# On Some Partitions of a Flag Manifold

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<tr>
<td><strong>Publisher</strong></td>
<td>International Press of Boston, Inc.</td>
</tr>
<tr>
<td><strong>Version</strong></td>
<td>Original manuscript</td>
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<tr>
<td><strong>Accessed</strong></td>
<td>Wed Jan 02 02:45:33 EST 2019</td>
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ON SOME PARTITIONS OF A FLAG MANIFOLD

G. Lusztig

INTRODUCTION

Let $G$ be a connected reductive group over an algebraically closed field $k$ of characteristic $p \geq 0$. Let $W$ be the Weyl group of $G$. Let $W$ be the set of conjugacy classes in $W$. The main purpose of this paper is to give a (partly conjectural) definition of a surjective map from $W$ to the set of unipotent classes in $G$ (see 1.2(b)). When $p = 0$, a map in the opposite direction was defined in [KL, 9.1] and we expect that it is a one sided inverse of the map in the present paper. The (conjectural) definition of our map is based on the study of certain subvarieties $B^w_g$ (see below) of the flag manifold $B$ of $G$ indexed by a unipotent element $g \in G$ and an element $w \in W$.

Note that $W$ naturally indexes ($w \mapsto O_w$) the orbits of $G$ acting on $B \times B$ by simultaneous conjugation on the two factors. For $g \in G$ we set $B_g = \{B \in B; g \in B\}$. The varieties $B_g$ play an important role in representation theory and their geometry has been studied extensively. More generally for $g \in G$ and $w \in W$ we set

$$B^w_g = \{B \in B; (B, gBg^{-1}) \in O_w\}.$$ 

Note that $B^1_g = B_g$ and that for fixed $g$, $(B^w_g)_{w \in W}$ form a partition of the flag manifold $B$.

For fixed $w$, the varieties $B^w_g$ ($g \in G$) appear as fibres of a map to $G$ which was introduced in [L3] as part of the definition of character sheaves. Earlier, the varieties $B^w_g$ for $g$ regular semisimple appeared in [L1] (a precursor of [L3]) where it was shown that from their topology (for $k = \mathbb{C}$) one can extract nontrivial information about the character table of the corresponding group over a finite field.

I thank David Vogan for some useful discussions.

1. The sets $S_g$

1.1. We fix a prime number $l$ invertible in $k$. Let $g \in G$ and $w \in W$. For $i, j \in \mathbb{Z}$ let $H^i_c(B^w_g, \mathbb{Q}_l)_j$ be the subquotient of pure weight $j$ of the $l$-adic cohomology space

Supported in part by the National Science Foundation

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\[ H^i_c(B^w_g, \mathbb{Q}_l) \] The centralizer \( Z(g) \) of \( g \) in \( G \) acts on \( B^w_g \) by conjugation and this induces an action of the group of components \( \tilde{Z}(g) \) on \( H^i_c(B^w_g, \mathbb{Q}_l) \) and on each \( H^i_c(B^w_g, \mathbb{Q}_l)_j \). For \( z \in \tilde{Z}(g) \) we set
\[
\Xi_{g,z}^w = \sum_{i,j \in \mathbb{Z}} (-1)^i \text{tr}(z, H^i_c(B^w_g, \mathbb{Q}_l)_j) v^j \in \mathbb{Z}[v]
\]
where \( v \) is an indeterminate; the fact that this belongs to \( \mathbb{Z}[v] \) and is independent of the choice of \( l \) is proved by an argument similar to that in the proof of [DL, 3.3].

Let \( l : W \to N \) be the standard length function. The simple reflections \( s \in W \) (that is the elements of length 1 of \( W \)) are numbered as \( s_1, s_2, \ldots \). Let \( w_0 \) be the element of maximal length in \( W \).

Let \( \mathcal{H} \) be the Iwahori-Hecke algebra of \( W \) with parameter \( v^2 \) (see [GP, 4.4.1]; in the definition in loc.cit. we take \( A = \mathbb{Z}[v, v^{-1}], a_s = b_s = v^2 \)). Let \( (T_w)_{w \in W} \) be the standard basis of \( \mathcal{H} \) (see [GP, 4.4.3, 4.4.6]). For \( w \in W \) let \( \hat{T}_w = v^{-2l(w)} T_w \). If \( s_{i_1} s_{i_2} \ldots s_{i_r} \) is a reduced expression for \( w \in W \) we write also \( \hat{T}_w = \hat{T}_{i_1 i_2 \ldots i_r} \).

For any \( g \in G, z \in \tilde{Z}(g) \) we set
\[
\Pi_{g,z} = \sum_{w \in W} \Xi_{g,z}^w \hat{T}_w \in \mathcal{H}.
\]
The following result can be proved along the lines of the proof of [DL, Theorem 1.6] (we replace the Frobenius map in that proof by conjugation by \( g \)); alternatively, for \( g \) unipotent, we may use 1.5(a).

(a) \( \Pi_{g,z} \) belongs to the centre of the algebra \( \mathcal{H} \).

According to [GP, 8.2.6, 7.1.7], an element \( c = \sum_{w \in W} c_w \hat{T}_w \) (\( c_w \in \mathbb{Z}[v, v^{-1}] \)) in the centre of \( \mathcal{H} \) is uniquely determined by the coefficients \( c_w (w \in W_{\text{min}}) \) and we have \( c_w = c_{w'} \) if \( w, w' \in W_{\text{min}} \) are conjugate in \( W \); here \( W_{\text{min}} \) is the set of elements of \( W \) which have minimal length in their conjugacy class. This applies in particular to \( c = \Pi_{g,z} \), see (a). For any \( C \in \mathbb{W} \) we set \( \Xi_{g,z}^C = \Xi_{g,z}^w \) where \( w \) is any element of \( C \cap W_{\text{min}} \).

Note that if \( g = 1 \) then \( \Pi_{g,1} = \sum_{w} v^{2l(w)} \hat{T}_w \). If \( g \) is regular unipotent then \( \Pi_{g,1} = \sum_{w} v^{2l(w)} \hat{T}_w \). If \( G = PGL_3(k) \) and \( g \in G \) is regular semisimple then \( \Pi_{g,1} = 6 + 3(v^2 - 1)(\hat{T}_1 + \hat{T}_2) + (v^2 - 1)^2(\hat{T}_{12} + \hat{T}_{21}) + (v^6 - 1)\hat{T}_{121} \); if \( g \in G \) is a transvection then \( \Pi_{g,1} = (2v^2 + 1) + v^4(\hat{T}_1 + \hat{T}_2) + v^6\hat{T}_{121} \).

For \( g \in G \) let \( cl(g) \) be the \( G \)-conjugacy class of \( g \); let \( \overline{cl(g)} \) be the closure of \( cl(g) \). Let \( S_g \) be the set of all \( C \in \mathbb{W} \) such that \( \Xi_{g,1}^C \neq 0 \) and \( \Xi_{g',1}^C = 0 \) for any \( g' \in \overline{cl(g)} - cl(g) \). If \( C \) is a conjugacy class in \( G \) we shall also write \( S_C \) instead of \( S_g \) where \( g \in C \).

We describe the set \( S_g \) and the values \( \Xi_{g,1}^C \) for \( C \in S_g \) for various \( G \) of low rank and various unipotent elements \( g \) in \( G \). We denote by \( u_n \) a unipotent element of \( G \) such that \( \dim B_{u_n} = n \). The conjugacy class of \( w \in W \) is denoted by \( (w) \).
$G$ of type $A_1$.

$$S_{u_1} = (1), S_{u_0} = (s_1); \Xi_{u_1,1}^1 = 1 + v^2, \Xi_{u_0,1}^{s_1} = v^2.$$ 

$G$ of type $A_2$.

$$S_{u_3} = (1), S_{u_1} = (s_1), S_{u_0} = (s_1 s_2).$$

$$\Xi_{u_3,1}^1 = 1 + 2v^2 + 2v^4 + v^6, \Xi_{u_1,1}^{s_1} = v^4, \Xi_{u_0,1}^{s_1 s_2} = v^4.$$ 

$G$ of type $B_2$, $p \neq 2$. (The simple reflection corresponding to the long root is denoted by $s_1$.)

$$S_{u_4} = (1), S_{u_2} = (s_1), S_{u_1} = \{(s_2), (s_1 s_2 s_1 s_2)\}, S_{u_0} = (s_1 s_2).$$

$$\Xi_{u_4,1}^1 = (1 + v^2)^2(1 + v^4), \Xi_{u_2,1}^{s_1} = v^4(1 + v^2), \Xi_{u_1,1}^{s_2} = 2v^4,$$

$$\Xi_{u_1,1}^{s_1 s_2 s_1 s_2} = v^6(v^2 - 1), \Xi_{u_0,1}^{s_1 s_2} = v^4.$$ 

$G$ of type $B_2$, $p = 2$. ($u'_2$ denotes a transvection; $u''_2$ denotes a unipotent element with $\dim B_{u''_2} = 2$ which is not conjugate to $u'_2$.)

$$S_{u_4} = (1), S_{u'_2} = (s_1), S(u''_2) = (s_2), S_{u_1} = (s_1 s_2 s_1 s_2), S_{u_0} = (s_1 s_2).$$

$$\Xi_{u_4,1}^1 = (1 + v^2)^2(1 + v^4), \Xi_{u_2,1}^{s_1} = v^4(1 + v^2), \Xi_{u'_2,1}^{s_2} = v^4(1 + v^2),$$

$$\Xi_{u_1,1}^{s_1 s_2 s_1 s_2} = v^8, \Xi_{u''_2,1}^{s_1 s_2} = v^4.$$ 

$G$ of type $G_2$, $p \neq 2, 3$. (The simple reflection corresponding to the long root is denoted by $s_2$.)

$$S_{u_6} = (1), S_{u_3} = (s_2), S_{u_2} = \{(s_1), (s_1 s_2 s_1 s_2 s_1 s_2)\}, S_{u_1} = (s_1 s_2 s_1 s_2),$$

$$S_{u_0} = (s_1 s_2).$$

$$\Xi_{u_6,1}^1 = (1 + v^2)^2(1 + v^4 + v^8), \Xi_{u_3,1}^{s_2} = v^6(1 + v^2), \Xi_{u_2,1}^{s_1} = v^4(1 + v^2),$$

$$\Xi_{u_2,1}^{s_1 s_2 s_1 s_2 s_1 s_2} = v^8(v^4 - 1), \Xi_{u_1,1}^{s_1 s_2 s_1 s_2} = 2v^8, \Xi_{u_0,1}^{s_1 s_2} = v^4.$$ 

$G$ is of type $A_3$. (The simple reflections are $s_1, s_2, s_3$ with $s_1 s_3 = s_3 s_1$.)

$$S_{u_6} = (1), S_{u_3} = (s_1), S_{u_2} = (s_1 s_3), S_{u_1} = (s_1 s_2), S_{u_0} = (s_1 s_2 s_3).$$

$$\Xi_{u_6,1}^1 = (1 + v^2)(1 + v^2 + v^4)(1 + v^2 + v^4 + v^6), \Xi_{u_3,1}^{s_1} = v^6 + v^8,$$

$$\Xi_{u_2,1}^{s_1 s_3} = v^6 + v^8, \Xi_{u_1,1}^{s_1 s_2} = v^6, \Xi_{u_0,1}^{s_1 s_2 s_3} = v^6.$$
G of type $B_3$, $p \neq 2$. (The simple reflection corresponding to the short root is denoted by $s_3$ and $(s_1 s_3)^2 = 1$.)

$$ S_{u_9} = (1), S_{u_5} = (s_1), S_{u_4} = \{(s_3), (s_2 s_3 s_2 s_3)\}, S_{u_3} = \{(s_1 s_3), (w_0)\}, $$

$$ S_{u_2} = (s_1 s_2), S_{u_1} = \{(s_2 s_3), (s_2 s_3 s_1 s_2 s_3)\}, S_{u_0} = (s_1 s_2 s_3). $$

$$ \Xi^{(s_1)}_{u_{9,1}} = (1 + v^2)^3(1 + v^4)(1 + v^4 + v^8), \Xi^{(s_3)}_{u_{5,1}} = v^8(1 + v^2)^2, $$

$$ \Xi^{(s_2 s_3 s_2 s_3)}_{u_{4,1}} = v^6(1 + v^2)(v^4 - 1), \Xi^{(s_1 s_3)}_{u_{3,1}} = 2v^6(1 + v^2)^2, $$

$$ \Xi^{(s_1 s_3)}_{u_{3,1}} = v^8(1 + v^2), \Xi^{(w_0)}_{u_{4,1}} = v^{14}(v^4 - 1), \Xi^{(s_1 s_2)}_{u_{2,1}} = 2v^8, $$

$$ \Xi^{(s_2 s_3)}_{u_{1,1}} = 2v^6, \Xi^{(s_2 s_3 s_1 s_2 s_3)}_{u_{0,1}} = v^6(v^2 - 1), \Xi^{(s_1 s_2 s_3)}_{u_{1,1}} = v^6. $$

G of type $C_3$, $p \neq 2$. (The simple reflection corresponding to the long root is denoted by $s_3$ and $(s_1 s_3)^2 = 1$; $u'_2$ denotes a unipotent element which is regular inside a Levi subgroup of type $C_2$; $u'_2$ denotes a unipotent element with $\dim B_{u'_2} = 2$ which is not conjugate to $u''_2$.)

$$ S_{u_9} = (1), S_{u_6} = (s_3), S_{u_4} = \{(s_3), (s_2 s_3 s_2 s_3)\}, S_{u_3} = \{(s_1 s_3), (w_0)\}, $$

$$ S_{u'_2} = (s_1 s_2), S_{u''_2} = (s_2 s_3), S_{u_1} = (s_2 s_3 s_1 s_2 s_3), S_{u_0} = (s_1 s_2 s_3). $$

$$ \Xi^{(s_1 s_3)}_{u_{9,1}} = (1 + v^2)^3(1 + v^4)(1 + v^4 + v^8), \Xi^{(s_3)}_{u_{6,1}} = v^6(1 + v^2)^2(1 + v^4), $$

$$ \Xi^{(s_2 s_3 s_2 s_3)}_{u_{4,1}} = v^{10}(v^4 - 1), \Xi^{(s_1)}_{u_{4,1}} = 2v^8(1 + v^2), $$

$$ \Xi^{(s_1 s_3)}_{u_{3,1}} = v^8(1 + v^2), \Xi^{(w_0)}_{u_{3,1}} = v^{14}(v^4 - 1), \Xi^{(s_1 s_2)}_{u''_2,1} = 2v^8, $$

$$ \Xi^{(s_2 s_3)}_{u''_2,1} = v^6(1 + v^2), \Xi^{(s_2 s_3 s_1 s_2 s_3)}_{u_{1,1}} = v^{10}, \Xi^{(s_1 s_2 s_3)}_{u_{0,1}} = v^6. $$

1.2. We expect that the following property of $G$ holds:

(a) \[ \mathcal{W} = \cup_u S_u \]

(u runs over a set of representatives for the unipotent classes in $G$).

The equality $\mathcal{W} = \cup_u S_u$ is clear since for a regular unipotent $u$ and any $w$ we have $\Xi_w^{(u,1)} = v^{2l(w)}$. Note that (a) holds for $G$ of rank $\leq 3$ if $p$ is not a bad prime for $G$ (see 1.1). We will show elsewhere that (a) holds for $G$ of type $A_n$ (any $p$) and of type $B_n, C_n, D_n$ ($p \neq 2$). When $G$ is simple of exceptional type, (a) should follow by computing the product of some known (large) matrices using 1.5(a).

Assuming that (a) holds we define a surjective map from $\mathcal{W}$ to the set of unipotent classes in $G$ by

(b) \[ C \mapsto c \]
where $C \in \mathfrak{W}$ and $C$ is the unique unipotent class in $G$ such that $C \in S_u$ for $u \in C$.

We expect that when $p = 0$ we have

\[(c) \quad c_u \in S_u\]

where for any unipotent element $u \in G$, $c_u$ denotes the conjugacy class in $\mathfrak{W}$ associated to $u$ in [KL, 9.1]. Note that $(c)$ holds for $G$ of rank $\leq 3$ (see 1.1). (We have used the computations of the map in [KL, §9], [S1], [S2].)

1.3. Assume that $G = Sp_{2n}(k)$ and $p \neq 2$. The Weyl group $W$ can be identified in the standard way with the subgroup of the symmetric group $S_{2n}$ consisting of all permutations of $[1, 2n]$ which commute with the involution $i \mapsto 2n + 1 - i$. We say that two elements of $W$ are equivalent if they are contained in the same conjugacy class of $S_{2n}$. The set of equivalence classes in $W$ is in bijection with the set of partitions of $2n$ in which every odd part appears an even number of times (to $C \in \mathfrak{W}$ we attach the partition which has a part $j$ for every $j$-cycle of an element of $C$ viewed as a permutation of $[1, 2n]$). The same set of partitions of $2n$ indexes the set of unipotent classes of $G$. Thus we obtain a bijection between the set of equivalence classes in $W$ and the set of unipotent classes of $G$. In other words we obtain a surjective map $\phi$ from $\mathfrak{W}$ to the set of unipotent classes of $G$ whose fibres are the equivalence classes in $\mathfrak{W}$. We will show elsewhere that for any unipotent class $C$ in $G$ we have $\phi^{-1}(C) = S_u$ where $u \in C$.

1.4. Recall that the set of unipotent elements in $G$ can be partitioned into "special pieces" (see [L5]) where each special piece is a union of unipotent classes exactly one of which is "special". Thus the special pieces can be indexed by the set of isomorphism classes of special representations of $\mathfrak{W}$ which depends only on $\mathfrak{W}$ as a Coxeter group (not on the underlying root system). For each special piece $\sigma$ of $G$ we consider the subset $S_{\sigma} := \cup_{C \in \sigma} S_C$ of $\mathfrak{W}$ (here $\sigma$ runs over the unipotent classes contained in $\sigma$). We expect that each such subset $S_\sigma$ depends only on the Coxeter group structure of $\mathfrak{W}$ (not on the underlying root system). As evidence for this we note that the subsets $S_\sigma$ for $G$ of type $B_3$ are the same as the subsets $S_\sigma$ for $G$ of type $C_3$. These subsets are as follows:

\[
\{1\}, \{(s_1), (s_3), (s_2s_3s_2s_3)\}, \{(s_1s_3), (w_0)\}, \{(s_1s_2)\}, \{(s_2s_3), (s_2s_3s_1s_2s_3)\}, \{(s_1s_2s_3)\}.
\]

1.5. Let $g \in G$ be a unipotent element and let $z \in \hat{Z}(g)$, $w \in W$. We show how the polynomial $\Xi_{g,w}^z$ can be computed using information from representation theory. We may assume that $p > 1$ and that $k$ is the algebraic closure of the finite field $F_p$. We choose an $F_p$ split rational structure on $G$ with Frobenius map $F_0 : G \to G$. We may assume that $q \in G^{F_0}$. Let $q = p^m$ where $m \geq 1$ is sufficiently divisible. In particular $F := F_0^m$ acts trivially on $\hat{Z}(g)$ hence $cl(g)^F$ is a
union of $G^F$-conjugacy classes naturally indexed by the conjugacy classes in $Z(g)$; in particular the $G^F$-conjugacy class of $g$ corresponds to $1 \in Z(g)$. Let $g_z$ be an element of the $G^F$-conjugacy class in $\text{cl}(g)^F$ corresponding to the $Z(g)$-conjugacy class of $z \in Z(g)$. The set $B_{g_z}^w$ is $F$-stable. We first compute the number of fixed points $|(B_{g_z}^w)^F|$. Let $\mathcal{H}_q = \mathcal{Q}_l \otimes \mathbb{Z}[v,v^{-1}] \mathcal{H}$ where $\mathcal{Q}_l$ is regarded as a $\mathbb{Z}[v,v^{-1}]$-algebra with $v$ acting as multiplication by $\sqrt{q}$. We write $T_w$ instead of $1 \otimes T_w$. Let $\text{Irr} \mathcal{W}$ be a set of representatives for the isomorphism classes of irreducible $\mathcal{W}$-modules over $\mathcal{Q}_l$. For any $E \in \text{Irr} \mathcal{W}$ let $E_q$ be the irreducible $\mathcal{H}_q$-module corresponding naturally to $E$. Let $\mathcal{F}$ be the vector space of functions $\mathcal{B}^F \rightarrow \mathcal{Q}_l$. We regard $\mathcal{F}$ as a $G^F$-module by $\gamma : f \mapsto f'$, $f'(B) = f(\gamma^{-1} B \gamma)$ for all $B \in \mathcal{B}^F$. We identify $\mathcal{H}_q$ with the algebra of all endomorphisms of $\mathcal{F}$ which commute with the $G^F$-action, by identifying $T_w$ with the endomorphism $f \mapsto f'$ where $f'(B) = \sum B' \in \mathcal{B}^F : (B,B') \in \mathcal{O}_w \ f(B)$ for all $B \in \mathcal{B}^F$. As a module over $\mathcal{Q}_l[G^F] \otimes \mathcal{H}_q$ we have canonically $\mathcal{F} = \oplus_{E \in \text{Irr} \mathcal{W}} \rho_E \otimes E_q$ where $\rho_E$ is an irreducible $G^F$-module. Hence if $\gamma \in G^F$ and $w \in \mathcal{W}$ we have $\text{tr}(\gamma T_w, \mathcal{F}) = \sum_{E \in \text{Irr} \mathcal{W}} \text{tr}(\gamma, \rho_E) \text{tr}(T_w, E_q)$. From the definition we have $\text{tr}(\gamma T_w, \mathcal{F}) = |\{B \in \mathcal{B}^F; (B, \gamma B \gamma^{-1}) \in \mathcal{O}_w\}| = |(B_{g_z}^w)^F|$. Taking $\gamma = g_z$ we obtain

\begin{equation}
(a) \quad |(B_{g_z}^w)^F| = \sum_{E \in \text{Irr} \mathcal{W}} \text{tr}(g_z, \rho_E) \text{tr}(T_w, E_q). \end{equation}

The quantity $\text{tr}(g_z, \rho_E)$ can be computed explicitly, by the method of [L4], in terms of generalized Green functions and of the entries of the non-abelian Fourier transform matrices [L2]; in particular it is a polynomial with rational coefficients in $\sqrt{q}$. The quantity $\text{tr}(T_w, E_q)$ can be also computed explicitly (see [GP], Ch.10,11); it is a polynomial with integer coefficients in $\sqrt{q}$. Thus $|(B_{g_z}^w)^F|$ is an explicitly computable polynomial with rational coefficients in $\sqrt{q}$. Substituting here $\sqrt{q}$ by $v$ we obtain the polynomial $\Xi_{g_z}^w$. This argument shows also that $\Xi_{g_z}^w$ is independent of $p$ (note that the pairs $(g, z)$ up to conjugacy may be parametrized by a set independent of $p$).

This is how the various $\Xi_{g,z}^w$ in 1.1 were computed, except in type $A_1, A_2, B_2$ where they were computed directly from the definitions. (For type $B_3, C_3$ we have used the computation of Green functions in [Sh]; for type $G_2$ we have used directly [CR] for the character of $\rho_E$ at unipotent elements.)

1.6. In this section we assume that $G$ is simply connected. Let $\tilde{G} = G(\mathbf{k}(\langle \epsilon \rangle))$ where $\epsilon$ is an indeterminate. Let $\tilde{\mathcal{B}}$ be the set of Iwahori subgroups of $\tilde{G}$. Let $\tilde{\mathcal{W}}$ the affine Weyl group attached to $\tilde{G}$. Note that $\tilde{\mathcal{W}}$ naturally indexes $(w \mapsto \mathcal{O}_w)$ the orbits of $\tilde{G}$ acting on $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$ by simultaneous conjugation on the two factors. For $g \in \tilde{G}$ and $w \in \tilde{\mathcal{W}}$ we set

\begin{equation}
\tilde{B}_g^w = \{ B \in \tilde{\mathcal{B}}; (B, gBg^{-1}) \in \mathcal{O}_w\}. \end{equation}
By analogy with [KL, §3] we expect that when \( g \) is regular semisimple, \( \tilde{B}_g^w \) has a natural structure of a locally finite union of algebraic varieties over \( k \) of bounded dimension and that, moreover, if \( g \) is also elliptic, then \( \tilde{B}_g^w \) has a natural structure of algebraic variety over \( k \). It would follow that for \( g \) elliptic and \( w \in \mathcal{W} \),

\[
\Xi_g^w = \sum_{i,j \in \mathbb{Z}} (-1)^i \dim H_c^i(\tilde{B}_g^w, \mathcal{Q}_l)_{v^j} \in \mathbb{Z}[v]
\]

is well defined; one can then show that the formal sum \( \sum_{w \in \mathcal{W}} \Xi_g^w \hat{T}_w \) is central in the completion of the affine Hecke algebra consisting of all formal sums \( \sum_{w \in \mathcal{W}} a_w \hat{T}_w \) (\( a_w \in \mathbb{Q}(v) \)) that is, it commutes with any \( \hat{T}_w \). (Here \( \hat{T}_w \) is defined as in 1.1 and the completion of the affine Hecke algebra is regarded as a bimodule over the actual affine Hecke algebra in the natural way.)

2. The sets \( s_g \)

2.1. In this section we assume that \( G \) is adjoint and \( p \) is not a bad prime for \( G \). For \( g \in G, z \in \tilde{Z}(g), w \in \mathcal{W} \) we set

\[
\xi_{g,z}^w = \Xi_{g,z}^w|_{v=1} = \sum_{i \in \mathbb{Z}} (-1)^i \text{tr}(z, H_c^i(\tilde{B}_g^w, \mathcal{Q}_l)) \in \mathbb{Z}.
\]

This integer is independent of \( l \). For any \( g \in G, z \in \tilde{Z}(g) \) we set

\[
\pi_{g,z} = \sum_{w \in \mathcal{W}} \xi_{g,z}^w w \in \mathbb{Z}[\mathcal{W}].
\]

This is the specialization of \( \Pi_{g,z} \) for \( v = 1 \). Hence from 2(a) we see that \( \pi_{g,z} \) is in the centre of the ring \( \mathbb{Z}[\mathcal{W}] \). Thus for any \( C \in \mathcal{W} \) we can set \( \xi_{g,z}^C = \xi_{g,z}^w \) where \( w \) is any element of \( C \). For \( g \in G \) let \( s_g \) be the set of all \( C \in \mathcal{W} \) such that \( \xi_{g,z}^C \neq 0 \) for some \( z \in \tilde{Z}(g) \) and \( \xi_{g',z}^C = 0 \) for any \( g' \in \text{cl}(g) - \text{cl}(g) \) and any \( z' \in \tilde{Z}(g') \).

We describe the set \( s_g \) and the values \( \xi_{g,z}^C = 0 \) for \( C \in s_g, z \in \tilde{Z}(g) \), for various \( G \) of low rank and various unipotent elements \( g \) in \( G \). We use the notation in 1.1. Moreover in the case where \( \tilde{Z}(g) \neq \{1\} \) we denote by \( z_n \) an element of order \( n \) in \( \tilde{Z}(g) \).

- \( G \) of type \( A_1 \).
  \[
s_{u_1} = (1), s_{u_0} = (s_1); \xi_{u_1,1} = 2, \xi_{u_0,1} = 1.
\]

- \( G \) of type \( A_2 \).
  \[
s_{u_3} = (1), s_{u_1} = (s_1), s_{u_0} = (s_1 s_2).
\]
  \[
\xi_{u_3,1} = 6, \xi_{u_1,1}^{(s_1)} = 1, \xi_{u_0,1}^{(s_1 s_2)} = 1.
\]
$G$ of type $B_2$.

$s_{u_4} = (1), s_{u_2} = (s_1), s_{u_1} = \{(s_2), (s_1s_2s_1s_2)\}, s_{u_0} = (s_1s_2)$.

\[
\begin{align*}
\xi_{u_4,1}^1 &= 8, \xi_{u_2,1}^{(s_1)} = 2, \xi_{u_1,1}^{(s_2)} = 2, \xi_{u_1,1}^{(s_1s_2s_1s_2)} = 0, \\
\xi_{u_1,1}^{(s_2)} &= 0, \xi_{u_1,2}^{(s_1s_2s_1s_2)} = 2, \xi_{u_0,1}^{(s_1s_2)} = 1.
\end{align*}
\]

$G$ of type $G_2$.

$s_{u_0} = (1), s_{u_3} = (s_2), s_{u_2} = (s_1), s_{u_1} = \{(s_1s_2s_1s_2s_1s_2), (s_1s_2s_1s_2s_1s_2)\}, s_{u_0} = (s_1s_2)$.

\[
\begin{align*}
\xi_{u_4,1}^1 &= 12, \xi_{u_2,1}^{(s_1)} = 2, \xi_{u_1,1}^{(s_1s_2s_1s_2s_1s_2)} = -3, \xi_{u_1,2}^{(s_1s_2s_1s_2s_1s_2)} = 3, \\
\xi_{u_1,2}^{(s_1s_2s_1s_2)} &= 0, \xi_{u_1,1}^{(s_1s_2s_1s_2)} = 2, \xi_{u_1,2}^{(s_1s_2s_1s_2)} = 0, \xi_{u_1,2}^{(s_1s_2s_1s_2)} = 2, \xi_{u_0,1}^{(s_1s_2)} = 1.
\end{align*}
\]

$G$ of type $B_3$.

$s_{u_9} = (1), s_{u_5} = (s_1), s_{u_4} = \{(s_3), (s_2s_3s_2s_3)\}, s_{u_3} = (s_1s_3), \\
 s_{u_2} = \{(s_1s_2), (w_0)\}, s_{u_1} = \{(s_2s_3), (s_2s_3s_1s_2s_3)\}, s_{u_0} = (s_1s_2s_3)$.

\[
\begin{align*}
\xi_{u_9,1}^1 &= 48, \xi_{u_5,1}^{(s_1)} = 4, \xi_{u_4,1}^{(s_2s_3s_2s_3)} = 0, \xi_{u_4,2}^{(s_2s_3s_2s_3)} = 4, \xi_{u_4,1}^{(s_3)} = 8, \\
\xi_{u_4,1}^{(s_3)} &= 0, \xi_{u_3,1}^{(s_1s_3)} = 2, \xi_{u_2,2}^{(w_0)} = 0, \xi_{u_2,2}^{(w_0)} = 6, \\
\xi_{u_2,2}^{(s_1s_2)} &= 2, \xi_{u_1,1}^{(s_1s_2)} = 0, \xi_{u_1,1}^{(s_2s_3)} = 2, \xi_{u_1,2}^{(s_2s_3)} = 0, \xi_{u_1,2}^{(s_2s_3)} = 2, \xi_{u_0,1}^{(s_1s_2s_3)} = 1.
\end{align*}
\]

$G$ of type $C_3$.

$s_{u_9} = (1), s_{u_6} = (s_3), s_{u_4} = \{(s_1), (s_2s_3s_2s_3)\}, s_{u_3} = (s_1s_3), \\
 s_{u_2} = (s_1s_2), s_{u_2}' = (s_2s_3), s_{u_1} = \{(s_2s_3s_1s_2s_3), (w_0)\} s_{u_0} = (s_1s_2s_3)$.

\[
\begin{align*}
\xi_{u_9,1}^1 &= 48, \xi_{u_6,1}^{(s_3)} = 8, \xi_{u_4,1}^{(s_2s_3s_2s_3)} = 0, \xi_{u_4,2}^{(s_2s_3s_2s_3)} = 4, \\
\xi_{u_4,1}^{(s_1)} &= 4, \xi_{u_4,1}^{(s_1)} = 0, \xi_{u_3,1}^{(s_1s_3)} = 2, \xi_{u_2,1}^{(s_1s_2)} = 2, \xi_{u_2,1}^{(s_2s_3)} = 2, \\
\xi_{u_1,1}^{(s_2s_3s_1s_2s_3)} &= 1, \xi_{u_1,2}^{(s_2s_3s_1s_2s_3)} = 1, \xi_{u_1,2}^{(w_0)} = -3, \xi_{u_1,2}^{(w_0)} = 3, \xi_{u_0,1}^{(s_1s_2s_3)} = 1.
\end{align*}
\]
2.2. For any unipotent element \( u \in G \) let \( n_u \) be the number of isomorphism classes of irreducible representations of \( \bar{Z}(u) \) which appear in the Springer correspondence for \( G \). Consider the following properties of \( G \):

(a) \[ \mathcal{W} = \sqcup_u s_u \]

\( (u \) runs over a set of representatives for the unipotent classes in \( G \)); for any unipotent element \( u \in G \),

(b) \[ |s_u| = n_u. \]

The equality \( \mathcal{W} = \sqcup_u s_u \) is clear since for a regular unipotent \( u \) and any \( w \) we have \( \xi_{u,1}^w = 1 \). Note that (a),(b) hold in the examples in 2.1. We will show elsewhere that (a),(b) hold if \( G \) is of type \( A \). We expect that (a),(b) hold in general.

Consider also the following property of \( G \): for any \( g \in G \), \( w \in \mathcal{W} \),

\[ \xi_{g,1}^w \]

is equal to the trace of \( w \) on the Springer representation \( \mathcal{W} \) of \( \bigoplus_i H^{2i}(B_g, \bar{\mathbb{Q}}_l) \).

Again (c) holds if \( G \) is of type \( A \) and in the examples in 2.1; we expect that it holds in general. Note that in (c) one can ask whether for any \( z \), \( \xi_{g,z}^w \) is equal to the trace of \( wz \) on the Springer representation of \( \mathcal{W} \times \bar{Z}(g) \) on \( \bigoplus_i H^{2i}(B_g, \bar{\mathbb{Q}}_l) \); but such an equality is not true in general for \( z \neq 1 \) (for example for \( G \) of type \( B_2 \)).

**References**


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