Non-Abelian string and particle braiding in topological order: Modular $SL(3,\mathbb{Z})$ representation and $(3 + 1)$-dimensional twisted gauge theory

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Non-Abelian string and particle braiding in topological order: Modular SL(3,\text{Z}) representation and (3 + 1)-dimensional twisted gauge theory

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(Received 31 July 2014; revised manuscript received 4 December 2014; published 29 January 2015)

String and particle braiding statistics are examined in a class of topological orders described by discrete gauge theories with a gauge group $G$ and a 4-cocycle twist $\omega_b$ of $G$'s cohomology group $\check{H}^4(G, \text{Z}/\text{Z})$ in three-dimensional space and one-dimensional time (3 + 1D). We establish the topological spin and the spin-statistics relation for the closed strings and their multi-string braiding statistics. The 3 + 1D twisted gauge theory can be characterized by a representation of a modular transformation group, SL(3,\text{Z}). We express the SL(3,\text{Z}) generators $S^{y\z}$ and $T^{y\gamma}$ in terms of the gauge group $G$ and the 4-cocycle $\omega_b$. As we compactify one of the spatial directions $z$ into a compact circle with a gauge flux $b$ inserted, we can use the generators $S^{y\z}$ and $T^{y\gamma}$ of an SL(2,\text{Z}) subgroup to study the dimensional reduction of the 3D topological order $C_0^{3D}$ to a direct sum of degenerate states of 2D topological orders $C_0^{2D}$ in different flux $b$ sectors: $C_0^{3D} = \oplus_b C_0^{2D}$. The 2D topological orders $C_0^{2D}$ are described by 2D gauge theories of the group $G$ twisted by the 3-cocycle $\omega_{(b)b}^3$, dimensionally reduced from the 4-cocycle $\omega_b$. We show that the SL(2,\text{Z}) generators, $S^{y\z}$ and $T^{y\gamma}$, fully encode a particular type of three-string braiding statistics with a pattern that is the connected sum of two Hopf links. With certain 4-cocycle twists, we discover that, by threading a third string through two-string unlink into a three-string Hopf-link configuration, Abelian two-string braiding statistics is promoted to non-Abelian three-string braiding statistics.

DOI: 10.1103/PhysRevB.91.035134

PACS number(s): 73.23.–b, 73.43.–f, 11.15.–q, 75.10.Kt

I. INTRODUCTION

In the 1986 Dirac Memorial Lectures, Feynman explained the braiding statistics of fermions by demonstrating the plate trick and the belt trick [1]. Feynman showed that the wave function of a quantum system obtains a mysterious $(-1)$ sign by exchanging two fermions, which is associated with the fact that an extra $2\pi$ twist or rotation is required to go back to the original state. However, it is known that there is richer physics in deconfined topological phases of 2 + 1D and 3 + 1D spacetime [2]. (Here $d + 1$D is $d$-dimensional space and one-dimensional time, while $2D$ is 2-dimensional space.) In 2 + 1D spacetime, there are “anyons” with exotic braiding statistics for point particles [3]. In 3 + 1D spacetime, Feynman only had to consider bosonic or fermionic statistics for point particles, without worrying about anyonic statistics. Nonetheless, there are string-like excitations, whose braiding process in 3 + 1D spacetime can enrich the statistics of deconfined topological phases. In this work, we aim to systematically address the string and particle braiding statistics in deconfined gapped phases of 3 + 1D topological orders. Namely, we aim to determine what statistical phase the wave function of the whole system gains under the string and particle braiding process.

Since the discovery of 2 + 1D topological orders [4–6] (see Ref. [7] for an overview), we have now gained quite systematic ways to classify and characterize them, by using the induced representations of the mapping class group (MCG) of the $\mathbb{T}^2$ torus (the modular group SL(2,\text{Z}) and the gauge/Berry phase structure of ground states [6,8,9]) and the topology-dependent ground-state degeneracy (GSD) [6,10,11], using the unitary fusion categories [12–19] and using simple current algebra [20–23], a pattern of zeros [24–29], and field theories [30–34]. Our better understanding of topologically ordered states also holds the promise of applying their rich quantum phenomena, including fractional statistics [3] and non-Abelian anyons, to topological quantum computation [35].

However, our understanding of 3 + 1 D topological orders is in its infancy and far from systematic. This motivates our work attempting to address question 1.

Q1. How do we (at least partially) classify and characterize 3D topological orders?

By classifying, we mean counting the number of distinct phases of topological orders and giving them a proper label. By characterizing, we mean describing their properties in terms of physical observables. Here our approach to studying dD topological orders is to simply generalize the above 2D approach and to use the GSD on the $d$ torus $\mathbb{T}^d = (S^1)^d$ and the associated representations of the MCG of $\mathbb{T}^d$ (recently proposed in Refs. [19] and [36]):

$$\text{MCG}(\mathbb{T}^d) = \text{SL}(d,\text{Z}).$$  

(1)

(Refer to Appendix A 4 and references cited therein for a brief review of the computation of 2D topological orders.) For three dimensions, the MCG SL(3,\text{Z}) is generated by the modular transformation $\hat{S}^{y\z}$ and $\hat{T}^{y\gamma}$ [37]:

$$\hat{S}^{y\z} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{T}^{y\gamma} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$  

(2)

What are examples of 3D topological orders? One class of them is described by a discrete gauge theory with a finite gauge group $G$. Another class is described by the twisted gauge theory [38], a gauge theory $G$ with a 4-cocycle twist $\omega_4 \in \check{H}^4(G, \text{Z}/\text{Z})$ of $G$’s fourth cohomology group. But the twisted
The non-Abelian three-string braiding statistics from a twisted and cocycles to study 3D topological order. The second is 3D orders. Borrow the more familiar 2D topological orders to understand (not in the real space; see Fig. 1). Then we will be able to topological orders are foreign and unfamiliar to us, we will first issue, our general philosophy is as follows. Since 3D gapped phases, these data can capture the gapped excitations such as particles and strings. (This strategy is widely used, especially in two dimensions.) There are two main issues that we focus on. The first is the dimensional reduction from three to two dimensions of SL(3, Z) modular data of the ground-state sectors of gapped phases, these data can capture the gapped excitations like string-like excitations. We aim to address question 2: the quasiexcitation basis, each sector carrying an index of \( b \), where a closed string (a red loop, shown in Fig. 2(c) excluding the background black).

In the most generic cases of topological orders (potentially without a gauge group description), the matrices \( S^{xy} \) and \( T^{xy} \) can still be block diagonalized as the sum of several sectors in the quasiexcitation basis, each sector carrying an index of \( b \):

\[
S^{xy} = \oplus_b S_b^{xy}, \quad T^{xy} = \oplus_b T_b^{xy}.
\]

The pair \((S_b^{xy}, T_b^{xy})\), generating an SL(2, Z) representation, describes a 2D topological order \( C_b^{2D} \). This leads to a dimension reduction of the 3D topological order \( C^{3D} \):

\[
C^{3D} = \oplus_b C_b^{2D}.
\]

In the more specific case, when the topological order allows a gauge group \( G \) description which we focus on here, we find that the \( b \) stands for the gauge flux for group \( G \) (that is, \( b \) is a group element for an Abelian \( G \), while \( b \) is a conjugacy class for a non-Abelian \( G \)).

The physical picture of the above dimensional reduction is the following (see Fig. 1): If we compactify one of the 3D spatial directions (say the \( z \) direction) into a small circle, the 3D topological order \( C^{3D} \) can be viewed as a direct sum of 2D topological orders \( C_b^{2D} \) with (accidental) degenerate ground states at the lowest energy.

In this work, we focus on a generic finite Abelian gauge group, \( G = \prod_i Z_{N_i} \) (isomorphic to products of cyclic groups), with generic cocycle twists from the group cohomology [38]. We examine the 3 + 1D twisted gauge theory twisted by 4-cocycle \( \omega_4 \in H^4(G, \mathbb{Z}/2) \) and reveal that it is a direct sum of 2 + 1D twisted gauge theories twisted by a dimensionally reduced 3-cocycle, \( \omega_3(b) \in H^3(G, \mathbb{Z}/2) \), of \( G \)'s third cohomology group, namely,

\[
C_{G, \omega_4}^{3D} = \oplus_b C_{Gb, \omega_3(b)}^{2D}.
\]

Surprisingly, even for an Abelian group \( G \), we find that such a twisted Abelian gauge theory can be dual to a twisted or untwisted non-Abelian gauge theory. We study this fact for 3D examples as an extension of the 2D examples in Ref. [40]. By this equivalence, we are equipped with (both untwisted and twisted) non-Abelian gauge theory to study its non-Abelian braiding statistics.

Non-Abelian three-string braiding statistics: We are familiar with the 2D braiding statistics: there is only particle-particle braiding, which yields bosonic, fermionic, or anyonic statistics by braiding a particle around another particle [3]. We find that the 3D topological order introduces both particle-like and string-like excitations. We aim to address question 2:

**Q2:** How do we characterize the braiding statistics of strings and particles in 3 + 1D topological orders?

The possible braiding statistics in three dimensions learned in the literature are as follows.

(i) **Particle-particle braiding**, which can only be bosonic or fermionic due to the absence of a nontrivial braid group in three dimensions for point particles.

(ii) **Particle-string braiding**, which is the Aharonov-Bohm effect of \( Z_N \) gauge theory, where a particle such as \( Z_N \) charges braiding around a string (or a vortex line) as \( Z_N \) flux, obtaining an \( e^{i \pi} \) phase of statistics [3,41].

(iii) **String-string braiding**, where a closed string (a red loop), shown in Fig. 2(c) excluding the background black
string, wraps around a blue loop. The related idea, known as loop-loop braiding, forming the loop braiding group, has been proposed mathematically \cite{42}. (See also some earlier studies in Refs. \cite{43} and \cite{44}.)

However, we address some extra new braiding statistics among three closed strings:

(iv) Three-string braiding, shown in Fig. 2(c), where a closed string (a red loop) wraps around another closed string (a blue loop) but the two loops are both threaded by a third loop (the black string). This braiding configuration was discovered recently in Ref. \cite{45}; Ref. \cite{46} is a related work for a twisted Abelian gauge theory.

The new ingredient of our work on braiding statistics can be summarized as follows: We consider the string and particle braiding of general twisted gauge theories with the most generic finite Abelian gauge group \( G = \prod N_i Z_{N_i} \), labeled by the data \((G,\omega)\). We provide a 3D-to-2D reduction approach to realize the three-string braiding statistics in Fig. 2. We first show that the SL(2, \( Z\)) representations \( (S_{\text{xy}}^{\text{yz}}, T_{\text{xy}}^{\text{yz}}) \) fully encode the particular type of Abelian three-closed-string statistics shown in Fig. 2. We further find that, for a twisted gauge theory with an Abelian \((Z_N)^d\) group, certain 4-cocycles (called type IV 4-cocycles) will make the twisted theory be a non-Abelian theory. More precisely, while the two-string braiding statistics of unlinks is Abelian, the three-string braiding statistics of Hopf links, obtained from threading the two strings with a third string, will become non-Abelian. We also demonstrate that \( S_{\text{xy}}^{\text{yz}} \) encodes this three-string braiding statistics.

Our article is organized as follows. In Sec. \( \text{II} \), we address the third question:

**Q3:** How do we formulate or construct certain 3+1D topological orders in the lattice?

We outline a lattice formulation of twisted gauge theories in terms of 3D twisted quantum double models, which generalize Kitaev’s 2D toric code and quantum double models. Our model is the lattice Hamiltonian formulation of Dijkgraaf-Witten theory \cite{38}, and we provide the spatial lattice as well as the spacetime lattice path integral pictures. In Sec. \( \text{III} \), we answer question 4:

**Q4:** What are the generic expressions of SL(3, \( Z\)) modular data?

We compute the modular SL(3, \( Z\)) representations of \( S \) and \( T \) matrices, using both the spacetime path integral approach and the representation (Rep) theory approach. In Secs. \( \text{III C} \) and \( \text{IV} \), we address question 5:

**Q5:** What is the physical interpretation of SL(3, \( Z\)) modular data in three dimensions?

We use the modular SL(3, \( Z\)) data to characterize the braiding statistics of particles and strings. In Sec. \( \text{V} \), we discuss the link and knot patterns of string braiding systematically and end with a conclusion. In addition to the text, we organize the following information in Appendix:\: (i) group cohomology and cocycles; (ii) projective representation; (iii) some examples of classification of topological orders; and (iv) direct calculations of \( S \) and \( T \) using cocycle path integrals.

\[ \text{Note: We adopt the name strings for the vision of incorporating the excitations from both closed strings (loops) and open strings. Such excitations can have a fusion or braiding process. In this work, however, we focus only on the closed string case. Our notation for a finite cyclic group is either} \ Z_N \text{ or} \ Z_{\text{N}}, \text{though they are equivalent mathematically. We use} \ Z_N \text{ to denote the gauge group} \ G, \text{the discrete gauge} \ Z_N \text{ flux, or the} \ Z_N \text{ variables, but} \ Z_N \text{ to denote only the classes of group cohomology or topological order classification. We denote} g_c(N_i, N_j) \equiv N_{ij}, g_c(N_i, N_j, N_k) \equiv N_{ijk} \text{ and} g_c(N_i, N_j, N_k, N_l) \equiv N_{ijkl}, \text{where} g_c \text{ stands for the greatest common divisor. We also have} \ |G| \text{ as the order of the group, and} \ \mathbb{R}/\mathbb{Z} = U(1). \text{ We may use subindex} n \text{ for} \omega_n \text{ to indicate an} n\text{-cocycle. In principle, we use types to count the number of cocycles in cohomology groups. But we use classes to count the number of distinct phases in topological orders.} \]
Normally the types outnumber the classes. We use the hat symbol $\hat{S}$ and $\hat{T}$ for modular matrices acting on the real space in the $x, y, z$ directions, so $\hat{S}^{xyz} \cdot (x, y, z) = (z, x, y)$ and $\hat{T}^{xyz} \cdot (x, y, z) = (x + y, y, z)$, while we use the symbols $S$ and $T$ to denote modular matrices in the quasiexcitation basis.

II. TWISTED GAUGE THEORY AND COCYCLES OF GROUP COHOMOLOGY

In this section, we aim to address question 3:

Q3: How do we formulate or construct certain 3+1D topological orders in the lattice?

We consider 3+1D twisted discrete gauge theories. Our motivation to study the discrete gauge theory is that it is topological and exhibits Aharonov-Bohm phenomena (see Refs. [3] and [41]). One approach to formulating a discrete gauge theory is the lattice gauge theory [47]. A famous example in both the high-energy and the condensed matter communities is the $Z_2$ discrete gauge theory in 2+1 dimensions (also called the $Z_2$ toric code, $Z_2$ spin liquids, or $Z_2$ topological order [48]). Kitaev’s toric code and quantum double model [49] provide a simple Hamiltonian,

$$H = -\sum_{\nu} A_{\nu} - \sum_{p} B_{p}, \quad (7)$$

where a space lattice formalism is used, and $A_{\nu}$ is the vertex operator acting on vertex $\nu$, $B_{p}$ is the plaquette (or face) term to ensure the zero-flux condition on each plaquette. Both $A_{\nu}$ and $B_{p}$ consist of only Pauli spin operators for the $Z_2$ model. Such ground states of the Hamiltonian are found to be $Z_2$ gauge theory with $|G|^2$ fourfold topological degeneracy on the $T^2$ torus. Its generalization to a twisted $Z_2$ gauge theory is the $Z_2$ double-semions model, captured by the framework of the Levin-Wen string-net model [12,48].

A. Dijkgraaf-Witten topological gauge theory

For a more generic twisted gauge theory, there is indeed another way using the spacetime lattice formalism to construct them by the Dijkgraaf-Witten topological gauge theory [38]. One can formulate the path integral $Z$ (or partition function) of a $(d + 1)$D gauge theory ($d$D space, 1D time) of a gauge group $G$ as

$$Z = \sum_{\gamma} e^{iS[\gamma]} = \sum_{\gamma} e^{i2\pi \langle M_{\mu} \rangle \epsilon_{\gamma} (M_{\mu}) \mod 2\pi}$$

$$= \frac{|G|}{|G|^N} \frac{1}{|G|} \prod_{[\epsilon_{\mu}]} \int d\omega \omega_{d+1} \cdot \langle (g_{ab}) \rangle|_{v_{a} \epsilon T_{i}}, \quad (8)$$

where we sum over all mappings $\gamma : M \rightarrow BG$, from the spacetime manifold $M$ to $BG$, the classifying space of $G$. In the second equality, we triangulate $M$ to $M_{\text{tri}}$, with the edge convex from the vertex $v_0$ to the vertex $v_p$. The action $\omega_{d+1} \cdot \langle (M_{\mu}) \rangle$ evaluates the cocycles $\omega_{d+1}$ in the spacetime $(d + 1)$-complex $M_{\text{tri}}$. By the relation between the topological cohomology class of $BG$ and the cohomology group of $G$: $H^{d+2}(BG, \mathbb{Z}) = H^{d+1}(G, \mathbb{R}/\mathbb{Z})$ [38,50], we can simply regard $\omega_{d+1}$ as the $d + 1$-cocycles of the cohomology group $H^{d+1}(G, \mathbb{R}/\mathbb{Z})$ (see more details in Appendix A). The group elements $g_{ab}$ are assigned at the edge $[v_{a}v_{b}]$. The $|G|^2/|G|^N$ factor is to mod out the redundant gauge equivalence configuration, with the number of vertices $N_{v}$. Another extra $|G|^{-1}$ factor modus out the group elements evolving in the time dimension. The cocycle $\omega_{d+1}$ is evaluated on all the $d + 1$-simplex $T_{i}$ (namely, a $d + 2$-cell) triangulations of the spacetime complex. In the case of our $3 + 1$ dimensions, we have the 4-cocycle $\omega_4$ evaluated at the 4-simplex (or 5-cell) as

Here the cocycle $\omega_4$ satisfies the cocycle condition, $\delta \omega_4 = 1$, which ensures that the path integral $Z$ on the 4-sphere $S^4$ (the surface of the 5-ball) will be trivial as 1. This is a feature of topological gauge theory. The $\epsilon$ is the sign of the orientation of the 4-simplex, which is determined by the sign of the volume determinant of the 4-simplex evaluated by $\epsilon = \text{sgn} (\det(01,02,03,04))$.

We utilize Eq. (8) to calculate the path integral amplitude from an initial state configuration $|\Psi_{\text{in}}\rangle$ on the spatial manifold evolving along the time direction to the final state $|\Psi_{\text{out}}\rangle$ (see Fig. 3). In general, the calculation can be done for the MCG on any spatial manifold $M_{\text{space}}$ as $\text{MCG}(M_{\text{space}})$. Here we focus on $M_{\text{space}} = T^3$ and $\text{MCG}(T^3) = SL(3, \mathbb{Z})$, as the modular transformation. We first note that $|\Psi_{\text{in}}\rangle = \tilde{O}|\Psi_{\text{R}}\rangle$, such a generic $SL(3, \mathbb{Z})$ transformation $\tilde{O}$ under the $SL(3, \mathbb{Z})$ representation can be absolutely generated by $\hat{S}^{xyz}$ and $\hat{T}^{xyz}$ of Eq. (2) [37], thus $\tilde{O} = \tilde{O}(\hat{S}^{xyz}, \hat{T}^{xyz})$ as a function of $\hat{S}^{xyz}$ and $\hat{T}^{xyz}$. Calculation of the modular $SL(3, \mathbb{Z})$ transformation from $|\Psi_{\text{in}}\rangle$ to $|\Psi_{\text{out}}\rangle = |\Psi_{\text{A}}\rangle$ by filling the 4-cocycles $\omega_4$ into the spacetime-complex triangulation renders the amplitude of

\begin{align*}
\text{FIG. 3. Illustration for } & \text{O}_{\text{A}(\text{B})} = \langle \Psi_{\text{A}} | \tilde{O} | \Psi_{\text{B}} \rangle. \text{ Evolution from an initial-state configuration } |\Psi_{\text{A}}\rangle \text{ on the spatial manifold (from the top) along the time direction (dashed line) to the final state } |\Psi_{\text{out}}\rangle \text{ (at the bottom). For the spatial } T^3 \text{ torus, the mapping class group } \text{MCG}(T^3) \text{ is the modular } SL(d, \mathbb{Z}) \text{ transformation. We show schematically the time evolution on the spatial } T^2 \text{ and } T^1. \text{ The } T^3 \text{ is shown as a } T^2 \text{ attached a } S^1 \text{ cycle on each point.}
\end{align*}
the matrix element \( \mathcal{O}(\alpha, \beta) \),
\[
\mathcal{O}(S^g, T^y)_{\alpha, \beta} = \langle \Psi_{\beta} | \hat{O}(S^g, T^y) | \Psi_{\alpha} \rangle,
\]
where both space and time are discretely triangulated, so this is a spacetime-lattice formalism.

**B. Canonical basis and the generalized twisted quantum double model \( D^+(G) \) to the 3D triple basis**

So far we have answered question 3 using the spacetime-lattice path integral. Our next goal is to construct its Hamiltonian on the space lattice and to find a good basis representing its quasiexcitations, such that we can efficiently read the information of \( \mathcal{O}(S^g, T^y) \) in this canonical basis. We outline the twisted quantum double model generalized to three dimensions as exactly soluble model in the next subsection, where the canonical basis can diagonalize its Hamiltonian.

1. Canonical basis

For a gauge theory with the gauge group \( G \), one may naively think that a good basis for the amplitude, Eq. (10), is the group elements \( \{g_x, g_y, g_z\} \) with \( g_i \in G \) as the flux labeling three directions of \( T^3 \). However, this flux-only label \( \{g_x, g_y\} \) is known to be improper on the \( T^3 \) torus already: the canonical basis labeling particles in two dimensions is \( |\alpha, a\rangle \), requiring both the charge \( \alpha \) (as the representation) and the flux \( a \) (the group element or the conjugacy class of \( G \)). We propose that the proper way to label excitations for a 3 + 1D twisted discrete gauge theory for any finite group \( G \) in the canonical basis requires one charge, \( \alpha \), and two fluxes, \( a \) and \( b \).

\[
|\alpha, a, b\rangle = \frac{1}{\sqrt{|G|}} \sum_{g_i \in G} \text{Tr} (\hat{\rho}_{a,b}^{\alpha, g_i}) |g_i, g_y, g_z\rangle,
\]
which is the finite-group discrete Fourier transformation on \( \{g_x, g_y, g_z\} \). This is a generalization of the 2D result in Ref. [40] and a very recent 3D Abelian case in Ref. [46]. Here \( \alpha \) is the charge of the representation (Rep) label, which is the \( C_{a,b}^{(2)} \) Rep of the centralizers \( Z_a, Z_b \) of the conjugacy classes \( C^a, C^b \). (For an Abelian \( G \), the conjugacy group is the group element, and the centralizer is the full \( G \).) \( C_{a,b}^{(2)} \) Rep means an inequivalent unitary irreducible projective representation of \( G \). \( \hat{\rho}_{a,b}^{\alpha, g_i}(c) \) labels this inequivalent unitary irreducible projective representation of \( G \). \( C_{a,b}^{(2)} \) Rep of \( G \) is an induced 2-cocycle, dimensionally reduced from the 4-cocycle \( \omega_4 \). We illustrate \( C_{a,b}^{(2)} \) in terms of geometric pictures in Eqs. (12) and (13):

\[
C_{\alpha, b}(c, d) = \text{Tr}[\hat{\rho}_{a,b}^{\alpha, g_i} |g_i, g_y, g_z\rangle |g_i, g_y, g_z\rangle],
\]

The reduced 2-cocycle \( C_{\alpha, b}(c, d) \) is from the 3-cocycle \( \omega_3 \) in Eq. (12), which triangulates a half of \( T^2 \), with a time interval \( I \). The reduced 2-cocycle \( C_{a,b}(c, d) \) is from 4-cocycle \( \omega_4 \) in Eq. (13), which triangulates a half of \( T^3 \) with a time interval \( I \). The dashed arrow stands for the time \( t \) evolution.

The \( \hat{\rho}_{a,b}^{\alpha, g_i}(c) \) values are determined by the \( C_{a,b}^{(2)} \) projective representation formula:

\[
\hat{\rho}_{a,b}^{\alpha, g_i}(c)d = C_{a,b}^{(2)}(c, d)\hat{\rho}_{a,b}^{\alpha, g_i}(cd).
\]

The trace term \( \text{Tr}(\hat{\rho}_{a,b}^{\alpha, g_i}(c)) \) is called the character in the math literature. One can view the charge \( \alpha \) along the \( x \) direction, and the flux \( a, b \) along the \( y, z \) directions. Other details and the calculations of \( C_{a,b}^{(2)} \) Rep, with many examples, are given in Appendix A.

We first recall that, in two dimensions, a reduced 2-cocycle \( C_{a,b}(c, d) \) comes from a slant product \( i_a \omega_3(b, c) \) of 3-cocycles [40], which is geometrically equivalent to filling three 3-cocycles in a triangular prism of Eq. (12). This is known to present the projective representation, \( \hat{\rho}_{a,b}^{\alpha, g_i}(c) \), because the induced 2-cocycle belongs to the second cohomology group \( H^2(G, \mathbb{R} / \mathbb{Z}) \) [40,51–53]. (See its explicit triangulation and a novel use of the projective representation in Sec VI B of Ref. [54].)

Similarly, in three dimensions, a reduced 2-cocycle \( C_{a,b}(c, d) \) comes from doing twice the slant products of 4-cocycles forming the geometry of Eq. (13) and renders

\[
C_{a,b}^{(2)} = i_b(C_{a}(d, c)) = i_b(i_a \omega(c,d)),
\]

presenting the \( C_{a,b}^{(2)} \) projective representation in Eq. (14), where \( \hat{\rho}_{a,b}^{\alpha, g_i}(c) \) labels the \( C_{a,b}^{(2)} \) projective representation in Eq. (12), where \( \hat{\rho}_{a,b}^{\alpha, g_i}(c) \) acts on \( (Z_a, Z_b) \rightarrow \text{GL}(Z_a, Z_b) \) can be written as a matrix in the general linear (GL) group. This 3D generalization for the canonical basis in Eq. (11) is not only natural, but also consistent with two dimensions when we turn off the flux along the \( z \) direction (e.g., set \( b = 0 \)), which reduces the 3D \( |\alpha, a, b\rangle \) to \( |\alpha, a\rangle \) in the 2D case.

2. Generalizing the 2D twisted quantum double model \( D^+(G) \) to the 3D twisted quantum triple model?

A natural way to combine the Dijkgraaf-Witten theory with Kitaev’s quantum double model Hamiltonian approach will enable us to study the Hamiltonian formalism for the twisted gauge theory, which is achieved in Refs. [55] and [53] for \( 2 + 1 \) dimensions, termed the twisted quantum double model. In two dimensions, the widely used notation \( D^+(G) \) implies the twisted quantum double model with its gauge group \( G \) and its cocycle twist \( \omega \). It is straightforward to generalize these results to \( 3 + 1 \) dimensions.

To construct the Hamiltonian on the 3D spatial lattice, we follow Ref. [55] with the form of the twisted quantum double model Hamiltonian of Eq. (7) and put the system on the \( T^3 \) torus. However, some modifications are adopted for three dimensions: the vertex operator \( A_e = |G|^{-1} \sum_{[\{i\} \in \mathcal{G}] A_e^i} \) acts on the vertices of the lattice by lifting the vertex point
and one computes the 4-cocycle filling amplitude as \( Z \) in Eq. (8). To evaluate Eq. (16)’s \( A_v \) acting on vertex \( v \), one effectively lifts 5 to \( 5' \), and fills 4-cocycles \( \omega \) into this geometry to compute the amplitude \( Z \) in Eq. (8). For this specific 3D spatial lattice surrounding vertex 5 with one, two, three, and four neighboring vertices, there are four 4-cocycles \( \omega \) filling in the amplitude of \( A_5^{[55]} \).

The plaquette operator \( B_p^{(1)} \) still enforces the zero-flux condition on each 2D face (a triangle \( p \)) spanned by three edges of a triangle. This will ensure zero flux on each face (along the Wilson loop of a 1-form gauge field). Moreover, zero-flux conditions are required if higher form gauge fluxes are presented. For example, for 2-form field, one adds an additional \( B^{(2)} \) to ensure zero flux on a 3-simplex (a tetrahedron \( p \)). Thus, \( \sum_p B_p \) in Eq. (7) becomes \( \sum_p B_p^{(1)} + \sum_p B_p^{(2)} + \ldots \).

Analogous to Ref. [55], the local operators \( A_v, B_p \) of the Hamiltonian have nice commuting properties: \( [A_v', A_v^b] = 0 \) if \( v \neq u \), \( [A_v', B_p^b] = [B_p, B_p^b] = 0 \), and also \( A_v^u v v' A_v^b = A_v^b \).

Note that \( A_v \) defines a ground-state projection operator \( P_v = |G|^{-1} \sum_v A_v^b \). If we consider a \( \mathbb{T}^3 \) torus triangulated in a cube with only a point \( v \) (all eight points are identified). It can be shown that both \( A_v \) and \( P \) as projection operators project other states to the ground state \( |a, a, b \rangle \), and \( P |a, a, b \rangle = |a, a, b \rangle \) \( \propto |a, a, b \rangle \). Since \( [A_v', B_p] = 0 \), one can simultaneously diagonalize the Hamiltonian, Eq. (7), by this canonical basis \( |a, a, b \rangle \) as the ground-state basis.

A similar 3D model was studied recently in Ref. [46]. There the zero-flux condition is imposed on both the vertex operator and the plaquette operator. Their Hilbert space thus is more constrained than that in Ref. [55] or ours. However, in the ground-state sector, we expect that the physics is the same. It is less clear to us whether the name twisted quantum double model and its notation, \( D^q(G) \), are still proper in three or higher dimensions. With the quantum double basis \( |a, a \rangle \) in two dimensions generalized to the triple basis \( |a, a, b \rangle \) in three dimensions, we are tempted to call it the twisted quantum triple model in three dimensions. It awaits mathematicians and mathematical physicists to explore more details in the future.

C. Cocycle of \( \mathcal{H}^4(G, \mathbb{R}/\mathbb{Z}) \) and its dimensional reduction

To study the twisted gauge theory of a finite Abelian group, we now provide the explicit data on cohomology group and 4-cocycles [56]. Here \( \mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z}) = \mathcal{H}^{d+1}(G, \mathbb{U}(1)) \) by \( \mathbb{R}/\mathbb{Z} = \mathbb{U}(1) \), as the \((d + 1)\)th cohomology group of \( G \) over \( G \) module \( \mathbb{U}(1) \). Each class in \( \mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z}) \) corresponds to a distinct \((d + 1)\)-cocycle. The different 4-cocycles label the distinct topological terms of \( 3 + 1 \)D twisted gauge theories.

(Having different topological terms may share the same data for topological orders, such as the same modular data \( S^{\Lambda_4} \) and \( T^{\Lambda_4} \). Thus different topological terms may describe the same topological order.) The 4-cocycles \( \omega_d \) are 4-cochains but, additionally, satisfy the cocycle condition \( \delta \omega = 0 \). The 4-cocain is a mapping \( \omega_d(a,b,c,d): (G)^4 \rightarrow \mathbb{U}(1) \), which inputs \( a,b,c,d \in G \) and outputs a \( \mathbb{U}(1) \) phase. Furthermore, distinct 4-cocycles are not identified by any 4-coboundary \( \delta \Omega_3 \). (Namely, distinct cocycles \( \omega_d \) and \( \omega_d' \) do not satisfy \( \omega_d/\omega_d' = \delta \Omega_3 \), for any 3-cocain \( \Omega_3 \).) The 4-cocain satisfies the group multiplication rule, \( \omega_d(a,b,c,d) \cdot \omega_d(b,c,d) = \omega_d(a,b,c,d) \cdot \omega_d(b,c,d) \) and thus forms a group \( \mathcal{C}^4 \), the 4-cocain further forms its subgroup \( \mathcal{C}^4 \), and the 4-cocain further forms the \( \mathbb{Z}^4 \) subgroup \( \mathcal{B}^4 \) (since \( \delta^2 = 1 \)). In short, \( \mathcal{B}^4 \subset \mathcal{C}^4 \subset \mathbb{C}^4 \). The fourth cohomology group is a kernel \( \mathcal{K}^4 \) (the group of 4-cocains) mod out the image \( \mathcal{B}^4 \) (the group of 4-coboundaries) relation: \( \mathcal{H}^4(G, \mathbb{R}/\mathbb{Z}) = \mathcal{K}^4/\mathcal{B}^4 \). We derive the fourth cohomology group of a generic finite Abelian group \( G = \prod_i \mathbb{Z}_{N_i} \) as

\[
\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z}) = \prod_{1 \leq i < j < l < m \leq k} (Z_{N_{ij}})^2 \times (Z_{N_{ijl}})^2 \times Z_{N_{ijlm}}. \tag{17}
\]

We construct generic 4-cocycles (not identified by 4-coboundaries) for each type, summarized in Table I.

We call type II first and type II second the 4-cocycles with topological term indices: \( p_{IIij}^{(1)}(1) \in Z_{N_{ij}} \) and \( p_{IIij}^{(2nd)}(1) \in Z_{N_{ij}} \) of Eq. (17). There are type III first and type III second 4-cocycles for topological term indices: \( p_{IIIij}^{(1)} \in Z_{N_{ii}} \) and \( p_{IIIij}^{(2nd)} \in Z_{N_{ii}} \). There is also a type IV 4-cocycle topological term index: \( p_{IVijlm} \in Z_{N_{ij}} \).

Since we earlier alluded to the relation, Eq. (5), \( \mathbb{C}^3 = \oplus_b C_3^b \), between 3D topological orders (described by 4-cocycles) as the direct sum of sectors of 2D topological orders (described by 3-cocycles), we wish to see how the dimensionally reduced 3-cocycle from 4-cocycles can hint at the \( C_3^b \) theory in two dimensions. The slant products \( C_0(a,c,d) \equiv i_0 \omega_1(a,c,d) \) are organized in the last column in Table I. The geometric interpretation of the induced 3-cocycle \( C_0(a,c,d) \equiv i_0 \omega_1(a,c,d) \) is derived from the 4-cocycle \( \omega_1 \),

\[
C_0(a,c,d) : \begin{align*}
v & \quad \begin{cases} \uparrow y \mid \downarrow \quad z \quad \begin{cases} \uparrow y' \mid \downarrow \quad z' \end{cases}
\end{align*} \tag{18}
\end{align*}
\]

The combination of Eq. (18) (with four 4-cocycles filling) times the contribution of Eq. (12) (with three 4-cocycles filling) will produce Eq. (13) with twelve 4-cocycles filling. Luckily, the types II and III \( \omega_d \)’s have a simpler form of \( C_0(a,c,d) = \omega_d(a,b,c,d) \cdot \omega_d(b,a,c,d) \cdot \omega_d(b,a,c,d) \), while the reduced form of type IV \( \omega_d \) is more involved [56].

This indeed promisingly suggests the relation in Eq. (6), \( c_3^G_{\omega_d} = \oplus_b C_3^G_{\omega_d} \), with \( G_3 \equiv G \) the original group. If we view \( b \) as the gauge flux along the \( z \) direction and compactify \( z \) into a circle, then a single winding around \( z \) acts as a monodromy defect carrying the gauge flux \( b \) (group elements or conjugacy classes) [54,57,58]. This implies the geometric picture in Fig. 4.
TABLE I. Cohomology group $H^4(G,\mathbb{R}/\mathbb{Z})$ and 4-cocycles $\omega_a$ for a generic finite Abelian group $G = \prod_{i=1}^n Z_{n_i}$. The first column lists the types in $H^4(G,\mathbb{R}/\mathbb{Z})$ of Eq. (17). The second column lists the topological term indices for the 3 + 1D twisted gauge theory. (When all indices $p_i = 0$, it becomes the normal untwisted gauge theory.) The third column lists the explicit 4-cocycle functions $\omega_a(a,b,c,d); (G)^4 \to U(1)$. Here $a = (a_1, a_2, \ldots, a_n)$, with $a_i \in G$ and $a_i \in Z_{n_i}$. (Same notation for $b,c,d$.) We define the $\mod N_j$ relation by $[c_j + d_j] \equiv [c_j + d_j] \pmod{N_j}$. The last column lists the induced 3-cocycles from the slant product $\omega_0(a,c,d) \equiv i_3 \omega_3(a,c,d)$ in terms of types I, II, and III 3-cocycles of $H^3(G,\mathbb{R}/\mathbb{Z})$ listed in Table XII.

<table>
<thead>
<tr>
<th>$H^4(G,\mathbb{R}/\mathbb{Z})$</th>
<th>4-cocycle name</th>
<th>4-cocycle form</th>
<th>Induced 3-cocycle $\omega_0(a,c,d)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_{n_i}$</td>
<td>Type II 1st $p_{(1b)}$</td>
<td>$\omega_{4(1b)}(a,b,c,d) = \exp \left( \frac{2\pi i(p_{(1b)})}{N_j} (a,b)(c_j + d_j - [c_j + d_j]) \right)$</td>
<td>Types I and II of $H^4(G,\mathbb{R}/\mathbb{Z})$</td>
</tr>
<tr>
<td>$Z_{n_j}$</td>
<td>Type II 2nd $p_{(2b)}$</td>
<td>$\omega_{4(2b)}(a,b,c,d) = \exp \left( \frac{2\pi i(p_{(2b)})}{N_j} (a,b)(c_j + d_j - [c_j + d_j]) \right)$</td>
<td>Types I and II of $H^4(G,\mathbb{R}/\mathbb{Z})$</td>
</tr>
<tr>
<td>$Z_{n_j}$</td>
<td>Type III 1st $p_{(1b)}$</td>
<td>$\omega_{4(1b)}(a,b,c,d) = \exp \left( \frac{2\pi i(p_{(1b)})}{N_j} (a,b)(c_j + d_j - [c_j + d_j]) \right)$</td>
<td>Types II of $H^4(G,\mathbb{R}/\mathbb{Z})$</td>
</tr>
<tr>
<td>$Z_{n_j}$</td>
<td>Type III 2nd $p_{(2b)}$</td>
<td>$\omega_{4(2b)}(a,b,c,d) = \exp \left( \frac{2\pi i(p_{(2b)})}{N_j} (a,b)(c_j + d_j - [c_j + d_j]) \right)$</td>
<td>Types II of $H^4(G,\mathbb{R}/\mathbb{Z})$</td>
</tr>
<tr>
<td>$Z_{n_{j\omega}}$</td>
<td>Type IV $p_{(2\omega)}$</td>
<td>$\omega_{4(2\omega)}(a,b,c,d) = \exp \left( \frac{2\pi i(p_{(2\omega)})}{N_j} a,b,c,d \right)$</td>
<td>Type III of $H^4(G,\mathbb{R}/\mathbb{Z})$</td>
</tr>
</tbody>
</table>

One can tentatively write the relation

$$C^{3D}_{G,\omega_4} = C^{2D}_{G,\omega_{4(1b)}} \oplus \phi_{b \neq 0} C^{2D}_{G,\omega_{4(b)}}.$$ 

There is a zero-flux $b = 0$ sector $C^{2D}_{G,\omega_{4(1b)}}$ (with $\omega_0 = 1$) where the 2D gauge theory with $G$ is untwisted. There are other direct sums of $C^{2D}_{G,\omega_{4(b)}}$ with nonzero $b$ flux insertion that have twisted $\omega_{4(b)}$.

However, different cocycles can represent the same topological order with the equivalent modular data. In the next section we examine Eq. (19) more carefully, not in terms of cocycles, but in terms of the modular data, $\hat{S}^{xyz}$ and $\hat{T}^{xy}$.

III. REPRESENTATION FOR $S^{xyz}$ and $T^{xy}$

The modular transformations $\hat{S}^{xyz}$, $\hat{T}^{xy}$, and $\hat{S}^{xyz}$ of Eqs. (2) and (3) act on the 3D real space as

$$\hat{S}^{xyz} \cdot (x,y,z) = (-y,-x,z),$$

$$\hat{T}^{xy} \cdot (x,y,z) = (x+y,y,z),$$

$$\hat{S}^{xyz} \cdot (x,y,z) = (z,x,y).$$

The modular transformation $SL(2,\mathbb{Z})$ is generated by $\hat{S}^{xy}$ and $\hat{T}^{xy}$, while $SL(3,\mathbb{Z})$ is generated by $\hat{S}^{xyz}$ and $\hat{T}^{xy}$. The dashed arrow represents the time evolution (as in Fig. 3) from $|\Psi_{in}\rangle$ to $|\Psi_{out}\rangle$ under $\hat{S}^{xyz}$, $\hat{T}^{xy}$, and $\hat{S}^{xyz}$, respectively. The $\hat{S}^{xy}$ and $\hat{T}^{xy}$ transformations on a $T^2$ torus’s $x,y$ plane with the $z$ direction untouched are equivalent to its transformations on a $T^2$ torus.

Q4: What are the generic expressions of $SL(3,\mathbb{Z})$ modular data?

First, in Sec. III A, we apply the cocycle approach using the spacetime path integral with $SL(3,\mathbb{Z})$ transformation acting along the time evolution to formulate the $SL(3,\mathbb{Z})$ modular data, and then in Sec. III B we use the more powerful representation (Rep) theory to determine the general expressions of those data in terms of $(G,\omega_4)$.

A. Path integral and cocycle approach

The cocycle approach uses the spacetime lattice formalism, where we triangulate the spacetime complex of a 4-
manifold $M = T^3 \times I$ (a $T^3$ torus times a time interval $I$) of Eqs. (23)–(25) into 4-simplexes. We then apply the path integral $Z$ in Eq. (8) and the amplitude form in Eq. (10) to obtain

$$T_{\alpha(a,b)} \equiv (|\Psi_A| T_{\alpha} |\Psi_B|),$$

$$S_{\alpha(a,b)} \equiv (|\Psi_A| S_{\alpha} |\Psi_B|),$$

$$S_{\alpha(a,b)} \equiv (|\Psi_A| S_{\alpha} |\Psi_B|),$$

$$GSD = Tr[P] = \sum_A (|\Psi_A| P |\Psi_A|).$$

Here $|\Psi_A|$ and $|\Psi_B|$ are the ground-state bases on the $T^4$ torus; for example, they are $|\alpha, a, b\rangle$ (with $\alpha$ charge and $\beta$ flux) in 2 + 1 dimensions and $|\alpha, a, b, c\rangle$ (with $\alpha, \beta, \gamma$ charges and $\delta, \epsilon$ fluxes) in 3 + 1 dimensions. We also include the data on GSD, where $P$ is the projection operator for ground states discussed in Sec. II B. In the case of $dD$ GSD on $T^4$ (e.g., 3D GSD on $T^3$), we simply compute the $Z$ amplitude filling in $T^4 \times S^1 = T^4$. There is no shortcut here except doing explicit calculations [56].

### B. Representation theory approach

The cocycle approach in Sec. III A provides nice physical intuition about the modular transformation process. However, the calculation is tedious. There is a powerful approach simply using Rep theory; we present the general formula of $\hat{S}^{T_{\alpha}}$, $\hat{T}^{T_{\alpha}}$, $\hat{S}^{T_{\alpha}}$ data in terms of $(G, \omega_G)$ directly. The three steps are outlined as follows: (i) Obtain Eq. (15)'s $C_{(a,b)}^2$ value by doing the slant product twice from the 4-cocycle $\omega_G$ or triangulating Eq. (12). (ii) Derive $\rho_{a,b}(c)$ of the $C_{(a,b)}^2$ projective Rep in Eq. (14), where $\rho_{a,b}(c)$ is the GL matrix. (iii) Write the modular data in the canonical basis $|\alpha, a, b\rangle, |\beta, c, d\rangle$ of Eq. (11).

After some long computations [56], we find the most general formula $\hat{S}^{T_{\alpha}}$ for a group $G$ (both Abelian or non-Abelian) with cocycle twist $\omega_G$:

$$\hat{S}^{T_{\alpha}} = \frac{1}{|G|} \sum_{\omega, \rho_{a,b}(c)} \rho_{a,b}(c) \rho_{\omega}^{\alpha} \rho_{\omega}^{\alpha} \rho_{\omega}^{\alpha} \rho_{\omega}^{\alpha},$$

$$\hat{T}^{T_{\alpha}} = \frac{1}{|G|} \sum_{\omega, \rho_{a,b}(c)} \rho_{a,b}(c) \rho_{\omega}^{\alpha} \rho_{\omega}^{\alpha} \rho_{\omega}^{\alpha} \rho_{\omega}^{\alpha},$$

$$\hat{S}^{T_{\alpha}} = \frac{1}{|G|} \sum_{\omega, \rho_{a,b}(c)} \rho_{a,b}(c) \rho_{\omega}^{\alpha} \rho_{\omega}^{\alpha} \rho_{\omega}^{\alpha} \rho_{\omega}^{\alpha},$$

$$\hat{GSD} = Tr[\rho_{a,b}(c)] = \sum_A (|\Psi_A| P |\Psi_A|).$$

Here $|\Psi_A|$ and $|\Psi_B|$ are the ground-state bases on the $T^4$ torus; for example, they are $|\alpha, a, b\rangle$ (with $\alpha$ charge and $\beta$ flux) in 2 + 1 dimensions and $|\alpha, a, b, c\rangle$ (with $\alpha, \beta, \gamma$ charges and $\delta, \epsilon$ fluxes) in 3 + 1 dimensions. We also include the data on GSD, where $P$ is the projection operator for ground states discussed in Sec. II B. In the case of $dD$ GSD on $T^4$ (e.g., 3D GSD on $T^3$), we simply compute the $Z$ amplitude filling in $T^4 \times S^1 = T^4$. There is no shortcut here except doing explicit calculations [56].

We write $\beta_G = \beta_G, \delta_G, \epsilon_G$ due to the coordinate identification under $\hat{S}^{T_{\alpha}}$. The assignments of the directions of gauge fluxes (group elements) are clearly expressed in the second line. We may use the first-line expression for simplicity.

We also provide the most general formula of $\hat{T}^{T_{\alpha}}$ in the $|\alpha, a, b\rangle$ basis:

$$\hat{T}^{T_{\alpha}} = \frac{1}{|G|} \sum_{\omega, \rho_{a,b}(c)} \rho_{a,b}(c) \rho_{\omega}^{\alpha} \rho_{\omega}^{\alpha} \rho_{\omega}^{\alpha} \rho_{\omega}^{\alpha}.$$
direction in Fig. 1. This $b$ flux can be viewed as the by-product of a monodromy defect causing a branch cut (a symmetry twist [55,57,58,60]), such that the wave function will gain a phase by winding around the compact $z$ direction. Now we further regard the $b$ flux as a string threading around in the background, so that winding around this background string [e.g., black string threading in Figs. 2(c), 6(c), and 7(c)] obtains the $b$ flux effect if there is a nontrivial winding in the compact direction $z$. The dashed arrow along the compact $z$ schematically indicates such a $b$ flux effect from the background string threading.

1. $T_b^{xy}$ and topological spin of a closed string

We apply the above idea to interpret $T_b^{xy}$, shown in Fig. 6. From Eq. (31), we have $T_b^{xy} = T_{a,b}^{xy} \equiv \exp(i\Theta_{a,b})$ with a fixed $b$, label for a given $b$ flux sector. For each $b$, $T_b^{xy}$ acts as a familiar 2D T matrix, $T_{a,i}$, which provides the topological spin of a quasiparticle $(\alpha,a)$ with charge $\alpha$ and flux $a$, in Fig. 6(a).

From the 3D viewpoint, however, this $(\alpha,a)$ particle is actually a closed string compactified along the compact $z$ direction. Thus, in Fig. 6(b), the self-2$\pi$ rotation of the topological spin of a quasiparticle $(\alpha,a)$ is indeed the self-2$\pi$ rotation of a framed closed string. (Physically we understand that there the string can be framed with arrows, because the inner texture of the string excitations are allowed in a condensed matter system, due to defects or the finite-size lattice geometry.) Moreover, from the equivalent 3D view in Fig. 6(c), we can view the self-2$\pi$ rotation of a framed closed string as the self-2$\pi$ flipping of a framed closed string, which flips the string inside-out and then outside-in, back to its original status. This picture works for both the $b = 0$ zero-flux sector and $b \neq 0$ under the background string threading. We thus propose $T_b^{xy}$ as the topological spin of a framed closed string, threaded by a background string carrying a monodromy $b$ flux.

2. $S_b^{xy}$ and three-string braiding statistics

Similarly, we apply the same philosophy to do 3D-to-2D reduction for $S_b^{xy}$, each effective two dimensions threading with a distinct gauge flux $b$. We can obtain $S_b^{xy}$ from Eq. (32) with SL(3, $Z$) modular data. Here we focus on interpreting $S_b^y$ in the Abelian topological order. Writing $S_b^{xy}$ in the canonical basis $|\alpha,a,b\rangle, |\beta,c,d\rangle$ of Eq. (11), we find that, true to Abelian topological order,

$$S_b^{xy} = S_{(\alpha,a,b)(\beta,c,d)} \equiv \frac{1}{|G|} S_{a,b}^{2D \alpha,\beta} \delta_{\alpha,d}.$$  

As we predict the generality in Eq. (4), the $S_b^{xy}$ here is diagonalized with the $b$ and $d$ identified (as the $z$-direction flux created by the background string threading). For a given fixed-$b$-flux sector, the only free indices are $|\alpha,a\rangle$ and $|\beta,c\rangle$, all collected in $S_{a,c}^{2D \alpha,\beta}$. (Explicit data are be presented in Sec. IV B.) Our interpretation is shown in Fig. 2. From a 2D viewpoint, $S_b^{xy}$ gives the full 2$\pi$ braiding statistics data for two quasiparticle $|\alpha,a\rangle$ and $|\beta,c\rangle$ excitations in Fig. 2(a). However, from the 3D viewpoint, the two particles are actually two closed strings compactified along the compact $z$ direction. Thus, the full-2$\pi$ braiding of two particles in Fig. 2(a) becomes that of two closed strings in Fig. 2(b). More explicitly, in the equivalent 3D view in Fig. 2(c), we identify the coordinates $x,y,z$ carefully to see that such a full-braiding process is one (red) string going inside to the loop of another (blue) string and then going back from the outside.
The above picture works again for both the $b = 0$ zero-flux sector and $b \neq 0$ under background string threading. When $b \neq 0$, the third (black) background string in Fig. 2(c) threads through the two (red and blue) strings. The third (black) string creates the monodromy defect/branch cut in the background and carries $b$ flux along $z$ acting on two (red and blue) strings which have nontrivial winding on the third string. This three-string braiding was first emphasized in a recent paper [45]; here we make further connections to the $S^3_{b}$ data and illuminate its physics in a 3D-to-2D reduction under $b$ flux sectors.

We have proposed and shown that $S^3_{b}$ can capture the physics of three-string braiding statistics with two strings threaded by a third background string causing $b$ flux monodromy, where the three strings have the linking configuration as the connected sum of two Hopf links $2_1^2#2_1^2$.

3. Spin-statistics relation for closed strings

Since a spin-statistics relation for 2D particles is shown in Fig. 5, we may wonder, by using our 3D-to-2D reduction picture, whether a spin-statistics relation for a closed string holds? To answer this question, we should compare the topological spin picture of $T^{y}_{b}$ with the exchange statistic picture of two closed strings in Fig. 7. Figure 7 essentially takes a half-braiding of the $S^3_{b}$ process in Fig. 2 and considers doing half-braiding on the same excitations in $|a,a,b,c,d\rangle$. In principle, one can generalize the framed worldline picture of particles in Fig. 5 to the framed worldsheet picture of closed strings. (The framed worldline is like a worldsheet; the framed worldsheet is like a worldvolume.) This interpretation shows that the topological spin in Fig. 6 and the exchange statistics in Fig. 7 carry the same data, namely,

$$T^{y}_{b} = T_{a,b}^{\alpha_{x,y,z}} = (S^{2D}_{\alpha_{x,y,z}})^{1/2} \text{ or } (S^{2D}_{\alpha_{x,y,z}})^{1/2}$$

from the data of Eqs. (31) and (34). The equivalence holds, up to a [complex conjugate (*)] sign caused by the orientation of the rotation and the exchange.

In Sec. IV B, we show, for the twisted gauge theory of Abelian topological orders, that such an interpretation of Eq. (35) is correct and agrees with our data. We term this the spin-statistics relation for closed strings.

In this section, we have obtained the explicit formulas of $S^3_{b}$, $T^{y}_{b}$, and $S^{y}$ (Secs. III A and III B) as well as capturing the physical meanings of $S^{3D}_{b}$ and $T^{y}_{b}$ (Sec. III C 3). Before concluding, we note that the full understanding of $S^{3}_{b}$ seems to be intriguingly related to its 3D nature. It is not obvious to us that the use of 3D-to-2D reduction can capture all the physics of $S^{3}_{b}$. We comment on this issue again in Sec. V.

IV. SL(3,$\mathbb{Z}$) MODULAR DATA AND MULTISTRING BRAIDING

A. Ground-state degeneracy and particle and string types

We now proceed to study the topology-dependent GSD, modular data $S$, $T$, of the $3+1$ twisted gauge theory with finite group $G = \prod Z_{N}$. We comment that the GSD on $\mathbb{T}^2$ of 2D topological order counts the number of quasiparticle excitations, which, from the representation theory, is simply counting the number of charges $a$ and fluxes $a$ forming the quasiparticle basis $|a,a\rangle$ spanning the ground-state Hilbert space. In two dimensions, GSD counts the number of types of quasiparticles (or anyons) as well as the number of bases $|a,a\rangle$. For higher dimensions, GSD on $\mathbb{T}^d$ of $d$-topological order still counts the number of canonical bases $|a,a,b,\ldots\rangle$, however, it may overcount the number of types of particles (with charge), strings (with flux), etc., excitations. From an untwisted $Z_N$ field theory perspective, the fluxed string may be described by a two-form $B$ field, and the charged particle may be described by a one-form $A$ field, with a BF action $\int BdA$. As we can see, fluxes $a$ and $b$ are overcounted.

We suggest that counting the number of types of particles of $d$ dimensions is equivalent to the process in Fig. 8, where we dig a ball $B^d$ with a sphere $S^{d-1}$ around particle $q$, which resides on $S^d$. And we connect it through an $S^1$ tunnel to its antiparticle $\bar{q}$. This process causes creation-annihilation from vacuum and counts how many types of $q$ sectors are equivalent to

$$\text{the number of particle types } = \text{GSD on } S^{d-1} \times I,$$  \hfill (36)
TABLE II. $S^\text{xyz}$ = $S^\text{xyz}_{a,b,c,d}$ modular data on the 3 + 1D twisted gauge theories with $G = Z_{N_1} \times Z_{N_2} \times Z_{N_3}$. In the last column, $\mathcal{H}^3$ stands for $\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$; the induced $S^\text{xyz}_b$ is listed in Table IV.

<table>
<thead>
<tr>
<th>$\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$</th>
<th>4-cocycle $S^\text{a,b,c}_{d,a,b}$</th>
<th>Induced $S^\text{xyz}_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_{N_1}$</td>
<td>Type II 1st</td>
<td>$\exp\left(\sum_k \frac{2\pi i}{N_1} (p_k a_k - a_k d_k)\right) \cdot \exp\left(\frac{2\pi i}{N_1} (a_1 b_1 - a_1 b_2)\right)$</td>
</tr>
<tr>
<td>$Z_{N_2}$</td>
<td>Type II 2nd</td>
<td>$\exp\left(\sum_k \frac{2\pi i}{N_2} (p_k a_k - a_k d_k)\right) \cdot \exp\left(\frac{2\pi i}{N_2} (a_1 b_1 - a_1 b_2)\right)$</td>
</tr>
<tr>
<td>$Z_{N_3}$</td>
<td>Type III 1st</td>
<td>$\exp\left(\sum_k \frac{2\pi i}{N_3} (p_k a_k - a_k d_k)\right) \cdot \exp\left(\frac{2\pi i}{N_3} (a_1 b_1 - a_1 b_2)\right)$</td>
</tr>
<tr>
<td>$Z_{N_3}$</td>
<td>Type III 2nd</td>
<td>$\exp\left(\sum_k \frac{2\pi i}{N_3} (p_k a_k - a_k d_k)\right) \cdot \exp\left(\frac{2\pi i}{N_3} (a_1 b_1 - a_1 b_2)\right)$</td>
</tr>
</tbody>
</table>

with $I \simeq S^I$ for this example. For the spacetime integral, one evaluates Eq. (29) with $M = S^{d-1} \times S^1 \times S^I$.

For closed string excitations, one may naïvely use $T^3$ to enclose a string, analogously to using $S^2$ to enclose a particle in three dimensions. Then one may deduce the number of string types = GSD on $T^3 \times S^1 \simeq T^3$, and that of spacetime integrals on $T^3$, as mentioned earlier, which is incorrect and overcounting. We suggest the number of string types = $S^\text{xyz}$ and $T^\text{xyz}$’s number of blocks, $b$.

(37)

whose blocks are labeled $b$ in the form of Eq. (4). We show explicit examples of counting using Eqs. (36) and (37) in Sec. IV B.

B. Abelian examples: 3D twisted $Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ gauge theories with types II and III 4-cocycle twists

We first study the most generic 3 + 1D finite Abelian twisted gauge theories with types II and III 4-cocycle twists in Table I. It is general enough for us to consider $G = Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ with nonvanishing gcd $N_i$, $N_j$. Types II and III (both the first and the second kinds) twisted gauge theories have GSD = $|G|^3$ on the spatial $T^3$ torus. As such, the canonical basis $[a, a, b]$ of the ground-state sector labels the charge $a$ (along x) and two fluxes $a$ (and b along y and z); each of the three has $|G|$ kinds. Thus, naturally from the Rep theory viewpoint, we have GSD = $|G|^3$. However, as mentioned in Sec. IV A, $|G|^3$ overcounts the number of strings and particles. By using Eqs. (36) and (37), we find that there are $|G|$ types of particles and $|G|$ types of strings. The canonical basis $[a, a, b]$ (GSD on $T^3$) counts twice the flux sectors.

In Table II, we list their $S^\text{xyz}$ values by computing Eq. (30), where we denote $a$ = $(a_1, a_2, a_3, \ldots)$, with $a_j \in Z_{N_j}$, and the same notation for other $b$, $c$, and $d$ fluxes.

Here we extract the $S^\text{xyz}_{a,b}$ part of $S^\text{xyz}$, ignoring the $|G|^{-1}$ factor:

$S^\text{xyz}_{a,b} = S^\text{xyz}_{a,b,c,d} \equiv \frac{1}{|G|} S^\text{xyz}_{a,b,c,d}$. (38)

The $S$ matrix reads $g_{xk} = d_k, g_{yk} = a_k$ in Eq. (30). In Table III, we show $T^\text{xyz}$. Here for an Abelian $G$, where $C^{(2)}_{a,b,c}(d,e)$ is a 2-coboundary (cohomologically trivial) and thus $\dim(\text{Rep}) = 1$, we compute $S^\text{xyz}$ by Eq. (33) and that reduces to Eq. (34), $S^\text{xyz}_{b} = (S^\text{xyz})_{a,b,c} \equiv \frac{1}{|G|} S^\text{xyz}_{a,b,c}$. In Table IV, we list $S^\text{xyz}$ in terms of $S^\text{xyz}_{a,b,c}$ for simplicity.

Several remarks follow. (i) For an untwisted gauge theory (topological term $p = 0$), which is the direct product of $Z_N$ gauge theory or $Z_N$ toric code, its statistics has the form $\exp(\sum_{a,b} (p_k a_k - a_k d_k)) \cdot \exp(\sum_{a,b} a_k b_k)$. This is described by the BF theory of $\int B dA$, action, with $\alpha, \beta$ as the charge of particles (1-form gauge field $A$) and $a, b$ as the flux of string (2-form gauge field $B$). This essentially describes the braiding between a pure particle and a pure string.

(ii) Both $S^\text{xyz}$ and $T^\text{xyz}$ have block diagonal forms $S^\text{xyz}_{b}$ and $T^\text{xyz}_{b}$ with respect to the $b$ flux (along $z$), which correctly reflects what Eq. (4) predicts.

(iii) $T^\text{xyz}$ is in the SL(3,$\mathbb{Z}$) canonical basis and fully diagonal, but $S^\text{xyz}$ may not be in the canonical basis for each block of $S^\text{xyz}_{b}$, due to its SL(2,$\mathbb{Z}$) nature. We can find the proper basis for each block by the method of Ref. [61]. Nevertheless, the eigenvalues of $S^\text{xyz}$ in Table IV are still proper and invariant regardless of the basis.

(iv) To characterize the topological orders, we can further compare the 3D $S^\text{xyz}_{b}$ data to the SL(2,$\mathbb{Z}$) data on the 2D $S^\text{xyz}$ of

TABLE III. $T^\text{xyz}_b$ modular data of the 3 + 1D twisted gauge theories with $G = Z_{N_1} \times Z_{N_2} \times Z_{N_3}$. We can view this in terms of the index $b$ for blocks of $T^\text{xyz}_{b}$ = $T^\text{xyz}_{a,b}$, with the flux $b$ along the compact $z$ direction.

| $\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$ | 4-cocycle | $T^\text{xyz}_{a,b}$ | Induced $T^\text{xyz}_{b}$ |
|--------------------------------------|-------------------------------|------------------|
| $Z_{N_1}$ | Type II 1st | $\exp\left(\sum_k \frac{2\pi i}{N_1} (p_k a_k - a_k d_k)\right) \cdot \exp\left(\frac{2\pi i}{N_1} (a_1 b_1 - a_1 b_2)\right)$ | Types I and II of $\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$ |
| $Z_{N_2}$ | Type II 2nd | $\exp\left(\sum_k \frac{2\pi i}{N_2} (p_k a_k - a_k d_k)\right) \cdot \exp\left(\frac{2\pi i}{N_2} (a_1 b_1 - a_1 b_2)\right)$ | Types I and II of $\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$ |
| $Z_{N_3}$ | Type III 1st | $\exp\left(\sum_k \frac{2\pi i}{N_3} (p_k a_k - a_k d_k)\right) \cdot \exp\left(\frac{2\pi i}{N_3} (a_1 b_1 - a_1 b_2)\right)$ | Two type IIs of $\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$ |
| $Z_{N_3}$ | Type III 2nd | $\exp\left(\sum_k \frac{2\pi i}{N_3} (p_k a_k - a_k d_k)\right) \cdot \exp\left(\frac{2\pi i}{N_3} (a_1 b_1 - a_1 b_2)\right)$ | Two type IIs of $\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$ |
TABLE IV. $S^{13}$ modular data on the 3+1D twisted gauge theories with $G = Z_{N_{1}} \times Z_{N_{2}} \times Z_{N_{3}}$. There are two more columns [$\mathcal{H}^{4}(G, \mathbb{R} / \mathbb{Z})$ and induced $S^{13}_{\rho}$] not shown here, since the data simply duplicate the first and fourth columns in Table II. The basis chosen here is not canonical for excitations, in the sense that particle braiding around a trivial vacuum still obtains a nontrivial statistic phase. Finding the proper canonical basis for each $b$ block of $S^{13}_{\rho}$ can be done by the method in Ref. [61].

\[
\omega_{k} \quad S^{2D}_{a,c(b)} = tr_{a,b}^{(k)}(a^{c^{-1}}tr_{a,b}^{(k)}(ac^{-2})
\]

| II 1st | $\exp(\sum_{a=0}^{2N_{1}}(\alpha_{a}(c_{a} - 2a_{a}) + \beta_{a}(a_{a} - 2c_{a}))) \cdot \exp(\sum_{b=0}^{2N_{2}}(\alpha_{b}(c_{b} - 2a_{b} + 2c_{b} - 2a_{b})) \cdot \exp(\sum_{c=0}^{2N_{3}}(\alpha_{c}(c_{c} - 2a_{c} + 2c_{c} - 2a_{c})))$ |
| II 2nd | $\exp(\sum_{a=0}^{2N_{1}}(\alpha_{a}(c_{a} - 2a_{a}) + \beta_{a}(a_{a} - 2c_{a}))) \cdot \exp(\sum_{b=0}^{2N_{2}}(\alpha_{b}(c_{b} - 2a_{b} + 2c_{b} - 2a_{b})) \cdot \exp(\sum_{c=0}^{2N_{3}}(\alpha_{c}(c_{c} - 2a_{c} + 2c_{c} - 2a_{c})))$ |
| III 1st | $\exp(\sum_{a=0}^{2N_{1}}(\alpha_{a}(c_{a} - 2a_{a}) + \beta_{a}(a_{a} - 2c_{a}))) \cdot \exp(\sum_{b=0}^{2N_{2}}(\alpha_{b}(c_{b} - 2a_{b} + 2c_{b} - 2a_{b})) \cdot \exp(\sum_{c=0}^{2N_{3}}(\alpha_{c}(c_{c} - 2a_{c} + 2c_{c} - 2a_{c})))$ |
| III 2nd | $\exp(\sum_{a=0}^{2N_{1}}(\alpha_{a}(c_{a} - 2a_{a}) + \beta_{a}(a_{a} - 2c_{a}))) \cdot \exp(\sum_{b=0}^{2N_{2}}(\alpha_{b}(c_{b} - 2a_{b} + 2c_{b} - 2a_{b})) \cdot \exp(\sum_{c=0}^{2N_{3}}(\alpha_{c}(c_{c} - 2a_{c} + 2c_{c} - 2a_{c})))$ |

$\mathcal{H}^{3}(G, \mathbb{R} / \mathbb{Z})$ in Table XII. (See Appendix A for data.) All of the dimensional reductions of these data ($S^{13}_{\rho}$ in Tables II and IV and $T_{k}$ in Table III) agree with the 3+1 cocycle (induced from the 4-cocycle $\omega_{k}$ in the final column in Table I. Combining all the data, we conclude that Eq. (19) becomes explicit. For example, type II twists for $G = (Z_{2})^{3}$ as

\[
C^{3D}_{(Z_{2})^{3},V} = 4C^{3D}_{(Z_{2})^{3},1}.
\]

\[
C^{3D}_{(Z_{2})^{3},a,b,IV} = C^{3D}_{(Z_{2})^{3},1} \otimes C^{3D}_{(Z_{2})^{3},a,1,1} \otimes 2C^{3D}_{(Z_{2})^{3},a,b}.
\]

Such a type II $a_{b,c}$ can produce a $b = 0$ sector of $(Z_{2})^{3}$ toric code of two dimensions as $C^{3D}_{(Z_{2})^{3},b}$, a $b \neq 0$ sector of $(Z_{2})^{3}$ toric code and another $b \neq 0$ sector $C^{3D}_{(Z_{2})^{3},a,b,c}$, for example. This procedure can be applied to other types of cocycle twists.

(v) To classify the topological orders, we interpret the decomposition in Eq. (19) as the implication of classification. Let us do the counting of the number of phases in the simplest example of type II, $G = Z_{2} \times Z_{2}$ twisted theory. There are four types in $[P^{11}_{1}P^{21}_{1}]$, $[P^{11}_{2}]$, and $[P^{21}_{2}]$. However, we find that there are only two topological orders of four. One is the trivial $(Z_{2})^{2}$ gauge theory as Eq. (39); the other is the nontrivial type as Eq. (40). There are two ways to see this: (i) from the full $S^{13}_{\rho}$, $T^{3}$ data and (ii) from the TQC theory of $S^{13}_{\rho}$, $T^{3}$ under distinct fluxes $b$, which is from an $\mathcal{H}^{3}(G, \mathbb{R} / \mathbb{Z})$ perspective. We should be aware that, in principle, tagging particles, strings, or gauge groups is not allowed, so one can identify many seemingly different orders by relabeling their excitations. We give more examples of counting 2D and 3D topological orders in Appendix A.

(vi) The spin-statistics relation of closed strings in Eq. (35) is verified to be correct here, while we can correct the complex conjugate in Eq. (35). This is why we draw the orientations in Figs. 6 and 7 oppositely. Interpreting $T^{3}$ as the topological spin also holds.

(vii) For all the above data (types II and III), there is a special cyclic relation for $S^{3+1}$ in three dimensions when the charge labels are equal, $\alpha = \beta$ for $S^{a,b}_{a,b,d}$ (e.g., for pure fluxes $\alpha = \beta = 0$, namely, for pure strings):

\[
S^{a,b}_{a,b,d} = S^{a,b}_{b,a,d} = S^{a,b}_{d,a,b} = 1.
\]

However, such a cyclic relation does not hold (even at zero charge) for $S^{2D}_{a,c(b)}$, namely, $S^{a,b}_{b,a,c} = S^{a,b}_{b,a,c} = S^{a,b}_{b,a,c} \neq 1$ in general. Some other cyclic relations have been studied recently in Refs. [45] and [46], with which we have not yet made detailed comparisons, but the perspectives may be different. In Ref. [46], their cyclic relation is determined by triple-linking numbers associated with the membrane operators. In Ref. [45], their cyclic relation is related to the loop braiding in Fig. 2. This is which is relevant instead for $S^{2D}_{a,c,b}$, not our cyclic relation of $S^{a,b}_{a,b,d}$ in three dimensions. We comment more on the difference and the subtlety of $S^{13}_{\rho}$ and $S^{3+1}$ in Sec. V B.

C. Non-Abelian examples: 3D twisted $(Z_{n})^{3}$ gauge theories with type IV 4-cocycles

We now study a more interesting example, the generic 3+1D finite Abelian twisted gauge theory with type IV 4-cocycle twists with $p_{j} \neq \not{0}$ in Table I. For generality, our formula also incorporates type IV twists together with the aforementioned types II and III twists. So all 4-cocycle twists are discussed in this subsection. Different from the example of Abelian topological order with Abelian statistics in Sec. IV B, we show that the type IV 4-cocycle $a_{b,c}$ will cause the gauge theory to become non-Abelian, having non-Abelian statistics even if the original $G$ is Abelian. Our inspiration, rooted in the 2D example of the type III 3-cocycle twist in Table XII, will cause a similar effect, discovered in Ref. [40]. In general, one can consider $G = Z_{N_{1}} \times Z_{N_{2}} \times Z_{N_{3}} \times Z_{N_{4}}$ with nonvanished gcd $N_{1234}$; however, we focus on $G = (Z_{n})^{4}$ with $N_{1234} = n$, with $n$ prime for simplicity. From $\mathcal{H}^{3}(G, \mathbb{R} / \mathbb{Z}) = Z_{n}$, we have $n^{21}$ types of theories; $n^{20}$ are Abelian gauge theories, and $n^{20}, (n - 1)$ types with type IV $a_{b,c}$ show non-Abelian statistics.

D. Ground-state degeneracy

We compute the GSD of gauge theories with a type IV twist on the spatial $T^{3}$ torus, truncated from $|G| = |n|^{3} = n^{12}$ to

\[
\text{GSD}_{T^{3},IV} = (n^{8} + n^{9} - n^{5}) + (n^{10} - n^{7} - n^{6} + n^{3})
\]

\[
\equiv \text{GSD}_{T^{3},IV}^{\text{Abel}} + \text{GSD}_{T^{3},IV}^{\text{Non-Abel}}.
\]

(We derive the above only for a prime $n$. The GSD truncation is less severe and is in between GSD$_{T^{3},IV}^{\text{Abel}}$ and $|G|^{4}$ for a nonprime $n$.) As such, the canonical basis $|a, b, c\rangle$ of the ground-state sector on $T^{3}$ no longer has $|G|^{3}$ labels with the $|G|$ number
charge and two pairs of $|G| \times |G|$ fluxes as in Sec. IV B. This truncation is due to the nature of non-Abelian physics of type IV $\omega_4T,IV$ twisted. We explain our notation in Eq. (43); the (n)Abel indicates the contribution from (non-)Abelian excitations. From the Rep theory viewpoint, we can recover the truncation back to $|G|^3$ by carefully reconstructing the quantum dimension of excitations. We obtain

$$|G|^3 = (\text{GSD}^\text{Abel}_{T,IV}) + (\text{GSD}^\text{Abel}_{T,IV}) \cdot n^2$$

$$= [n^4 + n^3 - n] \cdot n^2 \cdot (1)^3$$

$$+ [(n^2)^2 - n^4 + n^3 - n] \cdot n^2 \cdot (n)^2$$

$$= \{\text{Flux}_{IV}^\text{Abel}\} \cdot n^2 \cdot (\text{dim}_m)^2 + \{\text{Flux}_{IV}^\text{Abel}\} \cdot n^2 \cdot (\text{dim}_n)^2.$$  

(44)

dim$_m$ means that the dimension of Rep as dim(Rep) is $m$, which is also the quantum dimension of excitations. We have dimension 1 for Abelian and $n$ for non-Abelian. In summary, we understand the decomposition precisely in terms of each (non-)Abelian contribution as follows:

Flux sectors = $|G|^2 = [n^4]^2$

$$= \text{Flux}_{IV}^\text{Abel} + \text{Flux}_{IV}^\text{Abel}.$$  

GSD$_{T,IV}$ = GSD$^\text{Abel}_{T,IV}$ + GSD$^\text{Abel}_{T,IV}$

dim(Rep)$^2 = 1^2 \cdot n^2.$

Numbers of charge Rep = $n^4 \cdot n^2.$  

(45)

Actually, the canonical basis $|x,a,b,\alpha\rangle$ (GSD on $T^3$) still works, and the sum of Abelian Flux$^\text{Abel}_{IV}$ and non-Abelian Flux$^\text{Abel}_{IV}$ counts the flux number of $a,b$ as the unaltered $|G|^2$. The charge Rep $\alpha$ is unchanged with a number of $|G| = n^4$ for the Abelian sector with a rank 1 matrix (1-dim linear or projective) representation, however, the charge Rep $\alpha$ is truncated to a smaller number $n^2$ for the non-Abelian sector also, with a larger, rank $n$ matrix ($n$-dim projective) representation.

Another view of GSD$_{T,IV}$ can be inspired by a generic formula like Eq. (4),

$$\text{GSD}_{M' \times S^1} = \bigoplus_b \text{GSD}_{b,M'} = \sum_b \text{GSD}_{b,M'},$$  

(46)

where we sum over GSD in all $b$ flux sectors, with $b$ flux along $S^1$. Here we can take $M' \times S^1 = T^3$ and $M' = T^3$. For the non-type IV (untwisted, types II and III) $\omega_4$ case, we have $|G|$ sectors of $b$ flux and each has GSD$_{b,T^2}$ = $|G|^2$. For the type IV $\omega_4$ case $G = (Z_n)^2$ with a prime $n$, we have

GSD$_{T,IV}$

$$= |G|^2 + |G| - 1 \cdot |Z_n|^2 \cdot (1 \cdot |Z_n|^3 + (|Z_n|^2 - 1) \cdot n)$$

$$= n^8 + (n^4 - 1) \cdot n^2 \cdot (1 \cdot n^3 + (n^3 - 1) \cdot n).$$  

(47)

As we expect, the first part is from the zero-flux $b = 0$, which is the normal untwisted $2 + 1D (Z_n)^3$ gauge theory as $C_{(Z_n)^3}$ with $|G|^2 = n^8$ on the 2-torus. The remaining $(|G| - 1)$ copies are inserted with nonzero flux ($b \neq 0$) as $C_{(Z_n)^3}$ with type III 3-cocycle twists in Table XII. In some but not all cases, $C_{(Z_n)^3,\omega_4}$ is $C_{(Z_n)^3 \times (Z_n)^3}$ with type III 3-cocycle twists. In either case, the GSD$_{T,IV}$ for $b \neq 0$ has the same decomposition always, equivalent to an untwisted $Z_n$ gauge theory with $GSD_{T^2}$ = $n^2$ direct product with

$$GSD_{T^2,\omega_4} = (1 \cdot n^3 + (n^3 - 1) \cdot n)$$

$$\equiv GSD^\text{Abel}_{T^2,\omega_4} + GSD^\text{Abel}_{T^2,\omega_4}.$$  

(48)

from which we generalize the result derived for 2 + 1D type III $\omega_4$ twisted theory with $G = (Z_n)^3$ in Ref. [40] to $G = (Z_n)^3$ of a prime $n$. One can repeat the counting for 2 + 1 dimensions as in Eqs. (44) and (45); see Appendix A.

To summarize, from the GSD counting, we foresee that there exist non-Abelian strings in 3 + 1D type IV twisted gauge theory, with a quantum dimension $n$. These non-Abelian strings (fluxes) carry dim(Rep) = $n$ non-Abelian charges. Since charges are sourced by particles, these non-Abelian strings are not pure strings but are attached to non-Abelian particles. (For a projection perspective from three to two dimensions, a non-Abelian string of $C^3D$ is a non-Abelian dyon with both charge and flux of $C^3D$.)

E. Modular 3D $T^3$

We compute $T^3$ and $S^{3\times 2}$ using the formula derived in Sec. III B for type IV $\omega_4$ theory (for generality, we also include the twists by types II and III $\omega_4$). Due to the large GSD and the quantum dimension of a non-Abelian nature, we focus on the simplest example $G = (Z_n)^4$ theory to have the smallest number of data. By $T^3(G,R/Z) = Z_n^2$, we have two $2^2$ types of theories, where $2^2$ types of type IV are endorsed with non-Abelian statistics (while $2^2$ types are Abelian gauge theories of non-type IV, with their $T$ and $S$ data reported in Sec. IV B).

For $G = (Z_n)^4$, there are still GSD$_{T^3}$ = 1576. Thus both $T$ and $S$ are matrices of rank 1576. $T^{3\times 2}$ has 1576 components along the diagonal.

For $G = (Z_2)^4$, we first define a quantity $\eta_{g_1,g_2,g_3}$ of convenience from the $C_{(g_2,h_2,d,b)}$ in Eq. (15):

$$\eta_{g_1,g_2,g_3} = \begin{cases} 0 & \text{if } C_{(g_2,h_2,d)}(g_1,g_3) = +1; \\ 1 & \text{if } C_{(g_2,h_2,d)}(g_1,g_3) = -1. \end{cases}$$  

(50)

Below the $p_{lm}$ and $p_{llmn}$ are the shorthand for types II and III (both first and second) topological term labels; $p_{lm} f_m(a, b, c, d)$ and $p_{llmn} f_m(a, b, c)$ abbreviate the function forms in the exponents of types II and III $\omega_4^2$ in Table I. Namely, we regard their 4-cocycle $\omega_4(a,b,c,d)$ as a trivial 2-cocycle $c_{a,b,c}$, written as $c_{a,b,c}(d)$ where $c_{a,b,c}(d) = h_{a,b,c} = h_{a,b,c}$, where $h_{a,b,c}(d)$ is a 1-cochain: $h_{a,b,c} = \exp(i p_{lm} f_m(a,b,c)) = \exp(i p_{ll} f_m(a,b,c))$ for the type II case. $\eta_{a,b,c}$ = $\exp(i p_{llmn} f_m(a,b,c)) = \exp(i p_{lm} f_m(a,b,c))$ for the type III case. We derive $T^{3\times 2} = T_{(a,b,c)}^{3\times 2}$ of Eq. (31) in Table V.

F. Modular 3D $S^{3\times 2}$

The $S^{3\times 2}$ matrix has 1576 $\times$ 1576 components. We organize $S^{3\times 2}$ into four blocks, denoting by $n\text{Abel}(\text{Abel})$ (non-Abelian)/Abelian with 736 (840) components. Defining
Regarding dimensional reduction from 3D to 2D sectors, from the above, we go beyond Ref. [40] to give the complete classification of all 2M kinds of T^\gamma for the non-Abelian theories in H^4(G,R/Z) = Z_2^4 (half of 2^21). The ((\pm_\alpha,\pm_\beta)) pair makes up the numbers of charge Rep n^2 = 2^2 in Eq. (45). Details of the rank 2 matrix Rep are given in Appendix A.

<table>
<thead>
<tr>
<th>Excitations (a,b)</th>
<th>T^\gamma_{a,b}</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a, F(\text{Abel}))</td>
<td>\exp \left( \sum_{k=1}^{4} \pi i a k \right) \rightarrow \text{e.g., } \pm 1</td>
</tr>
<tr>
<td>((\pm_\alpha,\pm_\beta)) F(\text{Abel}))</td>
<td>\exp \left( \sum_{k=1}^{4} \pi i a k \right) \rightarrow \text{e.g., } \pm 1</td>
</tr>
</tbody>
</table>

The exp(\sum_{k=1}^{4} \pi i a k) factor in the top-left block shows the pure-particle pure-string braiding of untwisted Z_2 gauge theory (no \alpha_0 dependence). We define \delta_{\text{Abel}}(b,d,b,d) = 1 if a \in {b,d,b,d}; otherwise, \delta_{\text{Abel}}(b,d,b,d) = 0. Some other technical details follow: For G = (Z_2)^4, the constraint \delta_{\text{Abel}}(b,d,b,d) \cdot \delta_{\text{Abel}}(a,b) reduces to \delta_{\text{Abel}}(a,b). The survival nonzero \text{S}_{\text{Abel,Abel}} values occur in only two kinds of forms: either d = a or d \neq a:

\text{S}_{\text{Abel,Abel}} = \begin{cases} 1 & \text{if } d = a \text{ or } d = ab; \\ 0 & \text{else.} \end{cases}

Some remarks follow.

(i) Regarding dimensional reduction from 3D to 2D sectors with b flux, from the above S^{xy,\gamma} and T^{xy}, there is no difficulty deriving S^{xy} from Eq. (32). From all these modular S^{xy} and T^{xy} data, we find consistency with the dimensional reduction of 3D topological order by comparison with the induced 3-cocycle \omega_0 from \omega. Let us consider a specific example, given the type IV index, \rho_{1234} = 1, and other zero types II and III indices, \rho_{1} = \rho_{-} = 0:

\begin{align} C^{3D}_{(Z_2)^3,\omega_0,IV} & = \mathbb{B}^{3D}_{\omega_0} \\ & \mathbb{B}_{\omega_0} \oplus 10 \mathbb{C}_{(Z_2)^3}^{\omega_0} \oplus 10 \mathbb{C}_{(Z_2)^3,\omega_0,III}^{\omega_0} \oplus \mathbb{C}_{(Z_2)^3,\omega_0,III}^{\omega_0} \oplus \mathbb{C}_{(Z_2)^3,\omega_0,III}^{\omega_0} \oplus \ldots \\ & \mathbb{C}_{(Z_2)^3}^{\omega_0} \oplus 10 \mathbb{C}_{(Z_2)^3}^{\omega_0} \oplus 10 \mathbb{C}_{(Z_2)^3}^{\omega_0} \oplus \mathbb{C}_{(Z_2)^3,\omega_0,III}^{\omega_0} \oplus \mathbb{C}_{(Z_2)^3,\omega_0,III}^{\omega_0} \oplus \ldots \end{align}

The C^{3D}_{(Z_2)^4,\omega_0} again, is the normal \(Z_2^4\) gauge theory at \(b = 0\). The 10 copies of \(C^{3D}_{(Z_2)^x,\omega_0}\) have an untwisted dihedral \(D_4\) gauge theory \([D_4] = 8\) product with the normal \(Z_2\) gauge theory. The duality to \(D_4\) theory in two dimensions can be expected [40]; see Table VI. As a by-product of our work, we go beyond Ref. [40] to give the complete classification of all twisted 2D \(\omega_3\) of \(G = (Z_2)^3\) and their corresponding

\begin{align} S_{xy} & = \frac{1}{|G|} \left( \begin{array}{c} S_{\text{Abel,Abel}} \mid S_{\text{Abel,nAbel}} \mid S_{\text{Abel,nAbel}} \mid S_{\text{Abel,nAbel}} \end{array} \right) \begin{pmatrix} \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4, & a, & b \end{pmatrix} \\ & \begin{pmatrix} (\pm_\alpha, & \pm_\beta) \end{pmatrix} \\ \text{S}_{\text{Abel,Abel}} & = 1 \cdot \exp \left( \sum_k \frac{2\pi i}{N_k} ( -\alpha_d k + \beta_k a_k ) \right) \cdot \delta_{b,c} = \left( -1 \right)^{-\alpha_0 a_d + \beta_0 a} \cdot \delta_{b,c}, \\ \text{S}_{\text{Abel,nAbel}} & = \left( -1 \right)^{-\alpha_0 a_d} \cdot \exp \left( \sum_k \frac{2\pi i}{N_k} ( -\alpha_d k + \beta_k a_k ) \right) \cdot \delta_{b,c}, \\ \text{S}_{\text{nAbel,Abel}} & = \left( -1 \right)^{-\beta_0 a_d} \cdot \exp \left( \sum_k \frac{2\pi i}{N_k} ( -\beta_0 a_d + \alpha_0 d ) \right) \cdot \delta_{b,c}, \\ \text{S}_{\text{nAbel,nAbel}} & = \left( -1 \right)^{-\beta_0 a_d} \cdot \exp \left( \sum_k \frac{2\pi i}{N_k} ( -\beta_0 a_d + \alpha_0 d ) \right) \cdot \delta_{b,c}. \\ \end{align}

The \(\chi_{G,\mu}\) in Table VI. D^{(\gamma)}(G), the twisted quantum double model of G in 2 + 1 dimensions, and their 3-cocycle \(\omega_0\) (involving type III) types in \(C^{3D}_{(Z_2)^4,\omega_0}\). We classify the 64 types of 2D non-Abelian twisted gauge theories into five classes, which agree with Ref. [62]. Each class has distinct non-Abelian statistics. Both dihedral group \(D_4\) and quaternion group \(Q_8\) are non-Abelian groups of order 8, as |\(D_4| = |Q_8| = |Z_2^3| = 8\). D^{(\gamma)}(G) data can be found in Ref. [62]. Details are reeported in Appendix A.

\begin{table}[h]
\centering
\caption{SL(3, Z) modular data \(T^{\gamma} = T^{\gamma}_{ab}\) for the \((Z_2)^3\) theory with type IV \(\omega_0\). The formula of \(T^{\gamma}\) is separated into two sets: the first set, with 736 components (from the sector \(GSD_{\text{Abel}}^{\omega_0}\)), and another 840 components (from the sector \(GSD_{\text{nAbel}}^{\omega_0}\)). \(F = (a,b)\) are fluxes with eight components: \((a_1, a_2, a_3, a_4) \in (Z_2)^4\) and \((b_1, b_2, b_3, b_4) \in (Z_2)^4\). The number of distinct fluxes in \(F(\text{Abel})\) is 46 (= Flux_{\text{Abel}}^{\omega_0}); the number of distinct fluxes in \(F(\text{nAbel})\) is 210 (= Flux_{\text{nAbel}}^{\omega_0}). This table lists all 2^21 kinds of \(T^{\gamma}\) for the non-Abelian theories in \(H^4(G,R/Z) = Z_2^4\) (half of 2^21). The \((\pm_\alpha,\pm_\beta)\) pair makes up the numbers of charge Rep \(n^2 = 2^2\) in Eq. (45). Details of the rank 2 matrix Rep are given in Appendix A.

\begin{tabular}{|c|c|}
\hline
\textbf{Excitations} (\(a,b\)) & \textbf{\(T^{\gamma}_{a,b}\)} \\
\hline
\textbf{\((a, F(\text{Abel}))\)} & \textbf{\((\pm_\alpha,\pm_\beta) F(\text{Abel}))\)} \\
\hline
\textbf{\((\pm_\alpha,\pm_\beta)\)} & \textbf{\((\pm_\alpha,\pm_\beta)\)} \\
\hline
\end{tabular}
\end{table}
topological orders and twisted quantum double $D^\omega(G)$ in Appendix A.) The remaining 5 copies $C_{(Z_2)^3,\omega_{014}}^{3D}$ must contain the twist on the full group $(Z_2)^3$, not just its subgroup. This peculiar feature suggests the following remark.

(ii) Sometimes there may exist a duality between a twisted Abelian gauge theory and an untwisted non-Abelian gauge theory [40]; one may wonder whether one can find a dual non-Abelian gauge theory for $C_{(Z_2)^3,\omega_{014}}^{3D}$. We find, however, that $C_{(Z_2)^3,\omega_{014}}^{3D}$ cannot be dual to a normal gauge theory (neither Abelian nor non-Abelian) but must be a twisted (Abelian or non-Abelian) gauge theory. The reason is more involved. Let us first recall the more familiar 2D case. One can consider the $G = (Z_2)^3$ example with $C_{(Z_2)^3,\omega_{01}}^{2D}$, with $H^3(G,\mathbb{Z}/2) \approx (Z_2)^3$. There are $2^6$ non-Abelian types with type III $\omega_3$ (the other $2^6$ are Abelian without type III $\omega_3$). We find that the 64 non-Abelian types of 3-cocycles $\omega$ go to 5 classes, labeled $\omega_1[1]$, $\omega_3[3]$, $\omega_3[5]$, and $\omega_3[7]$, and their twisted quantum double model $D^\omega(G)$ values are listed in Table VI. The number in brackets following $\omega_3$ (first column) is related to the number of pairs of $\pm i$ in the $T$ matrix and the $d/i$ stand for the linear dependence ($d$)/independence ($i$) of $\omega_3$.

In Table VI, we show that two classes of 3-cocycles for $D^\omega((Z_2)^3)$ of two dimensions can have dual descriptions by gauge theory of a non-Abelian dihedral group $D_4$ and quaternion group $Q_8$. However, the other three classes of 3-cocycles for $D^\omega((Z_2)^3)$ do not have a dual (untwisted) non-Abelian gauge theory.

Now let us reconsider 3D $C_{G,\omega_{014}}^{3D}$, with $|Z_2|^4 = 16$. From Ref. [39], we know that 3 + 1D $D_4$ gauge theory undergoes decomposition by its five centralizers. Applying the rule of decomposition to other groups implies that for an untwisted group $G$ in 3D $C_{G,\omega}$, we can decompose it into sectors of $C_{G,b,b}$, where $G_b$ becomes the centralizer of conjugacy class (flux) $b$: $C_{G,b}^{3D} = \oplus b C_{G,b}^{2D}$. Some useful information is

$$C_{(Z_2)^3}^{3D} = 16C_{(Z_2)^2}^{2D},$$

$$C_{D_4}^{3D} = 2C_{D_4}^{2D} \oplus 2C_{(Z_2)^2}^{2D} \oplus C_{Z_2}^{2D},$$

$$C_{Z_2 \times Z_4}^{3D} = 4C_{Z_2 \times Z_4}^{2D} \oplus 4C_{Z_2}^{2D} \oplus 2C_{Z_2}^{2D},$$

$$C_{Q_8}^{3D} = 2C_{Q_8}^{2D} \oplus 3C_{Z_2}^{2D},$$

and we find that no such decomposition is possible for $|G| = 16$ groups to match Eq. (54)’s. Furthermore, if there exists a non-Abelian $G_{ nab}$, to have Eq. (54) those $(Z_2)^3$, $(Z_2)^2 \times (D_4)$ or the twisted $(Z_2)^3$ must be the centralizers of $G_{ nab}$. But one of the centralizers (the centralizer of the identity element as a conjugacy class $b = 0$) of $G_{ nab}$ must be $G_{ nab}$ itself, which was already ruled out from Eqs. (55) and (57). Thus, we prove that $C_{(Z_2)^3,\omega_{014}}^{3D}$ is not a normal 3 + 1D gauge theory (neither $Z_2 \times D_4$, nor Abelian, nor non-Abelian) but must only be a twisted gauge theory.

(iii) We discover that (see Fig. 9) for any twisted gauge theory $C_{G,\omega_{014}}^{3D}$ with type IV 4-cocycle $\omega_{014}$ (whose non-Abelian nature is not affected by adding other types II and III $\omega_{014}$,) by threading a third string through the two-string unlink $0^2_1$ into the three-string Hopf links $2_1^2 \theta^2_1$ configuration, Abelian two-string statistics is promoted to non-Abelian three-string statistics. We can see the physics from Eq. (54); the $C_{Z_2}^{2D}$ is Abelian in the $b = 0$ sector but non-Abelian in the $b \neq 0$ sector. The physics in Fig. 9 is then obvious; applying our discussion in Sec. III C about the equivalence between string threading and the $b \neq 0$ monodromy causes a branch cut.

(iv) Regarding the cyclic relation for non-Abelian $S^{xyz}$ in three dimensions, interestingly, for the $(Z_2)^3$ twisted gauge theory with non-Abelian statistics, we find that a similar cyclic relation, Eq. (41), still holds as long as two conditions are satisfied: (a) the charge labels are equivalent, $\alpha = \beta$; and (b) $\delta_{ab}(b,c,d,b) \cdot \delta_{cd}(a,b,a) \cdot \delta_{bc}(d,a,d) = 1$. However, Eq. (41) is modified by a factor depending on the dimensionality of $\text{Rep}\, \alpha$:

$$S_{\alpha,\beta}^{a,b,c} \cdot S_{\alpha,\beta}^{a,b,d} \cdot S_{\alpha,\beta}^{a,c} \cdot S_{\alpha,\beta}^{a,d} \cdot |\text{dim}(\alpha)|^{-3} = 1.$$

This identity should hold for any type IV non-Abelian strings. This is a cyclic relation of a 3D nature, instead of the dimensional-reducing 2D nature of $S_{\alpha,\beta}^{a,b,c}$ in Fig. 2.

V. CONCLUSION

A. Knot-and-link configuration

Throughout this paper, we have indicated that the mathematics of knots and links may be helpful in organizing our string-braiding patterns in three dimensions. Here we illustrate them more systematically. We use Alexander-Briggs notation for the knots and links (see Fig. 10).

The knots and links for our string-braiding patterns are organized in Table VII. We recall that, in Sec. III C, the topological spin for a closed string in the $b = 0$ flux sector of
$C^2_{b}$ does a self-2π flipping under the 0₁ unknot configuration. Due to our spin-statistics relation of a closed string, we can view the topological spin of the $b = 0$ sector as the exchange statistics of two identical strings in the 0₁ unknot configuration. On the other hand, for the topological spin in the $b \neq 0$ sector, we effectively thread a (black) string through the (blue) unknot, which forms a Hopf link, 2₁. Meanwhile, we can view the topological spin of the $b \neq 0$ sector as the exchange statistics of two identical strings threaded by a (blue) unknot in a connected sum of two Hopf links in the $2₁^#2₁$ configuration. Furthermore, we can promote two-string Abelian statistics under the 0₁ unknot of the $b = 0$ sector to three-string Abelian (Sec. IV B) or non-Abelian (in Sec. IV C) statistics under Hopf links 2₁ # 2₁ of the $b \neq 0$ sector.

Nothing prevents us from considering more generic knot-and-link patterns for three-string or multistring braiding. Our reason is this: From the full modular SL(3, Z) group viewpoint, $S^{xyz}$ is a necessary generator to access the complete data on the SL(3, Z) group. However, we have learned that our 3D-to-2D reduction by Eq. (4) contains the information on three fluxes (b = 0 sector) or with another background (b \neq 0 sector) as the exchange/braiding statistics of two loops without any knot is 0₁, and two unknots can form an unlink 0₁. A Hopf link is 2₁, and the connected sum of two Hopf links is 2₁#2₁.

**TABLE VII.** Various string-braiding patterns in terms of knots and links in Alexander-Briggs notation: the topological spin of a loop and the exchange/braiding statistics of two loops without any background string inserted (b = 0 sector) or with another background string inserted (b \neq 0 sector). Here we effectively view the string-braiding statistics of 3D topological order in terms of 2D sectors: $C^{2D} = \oplus b C^{2D}_b$.

<table>
<thead>
<tr>
<th>$C^{2D}_b$</th>
<th>Physics of strings</th>
<th>Knots and links</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b = 0$</td>
<td>Topological spin (T)</td>
<td>0₁</td>
</tr>
<tr>
<td></td>
<td>Exchange statistics</td>
<td>0₁</td>
</tr>
<tr>
<td></td>
<td>2-string braiding</td>
<td>2₁</td>
</tr>
<tr>
<td>$b \neq 0$</td>
<td>Topological spin (T)</td>
<td>2₁</td>
</tr>
<tr>
<td></td>
<td>Exchange statistics</td>
<td>2₁ # 2₁</td>
</tr>
<tr>
<td></td>
<td>3-string braiding</td>
<td>2₁ # 2₁#2₁,...</td>
</tr>
</tbody>
</table>

fig. 10. (Color online) Under Alexander-Briggs notation, an unknot is 0₁, and two unknots can form an unlink 0₁. A Hopf link is 2₁, and the connected sum of two Hopf links is 2₁#2₁.

Indeed, as noted in Sec. IV, the 3D $S^{xyz}$ matrix essentially contains the information on three fluxes $(d, a, b)$ = $(d_c, a_c, b_c)$ in Eq. (38), $S^{xyz} = S_{d,a,b|f(c,d)}(\text{top}) \cong \frac{1}{\text{top}} S_{d,a,b|f(b,c)}$. Since strings carry fluxes in three dimensions, this further suggests that we should look for the braiding involving three strings; three-loop braiding was also recently emphasized in Refs. [45] and [46].

The configuration we have studied so far, with three strings, is the Hopf link 2₁#2₁. We propose using more general three-string patterns, such as the link

\[ \lambda^3_m \]

or its connected sum, to study topological states. ($\lambda^3_m$ is in Alexander-Briggs notation; here 3 means that there are three closed loops, $\lambda$ is the crossing number, and $m$ is the label for different kinds of $\lambda^3$ linking.) For example, three-string braiding can include links of 6₁, 6₂, and 6₃ in Fig. 11. The configurations in Fig. 11 are potentially promising for study of the braiding statistics of strings to classify or characterize topological orders.

To examine whether multistring braiding is topologically well defined, we propose a way to check that (such as the braiding processes in Figs. 9 and 11) the path that one (red) loop A winds around another (blue) loop B along the time evolution is nontrivial in the complement space of loop B and base (black) loop C. Namely, the path of A needs to be a nontrivial element of the fundamental group

\[ \pi_1(\text{complement space}) \].

The trefoil knot is 3₁. Some other simplest three-string links (by Hopf links 2₁#2₁) are shown: 6₁, 6₂ (Borromean rings), 6₃. From the spin-statistics relation of a closed string discussed in Sec. III C, where the topological spin of certain knot/link configurations (0₁ for the monodromy flux $b = 0$ and 2₁ for $b \neq 0$) is equivalent to the exchange statistics of certain knot/link configurations (0₁ for $b = 0$ and 2₁ # 2₁ for $b \neq 0$) under Eq. (35). Therefore, we may further conjecture that the topological spin of a trefoil knot 3₁ may relate to the braiding statistics of 6₁, 6₂, 6₃.
for the complement space of B and C. Thus the path must be homotopically nontrivial.

Before concluding this subsection, another final remark is in order: In Sec. III C 3, we mention generalizing the framed worldline picture of particles in Fig. 5 to the framed worldsheet picture of closed strings. (Note: The framed worldline is like a worldsheet; the framed worldsheet is like a worldvolume.) Thus, it may be interesting to study how incorporating the framing of particles and strings (with worldline/worldsheet/worldvolume) can provide richer physics and textures in the knot-link pattern.

B. Cyclic identity for Abelian and non-Abelian strings

In Secs. IV B and IV C, we discuss cyclic identity for Abelian and non-Abelian strings, particularly for 3 + 1D twisted gauge theories. We find Eq. (60), the “cyclic identity of the 2D $S^1_{x,y}$ matrix of Eq. (38), $S^1_{x,y} = \frac{1}{|G|} S^1_{a,b,d} S^1_{b,d,a} S^1_{a,d,b} \cdot |\dim(a)|^{-3} = 1.$

For the Abelian case, the dimension of Rep is simply $\dim(a) = 1$, which reduces to Eq. (41).

On the other hand, we find that there is another cyclic identity, based on the 2D $S^0_{x,y} = S^0_{a,b,c} = \frac{1}{|G|} S^0_{a,c,b} S^0_{a,b,c} = 1.$

This Eq. (62) cyclic identity has two additional criteria: (i) Here $a = b = 0$ means that all strings must have 0 charges; and (ii) in addition, the $\prod Z_{N_k}$ flux labels $a_i, b_j, c_k$ must satisfy $a_i = |a|\hat{e}_i, b_j = |b|\hat{e}_j, c_k = |c|\hat{e}_k$, as a multiple of a single-unit flux, each only carrying one $\prod Z_{N_k}$ flux. Note that $\hat{e}_j \equiv (0, \ldots, 0, 1, 0, \ldots, 0)$ is defined to be a unit vector with a nonzero component as the $j$th component for $Z_{N_k}$ flux. Equation (62) is true even in the noncanonical basis, such as the case for the $b$ flux sector in Table IV. Thus, whether or not it is in the canonical basis [61] does not affect the identity, Eq. (62), at least for the example of Abelian types II and III 4-cocycles.

This 2D $S^0_{x,y}$ cyclic identity in Eq. (62) is indeed the cyclic relation in Ref. [45]. The fact that we associate the 2D $S^0_{x,y}$ matrix with the dimensional reduction of string braiding in Fig. 2 shows that the Abelian statistical angle $\theta_{a,c,i,(b_j)}$ can be defined, up to a basis, as $S^0_{a,c,i,(b_j)} = \exp(\i \theta_{a,c,i,(b_j)}).$

Thus Eq. (62) implies a cyclic relation for Abelian statistical angles:

$$\theta_{a,c,i,(b_j)} + \theta_{a,b,(c_k)} = 0 \pmod{2\pi}.$$ (64)

In contrast, the 3D cyclic relation works for both Abelian and non-Abelian strings, and it is not restricted to zero charge but only to equal charges, $a = \beta$. More importantly, Eq. (61) allows any flux for each $a$, $b$, and $c$, instead of being limited to a single-unit flux or a multiple of a single-unit flux in Eq. (62).

C. Main results

We have studied string and particle excitations in 3 + 1D twisted discrete gauge theories, which belong to a class of topological orders. These 3D theories are gapped topological systems with topology-dependent GSD. The twisted gauge theory contains the data on gauge group $G$ and 4-cocycle twist $\omega_4 \in \mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$ of the fourth cohomology group of $G$. Such data provide many types of theories, however, several types of theories belong to the same class of a topological order. To better characterize and classify topological orders, we use the MCG on the $T^3$ torus, as $\text{MCG}(T^3) = \text{SL}(d, \mathbb{Z})$, to extract the $\text{SL}(3, \mathbb{Z})$ modular data $\text{S}^+$ in the ground-state sectors, which, however, reveal information on gapped excitations of particles and strings. We have posed five main questions [Q1–Q5] and other subquestions throughout this work and have addressed each of them in some depth. We summarize our results and approaches below and make comparisons with some recent work.

1. Dimensional reduction

By inserting a gauge flux $b$ into a compactified circle $\mathcal{z}$ of 3D topological order $C^{3D}$, we can realize $C^{3D} = \bigoplus_i C^{3D}_i$, where $C^{3D}_i$ becomes a direct sum of degenerate states of 2D topological orders $C^{2D}_i$ in different flux $b$ sectors. We should emphasize that this dimensional reduction is not real-space decomposition along the $\mathcal{z}$ direction, but decomposition in the Hilbert space of ground states [excitation basis such as the canonical basis of Eq. (11)]. We propose that this decomposition in Eq. (5) will work for a generic topological order without a gauge group description. In the most general case, $b$ becomes the certain basis label of the Hilbert space. The recent study in Ref. [39] implements the dimensional reduction idea on the normal gauge theories described by the 3D Kitaev $Z_N$ toric code and 3D quantum double models without cocycle twists using the spatial Hamiltonian approach. In our work, we consider more generic twisted gauge theories with a lattice realization in the twisted 3D quantum double models under the framework of Dijkgraaf-Witten theory [38]. We apply both the spatial Hamiltonian approach and the spacetime path integral approach.

2. Modular data

We find explicit formula representations of the $\text{SL}(3, \mathbb{Z})$ modular data $\text{S}$ and $\text{T}$ using (i) the path integral and cocycle approach and (ii) the representation) theory approach. The Rep theory approach is convenient and perhaps contains more general and simplified expressions. While recent work either focuses on Abelian statistics [45,46] or focuses on normal gauge theories [39], our formula embodies generic non-Abelian twisted gauge theories and thus is the most powerful.

3. Classification and characterization

We use the modular data $\text{S}$ and $\text{T}$ to characterize the braiding statistics of some 2D and 3D topological orders. We can further use the modular data $\text{S}$ and $\text{T}$ taking into account excitation relabeling to classify (or partially classify) topological orders. Explicit 2D examples are $G = (\mathbb{Z}_2)^3$.
twisted gauge theories, and 3D examples are \( G = (\mathbb{Z}_2)^3 \) twisted gauge theories. Some of our results are compared with the mathematics literature in Appendix A. Some 2D results are compared to twisted quantum double models \( D^\omega(G) \).

Our result can also facilitate the study of symmetric protected topological states protected by a global symmetry \( G \). [51] By gauging the \( G \), symmetry of symmetric protected topological states, one can use the induced dynamical gauge theory to study the braiding of excitations and to characterize symmetric protected topological states [45,63–65].

4. Physics of string and particle braiding

We provide the physics meaning of the topological spin and spin-statistics relation for a closed string. We also interpret the three-string braiding statistics first studied in Ref. [45] from a new perspective: a dimensional reduction with \( b \) flux monodromy. We find that with the type IV 4-cocycle twist for \( b \) \( \neq 0 \) sectors of \( G \), symmetry of symmetric protected topological states, one can use the induced dynamical gauge theory to study the braiding of excitations and to characterize symmetric protected topological states [45,63–65].

Table VIII. Braiding statistics, Abelian or non-Abelian, in terms of \( (G, \omega_b) \), gauge group \( G \), and cocycle twist \( \omega_b \) of 3D topological order \( C^\text{3D}_{G,\omega_b} \). \( G_{\text{Abel}} \), Abelian \( G \); \( G_{\text{non-Abel}} \), non-Abelian \( G \); \( \omega \), statistics. The normal gauge theory has \( \omega_b = 1 \) with no cocycle twist. (Non-)Abelian statistics: either non-Abelian or pure Abelian statistics. (For example, any \( b \neq 0 \) sector of an untwisted \( S_3 \) gauge theory has pure Abelian statistics, because \( S_3 \) centralizers of nonindensity elements are Abelian, but some \( b \neq 0 \) sectors of untwisted \( D_3 \) and \( Q_8 \) gauge theories have non-Abelian statistics.) \( b \neq 0 \) two-string 0\(_2\) braiding is the process in Fig. 9(a); \( b \neq 0 \) three-string 2\(_1\)\#2\(_2\) braiding is the process in Fig. 9(b).

<table>
<thead>
<tr>
<th>Braiding statistics ( (G, \omega_b) ) of ( C^\text{3D}_{G,\omega_b} )</th>
<th>( b = 0 ) braiding:</th>
<th>( b \neq 0 ) braiding:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_{\text{Abel}} )</td>
<td>Abelian states</td>
<td>Abelian states</td>
</tr>
<tr>
<td>( G_{\text{non-Abel}} )</td>
<td>Abelian states</td>
<td>Non-Abelian states</td>
</tr>
<tr>
<td>( G_{\text{Abel}} )</td>
<td>Abelian states</td>
<td>Non-Abelian states</td>
</tr>
<tr>
<td>( G_{\text{non-Abel}} )</td>
<td>Non-Abelian states</td>
<td>Non-Abelian states</td>
</tr>
</tbody>
</table>

ACKNOWLEDGMENTS

J.W. would like to thank Tian Lan, as well as Ling-Yan Hung and Yidun Wan, for helpful conversations and warm encouragement. J.W. is grateful to Louis H. Kauffman for a long blackboard discussion about the theory of knots and links, at Perimeter Institute for Theoretical Physics, along with Yidun Wan. J.W. also thanks M. de Wild Propitius for e-mail correspondence in early 2013 and for expressing his interest in our work. J.W. acknowledges the use of the computational resources Compute and Titan at Perimeter Institute for Theoretical Physics. We thank Zheng-Cheng Gu for collaboration on a related work [60]. This research was supported by NSF Grant Nos. DMR-1005541, NSFC 11074140, and NSFC 11274192, the BMO Financial Group, and the John Templeton Foundation. J.W. was supported in part by the U.S. Department of Energy under cooperative research agreement Contract No. DE-FG02-05ER41360 and by the Isaac Newton Chair Fellowship. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research, Economic Development & Innovation.

APPENDIX A: GROUP COHOMOLOGY AND COCYCLES

1. Cohomology group

Here we review the cohomology group \( \mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z}) = \mathcal{H}^{d+1}(G, U(1)) \) by \( \mathbb{R}/\mathbb{Z} = U(1) \), as the \((d+1)\)-th cohomology group of \( G \) over the \( U(1) \) module \( U(1) \). Each class in \( \mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z}) \) corresponds to a distinct \((d+1)\)-cocycle.

The \((d+1)\)-cocycle is a mapping \( \omega(A_1, A_2, \ldots, A_n) : G^n \rightarrow U(1) \) which inputs \( A_i \in G, i = 1, \ldots, n \), and outputs a \( U(1) \) phase. The \((d+1)\)-cochains satisfy the group multiplication rule,

\[
\omega_1 \cdot \omega_2(A_1, A_2, \ldots, A_n) = \omega_1(A_1, \ldots, A_n) \cdot \omega_2(A_1, \ldots, A_n),
\]

and thus form a group. The coboundary operator \( \delta \),

\[
\delta \epsilon(g_1, g_2, \ldots, g_{n+1}) = \epsilon(g_2, \ldots, g_n g_{n+1}) \left( \epsilon(g_1, \ldots, g_n) \right)^{-1} c_{n+1}
\]

defines the \( n \)-cocycle condition \( \delta \omega = 1 \) (a pentagon relation in two dimensions). We check that the distinct \( n \)-cocycles are not equivalent by \( n \)-coboundaries. The \( n \)-cocycle forms a group \( C^n \), the \( n \)-cocycle forms a subgroup \( Z^n \) of \( C^n \), and the \( n \)-coboundary further forms a subgroup \( B^n \) of \( Z^n \) (since \( \delta^2 = 1 \)). Overall, this shows \( B^n \subset Z^n \subset C^n \). The \( n \)-cohomology group is exactly the relation of a kernel \( Z^n \) (the group of \( n \)-cocycles) modding out an image \( B^n \) (the group of \( n \)-coboundaries):

\[
\mathcal{H}^n(G, U(1)) = Z^n/B^n.
\]

To derive the expression of \( \mathcal{H}^n(G, U(1)) \) in terms of groups explicitly, we apply some key formulas, as follows.
The number of exponents can be systematically obtained by adding all the numbers in the previous column from the top row to the row etc., also for other, higher gcd's. Our definition of type $m$ group cohomology

\[
\mathcal{H}^d(G, M) = M
\]

and

\[
\mathcal{H}^d(G, M) \cong \mathcal{H}^d(G, U(1)) = G
\]

thus only $H^0(X', M) = M$ survives:

\[
H^d(X', M) \cong \otimes_{i=0}^d H^k(X, M) \otimes Z^{d-k}(X', M')
\]

(A7)

Using the universal coefficient theorem, we can rewrite Eq. (A5) as a decomposition below.

c. Decomposition

\[
H^d(X \times X', M) \cong \otimes_{i=0}^d H^k(X, M) \otimes Z^{d-k}(X', M')
\]

(A8)

The above is valid for both topological cohomology and group cohomology.

\[
\mathcal{H}^d(G \times G', M) \cong \otimes_{i=0}^d \mathcal{H}^k(G, M) \otimes Z^{d-k}(G', M')
\]

(A9)

provided that both $G$ and $G'$ are finite groups.

The expression of the Künneth formula is in terms of the tensor-product operation $\otimes_R$ and the torsion-product operation $\otimes_R^T$ of a base ring $R$, which we write $\otimes_R \equiv \otimes_R^T$ in shorthand. Their properties are

\[
\begin{align*}
\mathbb{M} \otimes \mathbb{M}' & \cong \mathbb{M}' \otimes \mathbb{M}, \\
\mathbb{M} \otimes Z \mathbb{M} & \cong \mathbb{M} \otimes Z \mathbb{M} = \mathbb{M}, \\
Z_n \otimes Z M & \cong \mathbb{M} \otimes Z Z_n = M/nM, \\
Z_n \otimes Z U(1) & \cong U(1) \otimes Z Z_n = 0, \\
Z_m \otimes Z Z_n & = Z_{gcd(m,n)}, \\
(\mathbb{M}' \otimes \mathbb{M}'') & \equiv M \otimes R M' \equiv (M'' \otimes \mathbb{M}'') \otimes R M', \\
\mathbb{M} \otimes R (\mathbb{M}' \otimes \mathbb{M}'') & \equiv (\mathbb{M} \otimes R \mathbb{M}') \otimes (\mathbb{M}' \otimes R \mathbb{M}'')
\end{align*}
\]

(A10)

The universal coefficient theorem can be derived from the Künneth formula, Eq. (A5), by taking $X = 0$ or $Z_1$ or a point, provided that both $G$ and $G'$ are finite groups.

\[
\begin{align*}
H^d(X, M) & \cong \otimes_{i=0}^d H^k(X, M) \otimes Z^{d-k}(X', M') \\
\mathcal{H}^d(G \times G', M) & \equiv \otimes_{i=0}^d \mathcal{H}^k(G, M) \otimes Z^{d-k}(G', M') \equiv \mathbb{M}' \otimes \mathbb{M}''
\end{align*}
\]

The universal coefficient theorem can be derived from the Künneth formula, Eq. (A5), by taking $X = 0$ or $Z_1$ or a point, provided that both $G$ and $G'$ are finite groups.
and
\[ \text{Tor}^p(M, M') \equiv M \otimes_R M', \]
\[ M \otimes_R M' \simeq M' \otimes_R M, \]
\[ Z \otimes_Z M = M \otimes_Z Z = 0, \]
\[ Z_n \otimes_Z M = \{ m \in M | nm = 0 \}, \]
\[ Z_n \otimes_Z U(1) = Z_n, \]
\[ Z_m \otimes_Z Z_n = Z_{(m,n)}, \]
\[ \text{Tor}^2(U(1), U(1)) = 0, \]
\[ M' \otimes M'' \otimes_R M = M' \otimes_R M \otimes M'' \otimes_R M, \]
\[ M \otimes_R M' \otimes M'' = M \otimes_R M' \otimes M \otimes_R B. \]

For other details, we refer the reader to Ref. [51] and references therein.

We summarize some useful facts in Table IX and some derived results in Table X.

2. Derivation of cocycles

To derive Table X, we find that by carrying out the K"unneth formula decomposition carefully for a generic finite Abelian group \( G = \prod Z_{N_i} \), some corresponding structure becomes transparent. See Table XI.

From the known field theory facts, we know that for 2 + 1D twisted gauge theories from \( \mathcal{H}^3(G, U(1)) = \prod_{1 \leq i < j < k \leq m} Z_{N_{ij}} \times Z_{N_{jk}} \times Z_{N_{ki}} \), the \( Z_{N_{ij}} \) classes are captured by a path integral \( \propto \exp(i \int A_i dA_j) \) up to some normal-ization factor. (Here we omit the wedge product, denoting \( A_i dA_j \equiv A_i \wedge dA_j \).) We also schematically denote the quantization factor \( k \); the details of \( k \)-level quantizations are given in Ref. [60].) The \( Z_{N_{ij}} \) classes are captured by a path integral \( \propto \exp(i \int A_i dA_j) \), where \( A \) is a 1-form gauge field. We deduce that the K"unneth formula decomposition in \( \mathcal{H}^{d+1}(G, U(1)) \) with the torsion product \( \text{Tor}^p \equiv \otimes_R \) suggests a wedge product \( \wedge \) structure in the corresponding field theory, while the tensor product \( \otimes_Z \) suggests appending an extra exterior derivative \( \wedge d \) structure in the corresponding field theory.

Example, \( \mathcal{H}^3(Z_{N_{ij}}, U(1)) \otimes_Z \mathcal{H}^3(Z_{N_{jk}}, U(1)) \rightarrow \exp(i \int A_i \wedge A_j) \), and \( \mathcal{H}^3(Z_{N_{ij}}, U(1)) \rightarrow \exp(i \int A_i) \), thus \( \mathcal{H}^3(Z_{N_{ij}}, U(1)) \otimes_Z \mathcal{H}^3(Z_{N_{jk}}, U(1)) \rightarrow \exp(i \int A_i \wedge A_j) \). This organization also shows the corresponding form of cocycles for 3 + 1 dimensions in Table I and 2 + 1 dimensions in Table XII. For example, the relation \( A_i \rightarrow a_i \) maps a 1-form field to a gauge flux \( a_i \) (or a group element). The relation \( dA_2 \rightarrow (b_2 + c_2) \) maps an exterior derivative to the operation, taking on different edges/vertices in the spacetime complex. We use this fact to determine whether two cocycles are the same forms or whether they are up to coboundaries. We comment that such a path integral is only suggestive so far, not yet being strongly evident enough to formulate a consistent field theoretic path integral. Thus we label them with speculative quotation marks in path integral forms in “fields.” The more systematic formulation in terms of field theoretic partition functions will be reported in Ref. [60] from the perspective of symmetric protected topological states.

TABLE XII. The cohomology group \( \mathcal{H}^3(G, R/\mathbb{Z}) \) and 3-cocycle \( \omega_3 \) for a generic finite Abelian group \( G = \prod Z_{N_i} \). The first column lists the classes in \( \mathcal{H}^3(G, R/\mathbb{Z}) \). The second column lists the topological term indices for the 2 + 1D twisted gauge theory. (When indices \( k = 0 \), it becomes the normal untwisted gauge theory.) The third column lists explicit 3-cocycle functions \( \omega_3(a, b, c): (G)^3 \rightarrow U(1) \). Here \( a = (a_1, a_2, \ldots, a_n) \), with \( a \in G \) and \( a_i \in Z_{N_i} \). The same notation is used for \( b, c, \) and \( d \). The last column lists induced 2-cocycles from the slant product \( C_{a}(b, c) \) using Eq. (A13).

<table>
<thead>
<tr>
<th>( \mathcal{H}^3(G, R/\mathbb{Z}) )</th>
<th>3-cocycle name</th>
<th>3-cocycle form</th>
<th>Induced ( C_{a}(b, c) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z_{N_i} ) Type I, ( k_{(ij)} )</td>
<td>( \omega_3^{(ij)}(a, b, c) = \exp \left( \frac{2\pi i k_{ij}}{N_i} a_i (b_i + c_i - (b_i + c_i)) \right) )</td>
<td>( \exp \left( \frac{2\pi i k_{ij}}{N_i} a_i (b_i + c_i - (b_i + c_i)) \right) )</td>
<td></td>
</tr>
<tr>
<td>( Z_{N_{ij}} ) Type II, ( k_{(ij)} )</td>
<td>( \omega_3^{(ij)}(a, b, c) = \exp \left( \frac{2\pi i k_{ij}}{N_i} a_i (b_i + c_i - (b_i + c_i)) \right) )</td>
<td>( \exp \left( \frac{2\pi i k_{ij}}{N_i} a_i (b_i + c_i - (b_i + c_i)) \right) )</td>
<td></td>
</tr>
<tr>
<td>( Z_{N_{ij}} ) Type III, ( k_{(ij)} )</td>
<td>( \omega_3^{(ij)}(a, b, c) = \exp \left( \frac{2\pi i k_{ij}}{N_i} a_i (b_i + c_i) \right) )</td>
<td>( \exp \left( \frac{2\pi i k_{ij}}{N_i} a_i (b_i + c_i) \right) )</td>
<td></td>
</tr>
</tbody>
</table>
3. Dimensional reduction from a slant product

In general, for dimensional reduction of cocycles, we can use the slant product mapping $n$-cochain $c$ to $(n - 1)$-cochain $i_c c$:

\[
i_c c(g_1, g_2, \ldots, g_{n-1}) = c(g, g_1, g_2, \ldots, g_{n-1})^{-1} g \cdot \prod_{j=1}^{n-1} c(g_1, \ldots, g_j, (g_1 \ldots g_j)^{-1} \cdot g \cdot (g_1 \ldots g_j), \ldots, g_{n-1})^{-1}.\]

Here we focus on the Abelian group $G$. For example, in $2 + 1$ dimensions, we have 3-cocycle to 2-cocycle:

\[
C_A(B, C) \equiv i_A \omega(B, C) = \frac{\omega(A, B, C)\omega(B, C, A)}{\omega(B, A, C)}.
\]

In $3 + 1$ dimensions, we have 4-cocycle to 3-cocycle:

\[
C_A(B, C, D) \equiv i_A \omega(B, C, D) = \frac{\omega(A, B, C, D)\omega(B, C, D, A)}{\omega(A, B, C, D)\omega(B, C, A, D)}.
\]

In order to study the projective representation (the second cohomology group $H^2$) from 4-cocycles, we do the slant product again:

\[
C'_{AB}(C, D) \equiv i_B C_A(C, D) = \frac{C_{A}(B,C)C_D(C,D,B)}{C_A(B,C,D)},
\]

\[
C'_{AB}(C, D) = \frac{\omega(A, B, C, D)\omega(B, C, D, A)}{\omega(A, B, C, D)\omega(B, C, A, D)} \cdot \frac{\omega(C, A, D, B)\omega(C, B, A, D)}{\omega(C, A, D, B)\omega(C, B, A, D)}.
\]

4. 2 + 1D topological orders of $H^3(G, \mathbb{R}/\mathbb{Z})$

4.1 Three-cocycles

We study the 2D MCG of $\mathbb{SL}(2, \mathbb{Z})$ modular data $S$ and $T$ using the rep theory approach.

4.2 Projective Rep $S$ and $T$ for Abelian topological orders

This subsection simply reviews some known facts for later convenience in discussing new results. Much of the discussion can be absorbed from Refs. [40, 55, 50, 66]. First, we study the Abelian topological orders from types I and II 3-cocycles $\omega$ in Table XII for $2 + 1$D topological orders. We can determine the $C_a$ projective representation (Rep) and $\hat{\rho}^\sigma_a(b)$:

\[
\hat{\rho}^\sigma_a(b) \hat{\rho}^\sigma_a(c) = C_a(b,c)\hat{\rho}^\sigma_a(bc).
\]

Given that $Z_a$ is the centralizer of $a \in G$, $C_a$ determines the projective Rep of $Z_a$. Each $C_a$ classifies a class of projective Rep called $C_a$ representations, $\hat{\rho}^\sigma$ of $Z_a$. In types I and II $\omega_a$, the irreducible $C_a$ representations $\hat{\rho}^\sigma_a$ of $Z_a$ are in one-to-one correspondence with the irreducible linear representations. The linear Rep originating from the normal twisted $\prod Z_{a_i}$ gauge theory/toric code is $exp(2\pi i(\sum_{i=1}^N \alpha_i a_i, h_i))$. It has pure charge $\alpha_i$ and flux $(h_i)$ coupling formulated by a BF theory in any dimension (a mutual Chern-Simons theory in $2 + 1$ dimensions). The full $C_a$ representation is

\[
\hat{\rho}^\sigma_a(h) = \exp\left(2\pi i \left(\sum_{i=1}^N \frac{1}{N_i} \alpha_i h_i\right)\right) \exp\left(2\pi i \left(\frac{1}{N^2_i} \sum_{i,j} p_i g_i h_j\right)\right) \times \exp\left(2\pi i \left(\sum_{i,j} \frac{1}{N_i N_j} p_i g_i h_j\right)\right).
\]

We interpret $(\alpha_1, g_1, \alpha_2, g_2, \alpha_3, g_3)$ and $(\beta_1, h_1, \beta_2, h_2, \beta_3, h_3)$ as the charges $\alpha$ and $\beta$ and fluxes $a$ and $b$ of particles in a doubled basis, $\{\alpha, g\}$, $\{\beta, h\}$. The generic $S$-matrix T-matrix formula for modular SL$(2, \mathbb{Z})$ data is [40, 55]

\[
T_{(a, A)(b, B)} = T_{(a, A)} \delta_{a, B} \delta_{A, B} = Tr_{\hat{\rho}^\sigma_a(g^a)}(h^a),
\]

where $T_{(a, A)} = e^{i\alpha_a}$ describes the exchange statistics of two identical particles or the topological spin of the same particle. On the other hand, the generic $S$-matrix formula in $2 + 1$ dimensions reads [40, 55]

\[
S_{(a, A)(b, B)}(p_j, p_i) = \frac{1}{|G|} \sum_{gh = hg} Tr_{\hat{\rho}^\sigma_a(h')} Tr_{\hat{\rho}^\sigma_b(g')} (g^a)^{k_i},
\]

yielding

\[
S_{(a, A)(b, B)}(p_j, p_i) = \frac{1}{|G|} \exp\left(-2\pi i \left(\frac{1}{N_i} \sum_{i,j} p_i g_i (a_i b_j)\right)\right) + 2 \sum_{j=1,2,3} \frac{1}{N^2_i} p_j (a_i b_j)
\]

\[
+ \sum_{ij=12,13,13} \frac{1}{N_i N_j} p_{ij} (a_i b_j + b_i a_j).
\]

One can use the $K$-matrix Chern-Simons theory of an action $S = \frac{i}{4\pi} \int K_{ij} a_i \wedge d a_j$ to encode the information on $\{a, g\}$, $\{\beta, h\}$ into quasiparticle vectors $l$ and $l'$, respectively, and formulate a $K$ with $S_{l,l'}(p_j, p_i) = \frac{1}{|G|} \exp(-2\pi i l'^T K^{-1} l')$. We
can use $S$ and $T$ to study the classifications of classes of topological orders. For example, for $G = (Z_2)^2$ twisted theories, simply using $T$ under basis (particle) relabeling, we find that the diagonal eigenvalues of $T$ can be labeled $(N_1, N_{-1}, N_0, N_{-0})$, as numbers of eigenvalues for $T = 1, -1, i, -i$. We show that using the data in Table XIII is enough to match the classes found in Ref. [64]. We denote by $(n_{z1}, n_{\bar{z}1})$ the numbers for (the pair of $\pm z$, the pair of $\pm \bar{z}$, the individual 1). Note that $N_1 + N_{-1} + N_0 + N_{-0} = 2n_{z1} + 2n_{\bar{z}1} + n_1 = GSD_{T^2} = |G|^2$. There are eight types of 3-cocycles, but there are only four classes in Table XIII. The number in brackets following $\omega_1$ (first column) indicates the number of $+i$ (or, equivalently, the number of pairs of $\pm i$, paired due to the nature of the twisted quantum double model). As another example, for $G = (Z_2)^3$ twisted theories, we find that, in Table XIV, by classifying and identifying the modular $S$ and $T$ data, the 64 Abelian-type 3-cocycles (all with Abelian statistics) in $\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$ are truncated to only four classes.

c. Projective Rep and $S$ and $T$ for non-Abelian topological orders

For $2 + 1D$ $G = (Z_2)^3$ twisted gauge theories of $\mathcal{H}^3((Z_2)^3, \mathbb{R}/\mathbb{Z}) = (Z_2)^3$, with 128 types of theories, we have shown that the 64 types of theories with Abelian statistics (from 64 types of 3-cocycles without type III twist) are truncated to four classes in Table XIV. Here we consider the remaining 64 types of 3-cocycles with type III twist in $\mathcal{H}^3((Z_2)^3, \mathbb{R}/\mathbb{Z})$. Although the gauge group $G$ is Abelian, the type III cocycle twist promotes the theory to having non-Abelian statistics. Our basic knowledge and formalism are rooted in Ref. [40], where the dual $D_4$ and $Q_8$ gauge theories are found for certain type III twists. Here we generalize the results in Ref. [40] to all kinds of 3-cocycle twists.

Our expression is the generalized case where 3-cocycles are based on type III’s but can include (or not include) types I and II 3-cocycles. There are 8 Abelian charged particles with zero flux and 14 non-Abelian charged particles (whose projective Rep $\tilde{\rho}_a^G(b)$ is 2D, described by a rank 2 matrix) without nonzero fluxes as dyons. For $a, b, c \in G = (Z_2)^3$, we label eight elements in $G = (Z_2)^3$ by $(0,0,0)$, $(1,0,0)$ as $(0,1,0)$, $(0,0,1)$, $(1,0,1)$, $(0,1,1)$, $(1,1,1)$. We denote these eight elements $F(0)$, $(F(1))$, $(F(2))$, $(F(3))$, $(F(4))$, $(F(5))$, $(F(6))$, and $(F(7))$, respectively. Let us recall that $\tilde{\rho}_a^G(g_b)$ contains $\alpha$, meaning the representation as charges; also, $g_b$ means the flux, and $g_b$ indicates, in general, the conjugacy class (i.e., flux) as basis. In short, our notation leads to $\tilde{\rho}_a^G(g_b)$ = $\tilde{\text{conj}}$-class/(flux)$\times$(representation/charge)

(i) $1 \cdot 8 = 8$ particles; $F(0)$; $(a_1, a_2, a_3)$. When the flux is 0, $a = F(0)$ is the conjugacy class $C_{F(0)}$. There are eight linear irreducible representations as charges. These charges can be labeled $(a_1, a_2, a_3)$, with $(a_1, a_2, a_3) \in (Z_2)^3$, $a_1, a_2, a_3 \in \{0, 1\}$. So we have

\[ \tilde{\rho}^{F(0)}_{F(0), (a_1, a_2, a_3)}(b) = \tilde{\rho}^{F(0)}_{F(0), (a_1, a_2, a_3)}(b_1, b_2, b_3) = \exp \left( \frac{2\pi i}{m^2} \sum_{j=1,2,3} a_j b_j \right) . \]

(A23)

(ii) $7 \cdot 2 = 14$ particles; $F(j)$; $\pm$. The remaining seven kinds of fluxes are $a = F(j)$ for $j = 1, \ldots, 7$. There are two kinds of representations for each. We can denote these two representations as $+$ or $-$. So together these give 14 more types of particles. In total there are $1 \cdot 8 + 7 \cdot 2 = 22$ quasiparticle excitations as the GSD on the $T^2$ torus. Generally, the representation is $\tilde{\rho}^{F(j)}_{F(j), \pm}(F(l))$ for some inserting flux $F(l)$. This is a 2D representation. The identity is always assigned $F(0)$; namely, $\tilde{\rho}^{F(j)}_{F(j), \pm}(F(0)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We list three more elements: $\tilde{\rho}^{F(j)}_{F(j), \pm}(F(1))$, $\tilde{\rho}^{F(j)}_{F(j), \pm}(F(2))$, and $\tilde{\rho}^{F(j)}_{F(j), \pm}(F(3))$. The remaining $\tilde{\rho}^{F(j)}_{F(j), \pm}(F(l))$ for $l = 4, \ldots, 7$ can be determined by Eq. (A17). The representations are adjusted by a 1D projective Rep for type I $\alpha_1$ and type II $\alpha_2, \alpha_3$-cocycles, with topological level quantized coefficients $p_1$, $p_2$, $p_3$ for type I and $p_{12}$, $p_{13}$, $p_{23}$ for type II. Under types I and II twists, the type III Rep adjusts to

\[ \tilde{\rho}^{F(j), \pm}_{F(j), \pm}(b) \rightarrow \tilde{\rho}^{F(j)}_{F(j), \pm}(b) e^{i \frac{\pi}{2} \sum_{i,j=1}^{1,2,3} p_{ij} a_i b_j + p_{0i} a_i b_0} . \]

(A24)
TABLE XV. The modular $T_a^q$ matrix for 2D twisted $(Z_2)^3$ theories with non-Abelian statistics. All 64 non-Abelian theories in $H^q((Z_2)^3, \mathbb{R}/\mathbb{Z})$ are listed.

<table>
<thead>
<tr>
<th>Particle</th>
<th>$T_a^q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\alpha \sigma, \beta \lambda, \gamma \phi) \equiv (F(0))$</td>
<td>1</td>
</tr>
<tr>
<td>$(\pm, F(1))$, $(\pm, F(2))$, $(\pm, F(3))$</td>
<td>$\pm i \rho_{1}^{\pm}, \pm i \rho_{2}^{\pm}, \pm i \rho_{3}^{\pm}$</td>
</tr>
<tr>
<td>$(\pm, F(4))$, $(\pm, F(5))$, $(\pm, F(6))$</td>
<td>$\pm i \rho_{1}^{\pm} \rho_{2}^{\pm} \rho_{3}^{\pm}, \pm i \rho_{1}^{\pm} \rho_{2}^{\pm}, \pm i \rho_{1}^{\pm} \rho_{2}^{\pm}$</td>
</tr>
<tr>
<td>$(\pm, F(7))$</td>
<td>$\pm i \rho_{1}^{\pm} \rho_{2}^{\pm} \rho_{3}^{\pm}$</td>
</tr>
</tbody>
</table>

(v) Two particles: $F(3)$; $\pm j = 3$. Here $(\alpha \sigma, \beta \lambda, \gamma \phi) \equiv (F(3) = (0,0,1), \rho_{F(j),\pm}^{(F(3))} = (0,1)_{1}^{1} e^{i \frac{\pi}{2} (\rho_{1}^{\alpha} + \rho_{2}^{\beta} + \rho_{3}^{\gamma})}.$

(vi) Two particles: $F(4)$; $\pm j = 4$. Here $(\alpha \sigma, \beta \lambda, \gamma \phi) \equiv (F(4) = (1,1,0), \rho_{F(j),\pm}^{(F(4))} = (0,1)_{1}^{1} e^{i \frac{\pi}{2} (\rho_{1}^{\alpha} + \rho_{2}^{\beta} + \rho_{3}^{\gamma})}.$

(vii) Two particles: $F(5)$; $\pm j = 5$. Here $(\alpha \sigma, \beta \lambda, \gamma \phi) \equiv (F(5) = (1,0,1), \rho_{F(j),\pm}^{(F(5))} = (0,1)_{1}^{1} e^{i \frac{\pi}{2} (\rho_{1}^{\alpha} + \rho_{2}^{\beta} + \rho_{3}^{\gamma})}.$

(viii) Two particles: $F(6)$; $\pm j = 6$. Here $(\alpha \sigma, \beta \lambda, \gamma \phi) \equiv (F(6) = (0,1,1), \rho_{F(j),\pm}^{(F(6))} = (0,1)_{1}^{1} e^{i \frac{\pi}{2} (\rho_{1}^{\alpha} + \rho_{2}^{\beta} + \rho_{3}^{\gamma})}.$

(ix) Two particles: $F(7)$; $\pm j = 7$. Here $(\alpha \sigma, \beta \lambda, \gamma \phi) \equiv (F(7) = (1,1,1), \rho_{F(j),\pm}^{(F(7))} = (0,1)_{1}^{1} e^{i \frac{\pi}{2} (\rho_{1}^{\alpha} + \rho_{2}^{\beta} + \rho_{3}^{\gamma})}.$

With the above projective Rep $\rho_{F(j),\pm}^{(F(b))}$, we can derive the analytic form of the modular data $S$ and $T$ in two dimensions. Here for $G = \langle (Z_2)^3 \rangle$,

\[ T_a^q = e^{i \frac{\pi}{2} \sum_{\rho_{1}, \rho_{2}, \rho_{3}} A_{a}^{\rho_{1}, \rho_{2}, \rho_{3}} (\pm)_{a}^{\rho_{1}, \rho_{2}, \rho_{3}} \rightarrow T_a^q = \pm 1 \text{ or } \pm i; \quad (A25) \]

\[ \eta_{b_1,b_2} = \left\{ \begin{array}{ll} 1 & \text{if } C_{b_1}(g_1,g_2) = +1, \\ 1 & \text{if } C_{b_1}(g_1,g_2) = -1. \end{array} \right. \quad (A26) \]

More explicitly, we compute $T_a^q$ in Table XV.

With the modular $S^{(\alpha \sigma, \beta \lambda, \gamma \phi)}$ matrix [of 64 types of 2D twisted $(Z_2)^3$ theories with non-Abelian statistics],

\[ S = \frac{1}{(G^2)} \begin{pmatrix} 1 & 2(-1)^{\beta_{a_1}+\beta_{a_2}+\beta_{a_3}} \delta_{b_1,b_4}(\cdot)(\cdot)_{a}^{\rho_{1},\rho_{2},\rho_{3}} & 2(-1)^{\beta_{a_1}+\beta_{a_2}+\beta_{a_3}} \delta_{b_1,b_4}(\cdot)(\cdot)_{a}^{\rho_{1},\rho_{2},\rho_{3}} \\ 2(-1)^{\beta_{a_1}+\beta_{a_2}+\beta_{a_3}} \delta_{b_1,b_4}(\cdot)(\cdot)_{a}^{\rho_{1},\rho_{2},\rho_{3}} & \sum_{j,k,l} (\cdot)(\cdot)_{a}^{\rho_{1},\rho_{2},\rho_{3}} & \sum_{j,k,l} (\cdot)(\cdot)_{a}^{\rho_{1},\rho_{2},\rho_{3}} \\ 2(-1)^{\beta_{a_1}+\beta_{a_2}+\beta_{a_3}} \delta_{b_1,b_4}(\cdot)(\cdot)_{a}^{\rho_{1},\rho_{2},\rho_{3}} & \sum_{j,k,l} (\cdot)(\cdot)_{a}^{\rho_{1},\rho_{2},\rho_{3}} & \sum_{j,k,l} (\cdot)(\cdot)_{a}^{\rho_{1},\rho_{2},\rho_{3}} \end{pmatrix} \quad (A27) \]

In Eq. (A27), the factor $(-1)^{\beta_{a_1}+\beta_{a_2}+\beta_{a_3}}$ is derived from a computation of $(i)^{\rho_{1},\rho_{1}} \cdot (i)^{\rho_{2},\rho_{3}} \delta_{b_1,b_4} = (-1)^{\rho_{1},\rho_{2}} \cdot \delta_{b_1,b_4}$. From Eq. (A26), we note that $\eta_{a,a} = 1$ is nonzero only when $a = (1,1,1) = F(7)$ for the $(Z_2)^3$ flux.

5. Classification of $2 + 1$D twisted $(Z_2)^3$ gauge theories, $D^q((Z_2)^3)$ and $H^q((Z_2)^3, \mathbb{R}/\mathbb{Z})$

The twisted $(Z_2)^3$ gauge theories dual to $D_4$, $Q_8$ non-Abelian gauge theories were first discovered in Ref. [40].

Here we present the three other classes which cannot be dual to any non-Abelian gauge theory, but can only be twisted (Abelian or non-Abelian) gauge theories themselves. We again label the diagonal eigenvalues of $T$ as $(N_1,N_{-1},N_0,N_{-i})$ and their number of eigenvalues as $T = 1, -1, i, -i$. We also use shorthand $\eta_{a_1,a_2,a_3}$ instead, which stands for the numbers for (the pair of $\pm i$, the pair of $\pm 1$, the individual 1) in the diagonal of $T$. Note that $N_1 + N_{-1} + N_0 + N_{-i} = 2n_{\pm 1} + 2n_{\pm i} + n_1 = \text{GSD}^2 = 22$. There are 64 types of 3-cocycles corresponding to theories with non-Abelian statistics, but there
are only 5 inequivalent classes in Table XIII. The number in brackets following $\omega_0$ (first column) indicates the number of +1 (or, equivalently, the number of pairs ±i, paired due to the nature of the quantum double model).

Although $\omega_0[3d]$ and $\omega_0[3i]$ share the same T-matrix data, they can still be distinguished by the linear dependency of the fluxes which carry three pairs of eigenvalues $i$. (And, of course, they can be distinguished by the more involved $S$ matrix.) There are 7 types in the $\omega_0[3d]$ class, whose ±i are generated by linear-dependent fluxes, and another 28 types in the $\omega_0[3i]$ class, whose ±i are generated by linear-independent fluxes. In this notation of linear (in)dependence, we have $\omega_0[1] = \omega_0[11]$, $\omega_0[5] = \omega_0[5d]$, $\omega_0[7] = \omega_0[7d]$. Such a concept is also used in the mathematics literature in Ref. [62], where the authors study the Frobenius-Schur indicators, Frobenius-Schur exponents, and support of cocycle twist, supp $\omega$, and use these data to classify the twisted quantum double model $D^\omega(G)$. Remarkably, we find that using our data is enough to match the classes found in the mathematics literature [62] in the quantum double and module category framework.

These findings, together with Appendix A4b, form a complete data set of $\mathcal{H}^3(G,\mathbb{Z}/2\mathbb{Z}) = (Z_2)^7$, where 128 types of 3-cocycles fall into four distinct classes of Abelian topological orders in Table XIII and five distinct classes of non-Abelian topological orders in Table XVI. In total there are nine distinct classes of topological orders within twisted $(Z_2)^3$ gauge theories. We note that $\omega_0[3i], \omega_0[3j],$ and $\omega_0[7]$ can only be twisted gauge theories, not dual to any untwisted non-Abelian gauge theory.

### 6. 3 + 1D topological orders of $\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$

This subsection continues the discussion and notations from $\mathcal{H}^3(G,\mathbb{R}/\mathbb{Z})$ of 2 + 1D to $\mathcal{H}^4(G,\mathbb{R}/\mathbb{Z})$ of 3 + 1D topological orders. Now we fill in some more information about the data on the projective Rep.

#### a. Projective Rep and $S$ and $T$ for Abelian topological orders

The data of $\rho_{a,b}^\omega(c)$ is organized below in Table XVII for $G = Z_{N_1} \times Z_{N_2} \times Z_{N_3}$. The modular $S$ and $T$ matrices for this Rep are presented in Tables II-IV. In the text, we provide an example of classifying 3D topological orders from 3 + 1D $(Z_2)^2$ twisted gauge theories of four types from $\mathcal{H}^3((Z_2)^2,\mathbb{R}/\mathbb{Z}) = (Z_2)^7$ and find out that the four types are truncated to only two distinct classes of topological orders.

#### b. Projective Rep and $S$ and $T$ for non-Abelian topological orders

Below we present the data on twisted gauge theories for those with non-Abelian statistics in $\mathcal{H}^4(G = (Z_2)^3,\mathbb{R}/\mathbb{Z})$ labeled by 4-cocycles $\omega_4$. Among $\mathcal{H}^4((Z_2)^3,\mathbb{R}/\mathbb{Z}) = (Z_2)^{21}$ types of theories, there are $2^{20}$ types that show non-Abelian statistics. In some cases, we write the formula in terms of a slightly generic $G = (Z_2)^n$, for a prime $n$.

Analogously to Appendix A4c, we recall that the 3D triple basis renders $\rho_{a,b}^\omega(g^i) = \rho_{\text{representation}}(\text{charge})$. So we understand that the representation $\rho(c)$ is constrained by the flux $a, b$. We consider type IV $\omega_{4,IV}$ twisted theories, but we include $\omega_{4,IV}$ further multiplied by type II $\omega_{4,II}$ and type III $\omega_{4,III}$ 4-cocycles. Thus, the representation also relates to their topological terms $p_{\text{flux}}$ of type II $\omega_{4,II}$ labeling $(Z_2)^2\mathbb{C}$ and $(Z_2)^{12}$ types of theories and $p_{\text{flux}}$ of type III $\omega_{4,III}$ labeling $(Z_2)^3\mathbb{C}$ and $(Z_2)^{12}$ types of theories. In total, all these 4-cocycles multiplied by $\omega_{4,IV}$ yield the $2^{20}$ types of theories showing non-Abelian statistics. Under types II and III twists, the type IV Rep is adjusted to $\rho_{a,b}^\omega(\pm \mathbb{C}) = \rho_{\text{Rep}}(\pm \mathbb{C}) = \rho_{\text{Rep}}(\pm \mathbb{C})$.

Note that the trace $\text{Tr}[\rho_{a,b}^\omega(\pm \mathbb{C})]$ is nonzero only when (i) $c = a$, $c = b$, or $c = ab$, with $\text{Tr}[\rho_{a,b}^\omega(\pm \mathbb{C})] \neq 0$, or (ii) $c = F(0) \text{ zero flux}$, i.e., $\text{Tr}[\rho_{a,b}^\omega(\pm \mathbb{C})] \neq 0$. Other cases have zero traces. Among the degeneracy sectors on the $\mathbb{T}^3$ torus, there are $256$ types, which among the $256$ configurations, there are $256$ types and those configurations are $256$ types and those configurations are $256$ types.

There are $(n^4 + n^5 - n)$ independent solutions for these sets of $a, b$. The conjugacy class $F(j_{ab})$ stands for fluxes. There are $n^4$ representations as charges; these can be labeled $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, with $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in $(Z_2)^4$, and $Z_2 = \{0, 1\}$. We write $\rho_{\text{Rep}}(\pm \mathbb{C})$. Equation (A28) becomes

\[
\rho_{\text{Rep}}(\pm \mathbb{C}) = \rho_{\text{Rep}}(\pm \mathbb{C}) = \rho_{\text{Rep}}(\pm \mathbb{C}) = \exp \left( \sum_{k=1}^{4} \frac{2\pi i}{N_k} \alpha_k \epsilon_k \right). \tag{A30}
\]

For $n = 2$, there are $(2^5 + 2^2 - 2) = 46$ (doubled) fluxes contributing Abelian excitations.

(ii) $(n^5 - (n^4 + n^5 - n)) = n^5 = 210 \times 4 = 840$ non-Abelian excitations. We denote $\rho_{\text{Rep}}(\pm \mathbb{C})$. For $n = 2$, there are $(n^5 - (n^4 + n^5 - n)) = 210$ (doubled) fluxes contributing non-Abelian excitations.
There are six $c_i d_m$ terms in type IV 3-cocycles:

\[
C^{(2)}_{a,b}(c,d) = \exp \left( \frac{2\pi i \rho^g_{1234}(a_0 b_3 - a_3 b_4) c_1 d_2}{N_{i,j,m}} \right) \\
+ \left( a_0 b_4 - a_2 b_2 \right) c_1 d_3 \\
+ \left( a_0 b_1 - a_1 b_3 \right) c_2 d_3 \\
+ \left( a_1 b_3 - a_3 b_1 \right) c_4 d_4 \\
+ \left( a_2 b_1 - a_1 b_2 \right) c_3 d_4.
\]  

(A31)

Below, each solution is multiplied by 6; due to $\binom{6}{2} = 30$, three terms, $a$, $b$, and $cb$, can choose 2 as the generator basis for $a$ and $b$. These terms have $\text{Tr} \left[ \rho^g_{ab,cb,k} \left( \chi \right) \right] \neq 0$ for $c = 0, a, b, ab$. And the permutation of $a, b$ results in an extra multiple of 2. We organize the solutions into the following six classes. Each style may contain dimensionally reduced 3-cocycles, as "type III 3-cocycle-like" or "mixed type III 3-cocycles." Here "type III 3-cocycle-like" means that the dimensionally reduced 2D theory has an induced 3-cocycle which is a type III 3-cocycle within a subgroup $(Z_2)^3$. "Mixed type III 3-cocycle" means that the dimensionally reduced 2D theory has an induced 3-cocycle which contains several type III 3-cocycles spanning the full group $(Z_2)^3$. The six styles of solutions are as follows.

**Style 1 (type III 3-cocycle-like).** $C^{(2)}_{a,b}(c,d)$ contains one $cd$ term: $\binom{6}{1} \times 6 = 36$ non-Abelian fluxes.

**Style 2 (type III 3-cocycle-like).** $C^{(2)}_{a,b}(c,d)$ contains two $cd$ terms: $\binom{6}{2} \times 3 \times 6 = 72$ non-Abelian fluxes. We have $\binom{6}{2}$ minus 3, because it is impossible to have nonzero coefficient $cd$ terms of $C^{(2)}_{a,b}(c,d)$ for any of the following terms:

**TABLE XVII.** $\rho^g_{a,b}(c)$ for a 3+1D twisted gauge theory with $G = Z_{N_1} \times Z_{N_2} \times Z_{N_3}$, of $\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$. We derive $\rho^g_{a,b}(c)$ from the equation introduced in the text. $\rho^g_{a,b}(c) \rho^{h,b}(d) = C^{(2)}_{a,b}(c,d) \rho^{h,b}(cd)$, presenting the projective representation, because the induced 2-cocycle belongs to the second cohomology group, $\mathcal{H}^2(G, \mathbb{R}/\mathbb{Z})$,

\[
\rho^g_{a,b}(c) = \exp \left( \sum_{i=1}^3 a_i c_i \right) \cdot \exp \left( \sum_{i=1}^3 a_i b_i d_i \right) \\
\rho^{h,b}(d) = \exp \left( \sum_{i=1}^3 b_i d_i \right)
\]
Styles 1–3 are pure type III 3-cocycle $\omega_3$-like, which $\tilde{\rho}_{a,b,(\pm,\pm,\pm)}(c)$ can be deduced from the result $G = (Z_2)^2$ in Appendix A 4 c. Styles 4–6 are mixed type III 3-cocycles in the whole $G = (Z_2)^4$ group, so one needs to assign the $\text{Rep}_{\tilde{\rho}_{a,b,(\pm,\pm,\pm)}}(c)$ in a slightly different manner. But it turns out that rank 2 matrices are always sufficient to encode the irreducible projective representation of $C_{ad}(c,d)$. After finding the $\tilde{\rho}_{a,b,(\pm,\pm,\pm)}(c)$, we analytically derive their 3D non-Abelian $S_{xyz}$ and $T^y$ presented in the text, in Table V, Eq. (51), and Eq. (52).

APPENDIX B: $S^{xyz}$ AND $T^y$ CALCULATION IN TERMS OF THE GAUGE GROUP $G$ AND 4-COCYCLE $\omega_4$

1. Unimodular group and $\text{SL}(N,\mathbb{Z})$

In the case of the unimodular group, there are unimodular matrices of rank $N$ forms $\text{GL}(N,\mathbb{Z})$, $S_U$ and $T_U$ have determinants $\det(S_U) = -1$ and $\det(T_U) = 1$ for any general $N$:

$$S_U = \begin{pmatrix} 0 & 0 & \cdots & (-1)^{N-1} & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \end{pmatrix},$$

(B1)

$$T_U = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \end{pmatrix}.$$  

Note that $\det(S_U) = -1$ in order to generate both determinant 1 and determinant $-1$ matrices.

For the $\text{SL}(N,\mathbb{Z})$ modular transformation, we denote their generators $S$ and $T$ for a general $N$ with $\det(S) = \det(T) = 1$:

$$S = \begin{pmatrix} 0 & 0 & \cdots & (-1)^{N-1} & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \end{pmatrix},$$

(B3)

$$T = T_U.$$  

(B4)

Here for simplicity, let us denote $S^{xyz}$ as $S_{3D}$, $S^y$ as $S_{2D}$, and $T^{xyz} = T_{3D} = T_{2D}$. Recall that $\text{SL}(3,\mathbb{Z})$ is fully generated by generators $S_{3D}$ and $T_{3D}$:

$$S_{3D} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \end{pmatrix},$$

$$T_{3D} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \end{pmatrix}.$$  

(B5)

By dimensional reduction (note $T_{2D} = T_{3D}$), we expect that

$$S_{2D}^4 = (S_{2D}T_{3D})^2 = 1,$$  

(B6)

$$S_{2D}T_{3D} = e^{2\pi i c} S_{2D}^2 = e^ {2\pi i c} C,$$  

(B7)

where $c$ carries the information on central charges. We can write

$$R = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ \end{pmatrix} = (T_{3D}S_{3D})^2T_{3D}^2S_{3D}^2T_{3D}S_{3D}T_{3D}S_{3D}T_{3D}S_{3D}.$$  

(B8)

One can check that

$$S_{3D}S_{3D}^\dagger = S_{3D}^2 = R^6 = (S_{3D}R)^4 = (RS_{3D})^4 = 1,$$  

(B9)

$$(S_{3D}R^2)^4 = (S_{3D}R^2)^2 = (R^2S_{3D}R^2)^2 = (R^3S_{3D})^3 = 1,$$  

(B10)

$$(S_{3D}R^2S_{3D})^2R^2 = R^2(S_{3D}R^2S_{3D})^2 = (mod 3).$$  

(B11)

Such expressions are known in the mathematic literature; some of them are listed in Ref. [37].

2. Rules for the path integral for the spacetime complex of cocycles

For the branching of a spacetime complex or a simplex, we define that, for any arrow that goes from a small number to a large number, the number ordering is $1 < 2 < 3 < 4 < \cdots < 0' < 1' < 2' < 2'' < 3' < 4' < 5' < 6' < 6'' < \cdots$. The time evolves along the fourth direction from the left to the right, or from a smaller number to a larger number. Also, we may write $[01],[12] = [02]$ or, equivalently, $g_{01}g_{12} = g_{02}$. If $[01] = g$ and $[12] = h$, then $[02] = gh$.

FIG. 12. (Color online) spacetime complex $T^y \times I$, where $I = [0,1]$ is the time direction, $T^y \times [0]$ and $T^y \times [1]$ are shown. Gray (blue) lines illustrate how the two $T^y$’s are connected for $t \in (0,1)$. Note that the two $T^y$’s differ by a rotation $S^{xyz}$. In other words, when time forms a loop, the two $T^y$’s are glued together as $1 \rightarrow 1', 2 \rightarrow 2', 3 \rightarrow 3', 4 \rightarrow 4', 5 \rightarrow 5', 6 \rightarrow 6', 7 \rightarrow 7', and 8 \rightarrow 8'$.  

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and the total amplitude for $M$ is

$$A = A_1 A_2.$$  \hspace{1cm} (B16)
Similarly, for $M_2$, we find that

$$A_2 = A'_2 A''_2,$$  \hspace{1cm} (B17)

where $A'_2$ is the amplitude for $M'_2$ (see Fig. 20),

$$A'_2 = \frac{\omega_4(g_{35}, g_{56}, g_{67}, g_{72}) \omega_2(g_{67}, g_{72}, g_{23}, g_{37})}{\omega_4(g_{35}, g_{56}, g_{67}, g_{72}) \omega_2(g_{67}, g_{72}, g_{23}, g_{37})} \times \frac{\omega_2(g_{46}, g_{67}, g_{72}, g_{26}) \omega_2(g_{72}, g_{24}, g_{46}, g_{67})}{\omega_2(g_{46}, g_{67}, g_{72}, g_{26}) \omega_2(g_{72}, g_{24}, g_{46}, g_{67})}$$  \hspace{1cm} (B18)

and $A''_2$ is the amplitude for $M''_2$ (see Fig. 21),

$$A''_2 = \omega_4(g_{23}, g_{34}, g_{46}, g_{67} g_{72}).$$  \hspace{1cm} (B19)

Here $g_{ij}$ is the group element on edge $(ij)$. We have

$g_{12} = g_{34} = g_{56} = g_{78} = g_z,$
$g_{13} = g_{24} = g_{57} = g_{68} = g_y,$
$g_{15} = g_{26} = g_{37} = g_{48} = g_z,$
$g_{23} = g_{67} = g_{34}^{-1} g_z,$
$g_{35} = g_{46} = g_{8y}^{-1} g_z,$
$g_{25} = g_{47} = g_{8y}^{-1} g_z,$
$g_{36} = g_{48} g_z.$  \hspace{1cm} (B20)

$$h_{12} = h_{34} = h_{56} = h_{78} = h_z,$$
$$h_{13} = h_{24} = h_{57} = h_{68} = h_y,$$
$$h_{15} = h_{26} = h_{37} = h_{48} = h_z,$$
$$h_{23} = h_{67} = h_y^{-1} h_z,$$
$$h_{35} = h_{46} = h_z^{-1} h_y,$$
$$h_{25} = h_{47} = h_y^{-1} h_z,$$
$$h_{36} = h_z^{-1} h_y h_z;$$  \hspace{1cm} (B21)

$g_{51} = g_{z}^{-1} w,$
$g_{62} = g_{z}^{-1} g_{x}^{-1} g_{w},$
$g_{84} = w h_{z}^{-1}.$

$g_{66} = g_{72} = g_{86} = w h_{y}^{-1}.$  \hspace{1cm} (B22)
Also, if the following conditions are not satisfied, the amplitude $A_{xyz}^{g_x, g_y, g_z; h_x, h_y, h_z; w}$ will be 0:

\[
g_x w = wh_z, \quad g_y w = wh_z, \quad g_z w = wh_x,
\]

\[
g_x g_y = g_y g_x, \quad g_y g_z = g_z g_y, \quad g_z g_x = g_x g_z,
\]

\[
h_x h_y = h_y h_x, \quad h_y h_z = h_z h_y, \quad h_z h_x = h_x h_z.
\]

(B23)

Note that the above has $g_x, g_y, g_z$ commute due to the identification on a $\mathbb{T}^3$ torus.

4. Explicit expression of $T^{xy}$ in terms of $(G, \omega_4)$

Similarly to $S^{xy}$, we can triangulate $T^{xy}$ on $\mathbb{T}^3 \times I$. It is easier to start with a $T^{xy}$ on $\mathbb{T}^2 \times I$ for two dimensions, which we denote $T_{2D}(w)$ and triangulate in the following $3! + 1 = 7$ tetrahedra (3-simplex). Here we have the vertex ordering for the arrows: $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 1' < 2' < 2'' < 3' < 5' < 6' < 6'' < 7'$.

\[
T_{2D}(w) = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 2 & 3' & 4 \\
1 & 2' & 3' & 4 \\
1 & 2' & 3 & 4
\end{array}
\]

The last extra piece is required to change the branching structure of the 3-simplex due to $T^{xy}$ transformation.

For $T_{3D}(w)$, we simply have seven pieces of slant products. Each slant product contains four 4-simplexes. So in total there are 28 pieces of 4-cocycles in $T_{3D}(w)$.

\[
T_{3D}(w) = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 \\
1 & 2 & 3' & 4 \\
1 & 2 & 3' & 4
\end{array}
\]

The constraints given by $T(w)$ are

\[
w^{-1} g_x w = h_x, \quad (B26)
\]

\[
w^{-1} g_y g_x w = h_y, \quad (B27)
\]

\[
w^{-1} g_z w = h_z. \quad (B28)
\]

Below we explicitly write seven $T_i$ values, where we omit the “w” arrow and do not draw it, which shall connect from the left 3-simplex to the right 3-simplex.

\[
(T_1) = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 \\
1 & 2' & 3' & 4 \\
1 & 2' & 3 & 4
\end{array} = \omega_4([12],[23],[35],[51']) \cdot \omega_4([23],[35],[56],[61']) \cdot \omega_4([35],[56],[67],[71']) \cdot \omega_4^{-1}(156,67,71',151'). \quad (B29)
\]
\[ (T_2) = \begin{array}{c}
\end{array} \]

\[ = \omega_2^{-1}([23],[36],[61^2],[1'2]) \cdot \omega_4([36],[67],[71^2],[1^2]) \cdot \omega_4^{-1}([67],[71^2],[1^2],[2^5]) \]
\[ \cdot \omega_4([67],[72^2],[2^5],[5^6]). \]  

(B30)

\[ (T_3) = \begin{array}{c}
\end{array} \]

\[ = \omega_4([37],[71^2],[1^2],[2^{2''}]) \cdot \omega_4^{-1}([71^2],[1^2],[2^{2''}],[2^{2''}]) \cdot \omega_4^{-1}([72^2],[2^{2''}],[2^{2''}],[5^6]) \]
\[ \cdot \omega_4^{-1}([72^2],[2^{2''}],[5^6],[6^6]). \]  

(B31)

\[ (T_4) = \begin{array}{c}
\end{array} \]

\[ = \omega_4^{-1}([23],[34],[46],[62^2]) \cdot \omega_4^{-1}([34],[46],[67],[72^2]) \cdot \omega_4^{-1}([46],[67],[78],[82^1]) \]
\[ \cdot \omega_4([67],[78],[82^2],[2^6]). \]  

(B32)

\[ (T_5) = \begin{array}{c}
\end{array} \]

\[ = \omega_4([34],[47],[72^2],[2^{2''}]) \cdot \omega_4^{-1}([47],[78],[82^2],[2^{2''}]) \]
\[ \cdot \omega_4([78],[82^2],[2^{2''}],[2^{4'}],[6^6]). \]  

(B33)

\[ (T_6) = \begin{array}{c}
\end{array} \]

\[ = \omega_4^{-1}([48],[82^2],[2^{2''}],[2^{4'}],[3^6]) \cdot \omega_4([82^2],[2^{2''}],[2^{4'}],[3^6]) \]
\[ \cdot \omega_4([82^2],[2^{4'}],[3^6],[6^6]). \]  

(B34)

For the tricky \( T_7 \), we shift \( 1' \) to a new later time slice, \( 1'' \), and shift \( 5' \) to a new later time slice, \( 5'' \):

\[ (T_7) = \begin{array}{c}
\end{array} \]

\[ = \omega_4^{-1}([1^2],[2^{2''}],[2^{4'}],[3^5]) \cdot \omega_4([2^{2''}],[2^{4'}],[3^5],[5^6]) \]
\[ \cdot \omega_4^{-1}([2^{4'}],[3^5],[5^6],[6^6]). \]  

(B35)

One can also define the projection operator on \( T^3 \) as

\[ P_{3D(w)} = (T_1)(T_2)(T_3)(T_4)(T_5)(T_6). \]  

(B36)

Once we have obtained the path integral of 4-cocycles, we can change the flux basis to the canonical basis and follow the procedure outlined in the Appendix of Ref. [55] to derive the Rep theory formula given in Sec. III B. One additional remark: An easier way to check the consistency of formulas for \( S \) and

\[ \sum_{b,d} \frac{1}{|G|} \text{tr} \rho_{b}^{a,a} (a) \text{tr} \rho_{b}^{a,a} (e) = \delta_{a,e}. \]  

(B37)

\[ \sum_{a,b,d} \frac{1}{|G|} \text{tr} \rho_{a}^{d,b} (d) \text{tr} \rho_{a}^{d,b} (d) = \delta_{a,y}. \]  

(B38)
Using the properties of $C_{a,b}^{(2)}(c,d)$ and the canonical basis \( \alpha, \beta \), we can justify that our formulas satisfy the rules (up to some projective representation's complex phases). See also Ref. [67] for the derivation.

Note Added in Proof. At the “Symmetry in Topological Phases” workshop at Princeton University, we became aware that the authors of Ref. [45] were working on the braiding statistics of 3 + 1D gapped phases; their studies intersect some of ours, but also further inspire our work. During the long process of preparing our manuscript, two works appeared (Refs. [45] and [46]) dealing with the Abelian braiding statistics of twisted gauge theories, as well as a preprint (Ref. [68]) considering the surface topological order of symmetric protected topological states with loop braiding statistics.

[56] See Appendix for further details, explicit examples and calculations.