightarrow U_\varepsilon$ such that $L^1 = f_1(L^1)$ transversely intersects $L^0$ at two points $z_\pm$ with the following properties:

- $f_1(pdq) = q_1 + d\theta$, where $\theta : L^1 \rightarrow \mathbb{R}$ is a compactly supported in Int $L^1$ function such that $\theta(z_\pm) = \mp \delta$ for a sufficiently small $\delta > 0$;
• the intersection indices of \( \tilde{L}^1 \) and \( L^0 \) at \( z_+ \) and \( z_- \) have opposite signs and can be chosen at our will.

**Proof.** For sufficiently small \( \delta_1, \delta_2, 0 < \delta_1 \ll \delta_2 \ll \varepsilon \), there exists a \( C^\infty \)-function \( \alpha : [-\varepsilon, \varepsilon] \to \mathbb{R} \) with the following properties:

* \( \alpha(t) = t \) for \( \delta_2 \leq |t| \leq \varepsilon \);
* \( \alpha(t) = t^3 - 3\delta_1^2t \) for \( |t| \leq \delta_1 \);
* the function \( \alpha \) has no critical points, other than \( \pm \delta_1 \);
* \( -\varepsilon < \alpha'(t) < 1 - \delta_2 \varepsilon \).

Let us also take a cut-off function \( \beta : [0, 1] \to [0, 1] \) which is equal to 0 near 1 and equal to 1 near 0. Take a quadratic form \( Q_j \) of index \( j - 1 \):

\[
Q_j(q_2, \ldots, q_n) = -\sum_{i=2}^{j} q_i^2 + \sum_{j+1}^{n} q_i^2, \quad j = 1, \ldots, n,
\]

and define a function \( \sigma : \{|q_i| \leq \varepsilon; i = 1, \ldots, n\} \to \mathbb{R} \) by the formula

\[
\sigma_j(q_1, q_2, \ldots, q_n) = q_1 + \delta_2 Q_j(q_2, \ldots, q_n) \beta \left( \frac{\rho}{\varepsilon} \right) \beta \left( \frac{|q_1|}{\varepsilon} \right) + (\alpha(q_1) - q_1) \beta \left( \frac{\rho}{\varepsilon} \right),
\]

where we denoted \( \rho := \max_{2 \leq i \leq n} |q_i| \). The function \( \sigma_j \) has two critical points \((-\delta_1, 0, \ldots, 0)\) and \((\delta_1, 0, \ldots, 0)\) of index \( j \) and \( j - 1 \), respectively. We note that

\[
-\varepsilon - Cn\delta_2\varepsilon \leq \frac{\partial \sigma_j}{\partial q_1} < 1 - \frac{\varepsilon}{2} + Cn\delta_2\varepsilon
\]

and

\[
\left| \frac{\partial \sigma_j}{\partial q_i} \right| \leq 2\delta_2\varepsilon + Cn\delta_2\varepsilon + \frac{C\delta_2}{\varepsilon}
\]

for \( i > 1 \), where \( C = ||\beta||_{C^1} \). In particular, if \( \delta_2 \) is chosen small enough we get

\[ -\varepsilon < \frac{\partial \sigma_j}{\partial q_i} < 1 + \varepsilon \]

and \( \left| \frac{\partial \sigma_j}{\partial q_i} \right| < \varepsilon \) for \( i = 2, \ldots, n \).

Assuming that \( L^1 \) is parameterized by the \( q \)-coordinates we define the required Lagrangian isotopy \( f_t : L^1 \to U_{\varepsilon} \) by the formula:

\[
f_t(q) = \left( q, 1 + t \left( \frac{\partial \sigma_j}{\partial q_1} - 1 \right), t \frac{\partial \sigma_j}{\partial q_2}, \ldots, t \frac{\partial \sigma_j}{\partial q_n} \right), \quad |q_i| < 2\varepsilon; \quad i = 1, \ldots, n.
\]
The Lagrangian manifold \( \widetilde{L}^1 = f_1(L^1) \) intersects \( L^0 \) at two points \( z_\pm \) with coordinates \( p = 0, q_1 = \pm \delta_1, q_2 = 0, \ldots, q_n = 0 \). The intersection index of \( \widetilde{L}^1 \) and \( L^0 \) at \( z_- \) is equal to \((-1)^j\), and to \((-1)^{j-1}\) at \( z_+ \). Thus by choosing \( j \) even or odd we can arrange the intersection to be positive at \( z_+ \) and negative at \( z_- \), or the other way around. The compactly supported function \( \theta \) determined from the equation \( f_1^*(pdq) = dq_1 + d\theta \) is equal to \( \sigma_j - q_1 \). In particular, \( \theta(z_\pm) = \mp 2\delta_1^3 \).

**Proof of Lemma 4.6** We denote \( \widetilde{J}_1^\pm := f_1(J_1^\pm) \), where \( f_t \) is the isotopy constructed in Lemma 4.7. Take any two points \( a, b \in D \subset \widetilde{D} := f(D) \subset \widetilde{L} := f(L) \) and connect them by a path \( \eta : [0, 1] \to \widetilde{D} \) such that \( \eta(0) = \tilde{b} := f(b) \) and \( \eta(1) = \tilde{a} := f(a) \). Denote \( B := \int \eta \lambda \).

For any real \( R \) there exists an embedded path \( \gamma : [0, 1] \to X \) connecting the points \( \gamma(0) = \tilde{a} \) and \( \gamma(1) = \tilde{b} \) in the complement of \( \tilde{L} \), homotopic to a path in \( \tilde{L} \) with fixed ends, and such that \( \int \gamma \lambda = R \). For a sufficiently small \( \varepsilon > 0 \) the embedding \( \gamma \) can be extended to a symplectic embedding \( \Gamma : U_\varepsilon \to X \) such that \( \Gamma^{-1}(\tilde{L}) = L^0 \cup L^1 \). Here we identify the domain \([0, 1]\) of the path \( \gamma \) with the interval

\[
I = \{ q_1 = -\varepsilon, q_j = 0, j = 2, \ldots, n; 0 \leq p_1 \leq 1, p_j = 0, j = 2, \ldots, n \} \subset \partial U_\varepsilon,
\]

so that we have \( \Gamma(c^0) = \tilde{a} \) and \( \Gamma(c^1) = \tilde{b} \).

The Lagrangian isotopy \( \widetilde{f}_t := \Gamma \circ f_t : L^1 \to X \), where \( f_t : L^1 \to U_\varepsilon \) is the isotopy constructed in Lemma 4.7, extends as a constant homotopy to the rest of \( L \) and provides us with a regular Lagrangian homotopy connecting the immersion \( f \) with a Lagrangian immersion \( L \to X \) which has two more transverse intersection points \( p_\pm := \Gamma(z_\pm) \) of opposite intersection index sign. See Figure 4.3. Consider the following loops \( \zeta_\pm \) in \( \tilde{L} \subset X \) based at the points \( p_\pm \). We start from the point \( p_\pm \) along the \( \Gamma \)-image of the oppositely oriented interval \( \widetilde{J}_1^\pm \) to the point \( \tilde{b} \), then follow the path \( \eta \) to the point \( \tilde{a} \), and finally follow along the \( \Gamma \)-image of the path \( J_0 \) back to \( p_\pm \).

Then we have

\[
\int_{\zeta_\pm} \lambda = -\int_{\widetilde{J}_1^\pm} \Gamma^*\lambda + \int_{\eta} \lambda + \int_{J_0^\pm} \Gamma^*\lambda = \left( -\int_{\widetilde{J}_1^\pm} \Gamma^*\lambda + \int_{\gamma} \lambda + \int_{J_0^\pm} \Gamma^*\lambda \right) + \left( \int_{\eta} \lambda - \int_{\gamma} \lambda \right)
\]
Fig. 4.3: The Lagrangian $f_1(L)$. The light curve represents $\gamma$.

$$= \left(-\int_{p_\pm} pdq - \int_{l} pdq + \int_{r_\pm} pdq\right) + (B + R) = -\varepsilon + B + R \mp 2\delta_1^3.$$ 

It remains to observe that there exists a sufficiently small $\varepsilon_0 > 0$ which can be chosen for any $R \in [A - C - 1, A - C + 1]$. Hence, by setting $R = A - C - \varepsilon_0$ and $\varepsilon = \varepsilon_0$ we arrange that the action of the intersection points $p_\pm$ is equal to $A \mp 2\delta_1^3$ while their intersection indices have opposite sign which could be chosen at our will. 

Lemma 4.8. Let $((0, \infty) \times Y, d(t \alpha))$ be the symplectization of a manifold $Y$ with a contact form $\alpha$. Let $\Lambda$ be a Legendrian submanifold and $L = (0, \infty) \times \Lambda$ the Lagrangian cylinder over it. Suppose that there exists a contact form preserving embedding $\Phi : (Q_C, \alpha_{st}) \to (Y, \alpha)$ and $\Gamma \subset Y$ an embedded isotropic arc connecting a point $b \in \Lambda$ with a point

$$\Phi(x_1 = 1, x_2 = 0, \ldots, x_{n-1} = 0, y_1 = 0, \ldots, y_n = 0, z = 0) \in \partial\Phi(Q_C).$$

Then there exists a Lagrangian isotopy $L_t \subset \mathbb{R} \times \Lambda$ supported in a neighborhood of $1 \times \Gamma \cup \Phi(Q_C)$, $t \in [0, 1]$, which begins at $L_0 = L$ such that
• $L_t$ transversely intersects $1 \times Y$ along a Legendrian submanifold $\Lambda_t$;

• $\Phi^{-1}(\Lambda_1) = \Lambda^0 \cup \Lambda^{-\varepsilon}$ for a sufficiently small $\varepsilon > 0$.

**Proof.** We use below the notation $I^k_a$, $a > 0$ for the cube $\{|x_i| \leq a, i = 1, \ldots, k\} \subset \mathbb{R}^k$. The embedding $\Phi$ can be extended to a slightly bigger domain $\hat{Q} = \{|x_i| \leq 1 + \sigma, |y_i| \leq C, i = 1, \ldots, n-1, |z| \leq C + \sigma\} \subset \mathbb{R}^{2n-1}$ for a sufficiently small $\sigma > 0$. The intersection $\hat{Q} \cap (\mathbb{R}^{n-1} = \{y = 0, z = 0\})$ is the cube $I^{n-1}_{1+\sigma} \subset \mathbb{R}^{n-1}$. We can assume that the intersection of the path $\Gamma$ with $\hat{Q}$ coincides with the interval $\{1 \leq x_1 \leq 1 + \sigma, x_j = 0, j = 2, \ldots, n-1\} \subset I^{n-1}_{1+\sigma}$. The Legendrian embedding $\Psi := \Phi|_{I^{n-1}_{1+\sigma}} : I^{n-1}_{1+\sigma} \to Y$ can be extended to a bigger parallelepiped

$$\Sigma = \{-1 - \sigma \leq x_1 \leq 2 + \sigma, |x_j| \leq 1 + \sigma, j = 2, \ldots, n-1\} \subset \mathbb{R}^{n-1}$$

such that the extended Legendrian embedding, still denoted by $\Psi$, has the following properties:

• $\Psi(\{1 \leq x_1 \leq 2, x_j = 0, j = 2, \ldots, n-1\}) = \Gamma$;

• $\Psi(\{x_1 = 2\}) \subset \Lambda$.

For a sufficiently small positive $\delta < C$ the Legendrian embedding can be further extended as a contact form preserving embedding

$$\hat{\Psi} : (\hat{P} := \{(x, y, z) \in \mathbb{R}^{2n-1}_{st}; x \in \Sigma, |y_i| \leq \delta, i = 1, \ldots, n-1, |z| \leq \delta, \alpha_{st}\} \to (Y, \alpha),$$

such that

• $\hat{\Psi}|_{\hat{P} \cap \hat{Q}} = \Phi|_{\hat{P} \cap \hat{Q}}$;

• the Legendrian manifold $\hat{\Lambda} := \hat{\Psi}^{-1}(\Lambda)$ is given by the formulas

$$\hat{\Lambda} := \{z = \pm(x_1 - 2)^{\frac{3}{2}}, y_1 = \pm\frac{3}{2}\sqrt{x_1 - 2}, x_1 \geq 2, y_j = 0, j = 2, \ldots, n-1\}$$

(note that any point on any Legendrian admits coordinates describing $\hat{\Lambda}$ as above).

Consider a cut-off $C^\infty$-function $\theta : [0, 1 + \sigma] \to [0, 1]$ such that $\theta(u) = 1$ if $u \leq 1$, $\theta(u) = 0$ if $u > 1 + \frac{\sigma}{2}$, $\theta' \leq 0$, and denote

$$\Theta(u_1, \ldots, u_{n-2}) := (3 + \sigma) \prod_{i=1}^{n-2} \theta(u_i), u_1, \ldots, u_{n-2} \in [0, 1 + \sigma].$$
Fig. 4.4: The function $g_s$.

For $s \in [0, 1]$ denote
$$
\Omega_s := \{2 - s\Theta(|x_2|, \ldots, |x_{n-1}|) \leq x_1 \leq 2 + \sigma\} \cap \Sigma \subset \mathbb{R}^{n-1}.
$$

We have $\Omega_1 \supset \{-1 - \sigma \leq x_1 \leq 2, |x_2|, \ldots, |x_{n-1}| \leq 1\} \supset I_1^{n-1}$ and $\Omega_0 = \{2 \leq x_1 \leq 2 + \sigma\} \cap \Sigma$.

For a sufficiently small positive $\varepsilon < \frac{3\sigma}{2}$ consider a family of piecewise smooth continuous functions $g_s : [2 - s, 2 + \sigma] \to [0, \sigma\frac{3}{2}]$, $s \in [0, 3 + \sigma]$ defined by the formulas
$$
g_s(u) = \begin{cases} 
(u - 2 + s)\frac{3}{2}, & u \leq 2 - s + \varepsilon\frac{3}{2}; \\
\varepsilon, & 2 - s + \varepsilon\frac{3}{2} < u < 2 + \varepsilon\frac{3}{2}; \\
(u - 2)\frac{3}{2}, & u \geq 2 + \varepsilon\frac{3}{2}.
\end{cases}
$$

See Figure 4.4. We can smooth $g_s$ near the points $2 + \varepsilon\frac{3}{2}$ and $2 - s + \varepsilon\frac{3}{2}$ in such away that the derivative is monotone near these points (i.e. decreasing near $2 - s + \varepsilon\frac{3}{2}$ and increasing near $2 + \varepsilon\frac{3}{2}$). We continue to denote the smoothened by $g_s$.

Next, define for $s \in [0, 1]$ a function $G_s : \Omega_s \to \mathbb{R}$ by the formula
$$
G_s(x_1, x_2, \ldots, x_{n-1}) = g_s\Theta(x_2, \ldots, x_{n-1})(x_1).
$$

Note that by decreasing $\varepsilon$ and $\sigma$ we can arrange that $\frac{\partial G_s}{\partial s}(x), \frac{\partial G_s}{\partial x_i}(x) < \delta, i = 1, \ldots, n - 1$, for all $s \in [0, 1]$ and $x \in \Omega_s$. We also observe that if $\frac{\partial G_s}{\partial x_i}(x) = 0$ then $G_s(x) = \varepsilon$. Choose a cut-off function $\mu : [1 - \delta, 1 + \delta] \to [0, 1]$ which is equal to 1 near 1 and equal to 0 near $1 \pm \delta$ and consider a family of Lagrangian submanifolds $N_s$, $s \in [0, 1]$, defined in the domain $([1 - \delta, 1 + \delta] \times \hat{P}, d(t\alpha_{st}))$ in the symplectization of $\hat{P}$ defined by the formulas
$$
z = \pm G_{s(t)}(x) \pm t \frac{\partial G_{s(t)}}{\partial t}(x), y_i = \pm \frac{\partial G_{s(t)}}{\partial x_i}(x),
$$
$$
x \in \Omega_{s(t)}, i = 1, \ldots, n - 1, t \in [1 - \delta, 1 + \delta].$$
First, let us check that \( N_s \) is Lagrangian for all \( s \in [0, 1] \). Indeed, we have \( d(t\alpha_{st}) = -d \left( zdt + \sum_{i=1}^{n-1} (ty_i)dx_i \right) \), and hence

\[
d(t\alpha_{st})|_N = \pm d \left( \left( G_{s\mu(t)} + t \frac{\partial G_{s\mu(t)}}{\partial t} \right) dt + \sum_{i=1}^{n-1} t \frac{\partial G_{s\mu(t)}}{\partial x_i} dx_i \right) = \pm d(d(tG_{s\mu(t)})) = 0.
\]

Next, we check that \( N_s \) is embedded. The only possible pairs of double points may be of the form \((x, y, z)\) and \((x, -y, -z)\), that is \( z = 0 \) and \( y = 0 \). But then \( \frac{\partial G_{s\mu(t)}}{\partial x_1} = 0 \), and hence \( G_{s\mu(t)}(x) = \varepsilon \) and \( \frac{\partial G_{s\mu(t)}}{\partial t}(x) = 0 \), which shows \( z = G_{s\mu(t)}(x) + t\frac{\partial G_{s\mu(t)}}{\partial t}(x) \neq 0 \).

We also note that \( N_s \cap \{ t = 1 \} \) is a Legendrian submanifold \( \{ z = \pm G_{s\mu(t)}(x), y_i = \pm \frac{\partial G_{s\mu(t)}}{\partial x_i}(x), i = 1, \ldots, n-1 \} \subset \tilde{P} \) and \( N_1 \) intersects \( Q_C \) along \( \Lambda^{-\varepsilon} \cup \Lambda^\varepsilon \). Near \( t = 1 \pm \delta \) the submanifold \( N_s \) coincides with the symplectization of the Legendrian submanifold \( \tilde{\Lambda} \) for all \( s \in [0, 1] \).

Let us remove from the Lagrangian cylinder \( L = (0, \infty) \times \Lambda \subset ((0, \infty) \times Y, t\alpha) \) the domain \([1 - \delta, 1 + \delta] \times \Lambda \) and replace it by \( \Psi(N_s) \). The resulted Lagrangian isotopy \( L_s \) has the following properties: \( L_0 = L \), \( L_1 \) intersects the contact slice \( 1 \times Y \) along a Legendrian submanifold \( \Lambda_1 \) and \( \Phi^{-1}(\Lambda_1) = \Lambda^{-\varepsilon} \cup \Lambda^\varepsilon \). Note that if we modify the embedding \( \Phi \) as \( \tilde{\Phi}(x, y, z) = \Phi(x, y, z - \varepsilon) \) we still get a contact form preserving embedding \( \tilde{\Phi} : (Q_C, \alpha_{st}) \to (Y, \alpha) \) for which \( \tilde{\Phi}^{-1}(\Lambda_1) = \Lambda^{-2\varepsilon} \cup \Lambda^0 \).

Proof of Proposition 4.1 for \( n > 3 \). Let \( X_- \) be a negative Liouville end of \( X \) bounded by a contact slice \( Y \subset X \) such that \( f \) is cylindrical below it. Denote \( \Lambda := f^{-1}(Y) \).

According to Lemma 4.4 for any \( \varepsilon \) there exists a Hamiltonian regular homotopy of \( f \) into a Lagrangian immersion with transverse self-intersection points of action \( < \varepsilon \). Moreover, the number of self-intersection points grows proportionally to \( \frac{1}{\varepsilon} \) when \( \varepsilon \to 0 \). For a sufficiently small \( C > 0 \) there exists a contact form preserving embedding \( (Q_C, \alpha_{st}) \to (Y \setminus \Lambda, \alpha := \lambda|_{Y}) \). Note that given an integer \( N > 0 \) and a positive \( \varepsilon < \frac{C}{N} \) there exists contact form preserving embeddings of \( N^n \) disjoint copies of \( (Q_{\varepsilon}, \alpha_{st}) \) into \( (Q_C, \alpha_{st}) \), i.e. when decreasing \( \varepsilon \) the number of domains \( (Q_{\varepsilon}, \alpha_{st}) \) which can be packed into \((Y \setminus \Lambda, \alpha)\) grows proportionally to \( \varepsilon^{-n} \), which is greater than \( \varepsilon^{-3} \) by assumption. Hence for a sufficiently small \( \varepsilon \) we can modify the Lagrangian immersion \( f \), so that the action of all its self-intersection points are \( < \varepsilon \), and at least \( \text{SI}(f) \) disjoint Darboux neighborhoods isomorphic to \( Q_{12\varepsilon} \) which do not intersect \( \Lambda \) can be packed into \((Y, \alpha)\). We will denote the number of self-intersection points by \( N \) and the corresponding \( Q_{12\varepsilon} \)-neighborhoods by \( U_1, \ldots, U_N \). Notice that
for a sufficiently small \( \theta > 0 \) there exists a Liouville form preserving embedding \((0, 1 + \theta) \times Y, t\alpha) \to (X, \lambda)\) which sends \( Y \times 1 \) onto \( Y \).

For each intersection point \( p_i \in f(L), i = 1, \ldots, N \), we will find a compactly supported Hamiltonian regular homotopy to balance each intersection point \( p_i \) without changing the action of the other intersection points. Recall \( 0 < a_{SI}(p_1, f) < \varepsilon \). Using Lemma 4.8 we isotope the Lagrangian cylinder \((0, 1 + \theta) \times \Lambda\) via a Lagrangian isotopy supported in a neighborhood of \( Y \times 1 \) so that:

- the deformed cylinder \( \tilde{\Lambda} \) intersects \( Y \) transversely along a Legendrian submanifold \( \tilde{\Lambda} \);

- for a sufficiently small \( \sigma > 0 \) and each \( i = 1, \ldots, N \), the cylinder \( \tilde{\Lambda} \) intersects \( U_i = Q_{12\varepsilon} \) along Legendrian planes \( \Lambda^0 = \{ y = 0, z = 0 \} \) and \( \Lambda^{-\sigma} = \{ z = -\sigma, y = 0 \} \).

We can further deform the Lagrangian \( \tilde{\Lambda} \) to make it cylindrical in \( [\frac{1}{2}, 1] \times Y \), and hence, we get embeddings \( (\frac{1}{2}, 1] \times Q_{12\varepsilon}, t\alpha_{st}) \to ((0, 1] \times Y, t\alpha) \) such that the intersections \( (\frac{1}{2}, 1] \times U_i, \alpha_{st}) \) with \( \tilde{\Lambda} \) coincide with the Lagrangians \( \Lambda^0 \) and \( \Lambda^{-\delta} \) from Lemma 4.5.

There are two cases, depending on the sign of the intersection; suppose first that the self-intersection index at the point \( p_i \) is negative. Then we apply Lemma 4.5 with \( k = 0 \) and construct a cylindrical at \(-\infty\) and fixed everywhere except \( \Lambda^{-\delta} \) and \( \Lambda^{-\delta} \times (0, \frac{1}{2}] \) Hamiltonian regular homotopy of the immersion \( f \) which deforms \( \Lambda^{-\delta} \to \tilde{\Lambda}^{-\delta} \) such that \( L^0 \) and \( \tilde{\Lambda}^{-\delta} \) positively intersect at 1 point \( B_0 \) of action \( a_{SI}(B_0, f) = a_{SI}(p_i, f) \). Hence, the point \( B_0 \) balances \( p_i \). Notice that this homotopes \( \Lambda \) to another Legendrian \( \tilde{\Lambda} \), and in fact \( \tilde{\Lambda} \) will never be Legendrian isotopic to \( \Lambda \) (after a balancing of a sigle intersection point; we show below that it will be isotopic after all intersection points are balaned).

If the self-intersection index of \( p_i \) is positive we first apply Lemma 4.6 to create two new intersection points \( p_+ \) and \( p_- \) of index 1 and \(-1\) and action equal to \( A - \sigma \) and \( A + \sigma \) respectively, for some \( A \in (a_{SI}(p_i, f), a_{SI}(p_i, f) + 4\varepsilon) \) and sufficiently small \( \sigma > 0 \). We then apply Lemma 4.5 with \( k = 2 \) and create 3 new intersection points \( B_0, B_1, B_2 \) of indices \(-1, -1, -1\) and of action \( A + \sigma \), \( A - \sigma \) and \( a_{SI}(p_i, f) \), respectively. Then \( (p_i, B_2), (p_+, B_1) \) and \( (p_-, B_0) \) are balanced Whitney pairs.

In the course of the above proof, \( \Lambda \) is homotoped to the Legendrian \( \tilde{\Lambda} \) at \(-\infty\). In order to make the constructed Hamiltonian homotopy of our Lagrangian fixed at \(-\infty\), it suffices to show that \( \Lambda \) is Legendrian isotopic to \( \tilde{\Lambda} \), because we can then apply Lemma 3.4 to undo this homotopy near \(-\infty\). Assume that \( \Lambda \) has a loose component and \( I(f) = 0 \). In the course of the above proof we only need to homotope
a single component of $\Lambda$ of our choosing; we choose the component of $\Lambda$ which is loose. Obviously we can also fix a universal loose Legendrian embedded in this component of $\Lambda$, thus the corresponding component of $\tilde{\Lambda}$ is also loose. Using part (ii) of Proposition 2.1, it only remains to show that $\Lambda$ is formally Legendrian isotopic to $\tilde{\Lambda}$. Because the algebraic count of self intersections of $f$ is zero the homotopy from $\Lambda$ to $\tilde{\Lambda}$ also has an algebraic count of zero self-intersections. This implies that they are formally isotopic; see Proposition 2.6 in [7].

To deal with the case $n = 3$ we will need an additional lemma. Let us denote by $P(C)$ the polydisc $\{p_i^2 + q_i^2 \leq \frac{C}{\pi}, i = 1, \ldots, n\} \subset \mathbb{R}_{st}^{2n}$.

Lemma 4.9. Let $(X, \omega)$ be a symplectic manifold with a negative Liouville end, $Y \subset X$ a contact slice, and $\lambda$ is the corresponding Liouville form on a neighborhood $\Omega \supset X - Y$ in $X$. Suppose that there exists a symplectic embedding $\Phi : P(C) \to X_+ \setminus Y$. Let $\Gamma$ be an embedded path in $X_+$ connecting a point $a \in Y$ with a point in $b \in \partial \tilde{P}$, $\tilde{P} := \Phi(P(C))$. Then for any neighborhoods $U \supset (\Gamma \cup \tilde{P})$ in $X_+$ there exists a Weinstein cobordism $(W, \omega, \tilde{X}, \phi)$ such that

(i) $W \subset X_+ \cap (U \cup \Omega)$, $\partial_- W = Y$;

(ii) the Liouville form $\tilde{\lambda} = \iota(\tilde{X})\omega$ coincides with $\lambda$ near $Y$ and on $\Omega \setminus U$;

(iii) $\phi$ has no critical points;

(iv) the contact manifold $(\tilde{Y} := \partial_+ W, \tilde{\alpha} := \tilde{\lambda}|_{\tilde{Y}})$ admits a contact form preserving embedding $(Q_a, \alpha_{st}) \to (\tilde{Y}, \tilde{\alpha})$ for any $a < \frac{C}{2}$.

Proof. For any $b \in (a, \frac{C}{2})$ the domain $U_b := \{|q_i| \leq 1, |p_i| < b; i = 1, \ldots, n\} \subset \mathbb{R}_{st}^{2n}$ admits a symplectic embedding $H : U_b \to \text{Int} P(C)$. Denote $\partial_n U_b := \{p_n = b\} \cap \partial U_b$.

Consider a Liouville form $\mu = \sum_{i=1}^{n} (1 - \sigma)p_idq_i - \sigma q_idp_i = \sum_{i=1}^{n} p_idq_i - \sigma d\left(\sum_{i=1}^{n} p_iq_i\right)$, where a sufficiently small $\sigma > 0$ will be chosen later. Then

$$\beta := \mu|_{\partial_n U_b} = d\left(\left(b - \sigma\right)q_n - \sigma \sum_{i=1}^{n-1} p_iq_i\right) + \sum_{i=1}^{n-1} p_idq_i.$$ 

Let us verify that for a sufficiently small $\sigma > 0$ there exists a contact form preserving embedding $(Q_a, \alpha_{st}) \to (\partial_n U_b, \beta)$. Consider the map $\Psi : Q_a \to \mathbb{R}_{st}^{2n}$ given by the
formulas
\[ p_i = -y_i, q_i = x_i, \quad i = 1, \ldots, n - 1, \quad p_n = b, \quad q_n = \frac{z}{b - \sigma} - \frac{\sigma}{b - \sigma} \sum_{1}^{n-1} x_i y_i. \]

Note that \(|q_n| \leq \frac{a + \sigma(n-1)}{b - \sigma} < 1\) if \(\sigma < \frac{b-a}{n}\). Hence, if \((x, y, z) \in Q_a\) we have
\[ |p_i| \leq a < b, \quad |q_i| \leq 1 \quad \text{for} \quad i = 1, \ldots, n - 1, \quad p_n = b, \quad |q_n| < 1, \]
i.e. \(\Psi(Q_a) \subset \partial_n U_b\). On the other hand
\[ \Psi^* \mu = \Psi^* \beta = d \left( \frac{z + \sigma}{b - \sigma} \sum_{1}^{n-1} x_i y_i - \frac{\sigma}{b - \sigma} \sum_{1}^{n-1} x_i y_i \right) - \sum_{1}^{n-1} y_i dx_i = \alpha_{\text{st}}. \]

There exists a domain \(\widehat{U}_b\), diffeomorphic to a ball with smooth boundary, such that

- \(U_b \subset \widehat{U}_b \subset U'\) for some \(b' \in (b, \frac{C}{2})\);
- \(\partial \widehat{U}_b \supset \partial_n U_b\);
- \(\widehat{U}_b\) is transverse to the Liouville field \(T\), \(\omega\)-dual to the Liouville form \(\mu\).

Note that there exists a Lyapunov function \(\psi : \widehat{U}_b \to \mathbb{R}\) for \(T\) such that \((\widehat{U}_b, \omega, T, \phi)\) is a Weinstein domain.

Denote \(\widehat{U}_b := \Phi(H(U_a)) \subseteq X_+\). We can assume that the path \(\Gamma\) connects a point on \(Y\) with a point on \(\partial \widehat{U}_b \setminus \Phi(H(\partial_n U_b))\).

We modify the Liouville form \(\lambda\), making it equal to 0 on the path \(\Gamma\) and equal to \(\Phi_* H_* \mu\) on \(\widehat{U}_b\). Next, we use Lemma \ref{lem:Liouville} to construct the required cobordism \((W, \omega, \tilde{X}, \phi)\) by connecting \(X_-\) and \(\widehat{U}_b\) via a Weinstein surgery along \(\Gamma\), and then apply Proposition \ref{prop:4.1} to cancel the zeroes of the Liouville field \(\tilde{X}\). As a result we ensure properties (i)--(iii). In fact, property (iv) also holds. Indeed, by construction \(\partial_+ W \supset \Phi(H(\partial_n U_b))\), and hence there exists a contact form preserving embedding \((Q_a, \alpha_{\text{st}}) \to (\partial_+ W, \tilde{\alpha} := \iota(\tilde{X}) \omega|_{\partial_+ W}). \)

**Proof of Proposition \ref{prop:4.1} for \(n = 3\).** The problem in the case \(n = 3\) is that we cannot get sufficiently many disjoint contact neighborhoods \(Q_c\) embedded into \(Y\) to balance all the intersection points. Indeed, both the number of intersection of action < \(\varepsilon\) and the number of \(Q_{12\varepsilon}\)-neighborhoods one can pack into contact slice \(Y\) grow as
\( \varepsilon^{-3} \) when \( \varepsilon \to 0 \). However, using the infinite Gromov width assumption we can cite Lemma 4.9 to modify \( Y \) so that it would contain a sufficient number of disjoint neighborhoods isomorphic to \( Q_{12\varepsilon} \). Indeed, suppose that there are \( N \) double points of action < \( \varepsilon \). By the infinite Gromov width assumption there exists \( N \) disjoint embeddings of polydiscs \( P(24\varepsilon) \) into \( X_+ \setminus f(L) \).

Using Lemma 4.9 we modify the Liouville form \( \lambda \) into \( \tilde{\lambda} \) away from \( f(L) \), so that \( (X, \tilde{\lambda}) \) admits a negative end bounded by a contact slice \( \tilde{Y} \) such that there exists \( N \) disjoint embeddings \( (Q_{12\varepsilon}, \alpha_{st}) \to (\tilde{Y}, \tilde{\alpha}) \) preserving the contact form. The rest of the proof is identical to the case \( n > 3 \).

\section{Proof of main theorems}

\textbf{Proof of Theorem 2.3.} We first use Proposition 4.1 to make the Lagrangian immersion \( f \) balanced and then use the following modified Whitney trick to eliminate each balanced Whitney pair.

Let \( p, q \in X \) be a balanced Whitney pair, \( p^0, p^1 \in L \) and \( q^0, q^1 \in L \) the pre-images of the self-intersection points \( p, q \), and \( \gamma^0, \gamma^1 : [0, 1] \to L \) are the corresponding paths such that \( \gamma^j(0) = p^j \), \( \gamma^j(1) = q^j \) for \( j = 0, 1 \), the intersection index of \( df(T_{p^j}L) \) and \( df(T_{p^j}L) \) is equal to 1 and the intersection index of \( df(T_{q^j}L) \) and \( df(T_{q^j}L) \) is equal to \( -1 \). Recall that according to our convention we are always ordering the pre-images of double points in such a way that their action is positive.

Choose a contact slice \( Y \), and consider a path \( \eta : [0, 1] \to L \) connecting a point in the loose component \( \Lambda \) of \( \partial L_+ \) with \( p \) such that \( \eta := f \circ \eta \) coincides with a trajectory of \( Z \) near the point \( \eta(0) \), and then modify the Liouville form \( \lambda \), keeping it fixed on \( X_- \), to make it equal to 0 on \( \eta \). We further modify \( \lambda \) in a neighborhood of \( \tau^0 \) making it 0 on \( \tau^0 \), where we use the notation \( \tau^0 := f \circ \gamma^0 \), \( \gamma^1 := f \circ \gamma^1 \). Note that this is possible because \( Y \cup \eta \cup \eta^0 \) deformation retracts to \( Y \). Assuming that this is done, we observe that \( \int_{\tau^1} \lambda = \int_{\tau^0} \lambda = 0 \).

Next, we use Lemma 3.2 to construct Darboux charts \( B_p \) and \( B_q \) centered at the points \( p \) and \( q \) such that the the intersecting branches in these coordinates look like coordinate Lagrangian planes \( \{ q = 0 \} \) and \( \{ p = 0 \} \) in the standard \( \mathbb{R}^{2n} \). Set \( \lambda_{st} := \frac{1}{2} \sum_{i=1}^{n} p_i dq_i - q_i dp_i \). Then the corresponding to it Liouville vector field \( Z_{st} = \frac{1}{2} \sum_{i=1}^{n} q_i \frac{\partial}{\partial q_i} + p_i \frac{\partial}{\partial p_i} \) is tangent to the Lagrangian planes through the origin.
We have $\lambda_{st} - \lambda = dH$ in $B_p \cup B_q$. Choosing a cut-off function $\alpha$ on $B_p \cup B_q$ which is equal to 1 near $p$ and $q$ and equal to 0 near $\partial B_p \cup \partial B_q$ we define $\lambda_1 := \lambda + d(\alpha H)$. The Liouville structure $\lambda_1$ coincides with the standard structure $\lambda_{st}$ in smaller balls around the points $p$ and $q$, and with $\lambda$ near $\partial B_p \cup \partial B_q$.

Next, we use Lemma 3.1 to modify the Liouville structure $\lambda_1$ in neighborhoods of paths $\gamma^0$ and $\gamma^1$ and create Weinstein domain $C$ by attaching handles of index 1 with $\gamma^0$ and $\gamma^1$ as their cores. The corresponding Lyapunov function on $C$ has two critical points of index 0, at $p$ and $q$, and two critical points of index 1, at the centers of paths $\gamma^0$ and $\gamma^1$. Note that the property $\int_{\gamma^j} \lambda_1 = 0$, $j = 0, 1$, is crucial in order to apply Lemma 3.1.

Next, we choose an embedded isotropic disc $\Delta \subset X_+ \setminus \text{Int} C$ with boundary in $\partial C$, tangent to $Z$ along the boundary $\partial \Delta$, and such that $\partial \Delta$ is isotropic, and homotopic in $C$ to the loop $\tilde{\gamma}^0 \cup \tilde{\gamma}^1$. We then again use Lemma 3.1 to attach to $C$ a handle of index 2 with the core $\Delta$. The resulted Liouville domain $\tilde{C}$ is diffeomorphic to the $2n$-ball. Moreover, according to Proposition 3.3 the Weinstein structure on $\tilde{C}$ is homotopic to the standard one via a homotopy fixed on $\partial \tilde{C}$. In particular, the contact structure induced on the sphere $\partial \tilde{C}$ is the standard one. The immersed Lagrangian manifold $f(L)$ intersects $\partial \tilde{C}$ along two Legendrian spheres $\Lambda^0$ and $\Lambda^1$, each of which is the standard Legendrian unknot which bounds an embedded Lagrangian disc inside $\tilde{C}$. These two discs intersect at two points, $p$ and $q$. Note that the Whitney trick allows us to disjoint these discs by a smooth (non-Lagrangian) isotopy fixed on their boundaries. In particular, the spheres $\Lambda^0$ and $\Lambda^1$ are smoothly unlinked. If they were unlinked as Legendrians we would be done. Indeed, the Legendrian unlink in $S^{2n-1}_{\text{std}}$ bounds two disjoint exact Lagrangian disks in $B^{2n}_{\text{std}}$. Unfortunately (or fortunately, because this would kill Symplectic Topology as a subject!), one can show that it is impossible to unlink $\Lambda^0$ and $\Lambda^1$ via a Legendrian isotopy.

The path $\tilde{\eta}$ intersects $\partial \tilde{C}$ at a point in $\Lambda^0$. Slightly abusing the notation we will continue using the notation $\tilde{\eta}$ for the part of $\tilde{\eta}$ outside the ball $\tilde{C}$. We then use Lemma 3.1 one more time to modify $\lambda_1$ by attaching a handle of index 1 to $X_- \cup \tilde{C}$ along $\tilde{\eta}$. As a result, we create inside $X_+$ a Weinstein cobordism $W$ which contains $\tilde{C}$, so that $\partial_- W = Y$ and $\tilde{Y} := \partial_+ W$ intersects $f(L)$ along a 2-component Legendrian link. One of its components is $\Lambda^1$, and the other one is the connected sum of the loose Legendrian $\Lambda$ and the Legendrian sphere $\Lambda^0$, which we denote by $\tilde{\Lambda}$. Again applying Proposition 3.3 we can deform the Weinstein structure on $W$ keeping it fixed on $\partial W$ to kill both critical points inside $W$. Hence all trajectories of the (new) Liouville vector field $Z$ inside $W$ begin at $Y$ and end at $\tilde{Y}$, and thus $W$ is Liouville isomorphic...
to $\tilde{Y} \times [0, T]$ for some $T$ (with Liouville form $e^t\lambda_1$, $t \in [0, T]$). We also note that the intersection of $f(L)$ with $W$ consists of two embedded Lagrangian submanifolds $A$ and $B$ transversely intersecting in the points $p, q$, where

- $A$ is diffeomorphic to the cylinder $\Lambda \times [0, 1]$, $A \cap Y = \Lambda$ and $A \cap \tilde{Y} = \tilde{\Lambda}$;
- $B$ is a disc bounded by the Legendrian sphere $\Lambda^1 = B \cap \tilde{Y}$.

The Legendrian $\tilde{\Lambda}$ is smoothly unlinked with $\Lambda^1$. Since $\tilde{\Lambda}$ is loose, Proposition 2.1 implies that there is a Legendrian isotopy of $\Lambda$ to $\hat{\Lambda}$ which is disjoint from a Darboux ball containing $\Lambda^1$. We realize this isotopy by a Lagrangian cobordism $A_1$ from $\Lambda$ to $\hat{\Lambda}$ using Lemma 3.4 and also realize the inverse isotopy by a Lagrangian cobordism $A_2$ from $\hat{\Lambda}$ to $\tilde{\Lambda}$. For some $\tilde{T}$, these cobordisms embed into $\tilde{Y} \times [0, \tilde{T}]$. Inside $\tilde{Y} \times [0, 2\tilde{T} + 2T]$, we define a cobordism $\tilde{A}$ from $\Lambda$ to $\tilde{\Lambda}$, built from the following pieces.

- $\tilde{A} \cap \tilde{Y} \times [0, T] = A$,
- $\tilde{A} \cap \tilde{Y} \times [T, \tilde{T} + T] = A_1$,
- $\tilde{A} \cap \tilde{Y} \times [\tilde{T} + T, \tilde{T} + 2T] = \hat{\Lambda} \times [\tilde{T} + T, \tilde{T} + 2T]$,
- $\tilde{A} \cap \tilde{Y} \times [\tilde{T} + 2T, 2\tilde{T} + 2T] = A_2$.

We then define $\tilde{B}$ by

- $\tilde{B} \cap \tilde{Y} \times [0, \tilde{T} + T] = \emptyset$,
- $\tilde{B} \cap \tilde{Y} \times [\tilde{T} + T, \tilde{T} + 2T] = B$,
- $\tilde{B} \cap \tilde{Y} \times [\tilde{T} + 2T, 2\tilde{T} + 2T] = \Lambda^1 \times [\tilde{T} + 2T, 2\tilde{T} + 2T]$.

A schematic of these cobordisms is given in Figure 5.1. After elongating $W$ (which can be achieved by choosing a contact slice closer to $-\infty$), $\tilde{A} \cup \tilde{B}$ can be deformed to $A \cup B$ via a Hamiltonian compactly supported regular homotopy fixed on the boundary. We then define $\tilde{f} : L \to X$ to be equal to $f$ everywhere, except the portions of $L$ which are mapped to $A$ and $B$ are instead mapped to $\tilde{A}$ and $\tilde{B}$, respectively. \qed
Proof of Theorem 2.2. We first use Gromov's $h$-principle for Lagrangian immersions \cite{6} to find a compactly supported regular homotopy starting at $f$ and ending at a Lagrangian immersion $\tilde{f}$ with the prescribed action class $A(f)$ (or the action class $a(f)$ in the Liouville case). More precisely, let us choose a triangulation of $L$. There are finitely many simplices of the triangulation which cover the compact part of $L$ where the embedding $f$ is not yet Lagrangian. Let $K$ be the polyhedron which is formed by these simplices. Using the $h$-principle for open Lagrangian immersions, we first isotope $f$ to an embedding which is Lagrangian near the $(n-1)$-skeleton of $K$, realizing the given (relative) action class. Let us inscribe an $n$-disc $D_i$ in each of the $n$-simplices of $K$, such that the embedding $f$ is already Lagrangian near $\partial D_i$. Next, we thicken $D_i$ to disjoint $2n$-balls $B_i \subset X$ intersecting $f(L)$ along $D_i$. We then apply Gromov's $h$-principle for Lagrangian immersions in a relative form to find for each $i$ a fixed near the boundary regular homotopy $D_i \to B_i$ of $D_i$ into a Lagrangian immersion. Note that all the self-intersection points of the resulted Lagrangian immersion $\tilde{f}$ are localized inside the ball $B_i$ and images of different discs $D_i$ and $D_j$ do not intersect.

Let us choose a negative end $X_-$, bounded by a contact slice $Y$ in such a way that the immersion $\tilde{f}$ is cylindrical in it and $X_- \cap \bigcup B_i = \emptyset$. Denote $L_- := \tilde{f}^{-1}(X_-), \Lambda_- = \Lambda(A_B)$.
∂L_. Let us choose a universal loose Legendrian U ⊂ Y for the Legendrian submanifold Λ_ ⊆ Y. Denote \( \tilde{\Lambda}_- = \Lambda_- \cap U \). Let \( V_- := \bigcup_0^\infty Z^{-s}(U) \subset X_- \) be the domain in \( X_- \) formed by all negative trajectories of Z intersecting \( U \). Let us choose disjoint paths \( \Gamma_i \) in \( L \setminus \mathrm{Int}(L_- \cup \bigcup_i D_i) \) connecting some points in \( \tilde{\Lambda}_- \) with points \( z_i \in \partial D_i \) for each \( n \)-simplex in \( K \). Choose small tubular neighborhoods \( U_i \) of \( \tilde{f}(\Gamma_i) \) in \( X_\) and \( \tilde{\Lambda}_- \).

Set

\[
\tilde{X} := V_- \cup \bigcup_i (B_i \cup U_i) \quad \text{and} \quad \tilde{L} := \tilde{f}^{-1}(\tilde{X}).
\]

The manifold \( \tilde{X} \) deformationaly retracts to \( V_- \) and hence \( \tilde{X} \) is contractible and the Liouville form \( \lambda|_{V_-} \) extends as a Liouville form for \( \omega \) on the whole manifold \( \tilde{X} \). We will keep the notation \( \lambda \) for the extended form. Thus \( \tilde{L} \) is an exact Lagrangian immersion into the contractible Liouville manifold \( \tilde{X} \), cylindrical at \( -\infty \) over a loose Legendrian submanifold of \( U \). Moreover, \( L \) is diffeomorphic to \( \mathbb{R}^n \), and outside a compact set the immersion is equivalent to the standard inclusion \( \mathbb{R}^n \hookrightarrow \mathbb{R}^{2n} \). We also note that \( I(\tilde{f}|_{\tilde{L}} : \tilde{L} \to \tilde{X}) = 0 \) since this immersion is regularly homotopic to the smooth embedding \( \tilde{f}|_{\tilde{L}} : \tilde{L} \to \tilde{X} \).

Applying Theorem 2.3 to \( \tilde{f}|_{\tilde{L}} \) we find an exact Lagrangian embedding \( \hat{f} \) which is regularly Hamiltonian homotopic to \( \tilde{f}|_{\tilde{L}} \) via a regular homotopy compactly supported in \( \tilde{X} \). We further note that the embeddings \( \hat{f} \) and \( f : \tilde{L} \to \tilde{X} \) are isotopic relative the boundary. Indeed, it follows from the \( h \)-cobordism theorem that an embedding \( \mathbb{R}^n \to \mathbb{R}^{2n} \) which coincides with the inclusion outside a compact set and which is regularly homotopic to it via a compactly supported homotopy is isotopic to the inclusion relative infinity.

Slightly abusing notation we define \( \hat{f} : L \to X \) to be equal to \( \tilde{f} \) on \( L \setminus \hat{L} \). This Lagrangian embedding is isotopic to \( f \) via an isotopy fixed outside a compact set. Finally we note that \( df : TL \to TX \) is homotopic to \( \Phi_1 \) since it is constructed with the \( h \)-principle for Lagrangian immersions, and \( d\hat{f} \) is homotopic to \( df \) since they are regularly Lagrangian homotopic.

Next, we deduce Theorem 1.1 from Theorem 2.2.

Proof of Theorem 1.1. Let \( B \) be the unit ball in \( \mathbb{R}^{2n} \). The triviality of the bundle \( T(L) \otimes \mathbb{C} \) is equivalent to existence of a Lagrangian homomorphism \( \Phi : TL \to T\mathbb{C}^n \).

We can assume that \( \Phi \) covers a map \( \phi : L \to \mathbb{C}^n \setminus \mathrm{Int} B \) such that \( \phi(\partial L) \subset \partial B \). Let \( v \in TL|_{\partial L} \) be the inward normal vector field to \( \partial L \) in \( L \), and \( v \) an outward normal to the boundary \( \partial B \) of the ball \( B \subset \mathbb{C}^n \). Homomorphism \( \Phi \) is homotopic to a
Lagrangian homomorphism, which will still be denoted by $\Phi$, sending $v$ to $\nu$. Indeed, the obstructions to that lie in trivial homotopy group $\pi_j(S^{2n-1})$, $j \leq n - 1$. Then $\Phi|_{\partial L}$ is a Legendrian homomorphism $T\partial L \to \xi$, where $\xi$ is the standard contact structure on the sphere $\partial B$ formed by its complex tangencies. Using Gromov’s $h$-principle for Legendrian embeddings we can, therefore, assume that $\phi|_{\partial L}: \partial L \to \partial B$ is a Legendrian embedding, and then, using Gromov’s $h$-principle for Lagrangian immersions deform $\phi$ to an exact Lagrangian immersion $\phi: L \to \mathbb{C}^n \setminus \text{Int} B$ with Legendrian boundary in $\partial B$ and tangent to $\nu$ along the boundary. Finally, we use Theorem 2.2 to make $\phi$ a Lagrangian embedding.

\section{Applications}

\subsection*{Lagrangian embeddings with a conical singular point}

Given a symplectic manifold $(X,\omega)$ we say that $L \subset M$ is a \textit{Lagrangian submanifold with an isolated conical point} if it is a Lagrangian submanifold away from a point $p \in L$, and there exists a symplectic embedding $f: B_\varepsilon \to X$ such that $f(0) = p$ and $f^{-1}(L) \subset B_\varepsilon$ is a Lagrangian cone. Here $B_\varepsilon$ is the ball of radius $\varepsilon$ in the standard symplectic $\mathbb{R}^{2n}$. Note that this cone is automatically a cone over a Legendrian sphere in the sphere $\partial B_\varepsilon$ endowed with the standard contact structure given by the restriction to $\partial B_\varepsilon$ of the Liouville form $\lambda_{st} = \frac{1}{2} \sum_{i=1}^{n} (p_i dq_i - q_i dp_i)$.

As a special case of Theorem 1.1 (when $\partial L$ is a sphere) we get

\textbf{Corollary 6.1.} Let $L$ be an $n$-dimensional, $n > 2$, closed manifold such that the complexified tangent bundle $T^*(L \setminus p) \otimes \mathbb{C}$ is trivial. Then $L$ admits an exact Lagrangian embedding into $\mathbb{R}^{2n}$, with exactly one conical point. In particularly a sphere admits a Lagrangian embedding to $\mathbb{R}^{2n}$ with one conical point for each $n > 2$.

\subsection*{Flexible Weinstein cobordisms}

The following notion of a flexible Weinstein cobordism is introduced in [1].

A Weinstein cobordism $(W,\omega, Z, \phi)$ is called \textit{elementary} if there are no $Z$-trajectories connecting critical points. In this case stable manifolds of critical points intersect $\partial_+ W$ along isotropic in the contact sense submanifolds. For each critical point $p$ we call the intersection $S_p$ of its stable manifold with $\partial_+ W$ the \textit{attaching sphere}. The attaching spheres for index $n$ critical points are Legendrian.
An elementary Weinstein cobordism \((W, \omega, Z, \phi)\) is called \textit{flexible} if the attaching spheres for all index \(n\) critical points in \(W\) form a loose Legendrian link in \(\partial_-W\).

A Weinstein cobordism \((W, \omega, Z, \phi)\) is called \textit{flexible} if it can be partitioned into elementary Weinstein cobordisms: \(W = W_1 \cup \cdots \cup W_N, W_j := \{c_{j-1} \leq \phi \leq c_j\}, j = 1, \ldots, N, m = c_0 < c_1 < \cdots < c_N = M\). Any subcritical Weinstein cobordism is by definition flexible.

\textbf{Theorem 6.2.} Let \((W, \omega, Z, \phi)\) be a flexible Weinstein domain. Let \(\lambda\) be the Liouville form \(\omega\)-dual to \(Z\), and \(\Lambda\) any other Liouville form such that the symplectic structures \(\omega, \Omega := d\Lambda\) are homotopic as non-degenerate (not necessarily closed) 2-forms. Then there exists an isotopy \(h_t : W \to W\) such that \(h_0 = \text{Id}\) and \(h_t^*\Lambda = \varepsilon \lambda + dH\) for a sufficiently small \(\varepsilon > 0\) and a smooth function \(H : W \to \mathbb{R}\). In particular, \(h_1\) is a symplectic embedding \((W, \varepsilon\omega) \to (W, \Omega)\).

Recall that a Weinstein cobordism \((W, \omega, Z, \phi)\) is called a \textit{Weinstein domain} if \(\partial_- W = \emptyset\).

\textbf{Corollary 6.3.} Let \((W, \omega, Z, \phi)\) be a flexible Weinstein domain, and \((X, \Omega)\) any symplectic manifold of the same dimension. If this dimension is 3 we further assume that \(X\) has infinite Gromov width. Then any smooth embedding \(f_0 : W \to X\), such that the form \(f_0^*\Omega\) is exact and the differential \(df : TW \to TX\) is homotopic to a symplectic homomorphism, is isotopic to a symplectic embedding \(f_1 : (W, \varepsilon\omega) \to (X, \Omega)\) for a sufficiently small \(\varepsilon > 0\). Moreover, if \(\Omega = d\Theta\) then the embedding \(f_1\) exists for an arbitrarily large constant \(\varepsilon\).

\textit{Proof of Theorem 6.2.} Let us decompose \(W\) into flexible elementary cobordisms: \(W = W_1 \cup \cdots \cup W_k\), where \(W_j := \{c_{j-1} \leq \phi \leq c_j\}\), \(j = 1, \ldots, k\) for a sequence of regular values \(c_0 < \min \phi < c_1 < \cdots < c_k = \max \phi\) of the function \(\phi\). Set \(V_j = \bigcup_i W_i\) for \(j \geq 1\) and \(V_0 = \emptyset\).

We will construct an isotopy \(h_t : W \to W\) beginning from \(h_0 = \text{Id}\) inductively over cobordisms \(W_j, j = 1, \ldots, k\). It will be convenient to parameterize the required isotopy by the interval \([0, 2k]\). Suppose that for some \(j = 1, \ldots, k\) we already constructed an isotopy \(h_t : W_j \to W, t \in [0, j - 1]\) such that \(h_{j-1}^*\Lambda = \varepsilon_{j-1}\lambda + dH\) on \(V_{j-1}\). Our goal is to extend it \([j - 1, j]\) to ensure that \(h_j\) satisfies this condition on \(V_j\). Without loss of generality we can assume that there exists only 1 critical point \(p\) of \(\phi\) in \(W_j\). Let \(\Delta\) be the stable disc of \(p\) in \(W_j\) and \(S := \partial \Delta \subset \partial_- W_j\)
the corresponding attaching sphere. By assumption, $S$ is subcritical or loose. The homotopical condition implies that there is a family of injective homomorphisms $\Phi_t: T\Delta \to TW$, $t \in [j - 1, j]$, such that $\Phi_{j-1} = dh_{j-1}|_{\Delta_j}$, and $\Phi_j: T\Delta_j \to (TW, \Omega)$ is an isotropic homomorphism. We also note that the cohomological condition implies that $\int_{\Delta} \Omega = 0$ when $\dim \Delta = 2$. Then, using Theorem 2.2 when $\dim \Delta = n$ and Gromov’s $h$-principle, \cite{6}, for isotropic embeddings in the subcritical case, we can construct an isotopy $g_t: \Delta \to W_j$, $t \in [j - 1, j]$, fixed at $\partial \Delta$, such that $g_{j-1} = h_{j-1}|_{\Delta}$ is the inclusion and the embedding $g_j: \Delta \to (W, \Omega)$ is isotropic. Furthermore, there exists a neighborhood $U \supset \Delta$ in $W_j$ such that the isotopy $g_t$ extends as a fixed on $W_{j-1}$ isotopy $G_t: W_{j-1} \cup U \to W$ such that $G_{t}|_{\Delta} = g_t$, $G_t|_{W_j} = h_{j-1}|_{W_{j-1}}$ for $t \in [j - 1, j]$, $G_{j-1}|_U = h_{j-1}|_U$ and $h_j: (W_{j-1} \cup U, \varepsilon_{j-1}\omega) \to (W, \Omega)$ is a symplectic embedding. Choose a sufficiently large $T > 0$ we have $Z^{-T}(W_j) \subset W_{j-1} \cup U_j$, and hence $h_j \circ e^{-T}|_{V_j}$ is a symplectic embedding $(W_j, \varepsilon_j\omega) \to (W, \Omega)$, where we set $\varepsilon_j := e^{-T}\varepsilon_{j-1}$. Then we can define the required isotopy $h_t: W \to W$, $t \in [j - 1, j]$, which satisfy the property that $h_j|_{V_j}$ is a symplectic embedding $(V_j, \varepsilon_j\omega) \to (W, \omega)$ by setting

$$h_t = \begin{cases} h_{j-1} \circ Z^{-2T(t-j+1)} & \text{for } t \in [j - 1, j - \frac{1}{2}], \\ G_t \circ Z^{-T} & \text{for } t \in [j - \frac{1}{2}, j]. \end{cases}$$

\end{proof}

References


